The Long-Moody construction and polynomial functors

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Abstract

In 1994, Long and Moody gave a construction on representations of braid groups which associates a representation of $\mathbf{B}_n$ with a representation of $\mathbf{B}_{n+1}$. In this paper, we prove that this construction is functorial: it gives an endofunctor, called the Long-Moody functor, between the category of functors from the homogeneous category associated with the braid groupoid to a module category. Then we study the effect of the Long-Moody functor on strong polynomial functors: we prove that it increases by one the degree of (very) strong polynomiality under an extra assumption.

Introduction

Linear representations of Artin braid group on n strands $\mathbf{B}_n$ is a rich subject which appears in diverse contexts in mathematics (see for example [5] or [20] for an overview). Even if, at the present time, a complete classification of these representations is probably out of reach, any new result which would allow us to gain a better understanding of them would be a useful contribution.

In 1994, in a joint work with Moody (see [18]), Long gave a method to construct from a linear representation $\rho : \mathbf{B}_{n+1} \longrightarrow GL(V)$ a new linear representation $lm(\rho) : \mathbf{B}_n \longrightarrow GL(V^{\oplus n})$ of $\mathbf{B}_n$. Moreover, the construction complexifies in a sense the initial representation. For example, applying it to a one dimensional representation of $\mathbf{B}_{n+1}$, the construction gives a mild variation of the unreduced Burau representation of $\mathbf{B}_n$. This method was in fact already implicitly present in two previous articles of Long dated 1989 (see [16, 17]). In the article [2] dated 2008, Bigelow and Tian are interested in the Long-Moody construction from a purely matricial point of view. They give alternative and purely algebraic proofs of some results of [18], and they slightly extend some of them. In addition, in a survey on braid groups (see the Open Problem 7 in [5]), Birman and Brendle underline the fact that the Long-Moody construction should be studied in greater detail.

Our work focuses on the study of the Long-Moody construction from a functorial point of view. More precisely, we consider the homogenous category $\mathfrak{U}\boldsymbol{\beta}$ associated with braid groups. This category is an example of a general construction introduced by Randal-Williams and Wahl in [21] on the braid groupoid. The category $\mathfrak{U}\boldsymbol{\beta}$ has $\mathbb{N}$ as objects and for each natural integer n, its automorphism group $Aut_{\mathfrak{U}\boldsymbol{\beta}}(n)$ is the braid group $\mathbf{B}_n$. Let $\mathbb{K}$-$\mathfrak{Mod}$ be the category of $\mathbb{K}$-modules, with $\mathbb{K}$ a commutative ring. A functor $F : \mathfrak{U}\boldsymbol{\beta} \longrightarrow \mathbb{K}$-$\mathfrak{Mod}$ gives by evaluation a family of representations of braid groups $\{F_n : \mathbf{B}_n \longrightarrow GL(F(n))\}_{n\in\mathbb{N}}$, which satisfies some compatibility properties. For instance, Randal-Williams and Wahl define in [21, Example 4.3] a functor $\mathfrak{Bur} : \mathfrak{U}\boldsymbol{\beta} \longrightarrow \mathbb{K}$-Mod such that the representation $\mathfrak{Bur}_n : \mathbf{B}_n \longrightarrow GL(\mathfrak{Bur}(n))$ is the unreduced Burau representation. In the same way, we define for instance in Example 2.26 a functor $\mathfrak{TYM} : \mathfrak{U}\boldsymbol{\beta} \longrightarrow \mathbb{K}$-Mod such that the representation $\mathfrak{TYM}_n : \mathbf{B}_n \longrightarrow GL(\mathfrak{TYM}(n))$ is the representation considered by Tong, Yang and Ma in [23].

In Proposition 3.7, we prove that the Long-Moody construction is functorial. More precisely, we show:

***Theorem A* (Proposition 3.7) .** *There is a functor* $\mathbf{LM} : \mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod} \longrightarrow \mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}$*, called the Long-Moody functor, which satisfies for* $\sigma \in \mathbf{B}_n$

$$\mathbf{LM}(F)(\sigma) = lm(F_n)(\sigma).$$

Among the functors in $\mathfrak{U}\boldsymbol{\beta}$-$Mod$, the strong polynomial functors play a key role. This notion extends the classical one of polynomial functors, which were first defined by Eilenberg and Mac Lane in [9]. Their definition uses cross effects and concerns module categories. This initial definition can be extended to monoidal categories where the monoidal unit is also a null object. Djament and Vespa introduce in [8] the definition of strong polynomial functors for symmetric monoidal categories with the monoidal unit being an initial object. Here, we will see that the category $\mathfrak{U}\boldsymbol{\beta}$ is not symmetric, nor braided, but pre-braided in the sense of [21]. However, the notion of strong polynomial functor may be extended to the wider context of pre-braided monoidal categories (see Definition 2.7). Therefore, we investigate the effects of the Long-Moody functor on strong polynomial functors. We establish the following theorem.

***Theorem B* (Corollary 4.20) .** *Let F be a very strong polynomial functor of $\mathfrak{U}\boldsymbol{\beta}$-$\mathfrak{Mod}$ of degree n. Then, the functor $\mathbf{LM}(F)$ is a very strong polynomial functor of $\mathfrak{U}\boldsymbol{\beta}$-$\mathfrak{Mod}$ of degree $n+1$.*

Thus, iterating the Long-Moody functor on a strong polynomial functor of $\mathfrak{U}\boldsymbol{\beta}$-$\mathfrak{Mod}$ of degree n, we generate polynomial functors of $\mathfrak{U}\boldsymbol{\beta}$-$Mod$, of any degree bigger than n. For instance, the functors $\mathfrak{Bur}$ and $\mathfrak{TYM}$ happen to be strong polynomial functors of degree one, and we prove that the functor $\mathfrak{Bur}$ is equivalent to a functor obtained by applying the Long-Moody construction. Strong polynomial functors turn out inter alia to be very useful for homological stability problems. For example, in [21], Randal-Williams and Wahl construct a general framework to prove homological stability for different families of groups. They obtain the stability for coefficients given by a specific kind of strong polynomial functors (namely coefficient systems of finite degree). Thus, the Long-Moody functor will provide new examples of twisted coefficients corresponding to the framework of Randal-Williams and Wahl.

This construction is extended in the upcoming work [22] for other families of groups, such as automorphism groups of free groups, braid groups of surfaces, mapping class groups of orientable and non-orientable surfaces or mapping class groups of 3-manifolds. The results proved here for (very) strong polynomial functors will also hold in the adapted categorical framework for these different families of groups.

The paper is organized as follows. In section 1, we first recall definitions, folklore facts and properties about braid groups and free groups, especially focusing on the link between them. Then, following [21], we aim at explaining the construction of a homogeneous category from a braided monoidal groupoid, using Quillen's construction. In Section 2, we review the notion of strong polynomial functors, and slightly extend the framework of [8] for pre-braided monoidal categories. We will especially dwell on the interesting case of the coefficient system of finite degree, which $\mathfrak{Bur}$ and $\mathfrak{TYM}$ happen to be examples. In Section 3, we prove Theorem A and give some properties of the Long-Moody functor. Section 4 is devoted to the proof of Theorem B and to some other remarkable properties of this functor.

Notation. We will consider a commutative ring $\mathbb{K}$ throughout this work (the classical example is $\mathbb{K}=\mathbb{C}$). For all natural integers n, considering a canonical basis $\{e_1,\ldots,e_n\}$ (respectively $\{f_1,\ldots,f_{n+1}\}$) of $\mathbb{K}^{\oplus n}$ (respectively of $\mathbb{K}^{\oplus n+1}$), we define $\gamma_n^{\mathbb{K}}:\mathbb{K}^{\oplus n}\hookrightarrow\mathbb{K}^{\oplus n+1}$ the canonical inclusion morphism by:

$$\forall i\in\{1,\ldots,n\}\,\gamma_n^{\mathbb{K}}(e_i)=f_i.$$

We define inductively $\gamma_{n,n'}^{\mathbb{K}}=\gamma_{n'-1}^{\mathbb{K}}\circ\cdots\circ\gamma_n^{\mathbb{K}}:\mathbb{K}^{\oplus n}\hookrightarrow\mathbb{K}^{\oplus n'}$ for all natural integers n and n' such that $n'\geq n$.

We denote by $\mathbb{K}$-$\mathfrak{Mod}$ the category of $\mathbb{K}$-modules. We will also consider another commutative ring R.

Let $\mathfrak{Cat}$ denote the category of small categories. For $\mathfrak{C}\in Obj(\mathfrak{Cat})$, the core $\mathscr{G}\mathfrak{r}(\mathfrak{C})$ is the subcategory of $\mathfrak{C}$ which has the same objects as $\mathfrak{C}$ and of which the morphisms are the isomorphisms of $\mathfrak{C}$. We denote by $\mathscr{G}\mathfrak{r}:\mathfrak{Cat}\longrightarrow\mathfrak{Cat}$ the functor which associates to a category its core.

Let $\mathfrak{C}$ be a category. For two objects A and B of the category $\mathfrak{C}$, if A embeds in B and if no explicit notation is given, then we denote by I_A^B the associated embedding. If 0 is an initial object in the category $\mathfrak{C}$, then we denote by $\iota_A:0\longrightarrow A$ the unique morphism from 0 to A.

For $\mathfrak{C}$ an object of $\mathfrak{Cat}$ and $\mathfrak{D}$ a category, we denote by $\mathbf{Fct}(\mathfrak{C},\mathfrak{D})$ the category of functors from $\mathfrak{C}$ to $\mathfrak{D}$. For the particular case where $\mathfrak{D}=\mathbb{K}$-$\mathfrak{Mod}$, we denote by $\mathfrak{C}$-$\mathfrak{Mod}$ the functor category $\mathbf{Fct}(\mathfrak{C},\mathbb{K}\text{-}\mathfrak{Mod})$. A monoidal category is denoted by $(\mathfrak{C},\natural,0,\alpha,\lambda,\rho)$ with $\natural$ the monoidal product, 0 the unit, α the associator, λ is the left unitor and ρ is the right unitor. If the category is braided, we denote by $b^{\mathfrak{C}}$ its braiding.

We denote by $\mathfrak{Gr}$ the category of groups. Let $\mathfrak{gr}$ be the full subcategory of $\mathfrak{Gr}$ of finitely generated free groups.

Let $\mathbb{N}$ denote the category of natural integers considered as a poset. For all natural integers n, we denote by γ_n the unique element of $Hom_{\mathbb{N}}(n, n+1)$. For all natural integers n and n' such that $n' \geq n$, we denote by $\gamma_{n,n'} : n \longrightarrow n'$ the unique element of $Hom_{\mathbb{N}}(n, n')$, composition of the morphisms $\gamma_{n'-1} \circ \gamma_{n'-2} \circ \cdots \circ \gamma_{n+1} \circ \gamma_n$. The addition defines a strict monoidal structure on $\mathbb{N}$, denoted by $(\mathbb{N}, +, 0)$.

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Contents

1 Recollections on braid and free groups and homogeneous categories

1.1 Braid groups and free groups

1.1.1 Generalities

The aim of this section is to describe the necessary tools for our study. First, we recall some classical facts about braid groups and their links with free groups. Then, we give some notions and properties about Quillen's construction from a monoidal groupoid, pre-braided categories and homogeneous categories introduced by Randal-Williams and Wahl in [21].

Basic notions We recall that the braid group on n strands denoted by $\mathbf{B}_n$ is the group generated by σ_1, ..., σ_{n-1} satisfying the relations:

- $\forall i \in \{1, \ldots, n-2\}$, $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$;
- $\forall i, j \in \{1, \ldots, n-1\}$ such that $\mid i-j \mid \geq 2$, $\sigma_i\sigma_j = \sigma_j\sigma_i$.

Definition 1.1. Let $\mathbf{B}_\infty$ denote the colimit of the family of groups $\{\mathbf{B}_n\}_{n\in\mathbb{N}}$ and $\mathbf{B}_\infty^{ab}$ its abelianisation.

Let us introduce the groupoid $\boldsymbol{\beta}$ associated with the family of braid groups.

Definition 1.2. The braid groupoid $\boldsymbol{\beta}$ is the groupoid with objects the natural integers $n \in \mathbb{N}$ and morphisms (for $n, m \in \mathbb{N}$):

$$Hom_{\boldsymbol{\beta}}(n,m) = \begin{cases} \mathbf{B}_n & \text{if } n = m \\ \emptyset & \text{if } n \neq m \end{cases}$$

We also consider the free group on n generators, which we denote by:

$$\mathbf{F}_n = \langle g_1, \ldots, g_n \rangle .$$

For each natural integer n, one can see $\mathbf{F}_n$ as a subgroup of $\mathbf{F}_{n+1}$.

Definition 1.3. For all natural integers n and n' such that $n' \geq n$, we consider morphisms $\gamma^{f,\bullet}_{n,n'} : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n'}$, defined by:

$$\forall i \in \{1, \ldots, n\}, \; \gamma^{f,\bullet}_{n,n'}(g_i) = g_{k_i}$$

with $g_{k_i} = g_{k_j}$ if and only if $i = j$.

Notation 1.4. For simplicity, the map $\gamma^{f,\bullet}_{n,n+1}$ will be denoted by $\gamma^{f,\bullet}_n$.

Example 1.5. Classical morphisms from $\mathbf{F}_n$ to $\mathbf{F}_{n+1}$ are the identifications of $\mathbf{F}_n$ as the subgroup of $\mathbf{F}_{n+1}$ generated by the n first respectively last copies of $\mathbf{F}_1$ in $\mathbf{F}_{n+1}$. They will be denoted by $\gamma^{f,f}_n : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n+1}$ respectively $\gamma^{f,l}_n : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n+1}$. Explicitly, these morphisms are defined for all $i \in \{1, \ldots, n\}$ by $\gamma^{f,f}_n(g_i) = g_i$ and $\gamma^{f,l}_n(g_i) = g_{i+1}$.

Definition 1.6. Let n and n' be natural integers such that $n' \geq n$. To a morphism $\gamma^{f,\bullet}_{n,n'} : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n'}$ defined in Definition 1.3, we associate the morphism $\bar{\gamma}^{f,\bullet}_{n,n'} : \mathbf{F}_{n'-n} \hookrightarrow \mathbf{F}_{n'}$ defined for all $i \in \{1, \ldots, n'-n\}$ by $\gamma^{f,\bullet}_{n,n'}(g_i) = g_{l_i}$ with $l_i \notin \{k_1, \ldots, k_n\}$ and $l_i < l_j$ if and only if $i < j$.

Notation 1.7. For simplicity, the map $\bar{\gamma}^{f,\bullet}_{n,n+1}$ will be denoted by $\bar{\gamma}^{f,\bullet}_n$.

Remark 1.8. In particular, since the free product is the coproduct in the category $\mathfrak{Gr}$, we have a group isomorphism $\mathbf{F}_{n'} \cong Im\left(\bar{\gamma}^{f,\bullet}_{n,n'}\right) * Im\left(\gamma^{f,\bullet}_{n,n'}\right)$.

Remark 1.9. In addition, for all natural integers n, we can extend elements of $Aut(\mathbf{F}_n)$ to elements of $Aut(\mathbf{F}_{n+1})$ using $\gamma^{f,\bullet}_n : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n+1}$. Namely, let φ be an element of $Aut(\mathbf{F}_n)$. Thanks to the universal property of the coproduct, one defines uniquely an element $\widetilde{\varphi}$ of $Aut(\mathbf{F}_{n+1})$ such that the following diagram is commutative.

$$\begin{tikzcd}
 & \mathbf{F}_{n+1} \arrow[dd, dotted, "\exists!\widetilde{\varphi}", "\cong"'] & \\
\mathbf{F}_1 \arrow[ur, hook, "\bar{\gamma}^{f,\bullet}_n"] \arrow[dr, hook, "\bar{\gamma}^{f,\bullet}_n"'] & & \mathbf{F}_n \arrow[ul, hook', "\gamma^{f,\bullet}_n"'] \arrow[dl, hook', "\gamma^{f,\bullet}_n\circ\varphi"] \\
 & \mathbf{F}_{n+1} &
\end{tikzcd}$$

Explicitly, the automorphism $\widetilde{\varphi}$ is defined by:

$$\widetilde{\varphi}(g_i) = \begin{cases} \gamma_n^{f,\bullet} \circ \varphi(g_i) & \text{if } i \in \{1, \ldots, n\} \\ g_{l_1} = \bar{\gamma}_n^{f,\bullet}(g_1) & \text{if } i = n+1. \end{cases}$$

Hence, for all natural integers n, we define $\widetilde{\gamma}_n^{f,\bullet} : Aut(\mathbf{F}_n) \hookrightarrow Aut(\mathbf{F}_{n+1})$ assigning $\widetilde{\gamma}_n^{f,\bullet}(\varphi) = \widetilde{\varphi}$ for all $\varphi \in Aut(\mathbf{F}_n)$.

Furthermore, we need to consider morphisms from the free groups $\mathbf{F}_n$ to the braid groups $\mathbf{B}_n$.

Definition 1.10. For all natural integers n, we consider $\varsigma_{n,\bullet} : \mathbf{F}_n \to \mathbf{B}_n$ a family of morphisms.

Example 1.11. A classical identification, based on what is called the pure braid local system in the literature (see [18, Remark p.223]) and denoted by $\varsigma_{n,1}$, is defined by the following assignment.

$$\begin{array}{rcl} \varsigma_{n,1} : \mathbf{F}_n & \hookrightarrow & \mathbf{B}_n \\ g_i & \longmapsto & \begin{cases} \sigma_1^2 & \text{if } i = 1 \\ \sigma_i \varsigma_{n,1}(g_{i-1}) \sigma_i^{-1} & \text{if } i \in \{2, \ldots, n-1\} \\ \sigma_1 \varsigma_{n,1}(g_{n-1}) \sigma_1^{-1} & \text{if } i = n \end{cases} \end{array}$$

1.1.2 Action of braid groups on automorphism groups of free groups

From a historical perspective (see for example [3] or [12]), the geometric point of view of topology gives us different actions of $\mathbf{B}_n$ on the free group $\mathbf{F}_n$. As a consequence, there are several ways to consider the group $\mathbf{B}_n$ as a subgroup of $Aut(\mathbf{F}_n)$. Geometrically speaking, it comes from the identification of $\mathbf{B}_n$ with the mapping class group of a n-punctured disk $\Sigma_{0,1}^n$: fixing a point y on the boundary of the disk $\Sigma_{0,1}^n$, each free generators g_i can be taken as a loop of the disk based on y turning around certain fixed points. Each element σ of $\mathbf{B}_n$, as an automorphism up to isotopy of the disk $\Sigma_{0,1}^n$, induces a well-defined action on the fundamental group $\pi_1\left(\Sigma_{0,1}^n\right) \cong \mathbf{F}_n$.

In the sequel, we will fix such a family of group actions of $\mathbf{B}_n$ on the free group $\mathbf{F}_n$ and denote by $a_{n,\bullet} : \mathbf{B}_n \to Aut(\mathbf{F}_n)$ the induced group morphism for all natural integers n.

Example 1.12. A first classical action is called the Artin representation (see for example [4, Section 1]) . It is defined for all elementary braids σ_i where $i \in \{1, \ldots, n-1\}$ by:

$$\begin{array}{rcl} a_{n,1}(\sigma_i) : \mathbf{F}_n & \longrightarrow & \mathbf{F}_n \\ g_j & \longmapsto & \begin{cases} g_{i+1} & \text{if } j = i \\ g_{i+1}^{-1} g_i g_{i+1} & \text{if } j = i+1 \\ g_j & \text{if } j \notin \{i, i+1\} \end{cases} \end{array}$$

Wada representations In 1992, Wada introduced in [25] a certain type of representations of braid groups. The Artin representation falls in fact into the framework of this study. Explicitly, Wada investiguated on representations $\rho : \mathbf{B}_n \longrightarrow Aut(\mathbf{F}_n)$ such that for all generators σ_i of $\mathbf{B}_n$ and g_j of $\mathbf{F}_n$:

$$[\rho(\sigma_i)](g_j) = \begin{cases} W(g_i, g_{i+1}) & \text{if } j = i \\ V(g_i, g_{i+1}) & \text{if } j = i+1 \\ g_j & \text{if } j \notin \{i, i+1\} \end{cases}$$

where $W(g_i, g_{i+1})$ and $V(g_i, g_{i+1})$ are reduced words on $\left\{g_i^{\pm 1}, g_{i+1}^{\pm 1}\right\}$. He conjectured a classification of these Wada-type representations. This conjecture was proved by Ito in [11]. We give here a functorial approach of their work. Let us formalize some of our tools before, so as to introduce this work.

Definition 1.13. Let $\mathbf{B}_\bullet : \mathbb{N} \longrightarrow \mathfrak{Gr}$, $\mathbf{F}_\bullet : \mathbb{N} \longrightarrow \mathfrak{Gr}$, $GL_\bullet : \mathbb{N} \longrightarrow \mathfrak{Gr}$ and $Aut_\bullet : \mathbb{N} \longrightarrow \mathfrak{Gr}$ be the functors defined by:

- Objects: for all natural integers n, $\mathbf{B}_\bullet(n) = \mathbf{B}_n$ the braid group on n strands, $\mathbf{F}_\bullet(n) = \mathbf{F}_n$ the free group on n generators, $GL_\bullet(n) = GL_n(\mathbb{K})$ the general linear group of degree n on $\mathbb{K}$ and $Aut_\bullet(n) = Aut(\mathbf{F}_n)$ the automorphism group of the free group on n generators;
- Morphisms: for all natural integers n, we define $\mathbf{B}_\bullet(\gamma_n) = \gamma_n^{b,\bullet}$, $\mathbf{F}_\bullet(\gamma_n) = \gamma_n^{f,\bullet}$, $GL_\bullet(\gamma_n) = \gamma_n^{\mathbb{K}}$ and $Aut_\bullet(\gamma_n) = \widetilde{\gamma}_n^{f,\bullet}$ (recall that $\widetilde{\gamma}_n^{f,\bullet}$ was defined in Remark 1.9).

Let us consider two words $W(g_1, g_2)$ and $V(g_1, g_2)$ on $\mathbf{F}_2$. Let $(W, V) : \mathbf{B}_2 \longrightarrow Aut(\mathbf{F}_2)$ be the morphism defined by:

$$[(W, V)(\sigma_1)](g_j) = \begin{cases} W(g_1, g_2) & \text{if } j = 1 \\ V(g_1, g_2) & \text{if } j = 2 \end{cases}$$

Two morphisms $(W, V) : \mathbf{B}_2 \longrightarrow Aut(\mathbf{F}_2)$ and $(W', V') : \mathbf{B}_2 \longrightarrow Aut(\mathbf{F}_2)$ are said to be:

- swap-dual if $W'(g_1, g_2) = V(g_2, g_1)$ and $V'(g_1, g_2) = W(g_2, g_1)$;
- backward-dual if $W'(g_1, g_2) = \left(W\left(g_1^{-1}, g_2^{-1}\right)\right)^{-1}$ and $V'(g_1, g_2) = \left(V\left(g_1^{-1}, g_2^{-1}\right)\right)^{-1}$;
- inverse if $(W', V') = (W, V)^{-1}$.

For all natural integers n, for all $i \in \{1, \ldots, n-1\}$, we denote by $incl_i^n : \mathbf{B}_2 \hookrightarrow \mathbf{B}_n$ the inclusion morphism induced by:

$$incl_i^n(\sigma_1) = \sigma_i.$$

Theorem 1.14. *[25, 11] Let $W(g_1, g_2)$ and $V(g_1, g_2)$ be two words on $\mathbf{F}_2$. Let $\mathcal{W} : \mathbf{B}_\bullet \longrightarrow Aut_\bullet$ be a natural transformation. It will be said to be of Wada-type if for all natural integers n, for all $i \in \{1, \ldots, n-1\}$, the following diagram is commutative.*

$$\begin{tikzcd} \mathbf{B}_n \arrow[r, "\mathcal{W}_n"] & Aut(\mathbf{F}_n) \\ \mathbf{B}_2 \arrow[u, "incl_i^n"] \arrow[r, "(W,V)"'] & Aut(\mathbf{F}_2) \arrow[u, "id_{\mathbf{F}_{n-i}} * - * id_{\mathbf{F}_{n-i-1}}"'] \end{tikzcd}$$

Note that therefore a Wada-type natural transformation is entirely determined by the choice of (W, V). Then, there are seven types of Wada-type natural transformation $\mathcal{W}$ up to the swap-dual, backward-dual and inverse equivalences, listed below.

1. $(W, V)(g_1, g_2) = \left(g_2, g_2^m g_1 g_2^{-m}\right)$ *where* $m \in \mathbb{Z}$*;*
2. $(W, V)(g_1, g_2) = (g_1, g_2)$*;*
3. $(W, V)(g_1, g_2) = \left(g_2, g_1^{-1}\right)$*;*
4. $(W, V)(g_1, g_2) = \left(g_2, g_2 g_1 g_2^{-1}\right)$*;*
5. $(W, V)(g_1, g_2) = \left(g_1^{-1}, g_2^{-1}\right)$*;*
6. $(W, V)(g_1, g_2) = \left(g_2^{-1}, g_2 g_1 g_2\right)$*;*
7. $(W, V)(g_1, g_2) = \left(g_1 g_2^{-1} g_1^{-1}, g_1 g_2^2\right)$*.*

Note that the action given by the first Wada representation is a generalization of the Artin representation.

Notation 1.15. The actions given by the i-th Wada-type natural transformation will be denoted by $a_{n,i} : \mathbf{B}_n \hookrightarrow Aut(\mathbf{F}_n)$. In particular, for $i = 1$, we will explicitly specify each time the parameter $m \in \mathbb{N}$ which is used.

We give a last action of $\mathbf{B}_n$ on $Aut(\mathbf{F}_n)$, which will be useful throughout our work and does not fall into the framework of the Wada classification.

Example 1.16. An action is defined for all elementary braids σ_i where $i \in \{1, \dots, n-1\}$ by:

$$\begin{array}{rcl} a_{n,8}(\sigma_i) : \mathbf{F}_n & \longrightarrow & \mathbf{F}_n \\ g_j & \longmapsto & \begin{cases} g_j & \text{if } j \neq i+1 \\ g_{j+2} g_{j+1}^{-1} g_j & \text{if } i \leq n-2 \text{ and } j = i+1 \\ g_n^{-1} g_{n-1} & \text{if } i = n-1 \text{ and } j = n \end{cases} \end{array}$$

1.1.3 Augmentation ideal

Finally, we need to focus on the augmentation ideal of the group ring $\mathbb{K}[\mathbf{F}_n]$.

Definition 1.17. The augmentation ideal of the group ring $\mathbb{K}[\mathbf{F}_n]$, denoted by $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$, is defined to be the kernel of the morphism:

$$\begin{array}{rcl} \mathbb{K}[\mathbf{F}_n] & \xrightarrow{\mathcal{K}} & \mathbb{K}. \\ \sum \lambda_g g & \longmapsto & \sum \lambda_g \end{array}$$

Proposition 1.18. *[26, Chapter 6, Proposition 6.2.6] The augmentation ideal $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ is a free $\mathbb{K}[\mathbf{F}_n]$-module with basis the set $\{(g_i - 1) \mid i \in \{1, \dots, n\}\}$.*

Remark 1.19. The proof in [26, Chapter 6, Proposition 6.2.9] is done there for $\mathbb{K} = \mathbb{Z}$, but the general case here is exactly the same.

Remark 1.20. An action $a_{n,\bullet} : \mathbf{B}_n \hookrightarrow Aut(\mathbf{F}_n)$ extends naturally to $\mathbb{K}[\mathbf{F}_n]$. Indeed, let $\sum \lambda_g g \in \mathbb{K}[\mathbf{F}_n]$ and $b \in \mathbf{B}_n$, then, one defines (abusing the notation) $a_{n,\bullet} : \mathbf{B}_n \hookrightarrow Aut(\mathbb{K}[\mathbf{F}_n])$ by:

$$a_{n,\bullet}(b)\left(\sum \lambda_g g\right) = \sum \lambda_g a_n(b)(g).$$

Hence, since $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ is a submodule of $\mathbb{K}[\mathbf{F}_n]$, this induces by restriction an action on $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ denoted by $a_{n,\bullet} : \mathbf{B}_n \hookrightarrow Aut\left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}\right)$ (by abusing the notation).

Example 1.21. We may compute the three corresponding actions for the examples $a_{n,\bullet} : \mathbf{B}_n \longrightarrow Aut(\mathbf{F}_n)$ in 1.12 and 1.16. For all elementary braids σ_i where $i \in \{1, \dots, n-1\}$:

$$\begin{array}{rcl} a_{n,1}(\sigma_i) : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} & \longrightarrow & \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \\ g_j - 1 & \longmapsto & \begin{cases} g_{i+1} - 1 & \text{if } j = i \\ g_{i+1}^{-1} g_i g_{i+1} - 1 = [g_i - 1] g_{i+1} + [g_{i+1} - 1]\left(1 - g_{i+1}^{-1} g_i g_{i+1}\right) & \text{if } j = i+1 \\ g_j - 1 & \text{if } j \notin \{i, i+1\} \end{cases} \end{array}$$

$$\begin{array}{rcl} a_{n,8}(\sigma_i) : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} & \longrightarrow & \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \\ g_j - 1 & \longmapsto & \begin{cases} g_j - 1 & \text{if } j \neq i+1 \\ \begin{aligned} g_{j+2} g_{j+1}^{-1} g_j - 1 \;=\; & [g_j - 1] - [g_{j+1} - 1] g_{j+1}^{-1} g_j \\ & + [g_{j+2} - 1] g_{j+1}^{-1} g_j \end{aligned} & \text{if } i \leq n-2 \text{ and } j = i+1 \\ [g_{n-1} - 1] + [g_n - 1]\left(-g_n^{-1} g_{n-1}\right) & \text{if } i = n-1 \text{ and } j = n \end{cases} \end{array}$$

1.2 The homogeneous category associated with a groupoid

This section focuses on the presentation and the study of the Quillen's construction, which associates to a groupoid a monoidal category where the unit is initial. Under some extra conditions, this construction has further properties: if the groupoid is braided and satisfies a no zero condition, then the Quillen's construction is a pre-braided category (see Section 1.2.2) and under two more assumptions it defines a homogeneous category (see Section 1.2.3). In this paper, we are particularly interested in the case of the groupoid associated with the family of braid groups.

1.2.1 Quillen's construction

In [21], Randal-Williams and Wahl study a construction due to Quillen in [10, p.219], for a monoidal category S acting on a category X in the case $S = X = \mathfrak{G}$ where $\mathfrak{G}$ is a groupoid. Our review here is based on [21, Section 1].

Definition 1.22. [21, Section 1.1] Let $(\mathfrak{G}, \natural, 0)$ be a strict monoidal groupoid. The Quillen's construction on the groupoid $\mathfrak{G}$, denoted by $\mathfrak{UG}$ is defined by:

- Objects: $Obj\,(\mathfrak{UG}) = Obj\,(\mathfrak{G})$;
- Morphisms: For A and B two objects of $\mathfrak{G}$, the morphisms from A to B in the category $\mathfrak{UG}$ are given by:
$$Hom_{\mathfrak{UG}}\,(A, B) = \underset{\mathfrak{G}}{colim}\,[Hom_{\mathfrak{G}}\,(-\natural A, B)]\,.$$
In other words, a morphism $[X, f] : A \longrightarrow B$ in the category $\mathfrak{UG}$ is an equivalence class of pairs (X, f) where:
  - X is an object of $\mathfrak{G}$;
  - $f : X\natural A \longrightarrow B$ is a morphism of $\mathfrak{G}$;
  - the equivalence relation $\sim$ is defined by $(X, f) \sim (X', f')$ if and only if there exists a morphism $g : X \longrightarrow X'$ in $\mathfrak{G}$ such that the following diagram commutes.
$$\begin{tikzcd} X\natural A \arrow[r, "f"] \arrow[d, "g\natural id_A"'] & B \\ X'\natural A \arrow[ur, "f'"'] & \end{tikzcd}$$
  - Composition: let $[X, f] : A \longrightarrow B$ and $[Y, g] : B \longrightarrow C$ be two morphisms in the category $\mathfrak{UG}$. Then, the composition is defined by:
$$[Y, g] \circ [X, f] = [Y\natural X, g \circ (id_Y\natural f)]\,.$$
  - Identity: for all objects X of $\mathfrak{UG}$, the identity morphism is given by $[0, id_X] : X \longrightarrow X$.

Remark 1.23. One easily checks that for all morphisms $[X', f] : X \longrightarrow A$, $[X'', g] : B \longrightarrow X$, $[D_1, \varphi_1] : C_1 \longrightarrow C_2$, $[D_2, \varphi_2] : C_2 \longrightarrow C_3$ and $[D_3, \varphi_3] : C_3 \longrightarrow C_4$ in the category $\mathfrak{UG}$:

- $[X', f] \circ [0, id_X] = [X', f]$ and $[0, id_X] \circ [X'', g] := [X'', g]$;
- $([D_1, \varphi_1] \circ [D_2, \varphi_2]) \circ [D_3, \varphi_3] = [D_1, \varphi_1] \circ ([D_2, \varphi_2] \circ [D_3, \varphi_3])$.

The Quillen's construction $\mathfrak{UG}$ has the additional following property.

Proposition 1.24. *[21, Proposition 1.8] Let $(\mathfrak{G}, \natural, 0)$ be a strict monoidal groupoid. Then, the unit 0 is initial in the category $\mathfrak{UG}$.*

1.2.2 Pre-braided monoidal categories

We present the notion of pre-braided category, introduced by Randal-Williams and Wahl in [21]. This is a generalization of braided category, which will prove to be necessary to understand the homogeneous category constructed from the groupoid $\boldsymbol{\beta}$.

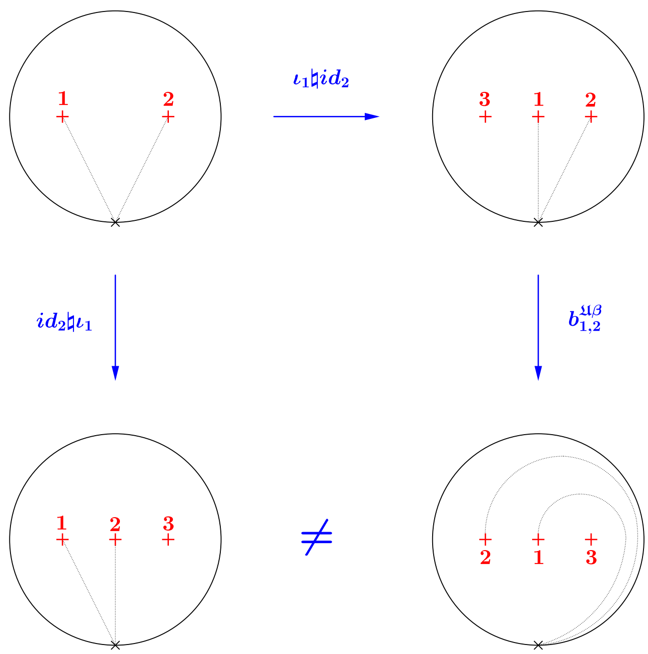
Definition 1.25. [21, Definition 1.5] Let $(\mathfrak{C}, \natural, 0, \alpha, \lambda, \rho)$ be a monoidal category such that the unit 0 is initial. We say that the monoidal category $(\mathfrak{C}, \natural, 0, \alpha, \lambda, \rho)$ is pre-braided if:

- the core $\mathscr{G}\mathfrak{r}\left(\left(\mathfrak{C},\natural,0,\alpha,\lambda,\rho\right)\right)$ is a braided monoidal category ;
- for all objects A and B of $\mathfrak{C}$, the groupoid braiding $b_{A,B}^{\mathfrak{C}}:A\natural B\longrightarrow B\natural A$ satisfies:

$$b_{A,B}^{\mathfrak{C}}\circ\left(id_{A}\natural\iota_{B}\right)=\iota_{B}\natural id_{A}:A\longrightarrow B\natural A.$$

Remark 1.26. A braided monoidal category is obviously pre-braided.

A monoidal product $\natural:\boldsymbol{\beta}\times\boldsymbol{\beta}\longrightarrow\boldsymbol{\beta}$ is defined assigning the usual addition for the objects and connecting two braids side by side for the morphisms, along with a braiding denoted $b_{-,-}^{\boldsymbol{\beta}}$. For more details, see for example [19, Chapter XI, Part 4] for the definition of the braiding. The pre-braiding defined on $\mathfrak{U}\boldsymbol{\beta}$ is not a braiding. Indeed, the following figure shows that $b_{1,2}^{\mathfrak{U}\boldsymbol{\beta}}\circ\left(\iota_{1}\natural id_{2}\right)\neq id_{2}\natural\iota_{1}$ whereas this two morphisms should be equal if $b_{-,-}^{\mathfrak{U}\boldsymbol{\beta}}$ was a braiding. This example shows in particular that a pre-braided monoidal category is not necessarily braided.



Under some assumption, the Quillen's construction $\mathfrak{U}\mathfrak{G}$ inherits a pre-braided property.

Proposition 1.27. *[21, Proposition 1.8] Let $\left(\mathfrak{G},\natural,0\right)$ be a strict monoidal groupoid. If the category $\mathfrak{G}$ is braided monoidal and has no zero divisors (i.e. for objects A and B of $\mathfrak{G}$, $A\natural B\cong0$ if and only if $A\cong B\cong0$), then the category $\left(\mathfrak{U}\mathfrak{G},\natural,0\right)$ is pre-braided monoidal. Moreover, the monoidal structure of $\mathfrak{U}\mathfrak{G}$ is such that the map $\mathfrak{G}\longrightarrow\mathfrak{U}\mathfrak{G}$ taking an isomorphism f to $[0,f]$ is monoidal.*

Remark 1.28. The monoidal structure on the category $\mathfrak{U}\mathfrak{G}$ is defined letting for $[X,f]\in Hom_{\mathfrak{U}\mathfrak{G}}\left(A,B\right)$ and $[Y,g]\in Hom_{\mathfrak{U}\mathfrak{G}}\left(C,D\right)$:

$$[X,f]\natural[Y,g]=\left[X\natural Y,\left(f\natural g\right)\circ\left(id_{X}\natural b_{A,Y}^{-1}\natural id_{C}\right)\right].$$

1.2.3 Homogeneous categories

The notion of homogeneous category is introduced by Randal-Williams and Wahl in [21, Section 1], inspired by the set-up of Djament and Vespa in [7, Section 1.2]. With two additional assumptions, the Quillen's construction $\mathfrak{U}\mathfrak{G}$ from a strict monoidal groupoid $\left(\mathfrak{G},\natural,0\right)$ will be endowed with an homogeneous category structure. First, we need to give basic definitions necessary to give the one of homogeneous category.

Definition 1.29. Let $\left(\mathfrak{C},\natural,0\right)$ be a strict monoidal category in which the unit 0 is also initial. For all objects A and B of $\mathfrak{C}$, we define:

- a preferred morphism: $\iota_A \natural id_B : (B = 0\natural B) \longrightarrow A\natural B$;
- a set of morphisms characterised by this preferred morphism:

$$Fix(B) = Fix(B, A\natural B) = \{\phi \in Aut(A\natural B) \mid \phi \circ (\iota_A \natural id_B) = \iota_A \natural id_B\}.$$

Remark 1.30. Since $(\mathfrak{C}, \natural, 0)$ is assumed to be small $Hom_{\mathcal{C}}(A, B)$ is a set and $Aut_{\mathcal{C}}(B)$ defines a group (with composition of morphisms as the group product). The group $Aut_{\mathcal{C}}(B)$ acts by post-composition on $Hom_{\mathcal{C}}(A, B)$:

$$\begin{array}{ccc} Aut_{\mathfrak{C}}(B) \times Hom_{\mathfrak{C}}(A, B) & \longrightarrow & Hom_{\mathfrak{C}}(A, B). \\ (\phi, f) & \longmapsto & \phi \circ f \end{array}$$

Now, we may introduce homogeneous categories.

Definition 1.31. Let $(\mathfrak{C}, \natural, 0)$ be a strict monoidal small category. This category is homogeneous if the unit 0 is initial in $\mathfrak{C}$ and if the two following assumptions are satisfied.

- (**H1**) : For all objects A and B of the category $\mathfrak{C}$, the action by post-composition of $Aut(B)$ on $Hom_{\mathfrak{C}}(A, B)$ is transitive.
- (**H2**) : For all objects A and B of the category $\mathfrak{C}$, the map

$$\begin{array}{ccc} Aut_{\mathfrak{C}}(A) & \longrightarrow & Aut_{\mathfrak{C}}(A\natural B) \\ f & \longmapsto & f\natural id_B \end{array}$$

  is injective with image $Fix(B) = \{\phi \in Aut_{\mathfrak{C}}(A\natural B) \mid \phi \circ (\iota_A \natural id_B) = \iota_A \natural id_B\}$.

Remark 1.32. A strict monoidal category $(\mathfrak{C}, \natural, 0)$ satisfying (**H1**) and (**H2**) is therefore determined by its core.

Let us focus on some elementary properties of homogeneous categories.

Proposition 1.33. *[21, Remark 1.4]Let $(\mathfrak{C}, \natural, 0)$ be a homogeneous category. Let A and B be two objects of this category. Then:*

1. $Hom_{\mathfrak{C}}(B, A\natural B) \cong Aut_{\mathfrak{C}}(A\natural B) / Aut_{\mathfrak{C}}(A)$.
2. $Hom_{\mathfrak{C}}(A, A) \cong Aut_{\mathfrak{C}}(A)$.
3. If $Hom_{\mathfrak{C}}(A, B) \neq \emptyset$ and $Hom_{\mathfrak{C}}(B, A) \neq \emptyset$, then $A \cong B$.

Remark 1.34. We will deal with objects in $\mathbf{Fct}(\mathfrak{C}, \mathcal{A})$ for $(\mathfrak{C}, \natural, 0)$ a homogeneous category and $\mathcal{A}$ an abelian category. In order to prove propositions for a functor F of $\mathbf{Fct}(\mathfrak{C}, \mathbb{K}\text{-Mod})$, according to the first property of 1.33, it is sufficient to restrict the work on morphisms to the automorphisms. In other words, proving a result dealing with F for all the automorphisms automatically extends to check this result on F for all the morphisms.

We should now give the two additional properties that a strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ may satisfy so as to the category $\mathfrak{UG}$ be homogeneous.

Definition 1.35. Let $(\mathfrak{G}, \natural, 0)$ be a strict monoidal groupoid. We define two assumptions.

- (**C**) : For all objects A, B and C of $\mathfrak{G}$, if $A\natural C \cong B\natural C$ then $A \cong B$. The category $\mathfrak{G}$ is then said to satisfy cancellation property.
- (**I**) : For all objects A, B of $\mathfrak{G}$, the following morphism is injective:

$$\begin{array}{ccc} Aut_{\mathfrak{G}}(A) & \longrightarrow & Aut_{\mathfrak{G}}(A\natural B). \\ f & \longmapsto & f\natural id_B \end{array}$$

Theorem 1.36. *[21, Theorem 1.10] Let $(\mathfrak{G}, \natural, 0)$ be a braided monoidal groupoid with no zero divisors.*

1. *The category satisfies* (**H1**) *if and only if the groupoid* $\mathfrak{G}$ *satisfies* (**C**).
2. *If the groupoid* $\mathfrak{G}$ *satisfies* (**I**), *then* $\mathfrak{U}\mathfrak{G}$ *satisfies* (**H2**).

In particular, if the groupoid $\mathfrak{G}$ *satisfies* (**C**) *and* (**I**), *then* $\mathfrak{U}\mathfrak{G}$ *is homogeneous.*

The relationship between the automorphisms of the groupoid $\mathfrak{G}$ and those of its associated Quillen's construction $\mathfrak{U}\mathfrak{G}$ is not always clear. Indeed, we intuitively expect that $\mathfrak{G}$ is the core of $\mathfrak{U}\mathfrak{G}$ but we need two more hypothesis in order to ensure this property.

Proposition 1.37. *[21, Proposition 1.7] Let* $(\mathfrak{G}, \natural, 0)$ *be a monoidal groupoid and let* $\mathfrak{U}\mathfrak{G}$ *denote the Quillen's construction on* $\mathfrak{G}$. *We assume that:*

- $Aut_{\mathfrak{G}}(0) = \{id_0\}$.
- *The groupoid* $\mathfrak{G}$ *has no zero divisors: if* $A\natural B \cong 0$ *in* $\mathfrak{G}$, *then* $A \cong 0$ *and* $B \cong 0$.

Then $\mathfrak{G}$ *is the core of* $\mathfrak{U}\mathfrak{G}$.

Example 1.38. The braid groupoid $\boldsymbol{\beta}$ is braided monoidal, it has no zero divisors. Moreover its monoidal structure clearly satisfies hypotheses (**C**) and (**I**), so its associated category $\mathfrak{U}\boldsymbol{\beta}$ is pre-braided homogeneous.

1.3 Coherent conditions

We consider two natural integers n and n' such that $n' \geq n$. Recall that we have defined four key group morphisms: $a_{n,\bullet} : \mathbf{B}_n \hookrightarrow Aut(\mathbf{F}_n)$, $\varsigma_{n,\bullet} : \mathbf{F}_n \hookrightarrow \mathbf{B}_{n+1}$, $\gamma^{b,\bullet}_{n,n'} : \mathbf{B}_n \hookrightarrow \mathbf{B}_{n'}$ and $\gamma^{f,\bullet}_{n,n'} : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n'}$ (with abreviations $\gamma^{b,\bullet}_n$ and $\gamma^{f,\bullet}_n$ when $n' = n+1$). We will need the following coherence conditions for our constructions.

Condition 1.39. We require $\gamma^{f,\bullet}_{n,n'} \circ (a_{n,\bullet}(\sigma)) = (a_{n',\bullet}(\sigma'\natural\sigma)) \circ \gamma^{f,\bullet}_{n,n'}$ for all elements σ of $\mathbf{B}_n$ and σ' of $\mathbf{B}_{n'-n}$, ie the following diagram is commutative.

$$\begin{tikzcd}
\mathbf{F}_n \arrow[r, "a_{n,\bullet}(\sigma)"] \arrow[d, "\gamma^{f,\bullet}_{n,n'}"'] & \mathbf{F}_n \arrow[d, "\gamma^{f,\bullet}_{n,n'}"] \\
\mathbf{F}_{n'} \arrow[r, "a_{n',\bullet}(\sigma'\natural\sigma)"'] & \mathbf{F}_{n'}
\end{tikzcd}$$

Remark 1.40. Condition 1.39 will give sufficient relations to define the Long-Moody functor on objects in Proposition 3.7. Moreover, Condition 1.39 will be used to prove Propositions 4.7 and 4.10.

Condition 1.41. Let m be a natural integer. We require $a_{m+n',\bullet}\left(\left(b^{\mathfrak{M}}_{m,n'-n}\right)^{-1}\natural id_n\right)$ to be the identity on the image of the homomorphism $\gamma^{f,\bullet}_{m+n,m+n'} \circ \bar{\gamma}^{f,\bullet}_{m,m+n} : \mathbf{F}_m \longrightarrow \mathbf{F}_{m+n'}$, ie for all element σ of $\mathbf{B}_n$ the following diagram is commutative.

$$\begin{tikzcd}
\mathbf{F}_m \arrow[r, "\bar{\gamma}^{f,\bullet}_{m,m+n'}"] \arrow[d, "\bar{\gamma}^{f,\bullet}_{m,m+n}"'] & \mathbf{F}_{m+n'} \\
\mathbf{F}_{m+n} \arrow[r, "\gamma^{f,\bullet}_{m+n,m+n'}"'] & \mathbf{F}_{m+n'} \arrow[u, "a_{m+n',\bullet}\left(\left(b^{\mathfrak{M}}_{m,n'-n}\right)^{-1}\natural id_n\right)"']
\end{tikzcd}$$

Condition 1.42. We require $a_{n',\bullet}(id_{n'-n}\natural -)$ to be the identity on the image of the homomorphism $\bar{\gamma}^{f,\bullet}_{n'-n,n'} : \mathbf{F}_{n'-n} \longrightarrow \mathbf{F}_{n'}$, ie for all element σ of $\mathbf{B}_n$ the following diagram is commutative.

$$\begin{tikzcd}
\mathbf{F}_{n'-n} \arrow[rr, "\bar{\gamma}^{f,\bullet}_{n'-n,n'}"] \arrow[dr, "\bar{\gamma}^{f,\bullet}_{n'-n,n'}"'] & & \mathbf{F}_{n'} \\
& \mathbf{F}_{n'} \arrow[ur, "a_{n',\bullet}\left(id_{n'-n}\natural\sigma\right)"'] &
\end{tikzcd}$$

In other words, for all element σ of $\mathbf{B}_n$, for all element f of $\mathbf{F}_{n'-n}$, $a_{n',\bullet}\left(id_{n'-n}\natural\sigma\right) \circ \bar{\gamma}^{f,\bullet}_{n,n'}(f) = \bar{\gamma}^{f,\bullet}_{n,n'}(f)$.

Remark 1.43. Condition 1.41 and 1.42 will be used to define the functor Υ_m in Proposition 4.3.

For the further constructions, we will have to make relevant choices for these four morphisms: the characterization of the choice will be encoded by the triplet $\left(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^f}\right)$.

Definition 1.44. A triplet $\left(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^f}\right)$ is said to be coherent if it satisfies the condition 1.39 for all natural integers n and n' such that $n' \geq n$. If it satisfies moreover conditions 1.41 and 1.42 for all natural integers n and n' such that $n' \geq n$, the coherent triplet $\left(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^f}\right)$ is said to be reliable.

Example 1.45. Considering the morphisms $a_{n,1}$, $\varsigma_{n,1}$ and $\gamma^{f,l}_{n,n'}$ for all natural integers n provide a reliable triplet $(1, 1, l)$.

2 Strong polynomial functors

We deal here with the concept of strong polynomial functor. This type of functor will be at the centre of our work in Section 4. We review (and in fact slightly extend) the definition and properties of a strong polynomial functor due to Djament and Vespa in [8] and also a particular case of coefficient systems of finite degree used by Randal-Williams and Wahl in [21].

In [8, Section 1], Djament and Vespa construct a framework to define strong polynomial functors in the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$, where $\mathfrak{M}$ is a symmetric monoidal category where the unit is an initial object and $\mathcal{A}$ is an abelian category. Here, we aim at generalizing this definition for functors from pre-braided monoidal categories having the same additional property. In particular, a notion of strong polynomial functor will be well-defined for the category $\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod} = \mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. The keypoint of this section is Proposition 2.4, in so far as it constitutes the crucial property necessary and sufficient to extend the definition of strong polynomial functor for the pre-braided case.

2.1 Strong polynomiality

Definition 2.1. We denote by $\mathfrak{Mon}^{\mathfrak{pb}}$ the category defined by the following assignment.

- Objects: the pre-braided strict monoidal small categories $(\mathfrak{M}, \natural, 0)$.
- Morphisms: the pre-braided strict monoidal functors $F : (\mathfrak{M}, \natural, 0) \longrightarrow (\mathfrak{N}, \natural, 0)$. Namely, these are strict monoidal functors F such that the functor $\mathscr{G}\mathfrak{r}(F)$ is braided.

Definition 2.2. We denote by $\mathfrak{Mon}^{\mathfrak{pb}}_{\mathfrak{ini}}$ the full subcategory of $\mathfrak{Mon}^{\mathfrak{pb}}$ whose objects are pre-braided strict monoidal small categories $(\mathfrak{M}, \natural, 0)$ such that the unit 0 is an initial object. We denote by $\mathfrak{Mon}^{\mathfrak{pb}}_{\mathfrak{null}}$ the full subcategory of $\mathfrak{Mon}^{\mathfrak{pb}}_{\mathfrak{ini}}$ whose objects are pre-braided strict monoidal small categories $(\mathfrak{M}, \natural, 0)$ such that the unit 0 is a null object.

We introduce the translation functor, which will play a central role in the definition of strong polynomiality.

Definition 2.3. Let $(\mathfrak{M}, \natural, 0)$ be an object of $\mathfrak{Mon}^{\mathfrak{pb}}$, let $\mathfrak{C}$ be a category and let x be an object of $\mathfrak{M}$. We define the endofunctor $x\natural id_{\mathfrak{M}} : \mathfrak{M} \longrightarrow \mathfrak{M}$ by:

- Objects: for all objects m of $\mathfrak{M}$, $x\natural id_{\mathfrak{M}}(m) = x\natural m$.

- Morphisms: for all morphism f of $Hom_{\mathfrak{M}}(m,m')$, $x\natural id_{\mathfrak{M}}(f)=id_x\natural f : x\natural m \longrightarrow x\natural m'$.

For x an object of $\mathfrak{M}$, we define the translation by x functor $\tau_x : \mathbf{Fct}(\mathfrak{M},\mathfrak{C}) \longrightarrow \mathbf{Fct}(\mathfrak{M},\mathfrak{C})$ to be the endofunctor of $\mathbf{Fct}(\mathfrak{M},\mathfrak{C})$ obtained by precomposition by the functor $x\natural id_{\mathfrak{M}}$.

The following proposition establishes the commutation of two translation functors associated with two objects of $\mathfrak{M}$. It is the keystone property to define polynomial functors.

Proposition 2.4. *Let $(\mathfrak{M},\natural,0)$ be an object of $\mathfrak{Mon}^{\mathfrak{pb}}$ such that for all object m of $\mathfrak{M}$, $Hom_{\mathfrak{M}}(m,m)=Aut_{\mathfrak{M}}(m)$. Let $\mathfrak{C}$ be a category. Let x and y be two objects of $\mathfrak{M}$. Then, there exist a natural isomorphism between functors from $\mathbf{Fct}(\mathfrak{M},\mathfrak{C})$ to $\mathbf{Fct}(\mathfrak{M},\mathfrak{C})$:*

$$\tau_x\circ\tau_y\cong\tau_y\circ\tau_x.$$

Proof. First, because of the associativity of the monoidal product $\natural$ and of the strictness of $\mathfrak{M}$, we have that $\tau_x\circ\tau_y=\tau_{x\natural y}$ and $\tau_y\circ\tau_x=\tau_{y\natural x}$. Let us denote by $b^{\mathfrak{M}}_{-,-}$ the pre-braiding of $\mathfrak{M}$. The key point is the fact that $b^{\mathfrak{M}}_{x,y} : x\natural y\xrightarrow{\cong} y\natural x$ is a braiding (in so far as it is a braiding defined on the core of $\mathfrak{M}$) and the asssumption that an endomorphism of $\mathfrak{M}$ is necessarily an automorphism. For all objects F of $\mathbf{Fct}(\mathfrak{M},\mathfrak{C})$, we define a morphism between $\tau_{x\natural y}(F)$ and $\tau_{y\natural x}(F)$ by:

$$\left(b^{\mathfrak{M}}_{x,y}\natural id_{\mathfrak{M}}\right)^{*}(F):\left[\tau_{x\natural y}(F)=F\left((x\natural y)\natural-\right)\right]\longrightarrow\left[\left(F\circ\left(b^{\mathfrak{M}}_{x,y}\natural id_{\mathfrak{M}}\right)\right)\left((x\natural y)\natural-\right)=F\left((y\natural x)\natural-\right)=\tau_{y\natural x}(F)\right].$$

Let $\lambda : F\Longrightarrow G$ be a natural transformation in $\mathbf{Fct}(\mathfrak{M},\mathfrak{C})$. Then:

- Let m be an object of $\mathfrak{M}$, since $b^{\mathfrak{M}}_{x,y}\natural id_m$ is a morphism of $\mathfrak{M}$ and λ is a natural transformation, we deduce that:

$$\left(\left(b^{\mathfrak{M}}_{x,y}\natural id_{\mathfrak{M}}\right)^{*}(G)(m)\right)\circ\tau_{x\natural y}(\lambda_m) \quad = \quad \tau_{y\natural x}(\lambda_m)\circ\left(\left(b^{\mathfrak{M}}_{x,y}\natural id_{\mathfrak{M}}\right)^{*}(F)(m)\right).$$

- Let us consider a morphism $f\in Hom_{\mathfrak{M}}(m,m')$. Since λ is a natural transformation, because of the functoriality of F and G and of composition rules of the monoidal product, we deduce from the result of the previous point:

$$\left(\left(b^{\mathfrak{M}}_{x,y}\natural id_{\mathfrak{M}}\right)^{*}(G)(f)\right)\circ\tau_{x\natural y}(\lambda_m)=\tau_{y\natural x}(\lambda_{m'})\circ\left(\left(b^{\mathfrak{M}}_{x,y}\natural id_{\mathfrak{M}}\right)^{*}(F)(f)\right).$$

Hence we have proved that $\left(b^{\mathfrak{M}}_{x,y}\natural id_{\mathfrak{M}}\right)^{*}(G)\circ\tau_{x\natural y}(\lambda)=\tau_{y\natural x}(\lambda)\circ\left(b^{\mathfrak{M}}_{x,y}\natural id_{\mathfrak{M}}\right)^{*}(F)$. Thus, we have defined a natural transformation $\left(b^{\mathfrak{M}}_{x,y}\natural id_{\mathfrak{M}}\right)^{*} : \tau_{x\natural y}\Longrightarrow\tau_{y\natural x}$. It happens to be an isomorphism in so far as we analogously construct a natural transformation $\left(\left(b^{\mathfrak{M}}_{x,y}\right)^{-1}\natural id_{\mathfrak{M}}\right)^{*} : \tau_{y\natural x}\Longrightarrow\tau_{x\natural y}$ and direct computations show that for all objects F of $\mathbf{Fct}(\mathfrak{M},\mathfrak{C})$ $\left(\left(b^{\mathfrak{M}}_{x,y}\right)^{-1}\natural id_{\mathfrak{M}}\right)^{*}(F)\circ\left(b^{\mathfrak{M}}_{x,y}\natural id_{\mathfrak{M}}\right)^{*}(F)=id_{\tau_{x\natural y}(F)}$ and $\left(b^{\mathfrak{M}}_{x,y}\natural id_{\mathfrak{M}}\right)^{*}(F)\circ\left(\left(b^{\mathfrak{M}}_{x,y}\right)^{-1}\natural id_{\mathfrak{M}}\right)^{*}(F)=id_{\tau_{y\natural x}(F)}$. □

Remark 2.5. In Proposition 2.4, the natural isomorphism is not unique: as the proof shows it, we could have used the morphism $\left(b^{\mathfrak{M}}_{y,x}\right)^{-1}\natural id_{\mathfrak{M}}$ instead to define an isomorphism between $\tau_{x\natural y}(F)$ and $\tau_{y\natural x}(F)$. This natural isomorphism becomes unique if we additionaly assume that the category $\mathfrak{M}$ is symmetric monoidal as done by Djament and Vespa in [8, Section 1].

In the remainder of this section, we consider an abelian category $\mathcal{A}$. Let us move on to the introduction of the evanescence and difference functors, which will characterize the (very) strong polynomiality of a functor in $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$.

If $\mathfrak{M}$ is a small category and $\mathcal{A}$ is an abelian category, then the functor category $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$ is an abelian category (see [19, Chapter VIII]).

From now until the end of Section 2, we will consider an object $(\mathfrak{M},\natural,0)$ of $\mathfrak{Mon}^{\mathfrak{pb}}_{\mathfrak{ini}}$, such that for all object m of $\mathfrak{M}$, $Hom_{\mathfrak{M}}(m,m)=Aut_{\mathfrak{M}}(m)$.

Remark 2.6. This condition is crucial so that Proposition 2.4 is satisfied.

In the remainder of this section, x will be an object of $\mathfrak{M}$. For all objects F of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$, we denote by $i_x(F) : \tau_0(F) \longrightarrow \tau_x(F)$ the natural transformations induced by the precomposition of F by the unique morphism $\iota_x : 0 \longrightarrow x$ of $\mathcal{M}$. Since $Id_{\mathbf{Fct}(\mathfrak{M},\mathcal{A})}$ and τ_x are endofunctors of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$, they induce $i_x : Id_{\mathbf{Fct}(\mathfrak{M},\mathcal{A})} \longrightarrow \tau_x$ a natural transformation of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$. Since the category $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$ is abelian, the kernel and cokernel of the natural transformation i_x exist.

Definition 2.7. We define $\kappa_x = \ker(i_x)$ and $\delta_x = \operatorname{coker}(i_x)$. The endofunctors κ_x and δ_x of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$ are called respectively evanescence and difference functor associated with x.

The following lemma present elementary properties of the translation, evanescence and difference functors. They are either straightfoward consequences of the definitions, or direct generalizations of the framework where $\mathfrak{M}$ is symmetric monoidal considered in [8]. Indeed, the proofs of the numbered properties are exactly the same as those of [8, Proposition 1.4]: everything works in the same way in so far as the commutation property of the translation endofunctor is still satisfied in the pre-braided case by Proposition 2.4 and if we use the convention to add objects on the right.

Lemma 2.8. *Let y be an object of $\mathfrak{M}$. Then the translation functor τ_x is exact and we have the following exact sequence in the category of endofunctors of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$:*

$$0 \longrightarrow \kappa_x \xrightarrow{\Omega_x} Id \xrightarrow{i_x} \tau_x \xrightarrow{\Delta_x} \delta_x \longrightarrow 0. \tag{1}$$

Moreover, considering a short exact sequence $0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$ in the category $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$, the snake lemma implies that we have the following exact sequence in the category $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$:

$$0 \longrightarrow \kappa_x(F) \longrightarrow \kappa_x(G) \longrightarrow \kappa_x(H) \longrightarrow \delta_x(F) \longrightarrow \delta_x(G) \longrightarrow \delta_x(H) \longrightarrow 0. \tag{2}$$

In addition :

1. *The translation endofunctors τ_x and τ_y of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$ commute up to natural isomorphism. They commute with limits and colimits.*

2. *The difference endofunctors δ_x and δ_y of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$ commute up to natural isomorphism. They commute with colimits.*

3. *The endofunctors κ_x and κ_y of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$ commute up to natural isomorphism. They commute with limits.*

4. *The natural inclusion $\kappa_x \circ \kappa_x \hookrightarrow \kappa_x$ is an isomorphism.*

5. *The translation endofunctor τ_x and the difference endofunctor δ_y commute up to natural isomorphism.*

6. *The translation endofunctor τ_x and the endofunctor κ_y commute up to natural isomorphism.*

7. *We have the following natural exact sequence in the category of endofunctors of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$:*

$$0 \longrightarrow \kappa_y \longrightarrow \kappa_{x\natural y} \longrightarrow \tau_x\kappa_y \longrightarrow \delta_y \longrightarrow \delta_{x\natural y} \longrightarrow \tau_y\delta_x \longrightarrow 0. \tag{3}$$

Thanks to lemma 2.8, we can define strong polynomial functors.

Definition 2.9. We recursively define on $n \in \mathbb{N}$ the category $\mathcal{P}ol_n(\mathfrak{M},\mathcal{A})$ of strong polynomial functors of degree smaller or equal to n to be the full subcategory of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$ as follows:

1. If $n < 0$, $\mathcal{P}ol_n(\mathfrak{M},\mathcal{A}) = \{0\}$;

2. if $n \geq 0$, the objects of $\mathcal{P}ol_n(\mathfrak{M},\mathcal{A})$ are the functors F such that for all objects x of $\mathfrak{M}$, the functor $\delta_x(F)$ is an object of $\mathcal{P}ol_{n-1}(\mathfrak{M},\mathcal{A})$.

The following three propositions are noteworthy properties of the framework in [8] adapted in the pre-braided case. Their proofs follow directly from those of their analogues in [8, Propositions 1.7, 1.8 and 1.9].

Proposition 2.10. *[8, Proposition 1.7] Let $\mathfrak{M}'$ be objects of $\mathfrak{Mon}_{\mathrm{ini}}^{\mathrm{pb}}$ such that for all object m of $\mathfrak{M}'$, $Hom_{\mathfrak{M}'}(m,m) = Aut_{\mathfrak{M}'}(m)$. Let $\alpha : \mathfrak{M} \longrightarrow \mathfrak{M}'$ be a strong monoidal functor. Then, the precomposition by α gives rise to a functor from $\mathcal{P}ol_n(\mathfrak{M}', \mathcal{A})$ to $\mathcal{P}ol_n(\mathfrak{M}, \mathcal{A})$.*

Proposition 2.11. *[8, Proposition 1.8]The category $\mathcal{P}ol_n(\mathfrak{M}, \mathcal{A})$ is closed under the translation endofunctor τ_x, under quotient, under extension and under colimit. Moreover, assuming that there exists a set $\mathfrak{E}$ of objects of $\mathfrak{M}$ such that:*

$$\forall m \in Obj(\mathfrak{M}),\ \exists \{e_i\}_{i\in I} \in Obj(\mathfrak{E})\ \textit{(where I is finite)},\ m \cong \underset{i\in I}{\natural} e_i,$$

then, an object F of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ belongs to $\mathcal{P}ol_n(\mathfrak{M}, \mathcal{A})$ if and only if $\delta_e(F)$ is an object of $\mathcal{P}ol_{n-1}(\mathfrak{M}, \mathcal{A})$ for all objects e of $\mathfrak{E}$.

Remark 2.12. The category $\mathcal{P}ol_n(\mathfrak{M}, \mathcal{A})$ is not necessarily closed under subobjects. It is the case if $\mathfrak{M}$ is an object of $\mathfrak{Mon}_{\mathrm{nul}}^{\mathrm{pb}}$, since then, for all objects x of $\mathfrak{M}$, κ_x is the null endofunctor of $\mathbf{Fct}\left(\mathfrak{Mon}_{\mathrm{nul}}^{\mathrm{pb}}, \mathcal{A}\right)$. Also, we will see in the next subsection that for very strong polynomial functors, as we force κ_x to be null for all objects x of $\mathfrak{M}$, very strong polynomial functors will be closed under kernel of an epimorphism. As a consequence, in the general case where $\mathfrak{M}$ is an object of $\mathfrak{Mon}_{\mathrm{ini}}^{\mathrm{pb}}$, a subfunctor of an object F of $\mathcal{P}ol_n(\mathfrak{M}, \mathcal{A})$ is not necessarily a strong polynomial functor.

Remark 2.13. If we consider $\mathfrak{M} = \mathfrak{U}\boldsymbol{\beta}$, then all object n (ie a natural integer) is clearly the addition of n times the object 1. Hence, because of the last statement of Proposition 2.11, when we will deal with strong polynomiality of objects in $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathcal{A})$, it will suffice to verify the polynomiality for τ_1.

Proposition 2.14. *[8, Proposition 1.9]Let F be an object of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$. Then, the functor F is an object of $\mathcal{P}ol_0(\mathfrak{M}, \mathcal{A})$ if and only if it the quotient of a constant functor of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$.*

Example 2.15. By Proposition 1.27, the category $\mathfrak{U}\boldsymbol{\beta}$ is a pre-braided monoidal category. This example is the first one which led us to extend the definition of [8]. Thus, we have a well-defined notion of strong polynomial functor on the category $\mathfrak{U}\boldsymbol{\beta}$.

Lemma 2.16. *Let n be a natural integer. Let F be a strong polynomial functor of degree n in the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Then a direct summand of F is necessarily an object of the category $\mathcal{P}ol_n(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.*

Proof. Let G be a direct summand of F, in other words we can write $F = G \oplus G'$. for all natural integers m, since the translation functor τ_m is an exact functor by Lemma 2.8 and the difference functor δ_m is a colimit in the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, we deduce that the difference functor commutes with the direct sum:

$$\delta_m F \cong \delta_m G \oplus \delta_m G'.$$

Let $d_0 \ldots, d_k$ be natural integers. Hence:

$$\delta_{d_0} \cdots \delta_{d_k} F \cong (\delta_{d_0} \cdots \delta_{d_k} G) \oplus (\delta_{d_0} \cdots \delta_{d_k} G').$$

A fortiori, if $\delta_{d_0} \cdots \delta_{d_n} F = 0$ then $\delta_{d_0} \cdots \delta_{d_n} G = 0$. ☐

2.2 Very strong polynomial functors

A certain type of functors, called coefficient systems of finite degree, closely related to the strong polynomial one, is used by Randal-Williams and Wahl in [21, Definition 4.10] for their homological stability theorems, generalizing the concept introduced by van der Kallen for general linear groups [24]. Let us define a new type of strong polynomial functor, related to coefficient systems of finite degree.

Definition 2.17. We define the category $\mathcal{VP}ol_n(\mathfrak{M}, \mathcal{A})$ of very strong polynomial functors of degree less or equal to n to be the full subcategory of $\mathcal{P}ol_n(\mathfrak{M}, \mathcal{A})$ as follows:

1. If $n < 0$, $\mathcal{VP}ol_n(\mathfrak{M}, \mathcal{A}) = \{0\}$;
2. if $n \geq 0$, a functor $F \in \mathcal{P}ol_n(\mathfrak{M}, \mathcal{A})$ is an object of $\mathcal{VP}ol_n(\mathfrak{M}, \mathcal{A})$ if for all objects x of $\mathfrak{M}$, $\kappa_x(F) = 0$ and the functor $\delta_x(F)$ is an object of $\mathcal{VP}ol_{n-1}(\mathfrak{M}, \mathcal{A})$.

Remark 2.18. Using the framework introduced by Randal-Williams and Wahl in [21, Definition 4.10], a coefficient system in every object x of $\mathfrak{M}$ of degree n at $N = 0$ is a very strong polynomial functor.

Proposition 2.19. *The category $\mathcal{VP}ol_n(\mathfrak{M}, \mathcal{A})$ is closed under the translation endofunctor τ_x, under kernel of epimorphism and under extension. Moreover, assuming that there exists a set $\mathfrak{E}$ of objects of $\mathfrak{M}$ such that:*

$$\forall m \in Obj(\mathfrak{M}),\ \exists \{e_i\}_{i\in I} \in Obj(\mathfrak{E})\ \textit{(where } I \textit{ is finite)},\ m \cong \underset{i\in I}{\natural} e_i,$$

then, an object F of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ *belongs to $\mathcal{VP}ol_n(\mathfrak{M}, \mathcal{A})$ if and only if $\kappa_e(F) = 0$ and $\delta_e(F)$ is an object of $\mathcal{VP}ol_{n-1}(\mathfrak{M}, \mathcal{A})$ for all objects e of $\mathfrak{E}$.*

Proof. The first point follows from the fact that for all objects x of $\mathfrak{M}$, the endofunctor τ_x commutes with the endofunctors δ_x and κ_x (see Lemma 2.8). For the second and third points, let us consider two short exact sequences of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$: $0 \longrightarrow G \longrightarrow F_1 \longrightarrow F_2 \longrightarrow 0$ and $0 \longrightarrow F_3 \longrightarrow H \longrightarrow F_4 \longrightarrow 0$ with F_i a very strong polynomial functor of degree n for all i. Let x be an object of $\mathfrak{M}$. We use the exact sequence (2) of Lemma 2.8 to obtain the two following exact sequences in the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$:

$$0 \longrightarrow \kappa_x(G) \longrightarrow 0 \longrightarrow 0 \longrightarrow \delta_x(G) \longrightarrow \delta_x(F_1) \longrightarrow \delta_x(F_2) \longrightarrow 0;$$

$$0 \longrightarrow 0 \longrightarrow \kappa_x(H) \longrightarrow 0 \longrightarrow \delta_x(F_3) \longrightarrow \delta_x(H) \longrightarrow \delta_x(F_4) \longrightarrow 0.$$

Therefore, $\kappa_x(F) = \kappa_x(H) = 0$ and the result follows directly by induction on the degree of polynomiality. For the last point, we consider the long exact sequence (3) of Lemma 2.8 applied to an object F of $\mathcal{VP}ol_n(\mathfrak{M}, \mathcal{A})$ to obtain the following exact sequence in the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$:

$$0 \longrightarrow \kappa_y(F) \longrightarrow \kappa_{x\natural y}(F) \longrightarrow \tau_x\kappa_y(F) \longrightarrow \delta_y(F) \longrightarrow \delta_{x\natural y}(F) \longrightarrow \tau_y\delta_x(F) \longrightarrow 0.$$

Hence, by induction on the length of objects as monoidal product of $\{e_i\}_{i\in I}$, we deduce that $\kappa_m(F) = 0$ for all objects m of $\mathfrak{M}$ if and only if $\kappa_e(F) = 0$ for all objects e of $\mathfrak{E}$. Moreover, since $\mathcal{VP}ol_n(\mathfrak{M}, \mathcal{A})$ is closed under extension and by the translation endofunctor τ_y, the result follows directly by induction on the degree of polynomiality n. $\square$

Proposition 2.20. *Let F be an object of* $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$*. The functor F is an object of $\mathcal{VP}ol_0(\mathfrak{M}, \mathcal{A})$ if and only if it is equivalent to a constant functor.*

Proof. Using the long exact sequence 1 of Lemma 2.8 applied to F, we deduce that F is an object of $\mathcal{VP}ol_0(\mathfrak{M}, \mathcal{A})$ if and only if $F \cong \tau_x F$ for all objects x of $\mathfrak{M}$. It is equivalent to say that for all objects x of $\mathfrak{M}$, the morphism $F(\iota_x)$ is an isomorphism. Hence, F is an object of $\mathcal{VP}ol_0(\mathfrak{M}, \mathcal{A})$ if and only if it is equivalent to the constant functor equals to $F(0)$. $\square$

There exist strong polynomial functors which are not very strong polynomial in any degree.

Example 2.21. Let us consider the category $\mathfrak{U}\boldsymbol{\beta}$ and n a natural integer. Let A be an object of $\mathcal{A}$ and let $0_{\mathcal{A}}$ be the null object of $\mathcal{A}$. We denote by $t_A : A \to 0$ the unique morphism in $Hom_{\mathcal{A}}(A, 0_{\mathcal{A}})$. Let $\mathfrak{A}_n$ be an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathcal{A})$, defined by:

- Objects: $\forall m \in \mathbb{N}$, $\mathfrak{A}_n(m) = \begin{cases} A & \text{if } n = m \\ 0_{\mathcal{A}} & \text{otherwise} \end{cases}$.
- Morphisms: let $[j-i, f]$ with $f \in \mathbf{B}_n$ be a morphism from i to j in the category $\mathfrak{U}\boldsymbol{\beta}$. Then:

$$\mathfrak{A}_n(f) = \begin{cases} id_A & \text{if } i = j = n \\ t_A & \text{if } i = n \leq j \\ \iota_A & \text{if } i \leq j = n \\ id_{0_{\mathcal{A}}} & \text{otherwise.} \end{cases}$$

The functor $\mathfrak{A}_n$ is called the atomic functor in A of degree n. For coherence, we fix $\mathfrak{A}_{-1}$ to be the null functor of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathcal{A})$. Then, it is clear that $i_p(\mathfrak{A}_n)$ is the zero natural transformation. On the one hand, we deduce the following natural equivalence $\kappa_1(\mathfrak{A}_n) \cong \mathfrak{A}_n$ and a fortiori $\mathfrak{A}_n$ is not a very strong polynomial functor. On the other hand, it is worth noting the natural equivalence $\delta_1(\mathfrak{A}_n) \cong \tau_1(\mathfrak{A}_n)$ and the fact that $\tau_1(\mathfrak{A}_n) \cong \mathfrak{A}_{n-1}$. Therefore, we recursively prove that $\mathfrak{A}_n$ is a strong polynomial functor of degree n.

Remark 2.22. On the contrary of $\mathcal{P}ol_n(\mathfrak{M}, \mathcal{A})$, a quotient of an object F of $\mathcal{VP}ol_n(\mathfrak{M}, \mathcal{A})$ is not necessarily a very strong polynomial functor. For example, for $\mathfrak{M} = \mathfrak{U}\boldsymbol{\beta}$ and A be an object of $\mathcal{A}$, let us consider the functor $\mathfrak{A}_0$ defined in Example 2.21, which we proved to be a strong polynomial functor of degree 0. Let $\mathfrak{A}$ be the constant object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathcal{A})$ equals to A. Then, we define a natural transformation $\alpha : \mathfrak{A} \Longrightarrow \mathfrak{A}_0$ assigning:

$$\forall n \in \mathbb{N},\ \alpha_n = \begin{cases} id_A & \text{if } n = 0 \\ t_A & \text{otherwise.} \end{cases}$$

Moreover, it is an epimorphism in the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathcal{A})$ since for all natural integers n, $coker(\alpha_n) = 0_{\mathcal{A}}$. We proved in Example 2.21 that $\mathfrak{A}_0$ is not a very strong polynomial functor of degree 0 whereas $\mathfrak{A}$ is a very strong polynomial functor of degree 0 by Proposition 2.20.

2.3 Examples of polynomial functors associated with braid representations

Different families of representations of braid group can be interpreted as very strong polynomial functors. First, we recall a result due to Tong, Yang and Ma.

Tong-Yang-Ma results In 1996, in the article [23], Tong, Yang and Ma interested in the representations of B_n where the ith generator is sent to a matrix of the form $Id_{i-1} \oplus T \oplus Id_{n-i-1}$, with T a $m \times m$ non-singular matrix and $m \geq 2$. In particular, for $m = 2$, they prove that there exist up to equivalence only two non trivial representations of this type. We give here the result and an interpretation of their work from a functorial point of view.

Theorem 2.23. *[23, Part II] Let $\eta : \mathbf{B}_\bullet \longrightarrow GL_\bullet$ be a natural transformation. Assume that for all natural integers n, for all $i \in \{1, \ldots, n-1\}$, the following diagram is commutative.*

$$\begin{tikzcd} \mathbf{B}_n \arrow[r, "\eta_n"] & GL_n(\mathbb{K}) \\ \mathbf{B}_2 \arrow[u, "incl_i^n"] \arrow[r, "\eta_2"'] & GL_2(\mathbb{K}) \arrow[u, "id_{i-1} * - * id_{m-i-1}"'] \end{tikzcd}$$

Two natural transformations $\eta$and η' will be said here to be equivalent if there exists a natural equivalence $\mu : GL_\bullet \longrightarrow GL_\bullet$ such that $\mu \circ \eta = \eta'$. Then, the natural transformation η is equivalent to the one of the following natural transformations.

1. *The trivial natural transformation, denoted by $\mathfrak{id}$: for every generator σ_i of $\mathbf{B}_n$, $\mathfrak{id}_n(\sigma_i) = Id_{GL_n(\mathbb{K})}$*
2. *The unreduced Burau natural transformation, denoted by $\mathfrak{bur}$: for all generators σ_i of $\mathbf{B}_n$,*
$$\mathfrak{bur}_{n,t}(\sigma_i) = Id_{i-1} \oplus B(t) \oplus Id_{n-i-1},$$
with
$$B(t) = \begin{bmatrix} 1-t & t \\ 1 & 0 \end{bmatrix}.$$
3. *A natural transformation based on a representation, which we call the Tong-Yang-Ma representation, denoted by $\mathfrak{tym}$: for every generator σ_i of $\mathbf{B}_n$ if $n \geq 2$,*
$$\mathfrak{tym}_{n,t}(\sigma_i) = Id_{i-1} \oplus TYM(t) \oplus Id_{n-i-1},$$
with
$$TYM(t) = \begin{bmatrix} 0 & 1 \\ t & 0 \end{bmatrix}.$$

Remark 2.24. The unreduced Burau representation (see [12, Section 3.1] or [5, Section 4.2] for more details about this family of representations) is reducible but indecomposable, whereas the Tong-Yang-Ma representation is irreducible [23, Part II]. We may also define a reduced Burau natural transformation (see [12, Section 3.3] for more details about this family of representations), denoted by $\overline{\mathfrak{bur}}$. For $n = 2$, one assigns $\overline{\mathfrak{bur}}(\sigma_1) := -t$. for all natural integers $n \geq 3$, we define for every Artin generator σ_i of $\mathbf{B}_n$ with $i \in \{2, \ldots, n-2\}$:

$$\overline{\mathfrak{bur}}_{n,t}(\sigma_i) = Id_{i-2} \oplus \overline{B}(t) \oplus Id_{n-i-2}$$

with

$$\overline{B}(t) = \begin{bmatrix} 1 & t & 0 \\ 0 & -t & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

and

$$\overline{\mathfrak{bur}}_{n,t}(\sigma_1) = \begin{bmatrix} -t & 0 \\ 1 & 1 \end{bmatrix} \oplus Id_{n-3} \quad ; \quad \overline{\mathfrak{bur}}_{n,t}(\sigma_{n-1}) = Id_{n-3} \oplus \begin{bmatrix} 1 & t \\ 0 & -t \end{bmatrix}.$$

Let us move on to a presentation of examples of polynomial functors associated with these families of braid representations. A first example is based on the family introduced by Tong, Yang and Ma.

Remark 2.25. Since, $\mathbf{B}_{\infty}^{ab}$ is isomorphic to $\mathbb{Z}$ as abelian groups, the group rings $R\left[\mathbf{B}_{\infty}^{ab}\right]$ and $R\left[\left\{t_i^{\pm 1} \mid i \in \{1, \ldots, n\}\right\}\right]$ are isomorphic as R-modules.

Example 2.26. Let $\mathfrak{TYM} : \mathfrak{U}\boldsymbol{\beta} \longrightarrow R\left[\mathbf{B}_{\infty}^{ab}\right]\text{-}\mathfrak{Mod}$ be the functor defined by:

- Objects: $\forall n \in \mathbb{N}$, $\mathfrak{TYM}(n) = R\left[\mathbf{B}_{\infty}^{ab}\right]^{\oplus n}$.
- Morphisms:
  - Automorphisms: for all interger $n \geq 2$, for every Artin generator σ_i of $\mathbf{B}_n$:

$$\mathfrak{TYM}(\sigma_i) = \mathfrak{tym}_{n,t_1}(\sigma_i).$$

  - General morphisms: let $n, n' \in \mathbb{N}$, such that $n' \geq n$, and $[n'-n, \sigma] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n, n')$. We define:

$$\mathfrak{TYM}([n'-n, \sigma]) = \mathfrak{TYM}(\sigma) \circ \gamma_{n,n'}^{R\left[\mathbf{B}_{\infty}^{ab}\right]}.$$

We call this functor the Tong-Yang-Ma functor. The assignment of $\mathfrak{TYM}$ defines a functor since $\mathfrak{TYM}(id_n) = id_{\mathbb{K}\left[\mathbf{B}_{\infty}^{ab}\right]^{\oplus n}}$ and for $i, j \in \{1, \ldots, n-1\}$, such that $j \geq i$, by our definition $\mathfrak{TYM}(\sigma_j \circ \sigma_i) = \mathfrak{TYM}(\sigma_j) \circ \mathfrak{TYM}(\sigma_i)$ (Remark (1.34) ensuring that assignment of $\mathfrak{TYM}$ defines properly a functor on $\mathfrak{U}\boldsymbol{\beta}$).

Another example naturally arises from the unreduced Burau representations.

Example 2.27. Let $\mathfrak{Bur} : \mathfrak{U}\boldsymbol{\beta} \longrightarrow R\left[\mathbf{B}_{\infty}^{ab}\right]\text{-}\mathfrak{Mod}$ be the functor defined by:

- Objects: $\forall n \in \mathbb{N}$, $\mathfrak{Bur}(n) = R\left[\mathbf{B}_{\infty}^{ab}\right]^{\oplus n}$.
- Morphisms:
  - Automorphisms: for all natural integers $n \geq 2$, for every Artin generator σ_i of $\mathbf{B}_n$:

$$\mathfrak{Bur}(\sigma_i) = \mathfrak{bur}_{n,t_1}(\sigma_i).$$

  - General morphisms: Let $n, n' \in \mathbb{N}$, such that $n' \geq n$, and $[n'-n, \sigma_i] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n, n')$. We define:

$$\mathfrak{Bur}([n'-n, \sigma_i]) = \mathfrak{Bur}(\sigma_i) \circ \gamma_{n,n'}^{R\left[\mathbf{B}_{\infty}^{ab}\right]}.$$

The functor $\mathfrak{Bur}$ already appears in [21, Example 4.3 and 4.15]. We call this functor the unreduced Burau functor.

The following example corresponds to the family of the reduced Burau representations.

Example 2.28. Let $\overline{\mathfrak{Bur}}:\mathfrak{U}\boldsymbol{\beta}\longrightarrow R\left[\mathbf{B}_{\infty}^{ab}\right]\text{-}\mathfrak{Mod}$ be the functor defined by:

- Objects: $\forall n\in\mathbb{N}^{*}$, $\overline{\mathfrak{Bur}}\left(n\right)=R\left[\mathbf{B}_{\infty}^{ab}\right]^{\oplus n-1}$ and $\overline{\mathfrak{Bur}}\left(0\right)=0$.
- Morphisms:
  - Automorphisms: for all interger $n\geq 2$, for every Artin generator σ_i of $\mathbf{B}_n$:
  $$\overline{\mathfrak{Bur}}\left(\sigma_i\right)=\overline{\mathfrak{bur}}_{n,t_1}\left(\sigma_1\right).$$
  - General morphisms: Let $n,n'\in\mathbb{N}$, such that $n'\geq n$, and $[n'-n,\sigma_i]\in Hom_{\mathfrak{U}\boldsymbol{\beta}}\left(n,n'\right)$. We define:
  $$\overline{\mathfrak{Bur}}\left([n'-n,\sigma_i]\right)=\overline{\mathfrak{Bur}}\left(\sigma_i\right)\circ\gamma_{n-1,n'-1}^{R\left[\mathbf{B}_{\infty}^{ab}\right]}.$$

We call this functor the reduced Burau functor. The assignment of $\overline{\mathfrak{Bur}}$ defines a functor since $\overline{\mathfrak{Bur}}\left(id_n\right)=id_{R\left[\mathbf{B}_{\infty}^{ab}\right]^{\oplus n-1}}$ and for $i,j\in\{1,\ldots,n-1\}$, such that $j\geq i$, by our definition $\overline{\mathfrak{Bur}}\left(\sigma_j\circ\sigma_i\right)=\overline{\mathfrak{Bur}}\left(\sigma_j\right)\circ\overline{\mathfrak{Bur}}\left(\sigma_i\right)$ (Remark (1.34) ensures that assignment of $\overline{\mathfrak{Bur}}$ defines properly a functor on $\mathfrak{U}\boldsymbol{\beta}$).

Proposition 2.29. *The functors $\mathfrak{Bur}$ and $\mathfrak{TYM}$ are very strong polynomial functors of degree* 1.

Proof. For the functor $\mathfrak{Bur}$, it is a consequence of [21, Example 4.15]. So we will focus on the case of the functor $\mathfrak{TYM}$. Let n be a natural integer. By the statement of Remark 2.13, it is sufficient to consider the application $i_1\mathfrak{TYM}\left([0,id_n]\right)=\gamma_{n,1+n}^{R\left[\mathbf{B}_{\infty}^{ab}\right]}$. This map is a monomorphism and its cokernel is $R\left[\mathbf{B}_{\infty}^{ab}\right]$. So $\kappa_1\mathfrak{TYM}=0$ is the null functor of $\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta},R\left[\mathbf{B}_{\infty}^{ab}\right]\text{-}\mathfrak{Mod}\right)$. Let n' be a natural integer such that $n'\geq n$ and let $\varphi_n^{n'}=[n'-n,\phi]\in Hom_{\mathfrak{U}\boldsymbol{\beta}}\left(n,n'\right)$. By naturality and the universal property of the cokernel, there exists a unique endomorphism of $R\left[\mathbf{B}_{\infty}^{ab}\right]$ such that the following diagram commutes, where the lines are exact. It is exactly the definition of $\delta_1\mathfrak{TYM}\left([n'-n,\phi]\right)$.

$$\begin{tikzcd}
0 \arrow[r] & R\left[\mathbf{B}_{\infty}^{ab}\right]^{n} \arrow[r, "\gamma_n^{R\left[\mathbf{B}_{\infty}^{ab}\right]}"] \arrow[d, "\mathfrak{TYM}\left(\varphi_n^{n'}\right)"'] & R\left[\mathbf{B}_{\infty}^{ab}\right]^{\oplus n+1} \arrow[r, "\pi_{n+1}"] \arrow[d, "\tau_1\left(\mathfrak{TYM}\right)\left(\varphi_n^{n'}\right)"] & R\left[\mathbf{B}_{\infty}^{ab}\right] \arrow[r] \arrow[d, dotted, "\exists!"] & 0 \\
0 \arrow[r] & R\left[\mathbf{B}_{\infty}^{ab}\right]^{\oplus n'} \arrow[r, "\gamma_{n'}^{R\left[\mathbf{B}_{\infty}^{ab}\right]}"'] & R\left[\mathbf{B}_{\infty}^{ab}\right]^{\oplus n'+1} \arrow[r, "\pi_{n'+1}"'] & R\left[\mathbf{B}_{\infty}^{ab}\right] \arrow[r] & 0
\end{tikzcd}$$

For all $(b,a)\in R\left[\mathbf{B}_{\infty}^{ab}\right]^{\oplus n}\oplus R\left[\mathbf{B}_{\infty}^{ab}\right]=R\left[\mathbf{B}_{\infty}^{ab}\right]^{\oplus n+1}$:

$$\tau_1\left(\mathfrak{TYM}\right)\left(\varphi_n^{n'}\right)(b,a)\quad=\quad\left(\mathfrak{TYM}\left([n'-n,\phi]\right)(b),a\right).$$

Therefore:

$$\left(\pi_{n'+1}\circ\tau_1\left(\mathfrak{TYM}\right)\left(\varphi_n^{n'}\right)\right)(b,a)=a=\pi_{n+1}\left(b,a\right).$$

Hence, $id_{R\left[\mathbf{B}_{\infty}^{ab}\right]}$ also makes the diagram commutative and thus $\delta_1\mathfrak{TYM}\left([n'-n,\phi]\right)=id_{R\left[\mathbf{B}_{\infty}^{ab}\right]}$. Hence, $\delta_1\mathfrak{TYM}$ is the constant functor equals to $R\left[\mathbf{B}_{\infty}^{ab}\right]$. A fortiori, because of Proposition 2.14, $\delta_1\mathfrak{TYM}$ is a degree 0 very strong polynomial functor. $\square$

Notation 2.30. We could have defined the same functors with another parameter as t_1. In this case, we put this other parameter $t'\in R\left[\mathbf{B}_{\infty}^{ab}\right]$ in index of the notation (e.g. $\mathfrak{Bur}_{t'}$). For $y\in\mathbb{K}$ and $F\in\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod}\right)$, let $yF:\mathfrak{U}\boldsymbol{\beta}\longrightarrow\mathbb{K}\text{-}\mathfrak{Mod}$ be the functor defined in the same way as F for objects, such that for all natural integers n, for every Artin generator σ_i of $\mathbf{B}_n$ (with $i\in\{1,\ldots,n-1\}$): $\left(yF\right)\left(\sigma_i\right)=yF\left(\sigma_i\right)$. Also, we could have defined the Burau functor with the assignement $\left(\mathbb{C}\left[\mathbf{B}_{\infty}^{ab}\right]\right)\left[\mathbf{B}_{\infty}^{ab}\right]^{\oplus n}$ on each object $n\in\mathbb{N}$ and the same assignement for morphisms. We will denote this version by $\breve{\mathfrak{Bur}}_{t_1}:\mathfrak{U}\boldsymbol{\beta}\longrightarrow\left(\mathbb{C}\left[\mathbf{B}_{\infty}^{ab}\right]\right)\left[\mathbf{B}_{\infty}^{ab}\right]\text{-}\mathfrak{Mod}$. We prove that $\breve{\mathfrak{Bur}}_{t_1}$ is also very strong polynomial of degree one exact exactly the same way as for $\mathfrak{Bur}$.

A link between the unreduced Burau and the reduced Burau functors can be established: the functor $\overline{\mathfrak{Bur}}$ is in fact a subfunctor of the functor $\mathfrak{Bur}$.

Definition 2.31. Let $\mathcal{T}_1 : \mathfrak{U}\boldsymbol{\beta} \longrightarrow R\left[\mathbf{B}_{\infty}^{ab}\right]\text{-}\mathfrak{Mod}$ be the functor defined by:

- Objects: $\forall n \in \mathbb{N}^*$, $\mathcal{T}_1(n) = R\left[\mathbf{B}_{\infty}^{ab}\right]$ and $\mathcal{T}_1(0) = 0$.
- Morphisms: for all elements b of $\mathbf{B}_n$, $\mathcal{T}_1(\emptyset) = id_0$ if $n = 0$ and $\mathcal{T}_1(b) = id_{R\left[\mathbf{B}_{\infty}^{ab}\right]}$ if $n \geq 1$.

Remark 2.32. The functor $\mathcal{T}_1$ is obviously a strong polynomial functor of degree 1. It is not very strong polynomial of degree 1 since $\delta_1\mathcal{T}_1 \cong \mathfrak{A}_0$. Nevertheless, it is worth noting that $\kappa_1\mathcal{T}_1 = 0$.

Lemma 2.33. *There exists a natural equivalence $\mathfrak{Bur} \overset{r}{\Longrightarrow} \tau_1\overline{\mathfrak{Bur}}$, defined for all natural integers nby:*

$$r_n = \overbrace{\begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & -1 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}}^{n}$$

Proof. Considering the objects, we know that $\mathfrak{Bur}(n) = R\left[\mathbf{B}_{\infty}^{ab}\right]^{\oplus n} = \tau_1\overline{\mathfrak{Bur}}(n)$ for all natural integers n. Moreover, it follows from straightforward matrix computations that $r_n \circ \mathfrak{Bur}(\sigma_i) = \tau_1\overline{\mathfrak{Bur}}(\sigma_i) \circ r_n$ for all $i \in \{1, \ldots, n-1\}$. $\square$

Proposition 2.34. *There exists a short exact sequence of* $\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta}, R\left[\mathbf{B}_{\infty}^{ab}\right]\text{-}Mod\right)$ *with the following form:*

$$0 \longrightarrow \overline{\mathfrak{Bur}} \longrightarrow \mathfrak{Bur} \longrightarrow \mathcal{T}_1 \longrightarrow 0\ .$$

It does not split.

Proof. One defines a natural transformation $r' : \overline{\mathfrak{Bur}} \Longrightarrow \tau_1\overline{\mathfrak{Bur}}$ by assigning for all natural integers n:

$$r'_n = \begin{bmatrix} 0 \\ Id_{n-1} \end{bmatrix} : R\left[\mathbf{B}_{\infty}^{ab}\right]^{\oplus n-1} \hookrightarrow R\left[\mathbf{B}_{\infty}^{ab}\right]^{\oplus n}.$$

We thus define the natural transformation:

$$0 \longrightarrow \overline{\mathfrak{Bur}} \xrightarrow{r^{-1}\circ r'} \mathfrak{Bur}\ .$$

Repeating mutatis mutandis the work done in the proof of Proposition 2.29, we conclude that for all natural integers n we have a short exact sequence, natural in n:

$$0 \longrightarrow \overline{\mathfrak{Bur}}(n) \longrightarrow \mathfrak{Bur}(n) \longrightarrow \mathcal{T}_1(n) \longrightarrow 0\ .$$

$\square$

Remark 2.35. Furthermore, it follows from the proof of Proposition 2.34, we have the equivalence of $\mathfrak{U}\boldsymbol{\beta}$-modules $\tau_1\mathfrak{Bur} \cong \tau_1\overline{\mathfrak{Bur}} \oplus \mathcal{T}_1$. Indeed, it follows directly from the definition of the Burau functor that for all generator σ_i of $\mathbf{B}_n$, $\tau_1\mathfrak{Bur}(\sigma_i) = \mathfrak{Bur}(\sigma_i) \oplus \mathcal{T}_1(\sigma_i)$. The verification for the objects being trivial and because of Remark 1.34, we directly deduce that we have an equality $\tau_1\mathfrak{Bur} = \mathfrak{Bur} \oplus \mathcal{T}_1$. Therefore, we obtain the desired equivalence.

Corollary 2.36. *The functor $\overline{\mathfrak{Bur}}$ is a strong polynomial functor of degree* 2.

Proof. According to Proposition 2.34 and Remark 2.32, the functor $\overline{\mathfrak{Bur}}$ is the kernel of the epimorphism $\mathfrak{Bur} \longrightarrow \mathcal{T}_1 \longrightarrow 0$. Thanks to the exact sequence (2) of Lemma 2.8, we obtain the following exact sequence in the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$:

$$0 \longrightarrow \delta_1\left(\overline{\mathfrak{Bur}}\right) \longrightarrow \delta_1(\mathfrak{Bur}) \longrightarrow \delta_1(\mathcal{T}_1) \cong \mathfrak{A}_0 \longrightarrow 0.$$

The following exact sequence is given applying again the exact sequence (2) of Lemma 2.8, Proposition 2.29 and Example 2.21:

$$0 \longrightarrow \kappa_1\delta_1\left(\overline{\mathfrak{Bur}}\right) \longrightarrow 0 \longrightarrow \mathfrak{A}_0 \longrightarrow \delta_1\delta_1\left(\overline{\mathfrak{Bur}}\right) \longrightarrow 0.$$

Hence, $\mathfrak{A}_0 \cong \delta_1\delta_1\left(\overline{\mathfrak{Bur}}\right)$ and the result follows from the fact that $\mathfrak{A}_0$ is a strong polynomial functor of degree 0 according to Example 2.21. ☐

A last example is given by the family of the Lawrence-Krammer representations (see [1, 14, 13]).

Example 2.37. Let $\mathfrak{LK} : \mathfrak{U}\boldsymbol{\beta} \longrightarrow \left(\mathbb{C}\left[\mathbf{B}_{\infty}^{ab}\right]\right)\left[\mathbf{B}_{\infty}^{ab}\right]\text{-}\mathfrak{Mod}$ be the functor defined by:

- Objects: for all natural integers $n \geq 2$, $\mathfrak{LK}(n) = \bigoplus_{1 \leq j < k \leq n} V_{j,k}$, with for all $1 \leq j < k \leq n$, $V_{j,k}$ is a free $\left(\mathbb{C}\left[\mathbf{B}_{\infty}^{ab}\right]\right)\left[\mathbf{B}_{\infty}^{ab}\right]$-module of rank one. Hence, $\mathfrak{LK}(n) \cong \left(\left(\mathbb{C}\left[\mathbf{B}_{\infty}^{ab}\right]\right)\left[\mathbf{B}_{\infty}^{ab}\right]\right)^{\oplus n(n-1)/2}$ as $\left(\mathbb{C}\left[\mathbf{B}_{\infty}^{ab}\right]\right)\left[\mathbf{B}_{\infty}^{ab}\right]$-modules. Moreover, one assigns $\mathfrak{LK}(1) = 0$ and $\mathfrak{LK}(0) = 0$.

- Morphisms:

  – Automorphisms: for all natural integers n, for every Artin generator σ_i of $\mathbf{B}_n$ (with $i \in \{1, \ldots, n-1\}$), for all $v_{j,k} \in V_{j,k}$ (with $1 \leq j < k \leq n$),

$$\mathfrak{LK}(\sigma_i)\, v_{j,k} = \begin{cases} v_{j,k} & \text{if } i \notin \{j-1, j, k-1, k\}, \\ t_1 v_{i,k} + \left(t_1^2 - t_1\right) v_{i,i+1} + (1-t_1)\, v_{i+1,k} & \text{if } i = j-1, \\ v_{i+1,k} & \text{if } i = j \neq k-1, \\ t_1 v_{j,i} + (1-t_1)\, v_{j,i+1} + \left(t_1^2 - t_1\right) t_2 v_{i,i+1} & \text{if } i = k-1 \neq j. \\ v_{j,i+1} & \text{if } i = k, \\ -t_2 t_1^2 v_{i,i+1} & \text{if } i = j = k-1. \end{cases}$$

  – General morphisms: let $n, n' \in \mathbb{N}$, such that $n' \geq n$, and $[n'-n, \sigma] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n, n')$. We define:

$$\mathfrak{LK}([n'-n, \sigma]) = \mathfrak{LK}(\sigma) \circ \gamma_{n(n-1)/2,\, n'(n'-1)/2}^{\left(R\left[\mathbf{B}_{\infty}^{ab}\right]\right)\left[\mathbf{B}_{\infty}^{ab}\right]}.$$

Remark 2.38. The Lawrence-Krammer representations are usually defined as modules over a group ring of type $\mathbb{C}\left\langle t^{\pm 1}, q^{\pm 1}\right\rangle$. However, since we have lead all our work for commutative rings, it is the reason why we have chosen to consider $\mathbb{C}[\mathbb{Z} \oplus \mathbb{Z}]$ which is restrictive, but does not change the frame of the Lawrence-Krammer representations.

Lemma 2.39. *There is a natural equivalence $\delta_1\mathfrak{LK} \overset{r''}{\Longrightarrow} \check{\mathfrak{Bur}}_{t_1}$, defined for all natural integers $n$by:*

$$r''_n = \begin{array}{c} \overbrace{\hphantom{0 \quad \cdots \quad 0 \quad 1}}^{n} \\ \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \end{array}.$$

Proof. Let $n$and n' be two natural integers such that $n' \geq n$. Let $\varphi_n^{n'} = [n'-n, \phi] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n, n')$. By naturality and because of the universal property of the cokernel, there exists a unique endomorphism of

$\left(\mathbb{C}\left[\mathbf{B}_{\infty}^{ab}\right]\right)\left[\mathbf{B}_{\infty}^{ab}\right]$-modules such that the following diagram commutes, where the lines are exact. It is exactly the definition of $\delta_1\mathfrak{LK}\left([n'-n,\phi]\right)$.

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus\limits_{1\leq j<k\leq n} V_{j,k} \arrow[r, "i_1\mathfrak{LK}([0,id_n])"] \arrow[d, "\mathfrak{LK}\left(\varphi_n^{n'}\right)"'] & \bigoplus\limits_{1\leq i<l\leq n+1} V_{i,l} \arrow[r, "\pi_n"] \arrow[d, "\tau_1(\mathfrak{LK})\left(\varphi_n^{n'}\right)"] & \bigoplus\limits_{1\leq i\leq n} V_{i,n+1} \arrow[r] \arrow[d, dotted, "\exists!"] & 0 \\
0 \arrow[r] & \bigoplus\limits_{1\leq j'<k'\leq n'} V_{j,k} \arrow[r, "i_1\mathfrak{LK}([0,id_{n'}])"'] & \bigoplus\limits_{1\leq i'<l'\leq n'+1} V_{i,l} \arrow[r, "\pi_{n'}"'] & \bigoplus\limits_{1\leq i'\leq n'} V_{i,n+1} \arrow[r] & 0
\end{tikzcd}$$

Let $i\in\{1,\ldots,n-1\}$, let $j\in\{1,\ldots,n-1\}$ and let $v_{j,n+1}$ be an element of $V_{j,n+1}$. Then we compute:

$$\mathfrak{LK}\left(\sigma_i\right)v_{j,n+1}=\begin{cases} v_{j,n+1} & \text{if } i\notin\{j-1,j\},\\ t_1v_{j-1,n+1}+\left(t_1^2-t_1\right)v_{j-1,j}+\left(1-t_1\right)v_{j,n+1} & \text{if } i=j-1,\\ v_{j+1,n+1} & \text{if } i=j\neq n.\end{cases}$$

We deduce that in the canonical basis $\{\mathbf{e}_{1,n+1},\mathbf{e}_{2,n+1},\ldots,\mathbf{e}_{n,n+1}\}$ of $\bigoplus\limits_{1\leq i\leq n} V_{i,n+1}$.

$$\delta_1\mathfrak{LK}\left(\sigma_i\right)=Id_{i-1}\oplus\widetilde{B}\left(t_1\right)\oplus Id_{n-i-1},$$

with:

$$\widetilde{B}\left(t_1\right)=\left[\begin{array}{cc} 0 & 1\\ t_1 & 1-t_1\end{array}\right].$$

□

Proposition 2.40. *The functor $\mathfrak{LK}$ is a very strong polynomial functor of degree 2.*

Proof. Let n be a natural integer. By Remark 2.13, we only have to consider the application $i_1\mathfrak{LK}\left([0,id_n]\right)$. This map is clearly a monomorphism and its cokernel is $\bigoplus\limits_{1\leq i\leq n} V_{i,n+1}$. Hence, $\kappa_1\mathfrak{LK}$ is the null constant functor of $\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta},\left(\mathbb{C}\left[\mathbf{B}_{\infty}^{ab}\right]\right)\left[\mathbf{B}_{\infty}^{ab}\right]\text{-}\mathfrak{Mod}\right)$. Since the functor $\mathfrak{B}\check{\mathfrak{ur}}_{t_1}$ is very strong polynomial of degree one by Proposition 2.29, we deduce from Lemma 2.39 that $\mathfrak{LK}$ is very strong polynomial of degree two. □

3 Functoriality of the Long-Moody construction

3.1 Review of the Long-Moody construction

The principle of the Long-Moody construction, corresponding to Theorem 2.1 of [18], is to build a linear representation of the braid group $\mathbf{B}_n$ starting from a representation $\rho:\mathbf{B}_{n+1}\to GL\left(V\right)$, with V a $\mathbb{K}$-module. Beforehand, remark that for $\varsigma_n:\mathbf{F}_n\to\mathbf{B}_{n+1}$ a group morphism, ρ induces a representation $\rho\circ\varsigma_n:\mathbf{F}_n\to GL\left(V\right)$ and therefore V is a $\mathbb{K}\left[\mathbf{F}_n\right]$-module. Moreover, since $\mathbf{B}_n$ can be viewed as a subgroup of $\mathbf{B}_{n+1}$, choosing an embedding $\gamma_n^b:\mathbf{B}_n\hookrightarrow\mathbf{B}_{n+1}$, V is also a $\mathbb{K}\left[\mathbf{B}_n\right]$-modul thanks to $\rho\circ\gamma_n^b:\mathbf{B}_n\to GL\left(V\right)$. Recall that the notation $a_{n,1}$ for the Artin representation (see Notation 1.15).

Theorem 3.1. *[18, Theorem 2.1] The original Long-Moody construction associated with ρ, defined as follows, is a representation of $\mathbf{B}_n$:*

$$\mathcal{LM}\left(\rho\right):\quad \mathbf{B}_n\quad\longrightarrow\quad GL\left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}\underset{\mathbb{K}[\mathbf{F}_n]}{\otimes}V\right)$$

where for all $\sigma\in\mathbf{B}_n$, $i\in\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $v\in V$:

$$\mathcal{LM}\left(\rho\right)\left(\sigma\right)\left(i\underset{\mathbb{K}[\mathbf{F}_n]}{\otimes}v\right)=a_{n,1}\left(\sigma\right)\left(i\right)\underset{\mathbb{K}[\mathbf{F}_n]}{\otimes}\rho\left(\gamma_n^b\left(\sigma\right)\right)\left(v\right).$$

We give here an alternative of the Long-Moody construction, which appears in the following Theorem. It is analogous to the one of [18, Theorem 2.1]. In Remark 4.8, we explain why we slightly modify the construction of Long and Moody. Analogously, it is worth noting that for $\rho : \mathbf{B}_n \to GL(V)$ a representation of $\mathbf{B}_n$, with V a $\mathbb{K}$-module, ρ induces a representation $\rho \circ \varsigma_{n,\bullet} : \mathbf{F}_n \to GL(V)$. As a consequence, V is equipped with a structure of $\mathbb{K}[\mathbf{F}_n]$-module.

Theorem 3.2. *[18, Alternative of Theorem 2.1] Let $\rho : \mathbf{B}_n \to GL(V)$ be a representation of $\mathbf{B}_n$, with V a $\mathbb{K}$-module, and let $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^f})$ be a coherent triplet. The alternative Long-Moody construction associated with ρ, which is defined as follows, is a representation of $\mathbf{B}_n$:*

$$lm_{(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f})}(\rho) : \quad \mathbf{B}_n \quad \longrightarrow \quad GL\left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} V\right)$$

where for all $\sigma \in \mathbf{B}_n$, $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $v \in V$:

$$lm_{(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f})}(\rho)(\sigma)\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right) = a_{n,\bullet}(\sigma)(i) \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} \rho(\sigma)(v).$$

Notation 3.3. When there is no ambiguity, once the triplet $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^f})$ is clearly given, we forget it in the notation $lm_{(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f})}$ for convenience (especially for proofs).

Proof. Let $\sigma, \sigma' \in \mathbf{B}_n$. For all elements v of V and i of $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$, since ρ is a representation of $\mathbf{B}_n$ and the map $a_{n,\bullet} : \mathbf{B}_n \to Aut(\mathbf{F}_n)$ is a group morphism, we directly prove that:

$$lm(\rho)(\sigma\sigma')\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right) \quad = \quad lm(\rho)(\sigma)\left(lm(\rho)(\sigma')\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right)\right).$$

□

Remark 3.4. For all $k \in \{1, \ldots, n\}$, we denote by $V_k = \mathbb{K}[(g_k - 1)] \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} V$ with g_k a generator of $\mathbf{F}_n$. The action of $\mathbf{B}_n$ on $\mathbf{F}_n$ extends naturally on $\mathbb{K}[\mathbf{F}_n]$. Then $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} V$ is isomorphic to $V^{\oplus n}$. Indeed, we have an isomorphism $\Lambda : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} V \longrightarrow \bigoplus_{k=1}^{n} V_k \cong V^{\oplus n}$ defined by:

$$\forall v \in V, \forall k \in \{1, \ldots, n\}, \ \Lambda\left((g_k - 1) \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right) := \left(0, \ldots, 0, \overbrace{v}^{k-th}, 0, \ldots, 0\right).$$

In fact, we may have a matricial point of view on this construction (see [18, Theorem 2.2]). In the same way, the study of Bigelow and Tian in [2] is performed from a purely matricial point of view.

Remark 3.5. In the original Long-Moody construction for representations, one could modify the construction adding an extra parameter $s \in \mathbb{K}^*$ (see [18, Corollary 2.6]). Indeed, considering $\rho : \mathbf{B}_n \to GL(V)$ a representation, we can define:

$$lm_{(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f})}(s\rho) : \quad \mathbf{B}_n \quad \longrightarrow \quad GL\left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} V\right)$$

assigning for all $\sigma \in \mathbf{B}_n$, $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $v \in V$:

$$lm_{(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}),q}(s\rho)(b)\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right) \quad = \quad \left(a_{n,\bullet}(b)(i) \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} s\rho(\sigma)(v)\right).$$

For $s = 1$, the construction is functorial as we will see in the next section. It is not the case if $s \neq 1$ because of the composition axiom which must be satisfied for a functor.

3.2 The Long-Moody functors

In this subsection, we prove that the Long-Moody construction given in Theorem 3.2 provides a functor $\mathbf{LM} : \mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod} \longrightarrow \mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}$. Let F be a $\mathfrak{U}\boldsymbol{\beta}$-module and n be a natural integer. For all $\sigma \in \mathbf{B}_n$, since $F(\sigma) \in GL(F(n))$, F defines a representation of $\mathbf{B}_n$:

$$F_{|Hom_{\mathfrak{U}\boldsymbol{\beta}}(n,n)} : \mathbf{B}_n \longrightarrow GL(F(n)).$$

A fortiori, the $\mathbb{K}$-module $F(n)$ is endowed with a $\mathbb{K}[\mathbf{B}_n]$-module structure with the action $F_{|Hom_{\mathfrak{U}\boldsymbol{\beta}}(n,n)}$. Thanks to the morphism $\varsigma_{n,\bullet} : \mathbf{F}_n \hookrightarrow \mathbf{B}_n$, it is also a $\mathbb{K}[\mathbf{F}_n]$-module. Hence, one can form the tensor product $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(n)$, which is isomorphic to $F(n)^{\oplus n}$ by the isomorphism Λ (see Remark 3.4).

Remark 3.6. Consider two natural integers n and n' such that $n' \geq n$, and $[n'-n,\sigma] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n,n')$. Let $\left(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^f}\right)$ be a coherent triplet. As $[n'-n,\sigma]$ belongs to $Hom_{\mathfrak{U}\boldsymbol{\beta}}(n,n')$, we remark that $F([n'-n,\sigma])$ belongs to $Hom_{\mathbb{K}\text{-}\mathfrak{Mod}}(F(n), F(n'))$. One can generalize the definition of the Long-Moody construction, defining the $\mathbb{K}$-module homomorphism:

$$lm(F)_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)}([n'-n,\sigma]) : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(n) \longrightarrow \mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'}]} \underset{\mathbb{K}[\mathbf{F}_{n'}]}{\otimes} F(n')$$

by $\forall i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$, $\forall v \in F(n)$:

$$lm_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)}(F)([n'-n,\sigma])\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right) = a_{n',\bullet}(\sigma)\left(\gamma^{f,\bullet}_{n,n'}(i)\right) \underset{\mathbb{K}[\mathbf{F}_{n'}]}{\otimes} F([n'-n,\sigma])(v).$$

Proposition 3.7. *For $F \in Obj(\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod})$ and $\left(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^f}\right)$ a coherent triplet, the following assignment defines a functor $\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)}(F) : \mathfrak{U}\boldsymbol{\beta} \longrightarrow \mathbb{K}\text{-}\mathfrak{Mod}$.*

- *Objects: $\forall n \in \mathbb{N}$, $\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)}(F)(n) = \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(n)$.*
- *Morphisms: Let $n, n' \in \mathbb{N}$, such that $n' \geq n$, and $[n'-n,\sigma] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n,n')$. One assigns:*

$$\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)}(F)([n'-n,\sigma]) = lm_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)}(F)([n'-n,\sigma]).$$

Notation 3.8. When there is no ambiguity, once the triplet $\left(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^f}\right)$ is clearly given, we forget it in the notation $\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)}$ for convenience (especially for proofs).

Proof. Let n be a fixed natural integer. Let $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $v \in F(n)$. On the one hand, according to our assignment and since $a_{n,\bullet}$ and $\gamma^{f,\bullet}_n$ are group morphisms, we easily prove that $\mathbf{LM}(F)(id_{\mathbf{B}_n}) = id_{\mathbf{LM}(F)(n)}$. On the other hand, let n, n' and n'' be natural integers such that $n'' \geq n' \geq n$, let $([n'-n,\sigma])$ and $([n''-n',\sigma'])$ be morphisms respectively in $Hom_{\mathfrak{U}\boldsymbol{\beta}}(n,n')$ and in $Hom_{\mathfrak{U}\boldsymbol{\beta}}(n',n'')$. Since $a_{n,\bullet}$ is a group morphism, we deduce from the definition that:

$$\begin{aligned} & \mathbf{LM}(F)(([n''-n',\sigma']) \circ ([n'-n,\sigma]))\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right) \\ = \quad & (a_{n'',\bullet}(\sigma') \circ a_{n'',\bullet}(id_{n''-n'}\natural\sigma))\left(\gamma^{f,\bullet}_{n',n''} \circ \gamma^{f,\bullet}_{n,n'}(i)\right) \underset{\mathbb{K}[\mathbf{F}_{n''}]}{\otimes} F([n''-n',\sigma'(id_{n''-n'}\natural\sigma)])(v). \end{aligned}$$

Condition 1.39 asserts that for all elements i' of $\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'}]}$:

$$\gamma^{f,\bullet}_{n',n''}(a_{n',\bullet}(\sigma)(i')) = a_{n'',\bullet}(id_{n''-n'}\natural\sigma)\left(\gamma^{f,\bullet}_{n',n''}(i')\right).$$

So, for all elements i of $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$: $\gamma^{f,\bullet}_{n',n''}\left(a_{n,\bullet}(\sigma)\left(\gamma^{f,\bullet}_{n,n'}(i)\right)\right) = a_{n'',\bullet}(id_{n''-n'}\natural\sigma)\left(\gamma^{f,\bullet}_{n',n''} \circ \gamma^{f,\bullet}_{n,n'}(i)\right)$. Therefore, the associativity axiom is satisfied. $\square$

In the following proposition, we define an endofunctor of $\mathfrak{U}\boldsymbol{\beta}$-$\mathfrak{Mod}$ corresponding to the Long-Moody construction. It will be called the Long-Moody functor.

Proposition 3.9. *For a coherent triplet $\left(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^f}\right)$, the following assignment defines a functor $\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)} : \mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod} \longrightarrow \mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}$.*

- *Objects: Let $F \in Obj\left(\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}\right)$, $\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)}(F)$ is defined as in Proposition 3.7.*
- *Morphisms: Let F and G be two $\mathfrak{U}\boldsymbol{\beta}$-modules, and $\eta : F \Longrightarrow G$ be a natural transformation. We define $\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)}(\eta) : \mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)}(F) \Longrightarrow \mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)}(G)$ for all natural integers n by:*

$$\forall i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}, \forall v \in F(n), \left(\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)}(\eta)\right)_n \left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right) = i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} \eta_n(v).$$

In other words:

$$\left(\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)}(\eta)\right)_n = id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} \eta_n.$$

Proof. The remaining point to check for $\mathbf{LM}$ to be well-defined is the consistency of our definition on morphims. Let F and G be two $\mathfrak{U}\boldsymbol{\beta}$-modules, and let $\eta : F \Longrightarrow G$ be a natural transformation. Let $n, n' \in \mathbb{N}$, such that $n' \geq n$, and $[n'-n, \sigma] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n, n')$. Since η is a natural transformation:

$$\forall v \in F(n), G([n'-n,\sigma])(\eta_n(v)) = \eta_{n'}(F([n'-n,\sigma])(v)).$$

Hence, it follows from the definition of $\mathbf{LM}$ that the following diagram commutes, and therefore $\mathbf{LM}$ is a morphism of $\mathfrak{U}\boldsymbol{\beta}$-modules.

$$\begin{tikzcd}
\mathbf{LM}(F)(n) \arrow[r, "\mathbf{LM}(\eta)_n"] \arrow[d, "\mathbf{LM}(F)\left([n'-n,\sigma]\right)"'] & \mathbf{LM}(G)(n) \arrow[d, "\mathbf{LM}(G)\left([n'-n,\sigma]\right)"] \\
\mathbf{LM}(F)(n') \arrow[r, "\mathbf{LM}(\eta)_{n'}"] & \mathbf{LM}(G)(n')
\end{tikzcd}$$

For F a $\mathfrak{U}\boldsymbol{\beta}$-module and $id_F : F \Longrightarrow F$ the identity natural transformation, it comes directly that $\mathbf{LM}(id_F) = id_{\mathbf{LM}(F)}$. Finally, let us check the associativity axiom. Let $\eta : F \Longrightarrow G$ and $\mu : G \Longrightarrow H$ be natural transformations. Let n be a natural integer, $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $v \in F(n)$. Yet, because μ and η are morphisms in the category of functors:

$$\mathbf{LM}(\mu\circ\eta)_n \left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right) = i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} (\mu_n \circ \eta_n)(v) = \mathbf{LM}(\mu)_n \circ \mathbf{LM}(\eta)_n \left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right).$$

□

Remark 3.10. For $\eta : F \Longrightarrow G$ a natural transformation, with Λ the isomorphism of Remark 3.4:

$$\forall n \in \mathbb{N}, \Lambda\left(\left(\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)}(\eta)\right)_n\right) = \eta_n^{\oplus n}.$$

Proposition 3.11. *For a coherent triplet $\left(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^f}\right)$, the functor $\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)} : \mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod} \longrightarrow \mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}$ is reduced and exact.*

Proof. Let $0_{\mathfrak{U}\boldsymbol{\beta}\text{-Mod}} : \mathfrak{U}\boldsymbol{\beta} \longrightarrow \mathbb{K}\text{-}\mathfrak{Mod}$ denotes the null functor. It comes straightfoward from the definition of the Long-Moody functor that $\mathbf{LM}\left(0_{\mathfrak{U}\boldsymbol{\beta}\text{-Mod}}\right) = 0_{\mathfrak{U}\boldsymbol{\beta}\text{-Mod}}$.
Let n be a natural integer. Since the augmentation ideal $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ is a free $\mathbb{K}[\mathbf{F}_n]$-module (see for example [26, Chapter 2, Proposition 6.2.6]), it is therefore a flat $\mathbb{K}[\mathbf{F}_n]$-module. Then, the result follows from the fact that the functor $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} - : \mathbb{K}\text{-}\mathfrak{Mod} \longrightarrow \mathbb{K}\text{-}\mathfrak{Mod}$ is an exact functor, the naturality for morphisms following straightforwardly from the definition of the Long-Moody functor (see Proposition 3.9). □

Remark 3.12. Assume that, for a coherent triplet $\left(\bullet_a, \bullet_\varsigma, \bullet_{\gamma f}\right)$ and an object F of the category $\mathfrak{U}\boldsymbol{\beta}$-$\mathfrak{Mod}$, $\mathbf{LM}_{\left(\bullet_a, \bullet_\varsigma, \bullet_{\gamma f}\right)}(F) = 0$. Then it follows directly from the definitions that F is the null functor.

Corollary 3.13. *For a coherent triplet $\left(\bullet_a, \bullet_\varsigma, \bullet_{\gamma f}\right)$, the functor $\mathbf{LM}_{\left(\bullet_a, \bullet_\varsigma, \bullet_{\gamma f}\right)}$ commutes with all finite limits and finite colimits.*

Proof. The functor $\mathbf{LM}$ is exact according to Proposition 3.11. The result is a property of exact functors (see for example [19, Chapter 8, section 3]). □

3.3 Evaluation of the Long-Moody functors on a constant functor

Let us give the examples of Long-Moody functors which arise using the actions that we have recorded in Theorem 1.14 and Example 1.16. We will always use the identification $\varsigma_{n,1} : \mathbf{F}_n \hookrightarrow \mathbf{B}_{n+1}$ of Example 1.11. Recall that we have denoted by $a_{n,i} : \mathbf{B}_n \hookrightarrow Aut\left(\mathbf{F}_n\right)$ the action given by the i-th Wada-type natural transformation in Notation 1.15 and $a_{n,8} : \mathbf{B}_n \hookrightarrow Aut\left(\mathbf{F}_n\right)$ the group morphism described in Example 1.16. For $\left(i, 1, \bullet_{\gamma f}\right)$ a coherent triplet associated with $a_{n,i}$ with $1 \leq i \leq 8$, we denote by $\mathbf{LM}_i : \mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod} \longrightarrow \mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}$ the corresponding Long-Moody functor defined in Proposition 3.9 for $i \in \{1, \ldots, 8\}$.

A first step to understand the effect of these endofunctors is to investigate on their effect on a constant functor.

Example 3.14. We denote by $\mathfrak{X} : \mathfrak{U}\boldsymbol{\beta} \longrightarrow R\left[\mathbf{B}_\infty^{ab}\right]\text{-}\mathfrak{Mod}$ the constant functor such that $\mathfrak{X}(n) = R\left[\mathbf{B}_\infty^{ab}\right]$ for all natural integers n.

For all $i \in \{1, \ldots, 8\}$, we have $\mathbf{LM}_i(\mathfrak{X})(n) \cong R\left[\mathbf{B}_\infty^{ab}\right]^{\oplus n}$. Moreover, once defined on automorphims, $\mathbf{LM}_i(\mathfrak{X})$ is defined on general morphisms by precomposition by $i_{R\left[\mathbf{B}_\infty^{ab}\right]^{\oplus n} \hookrightarrow R\left[\mathbf{B}_\infty^{ab}\right]^{\oplus n'}}$ for all natural integers n and n'.

3.3.1 Computations for $\mathbf{LM}_1$

If we assume that $m = 0$, then we obtain the functor $t_1^{-1}\mathbf{LM}_1(t_1\mathfrak{X}) : \mathfrak{U}\boldsymbol{\beta} \longrightarrow R\left[\mathbf{B}_\infty^{ab}\right]\text{-}\mathfrak{Mod}$, defined on automorphisms for all natural integers n, for every Artin generator σ_i of $\mathbf{B}_n$, by:

$$t_1^{-1}\mathbf{LM}_1(t_1\mathfrak{X})(\sigma_i) = \left(Id_{i-1} \oplus AD\left(t_1^2\right) \oplus Id_{n-i-1}\right),$$

with:

$$AD\left(t_1^2\right) = \begin{bmatrix} 0 & t_1^2 \\ t_1^2 & 0 \end{bmatrix}.$$

Now, let us assume that $m \neq 0$. We obtain the functor $t_1^{-1}\mathbf{LM}_{1,m\neq 0}(t_1\mathfrak{X}) : \mathfrak{U}\boldsymbol{\beta} \longrightarrow R\left[\mathbf{B}_\infty^{ab}\right]\text{-}\mathfrak{Mod}$, defined on automorphisms for all natural integers n, for every Artin generator σ_i of $\mathbf{B}_n$, by:

$$t_1^{-1}\mathbf{LM}_1(t_1\mathfrak{X})(\sigma_i) = Id_{i-1} \oplus \widetilde{\widetilde{B}}_m(t_1) \oplus Id_{n-i-1},$$

with:

$$\widetilde{\widetilde{B}}_m(t_1) = \begin{bmatrix} 0 & 1 \\ t_1^{2m} & 1 - t_1^{2m} \end{bmatrix}.$$

In fact, the functor $t_1^{-1}\mathbf{LM}_{\left(1,1,\bullet_{\gamma f}\right)}(t_1\mathfrak{X})$ is very similar to $\mathfrak{Bur}$: indeed, $t_1^{-1}\mathbf{LM}_{\left(1,1,\bullet_{\gamma f}\right)}(t_1\mathfrak{X}) \xrightarrow{r''} \mathfrak{Bur}_{t_1^2}$ with the natural equivalence r'' given in Lemma 2.39.

Recovering of the Lawrence-Krammer functor: Let us first introduce the following result due to Long in [18]. In this paragraph, we assume that $R = \mathbb{C}\left[\mathbf{B}_\infty^{ab}\right]$. For all natural integers n, we denote by $\mathfrak{x}_n$ the representation of $\mathbf{B}_n$ induced by $\mathfrak{X}$.

Proposition 3.15. *[18, special case of Corollary 2.10] Let n be a natural integer such that $n \geq 4$. Then, the Lawrence-Krammer representation $\mathfrak{LK}_{|\mathbf{B}_n}$ is a subrepresentation of $t_2^{-1}\mathbf{LM}_{(1,1,l)}\left(t_2\left(t_1^{-1}\mathbf{LM}_{(1,1,l)}(t_1\mathfrak{X})\right)\right)_{|\mathbf{B}_n}$.*

We first need to introduce new tools. Let n and m be two natural integers. Let $\underline{w}_n = (w_1, \ldots, w_n) \in \mathbb{C}^n$ such that $w_i \neq w_j$ if $i \neq j$. We define the configuration space:

$$Y_{\underline{w}_n,m} = \{(z_1, \ldots, z_m) \mid z_i \in \mathbb{C} \setminus \underline{w}_n,\ z_i \neq z_j \text{ if } i \neq j\}.$$

The two following results due to Long will be crucial to prove Proposition 3.15.

Proposition 3.16. *[18, Corollary 2.7] Let n be a natural integer and $\rho : \mathbf{B}_{n+1} \to GL(V)$ be a representation of $\mathbf{B}_n$ with V a $\left(\mathbb{C}\left[\mathbf{B}_\infty^{ab}\right]\right)\left[\mathbf{B}_\infty^{ab}\right]$-module. Then, the original representation defined by Long in [18, Theorem 2.1] is a group morphism:*

$$t_2^{-1}\mathcal{LM}(t_2\rho) : \mathbf{B}_n \to GL\left(H^1\left(Y_{\underline{w}_n,1}, E_\rho\right)\right)$$

for E_ρ a flat vector bundle associated with ρ(see [18, p. 225-226]).

Lemma 3.17. *[18, Lemma 2.9] For all natural integers m, there is an isomorphism of abelian groups:*

$$H^{m+1}\left(Y_{\underline{w}_n,m+1}, E_{t_1\mathfrak{x}_n}\right) \cong H^1\left(Y_{\underline{w}_n,1}, H^m\left(Y_{\underline{w}_{n+1},m}, E_{t_1\mathfrak{x}_n}\right)\right).$$

In particular, for $m = 1$:

$$H^2\left(Y_{\underline{w}_n,2}, E_{t_1\mathfrak{x}_n}\right) \cong H^1\left(Y_{\underline{w}_n,1}, H^1\left(Y_{\underline{w}_{n+1},2}, E_{t_1\mathfrak{x}_n}\right)\right).$$

Proof of Proposition 3.15. By Proposition 3.16, we can write as a representation:

$$t_2^{-1}\mathcal{LM}\left(t_2\left(t_1^{-1}\mathcal{LM}(t_1\mathfrak{X})\right)\right) : \mathbf{B}_n \to GL\left(H^1\left(Y_{\underline{w}_n,1}, E_{t_1^{-1}\mathcal{LM}(t_1\mathfrak{X})}\right)\right).$$

A fortiori by Lemma 3.17, $t_2^{-1}\mathcal{LM}\left(t_2\left(t_1^{-1}\mathcal{LM}(t_1\mathfrak{X})\right)\right)_{|\mathbf{B}_n}$ is an action of $\mathbf{B}_n$ on $H^2\left(Y_{\underline{w}_n,2}, E_{t_1\mathfrak{X}_{|\mathbf{B}_n}}\right)$. In particular, for $m = 2$ and $n \geq 4$, according to [15, Theorem 5.1], the representation of $\mathbf{B}_n$ factorizing through the Hecke algebra $H_n(t_1)$ corresponding to the Young diagram $(n-2, 2)$ (which is in fact the Lawrence-Krammer representation) is a subrepresentation of $t_2^{-1}\mathcal{LM}\left(t_2\left(t_1^{-1}\mathcal{LM}(t_1\mathfrak{x}_n)\right)\right)$. For all natural integers n, let us consider the embedding:

$$\begin{array}{rcl} \gamma_n^{f,l} \otimes Id : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbf{F}_n}{\otimes} \mathfrak{X}(n+1) & \hookrightarrow & \mathcal{I}_{\mathbb{K}[\mathbf{F}_{n+1}]} \underset{\mathbf{F}_{n+1}}{\otimes} \mathfrak{X}(n+1). \\ i \underset{\mathbf{F}_n}{\otimes} v & \longmapsto & \gamma_n^{f,l}(i) \underset{\mathbf{F}_n}{\otimes} v \end{array}$$

By the definition of the Long-Moody construction (see [18, Theorem 2.1]) and Condition 1.39, we deduce that $t_2^{-1}\mathcal{LM}\left(t_2\left(t_1^{-1}\mathcal{LM}(t_1\mathfrak{x}_n)\right)\right)$ is a subrepresentation of $t_2^{-1}\left(\tau_1\mathbf{LM}_{(1,1,l)}\right)\left(t_2\left(t_1^{-1}\mathbf{LM}_{(1,1,l)}(t_1\mathfrak{X})\right)\right)_{|\mathbf{B}_n}$. $\square$

We denote $\mathfrak{LK}^{\geq 4} : \mathfrak{U}\boldsymbol{\beta} \longrightarrow \left(\mathbb{C}\left[\mathbf{B}_\infty^{ab}\right]\right)\left[\mathbf{B}_\infty^{ab}\right]\text{-}\mathfrak{Mod}$ the subfunctor of the Lawrence-Krammer defined in Example 2.37 which is null on the objects such that $n < 4$. The result of Proposition 3.15 induces the existence of an intertwining $l_n : \left(\mathbb{C}\left[\mathbf{B}_\infty^{ab}\right]\right)\left[\mathbf{B}_\infty^{ab}\right]^{\oplus n(n-1)/2} \hookrightarrow \left(\mathbb{C}\left[\mathbf{B}_\infty^{ab}\right]\right)\left[\mathbf{B}_\infty^{ab}\right]^{\oplus n(n+1)}$ for all natural integers $n \geq 4$, ie a $\mathbb{C}\left[\mathbf{B}_\infty^{ab}\right]$-module morphism such that for all $\sigma \in \mathbf{B}_n$ the following diagram is commutative.

$$\begin{tikzcd} \left(\mathbb{C}\left[\mathbf{B}_\infty^{ab}\right]\right)\left[\mathbf{B}_\infty^{ab}\right]^{\oplus n(n-1)/2} \arrow[r, hook, "l_n"] \arrow[d, "\mathfrak{LK}(\sigma)"'] & \left(\mathbb{C}\left[\mathbf{B}_\infty^{ab}\right]\right)\left[\mathbf{B}_\infty^{ab}\right]^{\oplus (n+1)^2} \arrow[d, "\mathfrak{LK}(\sigma)"] \\ \left(\mathbb{C}\left[\mathbf{B}_\infty^{ab}\right]\right)\left[\mathbf{B}_\infty^{ab}\right]^{\oplus n(n-1)/2} \arrow[r, hook, "l_n"] & \left(\mathbb{C}\left[\mathbf{B}_\infty^{ab}\right]\right)\left[\mathbf{B}_\infty^{ab}\right]^{\oplus (n+1)^2} \end{tikzcd}$$

According to Remark 1.34, this intertwining defines in fact a natural transformation

$$l : \mathfrak{LK}^{\geq 4} \Longrightarrow t_2^{-1}\left(\tau_1\mathbf{LM}_{(1,1,l)}\right)\left(t_2\left(t_1^{-1}\mathbf{LM}_{(1,1,l)}(t_1\mathfrak{X})\right)\right)_{|\mathbf{B}_n}^{\geq 4}.$$

3.3.2 Computations for the other cases

If $i=2$, we obtain the functor $\mathbf{LM}_2(\mathfrak{X}):\mathfrak{U}\boldsymbol{\beta}\longrightarrow R\left[\mathbf{B}_{\infty}^{ab}\right]\text{-}\mathfrak{Mod}$, defined for $\sigma_i\in\mathbf{B}_n$ by:

$$\mathbf{LM}_2(F)(\sigma_i)=(F(\sigma_i))^{\oplus n}.$$

For $i=3$, we obtain the functor $t_1^{-1}\mathbf{LM}_3(t_1\mathfrak{X}):\mathfrak{U}\boldsymbol{\beta}\longrightarrow R\left[\mathbf{B}_{\infty}^{ab}\right]\text{-}\mathfrak{Mod}$, defined for $\sigma_i\in\mathbf{B}_n$ by:

$$t_1^{-1}\mathbf{LM}_3(t_1\mathfrak{X})(\sigma_i)\quad=\quad Id_{i-1}\oplus\begin{bmatrix}0 & -t_1^2\\ 1 & 0\end{bmatrix}\oplus Id_{n-i-1}.$$

Hence, the functor $t_1^{-1}\mathbf{LM}_3(t_1\mathfrak{X})$ is very similar to $\mathfrak{TYM}$: we directly obtain that $t_1^{-1}\mathbf{LM}_3(t_1\mathfrak{X})\overset{r''}{\cong}\mathfrak{TYM}_{-t_1^2}$.

For the case $i=8$, we assume moreover that we use the morphim $\gamma_n^{f,l}$. Therefore we use the coherent triplet $(8,1,l)$. We obtain the functor $t_1^{-1}\mathbf{LM}_8(t_1\mathfrak{X}):\mathfrak{U}\boldsymbol{\beta}\longrightarrow R\left[\mathbf{B}_{\infty}^{ab}\right]\text{-}\mathfrak{Mod}$, defined for $\sigma_i\in\mathbf{B}_n$ by:

$$t_1^{-1}\mathbf{LM}_8(t_1\mathfrak{X})(\sigma_i)\quad=\quad Id_{i-2}\oplus\overline{B}(1)\oplus Id_{n-i-2}$$

if $i\leq n-2$, and:

$$t_1^{-1}\mathbf{LM}_8(t_1\mathfrak{X})(\sigma_{n-1})\quad=\quad Id_{n-3}\oplus\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}$$

Hence, the functor $\mathbf{LM}_3(\mathfrak{X})$ is very similar to $\overline{\mathfrak{Bur}}_1$: indeed, direct computation shows that $\mathbf{LM}_3(\mathfrak{X})=\tau_1\left(\overline{\mathfrak{Bur}}_1\right)$.

For the actions given by the Wada-type natural transformation 4, 5, 6 and 7 in Theorem 1.14, the produced functors $\mathbf{LM}_i(\mathfrak{X}):\mathfrak{U}\boldsymbol{\beta}\longrightarrow R\left[\mathbf{B}_{\infty}^{ab}\right]\text{-}\mathfrak{Mod}$ are mild variations of what is given by the case $i=1$.

4 The Long-Moody functor applied to polynomial functors

Let us move on the effect of the Long-Moody functors on strong polynomial functors. Let $\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)$ be a reliable triplet, which we fix for all the work of this Section (in particular, we forget it in many notations).

4.1 The intermediary functors

We have to introduce two new functors which will play a key role in the main result of the study. First, let us recall the following crucial property of the augmentation ideal of a free product of groups. The proof of this proposition is a consequence of combining [6, Lemma 4.3] and [6, Theorem 4.7].

Proposition 4.1. *Let G and H be groups. Then, there is a natural $\mathbb{K}[G*H]$-module isomorphism:*

$$\mathcal{I}_{\mathbb{K}[G*H]}\cong\left(\mathcal{I}_{\mathbb{K}[G]}\underset{\mathbb{K}[G]}{\otimes}\mathbb{K}[G*H]\right)\bigoplus\left(\mathcal{I}_{\mathbb{K}[H]}\underset{\mathbb{K}[H]}{\otimes}\mathbb{K}[G*H]\right).$$

For a fixed natural integer m and F a $\mathfrak{U}\boldsymbol{\beta}$-module, let us consider the $\mathfrak{U}\boldsymbol{\beta}$*-module* $\tau_m\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)}(F)$. For all natural integers n, by Proposition 4.1, we have a $\mathbb{K}[\mathbf{F}_{m+n}]$-module isomorphism:

$$\begin{aligned}&\mathcal{I}_{\mathbb{K}[\mathbf{F}_{m+n}]}\underset{\mathbb{K}[\mathbf{F}_{m+n}]}{\otimes}F(m+n)\\ &\cong\left(\left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}\underset{\mathbb{K}[\mathbf{F}_m]}{\otimes}\mathbb{K}[\mathbf{F}_{m+n}]\right)\bigoplus\left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}\underset{\mathbb{K}[\mathbf{F}_n]}{\otimes}\mathbb{K}[\mathbf{F}_{m+n}]\right)\right)\underset{\mathbb{K}[\mathbf{F}_{m+n}]}{\otimes}F(m+n).\end{aligned}$$

Therefore, because of the distributivity of tensor product with respect to the direct sum, we prove the following proposition.

Proposition 4.2. *Let $F \in Obj\left(\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}\right)$ and m, n be natural integers. Then, we have the following $\mathbb{K}$-module isomorphism:*

$$\tau_m \mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)}\left(F\right)\left(n\right) \quad\cong\quad \left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_m]} \underset{\mathbb{K}[\mathbf{F}_m]}{\otimes} F\left(m+n\right)\right) \bigoplus \left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F\left(m+n\right)\right). \tag{4}$$

The aim of this section is in fact to show that this $\mathbb{K}$-module decomposition will lead to a $\mathfrak{U}\boldsymbol{\beta}$-module decomposition of $\tau_m \mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)}$ (see Theorem 4.13).

Proposition 4.3. *Let m be a fixed natural integer. For all natural integers n and $F \in Obj\left(\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}\right)$, the submodules $\mathcal{I}_{\mathbb{K}[\mathbf{F}_m]} \underset{\mathbb{K}[\mathbf{F}_m]}{\otimes} F\left(m+n\right)$ of $\tau_m \mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)}\left(F\right)\left(n\right)$ naturally define a subfunctor of $\tau_m \mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)}\left(F\right)$, which will be denoted by $\Upsilon_m^{\left(\bullet_\varsigma,\bullet_{\gamma^f}\right)}\left(F\right)$.*

This way, for all $F \in Obj\left(\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}\right)$, the subfunctors $\Upsilon_m^{\left(\bullet_\varsigma,\bullet_{\gamma^f}\right)}\left(F\right)$ of $\tau_m \mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)}\left(F\right)$ define a subfunctor of $\tau_m \mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)}$, which will be denoted by $\Upsilon_m^{\left(\bullet_\varsigma,\bullet_{\gamma^f}\right)}$.

Remark 4.4. As the notation suggests it, the functor $\Upsilon_m^{\left(\bullet_\varsigma,\bullet_{\gamma^f}\right)}$ does not depend on the morphism $a_{n,\bullet}$. We will see the reason why in the proof of Proposition 4.3.

Proof. First, let us show that $\Upsilon_m\left(F\right)$ is a subfunctor of $\tau_m \mathbf{LM}\left(F\right)$ for every $F \in Obj\left(\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}\right)$. For all natural integers n:

$$\Upsilon_m\left(F\right)\left(n\right) = \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]} \underset{\mathbb{K}[\mathbf{F}_m]}{\otimes} F\left(m+n\right).$$

Because of Proposition 4.2, the $\mathbb{K}$-module $\Upsilon_m\left(F\right)\left(n\right)$ is a submodule of the $\mathbb{K}$-module $\tau_m \mathbf{LM}\left(F\right)\left(n\right)$. Then, we define the monomorphism associated with the direct sum

$$\upsilon_{m,n}\left(F\right) = \bar{\gamma}^{f,\bullet}_{m,m+n} \underset{\mathbb{K}[\mathbf{F}_{m+n}]}{\otimes} id_{F(m+n+1)} : \Upsilon_m\left(F\right)\left(n\right) \hookrightarrow \tau_m \mathbf{LM}\left(F\right)\left(n\right)$$

by the assignment:

$$\forall i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]},\, \forall v \in F\left(m+n\right),\, \upsilon_{m,n}\left(F\right)\left(i \underset{\mathbb{K}[\mathbf{F}_m]}{\otimes} v\right) = \bar{\gamma}^{f,\bullet}_{m,m+n}\left(i\right) \underset{\mathbb{K}[\mathbf{F}_{m+n}]}{\otimes} v$$

where $\bar{\gamma}^{f,\bullet}_{m,m+n}$ is defined in Definition 1.6. Let n and n' be natural integers such that $n' \geq n$, and $[n'-n,\sigma] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}\left(n,n'\right)$ with $\sigma \in \mathbf{B}_{n'}$. Let $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}$ and $v \in F\left(m+n\right)$. We naturally define the functor $\Upsilon_m\left(F\right)$ on morphisms by:

$$\Upsilon_m\left(F\right)\left([n'-n,\sigma]\right)\left(i \underset{\mathbb{K}[\mathbf{F}_m]}{\otimes} v\right) \quad=\quad i \underset{\mathbb{K}[\mathbf{F}_m]}{\otimes} F\left(\left[n'-n,\left(id_m \natural \sigma\right) \circ \left(\left(b^{\mathfrak{U}\boldsymbol{\beta}}_{m,n'-n}\right)^{-1} \natural id_n\right)\right]\right)\left(v\right).$$

It remains to show that we define a natural transformation $\upsilon_m\left(F\right) : \Upsilon_m\left(F\right) \longrightarrow \tau_m \mathbf{LM}\left(F\right)$. We deduce from the defintion of monoidal structure on morphims in $\mathfrak{U}\boldsymbol{\beta}$ (see Remark (1.28)) and from the definition of the Long-Moody functor (see Proposition (3.7)) that:

$$\begin{aligned}
&\left(\tau_m \mathbf{LM}\left(F\right)\left([n'-n,\sigma]\right) \circ \upsilon_{m,n}\left(F\right)\right)\left(i \underset{\mathbb{K}[\mathbf{F}_m]}{\otimes} v\right)\\
=&\mathbf{LM}\left(F\right)\left(id_m \natural \left[n'-n,\sigma\right]\right)\left(\bar{\gamma}^{f,\bullet}_{m,m+n}\left(i\right) \underset{\mathbb{K}[\mathbf{F}_{m+n}]}{\otimes} v\right)\\
=&a_{m+n',\bullet}\left(id_m \natural \sigma\right)\left(a_{m+n',\bullet}\left(\left(b^{\mathfrak{U}\boldsymbol{\beta}}_{m,n'-n}\right)^{-1} \natural id_n\right)\left(\gamma^{f,\bullet}_{m+n,m+n'}\left(\bar{\gamma}^{f,\bullet}_{m,m+n}\left(i\right)\right)\right)\right)\\
&\underset{\mathbb{K}[\mathbf{F}_m]}{\otimes} F\left(\left[n'-n,\left(id_m \natural \sigma\right) \circ \left(\left(b^{\mathfrak{U}\boldsymbol{\beta}}_{m,n'-n}\right)^{-1} \natural id_n\right)\right]\right)\left(v\right).
\end{aligned}$$

It follows from Condition 1.41 that $a_{m+n',\bullet}\left(\left(b^{\mathfrak{U}\boldsymbol{\beta}}_{m,n'-n}\right)^{-1}\natural id_{n}\right)\circ\left(\gamma^{f,\bullet}_{m+n,m+n'}\circ\bar{\gamma}^{f,\bullet}_{m,m+n}\right)=\bar{\gamma}^{f,\bullet}_{m,m+n'}$. Since by Condition 1.42, $a_{m+n',\bullet}\left(id_{m}\natural\sigma\right)\circ\bar{\gamma}^{f,\bullet}_{m,m+n'}=\bar{\gamma}^{f,\bullet}_{m,m+n'}$ for all elements σ of $\mathbf{B}_{n'}$, we deduce that:

$$\left(\tau_{m}\mathbf{LM}\left(F\right)\left(\left[n'-n,\sigma\right]\right)\circ\upsilon_{m,n}\left(F\right)\right)\left(i\underset{\mathbb{K}\left[\mathbf{F}_{m}\right]}{\otimes}v\right)\quad=\quad\upsilon_{m,n'}\left(F\right)\circ\Upsilon_{m}\left(F\right)\left(\left[n'-n,\sigma\right]\right)\left(i\underset{\mathbb{K}\left[\mathbf{F}_{m}\right]}{\otimes}v\right).$$

Therefore, for every $F\in Obj\left(\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}\right)$, $\Upsilon_{m}\left(F\right)$ is a subfunctor of $\tau_{m}\mathbf{LM}\left(F\right)$. We define the functor Υ_{m} on natural transformations by:

$$\forall n\in\mathbb{N},\,\left(\Upsilon_{m}\left(\eta\right)\right)_{n}=id_{\mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{m}\right]}}\underset{\mathbb{K}\left[\mathbf{F}_{m}\right]}{\otimes}\eta_{m+n}.$$

It follows by straightfoward computations from our definitions that the family $\left\{\upsilon_{m}\left(F\right):\Upsilon_{m}\left(F\right)\longrightarrow\tau_{m}\mathbf{LM}\left(F\right)\right\}_{F\in Obj\left(\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}\right)}$ defines a natural transformation $\upsilon_{m}:\Upsilon_{m}\longrightarrow\tau_{m}\mathbf{LM}$ in the category $\mathfrak{U}\boldsymbol{\beta}$-$\mathfrak{Mod}$. □

The functor $\Upsilon_{m}^{\left(\bullet_{\varsigma},\bullet_{\gamma^{f}}\right)}$ satisfies some convenient properties.

Lemma 4.5. *For all natural integers m, the functor $\Upsilon_{m}^{\left(\bullet_{\varsigma},\bullet_{\gamma^{f}}\right)}$ is exact.*

Proof. The proof is similar as the one of Proposition 3.11. It is a consequence of the flatness of the augmentation ideal $\mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{m}\right]}$ as a $\mathbb{K}\left[\mathbf{F}_{m}\right]$-module. □

Proposition 4.6. *For all natural integers m and l, the functor $\Upsilon_{m}^{\left(\bullet_{\varsigma},\bullet_{\gamma^{f}}\right)}$ commutes with the difference functor δ_{l} and the evanescence functor κ_{l}.*

Proof. Let n be a natural integer and F be a $\mathfrak{U}\boldsymbol{\beta}$-module. For all natural integers l, it follows directly from the definition of i_{l} that

$$i_{l}\left(\Upsilon_{m}\left(F\right)\right)_{n}=\Upsilon_{m}\left(F\right)\left(\left[l,id_{l+n}\right]\right)=id_{\mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{m}\right]}}\underset{\mathbb{K}\left[\mathbf{F}_{m}\right]}{\otimes}F\left(\left[l,\left(b^{\mathfrak{U}\boldsymbol{\beta}}_{m,l}\right)^{-1}\natural id_{n}\right]\right).$$

and

$$\left(\Upsilon_{m}\left(i_{l}\left(F\right)\right)\right)_{n}=id_{\mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{m}\right]}}\underset{\mathbb{K}\left[\mathbf{F}_{m}\right]}{\otimes}F\left(\left[l,id_{l+m+n}\right]\right).$$

Since we know that the two following sequences are exact (because of Lemma 4.5 for the bottom one), the following diagram is commutative.

$$\begin{tikzcd}
0 \arrow[r] & \kappa_{l}\left(\Upsilon_{m}F\right)\left(n\right) \arrow[r] & \Upsilon_{m}F\left(n\right) \arrow[r, "i_{l}\left(\Upsilon_{m}\left(F\right)\right)_{n}"] \arrow[d, equal] & \tau_{l}\left(\Upsilon_{m}F\right)\left(n\right) \arrow[r] \arrow[d, "\cong"', "id_{\mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{m}\right]}}\underset{\mathbb{K}\left[\mathbf{F}_{m}\right]}{\otimes}F\left(\left[0,b^{\mathfrak{U}\boldsymbol{\beta}}_{m,l}\natural id_{n}\right]\right)"] & \delta_{l}\left(\Upsilon_{m}F\right)\left(n\right) \arrow[r] & 0 \\
0 \arrow[r] & \Upsilon_{m}\left(\kappa_{l}F\right)\left(n\right) \arrow[r] & \Upsilon_{m}F\left(n\right) \arrow[r, "\left(\Upsilon_{m}\left(i_{l}\left(F\right)\right)\right)_{n}"'] & \Upsilon_{m}\left(\tau_{l}F\right)\left(n\right) \arrow[r] & \Upsilon_{m}\left(\delta_{l}F\right)\left(n\right) \arrow[r] & 0
\end{tikzcd}$$

Therefore, by the universal properties of the kernel and the cokernel, we deduce that $\kappa_{l}\left(\Upsilon_{m}F\right)\left(n\right)\cong\Upsilon_{m}\left(\kappa_{l}F\right)\left(n\right)$ and $\delta_{l}\left(\Upsilon_{m}F\right)\left(n\right)\cong\Upsilon_{m}\left(\delta_{l}F\right)\left(n\right)$. The naturality in n follows directly from the definitions and our constructions. □

Proposition 4.7. *Let m be a natural integer and F be a $\mathfrak{U}\boldsymbol{\beta}$-module. Then, we have a natural monomorphism $\varepsilon_{m}\left(F\right):\mathbf{LM}_{\left(\bullet_{a},\bullet_{\varsigma},\bullet_{\gamma^{f}}\right)}\left(\tau_{m}F\right)\longrightarrow\tau_{m}\mathbf{LM}_{\left(\bullet_{a},\bullet_{\varsigma},\bullet_{\gamma^{f}}\right)}\left(F\right)$.*

This way, for all $F\in Obj\left(\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}\right)$, the subfunctors $\mathbf{LM}_{\left(\bullet_{a},\bullet_{\varsigma},\bullet_{\gamma^{f}}\right)}\left(\tau_{m}F\right)$ of $\tau_{m}\mathbf{LM}_{\left(\bullet_{a},\bullet_{\varsigma},\bullet_{\gamma^{f}}\right)}\left(F\right)$ define a subfunctor of $\tau_{m}\mathbf{LM}_{\left(\bullet_{a},\bullet_{\varsigma},\bullet_{\gamma^{f}}\right)}$, which will be denoted by $\mathbf{LM}_{\left(\bullet_{a},\bullet_{\varsigma},\bullet_{\gamma^{f}}\right)}\circ\tau_{m}$.

Proof. For all natural integers n, $\mathbf{LM}(\tau_m F)(n) = \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(m+n)$. Because of Proposition 4.2, the $\mathbb{K}$-module $\mathbf{LM}(\tau_m F)(n)$ is a submodule of the $\mathbb{K}$-module $\tau_m \mathbf{LM}(F)(n)$. Then, we define the monomorphism associated with the direct sum $\varepsilon_{m,n}(F) = \gamma^{f,\bullet}_{n,m+n} \underset{\mathbb{K}[\mathbf{F}_{m+n}]}{\otimes} id_{F(m+n)} : \mathbf{LM}(\tau_m F)(n) \hookrightarrow \tau_m \mathbf{LM}(F)(n)$ by the assignment:

$$\forall i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}, \forall v \in F(m+n), \varepsilon_{m,n}(F)\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right) = \gamma^{f,\bullet}_{n,m+n}(i) \underset{\mathbb{K}[\mathbf{F}_{m+n}]}{\otimes} v.$$

It remains to show that we define a natural transformation $\varepsilon_m(F) : \mathbf{LM}(\tau_m F) \longrightarrow \tau_m \mathbf{LM}(F)$. Let n and n' be natural integers such that $n' \geq n$, and $[n'-n, \sigma] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n, n')$ with $\sigma \in \mathbf{B}_{n'}$. Let $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $v \in F(m+n)$. The functor $\mathbf{LM}(\tau_m F)$ is defined on morphisms by:

$$\begin{aligned} & \mathbf{LM}(\tau_m F)([n'-n, \sigma])\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right) \\ = \quad & a_{n',\bullet}(\sigma)\left(\gamma^{f,\bullet}_{n,n'}(i)\right) \underset{\mathbb{K}[\mathbf{F}_{n'}]}{\otimes} F\left(\left[n'-n, (id_m \natural \sigma) \circ \left(\left(b^{\mathfrak{U}\boldsymbol{\beta}}_{m,n'-n}\right)^{-1} \natural id_n\right)\right]\right)(v). \end{aligned}$$

We remark that by Condition 1.39, we know that $a_{m+n',\bullet}\left(\left(b^{\mathfrak{U}\boldsymbol{\beta}}_{m,n'-n}\right)^{-1} \natural id_n\right) \circ \gamma^{f,\bullet}_{n,m+n'} = \gamma^{f,\bullet}_{n,m+n'}$. Then, naturality strightfowardly follows from the fact that $a_{m+n',\bullet}(id_m \natural \sigma) \circ \gamma^{f,\bullet}_{n',m+n'} = \gamma^{f,\bullet}_{n',m+n'} \circ a_{n',\bullet}(\sigma)$ for all elements σ of $\mathbf{B}_{n'}$, again by Condition 1.39. Therefore, for every $F \in Obj(\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod})$, $\mathbf{LM}(\tau_m F)$ is a subfunctor of $\tau_m \mathbf{LM}(F)$. It follows by straightfoward computations from our definitions that the family $\{\varepsilon_m(F) : \mathbf{LM}(\tau_m F) \longrightarrow \tau_m \mathbf{LM}(F)\}_{F \in Obj(\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod})}$ defines a natural transformation $\varepsilon_m : \mathbf{LM} \circ \tau_m \longrightarrow \tau_m \mathbf{LM}$ in the category $\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}$. $\square$

Remark 4.8. Using the original Long-Moody construction (see Theorem (3.1)), considering an extra family of morphisms $\gamma^{b,\bullet}_n : \mathbf{B}_n \to \mathbf{B}_{n+1}$ and adaptating the coherence Condition (1.39), we can define a Long-Moody functor $\mathscr{LM} : \mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod} \longrightarrow \mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}$ using a coherent triplet $\left(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^f}\right)$. However, Proposition 4.7 would not be satisfied. Indeed, in order to prove the naturality of $\varepsilon_m(F)$ for morphisms of $\mathfrak{U}\boldsymbol{\beta}$, we would need the family of morphisms $\gamma^{b,\bullet}_n : \mathbf{B}_n \to \mathbf{B}_{n+1}$ to satisfy an additional coherence condition which would contradicts Condition 1.39.

Proposition 4.9. *Let m be a natural integer. Then, there is an isomorphism in the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}, \mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod})$:*

$$\tau_m \mathbf{LM}_{\left(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^f}\right)} \cong \Upsilon_m^{\left(\bullet_\varsigma, \bullet_{\gamma^f}\right)} \bigoplus \left(\mathbf{LM}_{\left(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^f}\right)} \circ \tau_m\right).$$

Proof. Let F be a $\mathfrak{U}\boldsymbol{\beta}$-module. By Propositions 4.3 and 4.7, by the universal property of the direct sum and the naturality which comes directly, we deduce that $\Upsilon_m F \bigoplus \mathbf{LM}(\tau_m F)$ is a subfunctor of $\tau_m \mathbf{LM}(F)$. For all natural integers n, we have an isomorphism of $\mathbb{K}$-modules according to Propostion 4.2:

$$\Upsilon_m F(n) \bigoplus \mathbf{LM}(\tau_m F)(n) \cong \tau_m \mathbf{LM}(F)(n).$$

Hence by the universal property of the direct sum, for all natural integers n, there exists a unique $\mathbb{K}$-module isomorphism $\Xi_{\oplus,n,F} = \upsilon_{m,n}(F) \oplus \varepsilon_{m,n}(F) : \Upsilon_m F(n) \bigoplus \mathbf{LM}(\tau_m F)(n) \xrightarrow{\cong} \tau_m \mathbf{LM}(F)(n)$ such that the following diagram is commutative.

$$\begin{tikzcd}
 & \mathbf{LM}(\tau_m F)(n) \arrow[dl, hook', "0 \oplus id_{\mathbf{LM}(\tau_m F)}"'] \arrow[dr, hook, "\varepsilon_{m,n}(F)"] & \\
\Upsilon_m F(n) \bigoplus \mathbf{LM}(\tau_m F)(n) \arrow[rr, "\cong", "\Xi_{\oplus,n,F}"'] & & \tau_m \mathbf{LM}(F)(n) \\
 & \Upsilon_m F(n) \arrow[ul, hook, "id_{\Upsilon_m F} \oplus 0"] \arrow[ur, hook', "\upsilon_{m,n}(F)"'] &
\end{tikzcd}$$

The naturality for morphims of $\mathfrak{U}\boldsymbol{\beta}$ directly follows from the functoriality of $\tau_m\mathbf{LM}(F)$, $\mathbf{LM}(\tau_m F)$ and $\Upsilon_m F$ and from the fact that $\upsilon_m(F)$ and $\varepsilon_m(F)$ are natural transformations. Hence we have a natural equivalence:

$$\Xi_{\oplus,F} = \upsilon_m(F) \oplus \varepsilon_m(F) : \Upsilon_m(F) \bigoplus \mathbf{LM}\tau_m(F) \xrightarrow{\cong} \tau_m\mathbf{LM}(F).$$

It remains to show the naturality for morphims of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}, \mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod})$. Let F and G be two $\mathfrak{U}\boldsymbol{\beta}$-modules, and $\eta : F \Longrightarrow G$ be a natural transformation. Then, it follows from our definitions and constructions that for all natural integers n, for all $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}$, $j \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $v, w \in F(m+n)$:

$$\begin{aligned}
&\left(\Upsilon_m(\eta) \bigoplus \mathbf{LM}\tau_m(\eta)\right)_n \left(\left(i \underset{\mathbb{K}[\mathbf{F}_m]}{\otimes} v\right) \oplus \left(j \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} w\right)\right) \\
= \quad &\left(\Xi^{-1}_{\oplus,n,G} \circ (\tau_m\mathbf{LM}(\eta))_n \circ \Xi_{\oplus,n,F}\right) \left(\left(i \underset{\mathbb{K}[\mathbf{F}_m]}{\otimes} v\right) \oplus \left(j \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} w\right)\right).
\end{aligned}$$

Hence, we have a natural equivalence $\Xi_{\oplus} = \upsilon_m \oplus \varepsilon_m$ in the category $\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}$. □

4.2 Splitting of the difference functor

Our aim is to study the effect of the Long-Moody construction on strong polynomial functors. To that purpose, we focus on the behaviour of the difference functor on the Long-Moody construction. Thus, we will be interested in the study of the cokernel of the map $i_m\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma f}\right)}(F)$. Let us recall some properties of this map.

- For all natural integers n, $i_m(\mathbf{LM}(F))_n = \mathbf{LM}(F)([m, id_{m+n}])$. Explicitly for all elements k of $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$, for all elements w of $F(n)$:

$$\mathbf{LM}(F)([m, id_{m+n}]) \left(k \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} w\right) = \gamma^{f,\bullet}_{n,m+n}(k) \underset{\mathbb{K}[\mathbf{F}_{m+n}]}{\otimes} F([m, id_{m+n}])(w).$$

- we have the natural exact sequence:hom (a, b)

$$0 \longrightarrow \kappa_m\mathbf{LM}(F)(n) \longrightarrow \mathbf{LM}(F)(n) \xrightarrow{i_m\mathbf{LM}(F)(n)} \tau_m\mathbf{LM}(F)(n) \longrightarrow \delta_m\mathbf{LM}(F)(n) \longrightarrow 0\,.$$

Lemma 4.10. *Let m be a natural integer and F be a $\mathfrak{U}\boldsymbol{\beta}$-module. The submodules $Im\left[\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma f}\right)}(F)([m, id_{m+n}])\right]$ of $\tau_m\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma f}\right)}(F)(n)$ define a subfunctor of $\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma f}\right)}\tau_m(F)$, which will be denoted by $Im_m(F)$.*

This way, for all $F \in Obj(\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod})$, the subfunctors $Im_m(F)$ of $\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma f}\right)}\tau_m(F)$ define a subfunctor of $\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma f}\right)}\tau_m$, which will be denoted by Im_m.

Proof. Let n be natural integer. Let $k \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $w \in F(n)$. The module

$$Im_m(F)(n) = Im\left[\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma f}\right)}(F)([m, id_{m+n}])\right]$$

is the submodule of $\tau_m\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma f}\right)}(F)(n)$ with elements of the form $\gamma^{f,\bullet}_{n,m+n}(k) \underset{\mathbb{K}[\mathbf{F}_{m+n}]}{\otimes} F([m, id_{m+n}])(w)$.
We denote by

$$im_m(F)_n : Im_m(F)(n) \hookrightarrow \tau_m\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma f}\right)}(F)(n)$$

the associated inclusion morphism. Recalling $\varepsilon_m : \mathbf{LM}\tau_m \Longrightarrow \tau_m\mathbf{LM}$ the natural transformation defined in Proposition 4.7, we have:

$$\mathbf{LM}(F)([m, id_{n+m}]) \left(k \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} w\right) \quad = \quad \varepsilon_{m,n}(F) \circ (\mathbf{LM}(\iota_m F)([0, id_n])) \left(k \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} w\right).$$

Hence, by the definition of $\varepsilon_{m,n}(F)$, using the decomposition of Proposition 4.2, the projection on the summand $\Upsilon_m F(n)$ is zero. By the universal property of the kernel, we deduce that there exists a unique morphism of $\mathbb{K}$-modules:

$$\vartheta_{m,n}(F) : Im_m(F)(n) \longrightarrow \mathbf{LM}(\tau_m F)(n)$$

such that the following diagram is commutative.

$$\begin{tikzcd} Im_m(F)(n) \arrow[rr, hook, "im_m(F)_n"'] \arrow[dr, "\vartheta_{m,n}(F)"'] & & \tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^f})}(F)(n) \\ & \mathbf{LM}(\tau_m F)(n) \arrow[ur, hook, "\varepsilon_{m,n}(F)"'] & \end{tikzcd}$$

A fortiori, $\vartheta_{m,n}(F)$ is a monomorphism and for all $k \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$, and for all $w \in F(n)$:

$$\vartheta_{m,n}(F)\left(\gamma^{f,\bullet}_{n,m+n}(k) \underset{\mathbb{K}[\mathbf{F}_{m+n}]}{\otimes} \iota_m F([0, id_n])(w)\right) \quad = \quad k \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F([m, id_{m+n}])(w).$$

Hence, the $\mathbb{K}$-module $Im_m(F)(n)$ is a submodule of the $\mathbb{K}$-module $\mathbf{LM}(\tau_m F)(n)$ for all natural integers n. Let n and n' be natural integers such that $n' \geq n$, and $[n'-n, \sigma] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n, n')$. Let $k \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $w \in F(n)$. We naturally define the functor $Im_m(F)$ on morphisms by:

$$\begin{aligned} & Im_m(F)([n'-n, \sigma])\left(\gamma^{f,\bullet}_{n,m+n}(k) \underset{\mathbb{K}[\mathbf{F}_{m+n}]}{\otimes} F([m, id_{m+n}])(w)\right) \\ = \quad & a_{m+n',\bullet}\left((id_m \natural \sigma) \circ \left(\left(b^{\mathfrak{U}\boldsymbol{\beta}}_{m,n'-n}\right)^{-1} \natural id_n\right)\right)\left(\gamma^{f,\bullet}_{n,m+n'}(k)\right) \\ & \underset{\mathbb{K}[\mathbf{F}_{m+n'}]}{\otimes} F\left(\left[m+n'-n, (id_m \natural \sigma) \circ \left(\left(b^{\mathfrak{U}\boldsymbol{\beta}}_{m,n'-n}\right)^{-1} \natural id_n\right)\right]\right)(w). \end{aligned}$$

Let us check that $Im_m(F)$ is a subfunctor of $\mathbf{LM}(\tau_m F)$ for every $F \in Obj(\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod})$. First, we remark that by Condition 1.39, we know that $a_{m+n',\bullet}\left(\left(b^{\mathfrak{U}\boldsymbol{\beta}}_{m,n'-n}\right)^{-1} \natural id_n\right) \circ \gamma^{f,\bullet}_{n,m+n'} = \gamma^{f,\bullet}_{n,m+n'}$. By Condition 1.39, $a_{m+n',\bullet}(id_m \natural \sigma) \circ \gamma^{f,\bullet}_{n,m+n'} = \gamma^{f,\bullet}_{n',m+n'} \circ a_{n',\bullet}(\sigma) \circ \gamma^{f,\bullet}_{n,n'}$ for all elements σ of $\mathbf{B}_{n'}$, so we deduce the following equality:

$$\begin{aligned} & \left(\vartheta_{m,n'}(F) \circ Im_m(F)([n'-n, \sigma])\right)\left(\gamma^{f,\bullet}_{n,m+n}(k) \underset{\mathbb{K}[\mathbf{F}_{m+n}]}{\otimes} F([m, id_{m+n}])(w)\right) \\ = \quad & \left(\mathbf{LM}(\tau_m F)([n'-n, \sigma]) \circ \vartheta_{m,n}(F)\right)\left(\gamma^{f,\bullet}_{n,m+n}(k) \underset{\mathbb{K}[\mathbf{F}_{m+n}]}{\otimes} F([m, id_{m+n}])(w)\right). \end{aligned}$$

Let us show that we may define a natural transformation $\vartheta_m : Im_m \Longrightarrow \mathbf{LM}\tau_m$ in the category $\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}$. Let F and G be two $\mathfrak{U}\boldsymbol{\beta}$-modules, and $\eta : F \Longrightarrow G$ a natural transformation. We naturally define the functor Im_m on natural transformations by:

$$\forall n \in \mathbb{N},\ \left(Im_m(\eta)\right)_n = id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{m+n}]}} \underset{\mathbb{K}[\mathbf{F}_{m+n}]}{\otimes} \eta_{m+n}.$$

Then, it follows from our definitions that for all natural integers n, for all $k \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $w \in F(m+n)$:

$$\begin{aligned} & \left(\vartheta_{m,n}(G) \circ \left(Im_m(\eta)\right)_n\right)\left(\gamma^{f,\bullet}_{n,m+n}(k) \underset{\mathbb{K}[\mathbf{F}_{m+n}]}{\otimes} F([m, id_{m+n}])(w)\right) \\ = \quad & \left(\left(\mathbf{LM}\tau_m(\eta)\right)_n \circ \vartheta_{m,n}(F)\right)\left(\gamma^{f,\bullet}_{n,m+n}(k) \underset{\mathbb{K}[\mathbf{F}_{m+n}]}{\otimes} F([m, id_{m+n}])(w)\right). \end{aligned}$$

$\square$

The following proposition is thus a direct consequence of this lemma and Proposition 4.9.

Proposition 4.11. *Let m be a natural integer. For a reliable triplet $\left(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^f}\right)$, there is an isomorphism in the category* $\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}, \mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}\right)$*:*

$$\delta_m \mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)} \cong \Upsilon_m^{\left(\bullet_\varsigma,\bullet_{\gamma^f}\right)} \bigoplus \frac{\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)} \circ \tau_m}{Im_m}.$$

We may refine this proposition using the following lemma.

Lemma 4.12. *Let m be a natural integer and $\left(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^f}\right)$ be a reliable triplet. There is an isomorphism in the category* $\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}, \mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}\right)$*:*

$$\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)} \circ \delta_m \cong \frac{\mathbf{LM}_{\left(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f}\right)} \circ \tau_m}{Im_m}.$$

Proof. Let F be a $\mathfrak{U}\boldsymbol{\beta}$-module. Let us consider the exact sequence (1) of Lemma 2.8 applied to F:

$$0 \longrightarrow \kappa_m(F) \xrightarrow{\Omega_m(F)} F \xrightarrow{i_m(F)} \tau_m(F) \xrightarrow{\Delta_m(F)} \delta_m(F) \longrightarrow 0\ .$$

Since $\mathbf{LM}$ is an exact functor (see Proposition 3.11), we deduce that the following sequence is exact:

$$0 \longrightarrow \mathbf{LM}(\kappa_m F) \xrightarrow{\mathbf{LM}(\Omega_m(F))} \mathbf{LM}(F) \xrightarrow{\mathbf{LM}(i_m(F))} \mathbf{LM}(\tau_m(F)) \xrightarrow{\mathbf{LM}(\Delta_m(F))} \mathbf{LM}(\delta_m(F)) \longrightarrow 0\ .$$

Therefore, because of the exactness of this sequence and Lemma 4.10, for all natural integers n:

$$\ker\left(\mathbf{LM}\left(\Delta_m(F)\right)_n\right) = Im\left(\mathbf{LM}\left(i_m(F)\right)_n\right) \cong Im_m(F)(n)\,.$$

The first row of the following diagram is a short exact sequence. The universal property of the cokernel ensures that there exists a unique $\mathbb{K}$-module morphism $\Psi_{n,F} : \frac{\mathbf{LM}(\tau_m F)(n)}{Im_m(F)(n)} \longrightarrow \mathbf{LM}(\delta_m F)(n)$ such that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & Im_m(F)(n) \arrow[r, "\vartheta_{m,n}(F)"] \arrow[dr, "0"'] & \mathbf{LM}(\tau_m F)(n) \arrow[r, "\pi_n(F)"] \arrow[d, "\mathbf{LM}(\Delta_m(F))_n"] & \frac{\mathbf{LM}(\tau_m F)(n)}{Im_m(F)(n)} \arrow[r] \arrow[dl, dotted, "\exists!\Psi_{n,F}"] & 0 \\
 & & \mathbf{LM}(\delta_m(F))(n) & &
\end{tikzcd}$$

Moreover, since $\ker\left(\mathbf{LM}\left(\Delta_m(F)\right)_n\right) = Im_m(F)(n)$, we know that $\Psi_{n,F}$ is an isomorphism of $\mathbb{K}$-modules.

Let us check the naturality on n. Let $n, n' \in \mathbb{N}$, such that $n' \geq n$, and $[n'-n, \sigma] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n, n')$. Since the functor $Im_m(F)$ is a subfunctor of $\mathbf{LM}(\tau_m F)$ by Lemma 4.10, we define the quotient functor $\frac{\mathbf{LM}(\tau_m F)}{Im_m(F)}$ on morphisms by the universal property of cokernel:

$$\begin{tikzcd}
0 \arrow[r] & Im_m(F)(n) \arrow[r, "\vartheta_{m,n}(F)"] \arrow[d, "Im_m(F)\left(\left[n'-n,\sigma\right]\right)"'] & \mathbf{LM}(\tau_m(F))(n) \arrow[r, "\pi_n(F)"] \arrow[d, "\mathbf{LM}(\tau_m(F))\left[n'-n,\sigma\right]"'] & \frac{\mathbf{LM}(\tau_m F)(n)}{Im_m(F)(n)} \arrow[r] \arrow[d, dotted, "\exists!\frac{\mathbf{LM}(\tau_m F)}{Im_m(F)}\left(\left[n'-n,\sigma\right]\right)"] & 0 \\
0 \arrow[r] & Im_m(F)(n') \arrow[r, "\vartheta_{m,n'}(F)"] & \mathbf{LM}(\tau_m(F))(n') \arrow[r, "\pi_{n'}(F)"] & \frac{\mathbf{LM}(\tau_m F)(n')}{Im_m(F)(n')} \arrow[r] & 0
\end{tikzcd}$$

Thus, the following diagram commutes.

$$\begin{tikzcd}
\frac{\mathbf{LM}(\tau_m F)(n)}{Im_m(F)(n)} \arrow[r, "\Psi_{n,F}"] \arrow[d, "\frac{\mathbf{LM}(\tau_m F)}{Im_m(F)}\left(\left[n'-n,\sigma\right]\right)"'] & \mathbf{LM}(\delta_m(F))(n) \arrow[d, "\mathbf{LM}(\delta_m(F))\left(\left[n'-n,\sigma\right]\right)"] \\
\frac{\mathbf{LM}(\tau_m F)(n')}{Im_m(F)(n')} \arrow[r, "\Psi_{n',F}"'] & \mathbf{LM}(\delta_m(F))(n')
\end{tikzcd}$$

So, we obtain that $\Psi_F : \frac{\mathbf{LM}(\tau_m F)}{Im_m(F)} \longrightarrow \mathbf{LM}(\delta_m F)$ is an isomorphism of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}, \mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod})$.

It remains to show that we can define a natural isomorphism $\Psi : \frac{\mathbf{LM}\circ\tau_m}{Im_m} \xrightarrow{\cong} \mathbf{LM}\circ\delta_m$ in the category $\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}$. Let F and G be two $\mathfrak{U}\boldsymbol{\beta}$-modules, and $\eta : F \Longrightarrow G$ a natural transformation. We naturally define the quotient functor $\frac{\mathbf{LM}\circ\tau_m}{Im_m}$ on natural transformations to be the quotient:

$$\begin{tikzcd}
\mathbf{LM}(\tau_m(F))(n) \arrow[r, two heads, "\pi_n(F)"] \arrow[d, "(\mathbf{LM}\tau_m(\eta))_n"'] & \frac{\mathbf{LM}(\tau_m F)(n)}{Im_m(F)(n)} \arrow[d, "\left(\frac{\mathbf{LM}\circ\tau_m}{Im_m}(\eta)\right)_n"] \\
\mathbf{LM}(\tau_m(G))(n) \arrow[r, two heads, "\pi_n(G)"'] & \frac{\mathbf{LM}(\tau_m G)(n)}{Im_m(G)(n)}
\end{tikzcd}$$

Moreover, by naturality in objects of $\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}$ of the exact sequence (1) of Lemma 2.8, because of the exactness of the Long-Moody functor (see Proposition 3.11) and by definition of $\Psi_{n,F}$, the natural transformation $\mathbf{LM}\circ\delta_m(\eta)$ is defined to make the following square commutative for all natural integers n.

$$\begin{tikzcd}
\frac{\mathbf{LM}(\tau_m F)(n)}{Im_m(F)(n)} \arrow[r, "\Psi_{n,F}"] \arrow[d, "\left(\frac{\mathbf{LM}\circ\tau_m}{Im_m}(\eta)\right)_n"'] & \mathbf{LM}(\delta_m(F))(n) \arrow[d, "(\mathbf{LM}\circ\delta_m(\eta))_n"] \\
\frac{\mathbf{LM}(\tau_m G)(n)}{Im_m(G)(n)} \arrow[r, "\Psi_{n,G}"'] & \mathbf{LM}(\delta_m(G))
\end{tikzcd}$$

□

Lemma 4.12 leads therefore to the following result.

Theorem 4.13. *Let m be a natural integer and $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^f})$ be a reliable triplet. There is a natural isomorphism in the category $\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}$:*

$$\delta_m \mathbf{LM}_{(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f})} \cong \Upsilon_m^{(\bullet_\varsigma,\bullet_{\gamma^f})} \bigoplus \left(\mathbf{LM}_{(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f})} \circ \delta_m\right).$$

Furthermore, we can determine the behaviour of the evanescence functor.

Theorem 4.14. *Let m be a natural integer. Then, the endofunctor κ_m commutes with the endofunctor $\mathbf{LM}_{(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^b},\bullet_{\gamma^f})}$ of the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}, \mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod})$. In other words, there is a natural isomorphism in the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}, \mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod})$:*

$$\kappa_m \circ \mathbf{LM}_{(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f})} \cong \mathbf{LM}_{(\bullet_a,\bullet_\varsigma,\bullet_{\gamma^f})} \circ \kappa_m.$$

Proof. Since the Long-Moody functor is exact (see Proposition 3.11), we have the following exact sequence.

$$0 \longrightarrow \mathbf{LM}\circ\kappa_m \xrightarrow{\mathbf{LM}\circ\Omega_m} \mathbf{LM} \xrightarrow{\mathbf{LM}\circ i_m} \mathbf{LM}\circ\tau_m \xrightarrow{\mathbf{LM}\circ\Delta_m} \mathbf{LM}\circ\delta_m \longrightarrow 0$$

Therefore, the image functor $Im(\mathbf{LM}\circ i_m)$ is isomorphic to the kernel functor $Ker(\mathbf{LM}\circ\Delta_m)$. Applying the exact sequence 1 of Lemma 2.8 to $\mathbf{LM}(F)$, we obtain the following diagram, where the first line is an exact sequence and the square is commutative.

$$\begin{tikzcd}
0 \arrow[r] & \kappa_m\circ\mathbf{LM} \arrow[r, "\Omega_m\circ\mathbf{LM}"] & \mathbf{LM} \arrow[r, "i_m\circ\mathbf{LM}"] & \tau_m\mathbf{LM} \arrow[r, "\Delta_m\circ\mathbf{LM}"] \arrow[d, "\text{by Proposition 4.9}"', "\cong"] & \delta_m\mathbf{LM} \arrow[r] \arrow[d, "\cong\text{ by Theorem 4.13}"] & 0 \\
 & & & \Upsilon_m\bigoplus(\mathbf{LM}\circ\tau_m) \arrow[r, "id\bigoplus(\mathbf{LM}\circ\Delta_m)", "\text{by Lemma 4.10}"'] & \Upsilon_m\bigoplus(\mathbf{LM}\circ\delta_m) &
\end{tikzcd}$$

By the exactness property of the first row, we deduce that:

$$Im(\mathbf{LM}\circ i_m) \cong Ker(\mathbf{LM}\circ\Delta_m) \cong Im(i_m\circ\mathbf{LM}).$$

As the functor $\kappa_m \circ \mathbf{LM}$ (resp. $\mathbf{LM} \circ \kappa_m$) is defined to be the kernel of $\mathbf{LM} \xrightarrow{i_m \circ \mathbf{LM}} \tau_m \mathbf{LM}$ (resp. of $\mathbf{LM} \xrightarrow{\mathbf{LM} \circ i_m} \mathbf{LM} \circ \tau_m$), by the unicity up to isomorphism of the kernel, we conclude that $\kappa_m \circ \mathbf{LM} \cong \mathbf{LM} \circ \kappa_m$. ☐

4.3 Increasing of the polynomial degree

Proposition 4.15. *Let m be a natural integer and F be a non-null $\mathfrak{U}\boldsymbol{\beta}$-module. If the functor F is strong polynomial of degree n, then:*

1. *the functor $\Upsilon_m^{\left(\bullet_{\varsigma},\bullet_{\gamma f}\right)}(F)$ belongs to $\mathcal{P}ol_n\left(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}\right)$;*
2. *the functor $\mathbf{LM}_{\left(\bullet_a,\bullet_{\varsigma},\bullet_{\gamma f}\right)}(F)$ belongs to $\mathcal{P}ol_{n+1}\left(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}\right)$.*

Proof. We prove these two results by induction on the degree of polynomiality. For the first result, it follows from Proposition 4.6. For the second result, let us first consider F a strong polynomial functor of degree 0, by Theorem 4.13, we obtain that:

$$\delta_1 \mathbf{LM}(F) \cong \Upsilon_1(F).$$

Therefore $\mathbf{LM}(F)$ is a strong polynomial functor of degree less or equal to 1. Now, assume that F is a strong polynomial functor of degree $n \geq 0$. By Theorem 4.13:

$$\delta_1 \mathbf{LM}(F) \cong \mathbf{LM}(\delta_1 F) \bigoplus \Upsilon_1(F).$$

By the inductive hypothesis and the result on Υ_1, we deduce that $\mathbf{LM}(F)$ is a strong polynomial functor of degree less or equal to $n+1$. ☐

Corollary 4.16. *For all natural integers n, the endofunctor $\mathbf{LM}_{\left(\bullet_a,\bullet_{\varsigma},\bullet_{\gamma f}\right)}$ of $\mathfrak{U}\boldsymbol{\beta}$-$\mathfrak{Mod}$ restricts to a functor:*

$$\mathbf{LM}_{\left(\bullet_a,\bullet_{\varsigma},\bullet_{\gamma f}\right)} : \mathcal{P}ol_n\left(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}\right) \longrightarrow \mathcal{P}ol_{n+1}\left(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}\right).$$

Corollary 4.17. *Let n be a natural integer and F be an object of $\mathcal{P}ol_n\left(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}\right)$ such that the degree of strong polynomiality of $\Upsilon_1^{\left(\bullet_{\varsigma},\bullet_{\gamma f}\right)}(F)$ is equal to n. Then, the functor $\mathbf{LM}_{\left(\bullet_a,\bullet_{\varsigma},\bullet_{\gamma^b},\bullet_{\gamma f}\right)}(F)$ is a strong polynomial functor of degree equals to $n+1$.*

Lemma 4.18. *Let n be a natural integer and F be an object of $\mathcal{VP}ol_n\left(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}\right)$. Then the functor $\Upsilon_1^{\left(\bullet_{\varsigma},\bullet_{\gamma f}\right)}(F)$ is very strong polynomial of degree equals to the one of F.*

Proof. We proceed by induction on the degree of polynomiality of F. First, if we assume that F belongs to $\mathcal{VP}ol_0\left(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}\right)$, then according to Proposition 2.20, there exists a constant functor C such that $F \cong C$. It follows directly from the definition of Υ_1 (see Proposition 4.3) that $\Upsilon_1(C) = C$. Hence, $\Upsilon_1(F) \cong F$ is a degree 0 very strong polynomial functor. Now, assume that F is a very strong polynomial functor of degree $n \geq 0$. Thanks to Proposition 4.6, we deduce that $\kappa_1 \Upsilon_1(F) \cong \Upsilon_1(\kappa_1 F) = 0$ and $\delta_1 \Upsilon_1(F) \cong \Upsilon_1(\delta_1 F)$. Since the functor $\delta_1 F$ is a degree $n-1$ very strong polynomial functor, the result follows from the inductive hypothesis. ☐

Remark 4.19. The previous proof does not work for strong polynomial functors since the initialisation fails. Indeed, considering the atomic functor $\mathfrak{A}_0$, which is strong polynomial of degree 0 (see Example 2.21), then $\Upsilon_1(\mathfrak{A}_0) = 0$.

Theorem 4.20. *Let n be a natural integer and F be an object $\mathcal{VP}ol_n\left(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}\right)$ of degree equals to n. Then, the functor $\mathbf{LM}_{\left(\bullet_a,\bullet_{\varsigma},\bullet_{\gamma f}\right)}(F)$ is a very strong polynomial functor of degree equals to $n+1$.*

Proof. Thanks to Lemma 4.18, it follows from Corollary 4.17 that $\mathbf{LM}(F)$ is a strong polynomial functor of degree equals to $n+1$. Since the functor $\mathbf{LM}(F)$ commutes with the evanescence functor κ_m by Theorem 4.14, we deduce that $\kappa_m \circ \mathbf{LM}(F) \cong \mathbf{LM}(\kappa_m(F)) = 0$. Moreover, thanks to Theorem 4.13, we know that:

$$\kappa_m(\delta_m \mathbf{LM}(F)) \cong \kappa_m(\Upsilon_m(F)) \bigoplus \kappa_m(\mathbf{LM}(\delta_m(F))).$$

Therefore, the fact that $\Upsilon_m(F)$ commutes with the evanescence functor κ_m by Corollary 4.6 and again Theorem 4.14 imply that:

$$\kappa_m(\delta_m \mathbf{LM}(F)) \cong \Upsilon_m(\kappa_m F) \bigoplus \mathbf{LM}(\kappa_m(\delta_m(F))).$$

The results then follows from the fact that F is an object $\mathcal{VP}ol_n(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ and Υ_m is a reduced endofunctor of the category $\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}$. □

Example 4.21. By Proposition 2.20, $\mathfrak{X}$ is a very strong polynomial functor of degree 0. Now applying the Long-Moody functor, we proved (see Section 3.3.1) that $\mathbf{LM}_{(1,1,\bullet_{\gamma^f})}(\mathfrak{X})$ is isomorphic to $\mathfrak{Bur}_1$, which is very strong polynomial of degree 1 by Proposition (2.29).

4.4 Other properties of the Long-Moody functors

We have proven in the previous subsection that a Long-Moody functor sends (very) strong polynomial functors on (very) strong polynomial functors. We can also prove that a (very) strong polynomial functor in the essential image of a Long-Moody functor is necessarily the image of another strong polynomial functor.

Lemma 4.22. *Let n be a natural integer and F be an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ such that $\Upsilon_1^{(\bullet_\varsigma, \bullet_{\gamma^f})}(F)$ is an object of $\mathcal{P}ol_n(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Then, F is an object of $\mathcal{P}ol_{n+1}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.*

Proof. We proceed by induction on the degree of polynomiality of $\Upsilon_1(F)$. First, assuming that $\Upsilon_1(F)$ belongs to $\mathcal{P}ol_0(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, we deduce from Proposition 4.6 that $\Upsilon_1(\delta_1 F) = 0$. It follows from the definition of $\Upsilon_1(F)$ (see Proposition 4.3) that for all $n \geq 1$, $\delta_1 F(n) = 0$. Hence, $\delta_1(\delta_1 F) \cong 0$ and therefore F is an object of $\mathcal{P}ol_1(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Now, assume that $\Upsilon_1(F)$ is a strong polynomial functor of degree $n \geq 0$. Since $\Upsilon_1(\delta_1 F) \cong \delta_1 \Upsilon_1(F)$ by Proposition 4.6, $\Upsilon_1(\delta_1 F)$ is an object of $\mathcal{P}ol_{n-1}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. The inductive hypothesis implies that $\delta_1 F$ is an object of $\mathcal{P}ol_n(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. □

Remark 4.23. Let us consider the atomic functor $\mathfrak{A}_n$ (with $n > 0$), which is strong polynomial of degree n (see Example 2.21). Then $\Upsilon_1(\mathfrak{A}_n) \cong \mathfrak{A}_{n-1}^{\oplus n}$ is strong polynomial of degree $n-1$. This illustrates the fact that $n+1$ is the best boundary for the degree of polynomiality in Lemma 4.22.

Proposition 4.24. *Let n be a natural integer and $(\bullet_a, \bullet_{\gamma^b}, \bullet_{\gamma^f})$ be a reliable triplet. Let F be a strong polynomial functor of degree n in the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Assume that there exists an object G of the category $\mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}$ such that $\mathbf{LM}_{(\bullet_a, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(G) = F$. Then, the functor G is a strong polynomial functor of degree less or equal to n in the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.*

Proof. It follows from Theorem 4.13 that:

$$\delta_1 F \cong \Upsilon_1 G \oplus (\mathbf{LM}(\delta_1 G)).$$

According to Lemma 2.16, the functor $\varUpsilon_1 G$ is an object of the category $\mathcal{P}ol_{n-1}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, and because of Lemma 4.22 the functor G is an object of the category $\mathcal{P}ol_n(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. □

Proposition 4.25. *Let $(\bullet_a, \bullet_\varsigma \bullet_{\gamma^f})$ be a coherent triplet. The functor $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^f})} : \mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod} \longrightarrow \mathfrak{U}\boldsymbol{\beta}\text{-}\mathfrak{Mod}$ is not essentially surjective.*

Proof. Let d be a natural integer. Let $E_d : \mathfrak{U}\boldsymbol{\beta} \longrightarrow \mathbb{K}\text{-}\mathfrak{Mod}$ be the functor which factorizes through the category $\mathbb{N}$, such that $E_d(n) = \mathbb{K}^{\oplus n^d}$ for all natural integers n and for all $[n'-n, \sigma] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n, n')$ (with n, n' natural integers such that $n' \geq n$), $E_d([n'-n, \sigma]) = \gamma^{\mathbb{K}}_{n^d, n'^d}$. In particular, fo all natural integers n, for every Artin generator σ_i of $\mathbf{B}_n$, $E_d(\sigma_i) = id_{\mathbb{K}^{\oplus n^d}}$.

It inductively follows from this definition and direct computations that E_d is a very strong polynomial functor of degree d.

Let us assume that $\mathbf{LM}_{(\bullet_a,\bullet_\varsigma,\bullet_{\gamma f})}$ is essentially surjective. So, there exists an object F of $\mathfrak{U}\boldsymbol{\beta}$-$\mathfrak{Mod}$ such that $\mathbf{LM}(F) \cong E_d$. Because of the definition of $\mathbf{LM}(F)$ on morphisms (see Proposition 3.7), this implies that for all natural integer n and for all $\sigma \in \mathbf{B}_n$, $a_{n,\bullet}(\sigma) = id_n$.

Also, if $\mathbf{LM}_{(\bullet_a,\bullet_\varsigma,\bullet_{\gamma f})}$ is essentially surjective, there exists an object T of the category $\mathfrak{U}\boldsymbol{\beta}$-$\mathfrak{Mod}$ such that we can recover the Burau functor from $\mathbf{LM}(T)$ (ie something like $\alpha\mathbf{LM}(T)$ with α a constant). According to Proposition 4.24, the functor T is a strong polynomial functor of degree less or equal to one since the Burau functor is very strong polynomial of degree one (see Proposition 2.29). We deduce from the definition of $\mathbf{LM}(T)$ on objects and morphisms (see Proposition 3.7) that for all $n \geq 1$, $T(n) = \mathbb{K}$ and for all generator σ_i of $\mathbf{B}_n$:

$$\mathbf{LM}(T)(\sigma_i) = T(\sigma_i) \cdot Id_n.$$

Then necessarily, for all $i \in \{1, \ldots, n\}$, $T(\sigma_i) = \delta$ such that $\delta^2 = t$ and we consider $\delta^{-1}\mathbf{LM}(T)$. We deduce that there exists a natural transformation $\omega : \delta^{-1}\mathbf{LM}(T) \overset{\cong}{\to} \mathfrak{Bur}$. This contradicts the fact that for all $\sigma \in \mathbf{B}_n$, $a_{n,\bullet}(\sigma) = id_n$. □

Remark 4.26. The proof of Proposition 4.25 shows in particular that a Long-Moody functor $\mathbf{LM}_{(\bullet_a,\bullet_\varsigma,\bullet_{\gamma f})}$ for a coherent triplet $(\bullet_a,\bullet_\varsigma,\bullet_{\gamma f})$ is not essentially surjective on very strong polynomial functors in any degree.

In [5, Section 4.7, Open Problem 7], Birman and Brendle ask "wether all finite dimensional unitary matrix representations of $\mathbf{B}_n$ arise in a manner which is related to the construction" recalled in Theorem 3.1. Since the Tong-Yang-Ma and unreduced Burau representations recalled in Theorem 2.23 are unitary representations, the proof of Proposition 4.25 shows that any Long-Moody functor (and especially the one based on the construction of Theorem 3.1) cannot provide all the functors encoding unitary representations. Therefore, we refine the problem asking wether all functors encoding families of unitary representations of braid groups lies in the image of a Long-Moody functor.

Remark 4.27. Another question is to ask wether we can obtain the reduced Burau functor $\overline{\mathfrak{Bur}}$ by a modified Long-Moody functor: the modification would be to take the tensor product with $\mathcal{I}_{\mathbf{F}_{n-1}}$ on $\mathbf{F}_{n-1}$, since for F an element of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, the $\mathbb{K}$-module $F(n+1)$ is in fact a $\mathbb{K}[\mathbf{F}_{n-1}]$-module thanks to the morphism $\varsigma_n \circ \varsigma_{n-1} : \mathbf{F}_{n-1} \hookrightarrow \mathbf{B}_{n+1}$ for all natural integers n.

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