ON THE MOD-2 COHOMOLOGY OF $SL_3(\mathbb{Z}[\frac{1}{2},i])$

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ABSTRACT. Let $\Gamma = SL_3(\mathbb{Z}[\frac{1}{2},i])$, let X be any mod-2 acyclic Γ-CW complex on which Γ acts with finite stabilizers and let X_s be the 2-singular locus of X. We calculate the mod-2 cohomology of the Borel construction of X_s with respect to the action of Γ. This cohomology coincides with the mod-2 cohomology of Γ in cohomological degrees bigger than 8 and the result is compatible with a conjecture of Quillen which predicts the structure of the cohomology ring $H^*(\Gamma;\mathbb{F}_2)$.

1. Introduction

A major motivation for studying the mod-2 cohomology of $SL_3(\mathbb{Z}[\frac{1}{2},i])$ comes from a conjecture of Quillen (Conjecture 14.7 of [Q1]) which concerns the structure of the mod-p cohomology of $GL_n(\Lambda)$ where Λ is a ring of S-integers in a number field such that p is invertible in Λ and Λ contains a primitive p-th root of unity ζ_p. The conjecture stipulates that under these assumptions $H^*(GL_n(\Lambda);\mathbb{Z}/p)$ is free over the polynomial algebra $\mathbb{Z}/p[c_1,\ldots,c_n]$ where the c_i are the mod-p Chern classes associated to an embedding of Λ into the complex numbers. In the sequel we will denote this conjecture by $C(n,\Lambda,p)$.

We will show in Theorem 5.1 that for $\Lambda = \mathbb{Z}[\frac{1}{2},i]$ conjecture $C(n,\Lambda,2)$ is equivalent to the existence of an isomorphism

$$H^*(GL_n(\mathbb{Z}[\frac{1}{2},i]);\mathbb{F}_2) \cong \mathbb{F}_2[c_1,\ldots,c_n]\otimes E(e_1,e_1',\ldots,e_{2n-1},e_{2n-1}')$$

where the classes c_i are the Chern classes of the tautological n-dimensional complex representation of $GL_n(\mathbb{Z}[\frac{1}{2},i])$, E denotes an exterior algebra and the classes e_{2i-1}, e_{2i-1}' are of cohomological degree $2i-1$ for $i=1,\ldots,n$.

Conjecture $C(n,\mathbb{Z}[\frac{1}{2},i],2)$ is trivially true for $n=1$ and has been verified for $n=2$ in [W]. On the other hand, Dwyer's method in [D] using étale approximations X_n for the homotopy type of the 2-completion of $BGL_n(\mathbb{Z}[\frac{1}{2}])$ and comparing the set of homotopy classes of $[BP,X_n]$ with that of $[BP,BGL_n(\mathbb{Z}[\frac{1}{2}])]$ for suitable cyclic groups of order 2^n can be adapted to disprove $C(16,\mathbb{Z}[\frac{1}{2},i],2)$. We will not dwell on this in this paper. However, we note that étale approximations can also be used to show that if $C(n,\mathbb{Z}[\frac{1}{2},i],2)$ fails then $C(2n,\mathbb{Z}[\frac{1}{2}],2)$ fails as well [HL]. We also note that $C(n,\mathbb{Z}[\frac{1}{2}],2)$ is known to be true for $n=2$ by [M] and $n=3$ by [H2] but is known to be false for $n=32$ by [D] and even for $n\geq 14$ [HL].

In this paper we give a partial calculation of $H^*(SL_3(\mathbb{Z}[\frac{1}{2},i]);\mathbb{F}_2)$ and make a first step in an attempt to study conjecture $C(3,\mathbb{Z}[\frac{1}{2},i],2)$. We propose the same strategy as the one which was used in the case of $SL_3(\mathbb{Z}[\frac{1}{2}])$. In a first step one uses a centralizer spectral sequence introduced in [H1] in order to calculate the mod-2 Borel cohomology $H^*_G(X_s;\mathbb{F}_2)$ where X is any mod-2 acyclic G-CW complex on which a suitable discrete group G acts with finite stabilizers and X_s is the 2-singular locus of X, i.e. the subcomplex consisting of all points for which the isotropy group of the action of G is of even order. For $G = SL_3(\mathbb{Z}[\frac{1}{2}])$

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this step was carried out in [H1] and for $G = SL_3(\mathbb{Z}[\frac{1}{2}, i])$ it is carried out in this paper. The precise form of X does not really matter in this step.

The second step involves a very laborious analysis of the relative mod-2 Borel cohomology $H^*_G(X, X_s; \mathbb{F}_2)$ and of the connecting homomorphism for the Borel cohomology of the pair (X, X_s). In the case of $G = SL_3(\mathbb{Z}[\frac{1}{2}])$ this was carried out by hand in [H2]. A by hand calculation looks forbidding in the case of $G = SL_3(\mathbb{Z}[\frac{1}{2}, i])$ and this paper makes no attempt on such a calculation. However, we do make some comments on what is likely to be involved in such an attempt.

Here are the main results of this paper. In these results the elements b_2 respectively b_3 are of degree 4 resp. 6. They are given as Chern classes of the tautological 3-dimensional complex representation of $SL_3(\mathbb{Z}[\frac{1}{2}, i])$. The indices of the other elements give their cohomological degrees. These elements come from Quillen's exterior cohomology classes in the cohomology of $GL_3(\mathbb{F}_p)$ for suitable primes p, for example for $p = 5$ (cf. section 3.2 for more details). Furthermore Σ^n denotes n-fold suspension so that $\Sigma^4\mathbb{F}_2$ is a one dimensional $\mathbb{F}_2$-vector space concentrated in degree 4.

Theorem 1.1. *Let $\Gamma = SL_3(\mathbb{Z}[\frac{1}{2}, i])$ and let X be any mod-2 acyclic Γ-CW complex such that the isotropy group of each cell is finite. Then the centralizer spectral sequence of* [H1]

$$\lim{}^s_{\mathcal{A}_*(\Gamma)} H^t C_\Gamma(E); \mathbb{F}_2) \Longrightarrow H^{s+t}_\Gamma(X_s; \mathbb{F}_2)$$

collapses at E_2 and gives a short exact sequence

$$0 \to \Sigma^4\mathbb{F}_2 \oplus \Sigma^4\mathbb{F}_2 \oplus \Sigma^7\mathbb{F}_2 \to H^*_\Gamma(X_s; \mathbb{F}_2) \to \mathbb{F}_2[b_2, b_3] \otimes E(d_3, d_3', d_5, d_5') \to 0$$

in which the second map is a map of graded algebras.

Next let

$$\psi : H^*(\Gamma; \mathbb{F}_2) = H^*_\Gamma(X; \mathbb{F}_2) \to H^*_\Gamma(X_s; \mathbb{F}_2) \to \mathbb{F}_2[b_2, b_3] \otimes E(d_3, d_3', d_5, d_5')$$

be the composition of the map induced by the inclusion $X_s \subset X$ and the epimorphism of Theorem 1.1.

Theorem 1.2. *Let $\Gamma = SL_3(\mathbb{Z}[\frac{1}{2}, i])$ and X be as in the previous theorem.*

a) If $SD_3(\mathbb{Z}[\frac{1}{2}, i])$ denotes the subgroup of diagonal matrices of Γ then the target of ψ can be identified with a subalgebra of $H^*(SD_3(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$ and ψ is induced by the restriction homomorphism $H^*(B\Gamma; \mathbb{F}_2) \to H^*(SD_3(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$.

b) There exists a map of graded $\mathbb{F}_2$-algebras

$$\varphi : \mathbb{F}_2[c_2, c_3] \otimes E(e_3, e_3', e_5, e_5') \to H^*(\Gamma; \mathbb{F}_2)$$

with e_i and e_i' of degree $2i - 1$ such that the composition of φ with ψ is the isomorphism which sends c_i to b_i, $i = 2, 3$, e_i to d_i and e_i' to d_i', $i = 3, 5$.

c) The homomorphism ψ is surjective in all degrees, an isomorphism in degrees $* > 8$ and its kernel is finite dimensional in degrees $* \leq 8$.

Remark 1.3. In section 5 we will discuss the relation of Theorem 1.2 with a conjecture of Quillen on the structure of the cohomology of $H^*(GL(n, \Lambda); \mathbb{F}_2)$ for rings of S-integers Λ in a number field satisfying suitable assumptions (cf. 14.7 of [Q1]). This conjecture would hold in the case of $n = 3$ and $\Lambda = \mathbb{Z}[\frac{1}{2}, i]$ if the maps ψ and ψ of part (b) of Theorem 1.2 turned out to be isomorphisms (cf. Proposition 5.5).

The following result is an immediate consequence of Theorem 1.2.

Corollary 1.4. *Let $\Gamma = SL_3(\mathbb{Z}[\frac{1}{2}, i])$ and X be as in Theorem 1.1. Then the following conditions are equivalent.*

a) The restriction homomorphism $H^*(B\Gamma; \mathbb{F}_2) \to H^*(SD_3(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$ is injective and $H^*(B\Gamma; \mathbb{F}_2)$ is isomorphic as a graded $\mathbb{F}_2$-algebra to $\mathbb{F}_2[b_2, b_3] \otimes E(d_3, d_3', d_5, d_5')$.

b) There is an isomorphism

$$H^*_\Gamma(X, X_s; \mathbb{F}_2) \cong \Sigma^5\mathbb{F}_2 \oplus \Sigma^5\mathbb{F}_2 \oplus \Sigma^8\mathbb{F}_2$$

and the connecting homomorphism $H^*_\Gamma(X_s; \mathbb{F}_2) \to H^{*+1}_\Gamma(X, X_s; \mathbb{F}_2)$ is surjective. □

The paper is organized as follows. In section 2 we recall the centralizer spectral sequence and in section 3 we prove Theorem 1.1 and Theorem 1.2. In Section 4 we make some comments on step 2 of the program of a complete calculation of $H^*(\Gamma; \mathbb{F}_2)$. Finally in section 5 we discuss the relation with Quillen's conjecture.

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2. The centralizer spectral sequence

We recall the centralizer spectral sequence introduced in [H1].

Let G be a discrete group and let p be a fixed prime. Let $\mathcal{A}(G)$ be the category whose objects are the elementary abelian p-subgroups E of G, i.e. subgroups which are isomorphic to $(\mathbb{Z}/p)^k$ for some integer k; if E_1 and E_2 are elementary abelian p-subgroups of G, then the set of morphisms from E_1 to E_2 in $\mathcal{A}(G)$ consists precisely of those group homomorphisms $\alpha : E_1 \to E_2$ for which there exists an element $g \in G$ with $\alpha(e) = geg^{-1}$ for all $e \in E_1$. Let $\mathcal{A}_*(G)$ be the full subcategory of $\mathcal{A}(G)$ whose objects are the non-trivial elementary abelian p-subgroups.

For an elementary abelian p-subgroup we denote its centralizer in G by $C_G(E)$. Then the assignment $E \mapsto H^*(C_G(E); \mathbb{F}_p)$ determines a functor from $\mathcal{A}_*(G)$ to the category $\mathcal{E}$ of graded $\mathbb{F}_p$-vector spaces. The inverse limit functor is a left exact functor from the functor category $\mathcal{E}^{\mathcal{A}_*(G)}$ to $\mathcal{E}$. Its right derived functors are denoted by $\lim^s$. The p-rank $r_p(G)$ of a group G is defined as the supremum of all k such that G contains a subgroup isomorphic to $(\mathbb{Z}/p)^k$.

For a G-space X and a fixed prime p we denote by X_s the p-singular locus, i.e. the subspace of X consisting of points whose isotropy group contains an element of order p. Let EG be the total space of the universal principal G-bundle. The mod-p cohomology of the Borel construction $EG \times_G X$ of a G space X will be denoted $H^*_G(X; \mathbb{F}_p)$. The following result is a special case of part (a) of Corollary 0.4 of [H1].

Theorem 2.1. *Let G be a discrete group and assume there exists a finite dimensional mod-p acyclic G-CW complex X such that the isotropy group of each cell is finite. Then there exists a cohomological second quadrant spectral sequence*

$$E_2^{s,t} = \lim\nolimits^s_{\mathcal{A}_*(G)} H^t(C_G(E); \mathbb{F}_p) \Longrightarrow H^{s+t}_G(X_s; \mathbb{F}_p)$$

with $E_2^{s,t} = 0$ if $s \geq r_p(G)$ and $t \geq 0$.

Remark 2.2. The edge homomorphism in this spectral sequence is a map of algebras

$$H^*_G(X_s; \mathbb{F}_p) \to \lim\nolimits_{\mathcal{A}_*(G)} H^*(C_G(E); \mathbb{F}_p)$$

which is given as follows.

Let X^E be the fixed points for the action of E on X. The G-action on X restricts to an action of the centralizer $C_G(E)$ on X^E and the G-equivariant maps

$$G \times_{C_G(E)} X^E \to X_s, \quad (g,x) \mapsto gx \ .$$

for $E \in \mathcal{A}_*(G)$ induce compatible maps in Borel cohomology

$$H^*_G(X_s;\mathbb{F}_2) \to H^*_G(G \times_{C_G(E)} X^E;\mathbb{F}_2) \cong H^*_{C_G(E)}(X^E;\mathbb{F}_2) \cong H^*(C_G(E);\mathbb{F}_2)$$

which assemble to give the map to the inverse limit. Here we have used that by classical Smith theory X^E is mod p-acyclic if X is mod-p acyclic and hence we get canonical isomorphisms $H^*_{C_G(E)}(X^E;\mathbb{F}_2) \cong H^*_{C_G(E)}(*;\mathbb{F}_2) \cong H^*(C_G(E);\mathbb{F}_2)$.

Furthermore the composition

$$H^*(G;\mathbb{F}_p) = H^*_G(X;\mathbb{F}_p) \to H^*_G(X_s;\mathbb{F}_2) \to H^*(C_G(E);\mathbb{F}_2) \tag{2.1}$$

is induced by the inclusions $C_G(E) \to G$ as E varies through $\mathcal{A}_*(G)$.

In [H1] we have used this spectral sequence in the case $p = 2$ and $G = SL_3(\mathbb{Z})$. Here we will use it in the case $p = 2$ and $G = SL(3,\mathbb{Z}[\frac{1}{2},i])$. In both cases we have $r_2(G) = 2$ and hence the spectral sequence collapses at E_2 and degenerates into a short exact sequence

$$0 \to \lim\nolimits^1_{\mathcal{A}_*(G)} H^t(C_G(E);\mathbb{F}_2) \to H^{t+1}_G(X_s;\mathbb{F}_2) \to \lim\nolimits_{\mathcal{A}_*(G)} H^{t+1}(C_G(E);\mathbb{F}_2) \to 0 \ . \tag{2.2}$$

3. The centralizer spectral sequence for $SL_3(\mathbb{Z}[\frac{1}{2},i])$

3.1. **The Quillen category.** Let K be any number field, let $\mathcal{O}_K$ be its ring of integers and consider the ring of S-integers $\mathcal{O}_K[\frac{1}{2}]$. Then, up to equivalence, the Quillen category of $G := SL_3(\mathcal{O}_K[\frac{1}{2}])$ for the prime 2 is independant of K. In fact, because 2 is invertible every elementary abelian 2-subgroup is conjugate to a diagonal subgroup, and hence $\mathcal{A}_*(G)$ has a skeleton, say $\mathcal{A}$, with exactly two objects, say E_1 and E_2 of rank 1 and 2, respectively. We take E_1 to be the subgroup generated by the diagonal matrix whose first two diagonal entries are -1 and whose third diagonal entry is 1, and E_2 to be the subgroup of all diagonal matrices with diagonal entries 1 or -1 and determinant 1.

The automorphism group of E_1 is trivial, of course, while $\mathrm{Aut}_{\mathcal{A}}(E_2)$ is isomorphic to the group of all abstract automorphisms of E_2 which we can identify with $\mathfrak{S}_3$, the symmetric group on three elements. There are three morphisms from E_1 to E_2 and $\mathrm{Aut}_{\mathcal{A}}(E_2)$ acts transitively on them.

3.2. **The centralizers and their cohomology.** For the centralizers in $H := GL_3(\mathcal{O}_K[\frac{1}{2}])$ we find $C_H(E_1) \cong GL_2(\mathcal{O}_K[\frac{1}{2}]) \times GL_1(\mathcal{O}_K[\frac{1}{2}])$ resp. $C_H(E_2) \cong D_3(\mathcal{O}_K[\frac{1}{2}])$ if $D_n(\mathcal{O}_K[\frac{1}{2}])$ denotes the subgroup of diagonal matrices in $GL_n(\mathcal{O}_K[\frac{1}{2}])$. This implies

$$C_G(E_1) \cong GL_2(\mathcal{O}_K[\frac{1}{2}]), \quad C_G(E_2) \cong D_2(\mathcal{O}_K[\frac{1}{2}]) \cong \mathcal{O}_K[\frac{1}{2}]^\times \times \mathcal{O}_K[\frac{1}{2}]^\times \ .$$

From now on we specialize to the case $K = \mathbb{Q}_2[i]$ where we have $\mathcal{O}_K[\frac{1}{2}] = \mathbb{Z}[\frac{1}{2},i]$. In this case the cohomology of the centralizers is explicitly known. In the sequel we abbreviate $SL_3(\mathbb{Z}[\frac{1}{2},i])$ by Γ.

3.2.1. *The cohomology of $C_\Gamma(E_2)$.* There is an isomorphism of groups

$$\mathbb{Z}/4 \times \mathbb{Z} \cong \mathbb{Z}[\frac{1}{2}, i]^\times, \quad (n, m) \mapsto i^n(1+i)^m$$

and therefore we get an isomorphism

$$H^*(C_\Gamma(E_2); \mathbb{F}_2) \cong H^*(\mathbb{Z}[\frac{1}{2}, i]^\times \times \mathbb{Z}[\frac{1}{2}, i]^\times; \mathbb{F}_2) \cong \mathbb{F}_2[y_1, y_2] \otimes E(x_1, x_1', x_2, x_2') \tag{3.1}$$

with y_1 and y_2 in degree 2 and the other generators in degree 1. We agree to choose the generators so that y_1, x_1 and x_1' come from the first factor with x_1 and x_1' being the dual basis to the basis of

$$H_1(\mathbb{Z}[\frac{1}{2}, i]^\times; \mathbb{F}_2) \cong \mathbb{Z}[\frac{1}{2}, i])^\times / \big(\mathbb{Z}[\frac{1}{2}, i])^\times\big)^2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$$

given by the image of i and $(1+i)$ in the mod-2 reduction of the abelian group $GL_1(\mathbb{Z}[\frac{1}{2}, i])$ and y_1 coming from $H^2(\mathbb{Z}/4; \mathbb{F}_2)$; likewise with y_2, x_2 and x_2' coming from the second factor.

3.2.2. *The cohomology of $C_\Gamma(E_1)$.* This cohomology has been calculated in [W]. In fact, from Theorem 1 of [W] we know

$$H^*(C_\Gamma(E_1); \mathbb{F}_2) \cong H^*(GL_2(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2) \cong \mathbb{F}_2[c_1, c_2] \otimes E(e_1, e_1', e_3, e_3')\ . \tag{3.2}$$

In the sequel we give a short summary of this calculation. The classes e_1, e_1', e_3 and e_3' are pulled back from Quillen's exterior classes q_1 and q_3 [Q2] in

$$H^*(GL_2(\mathbb{F}_5); \mathbb{F}_2) \cong \mathbb{F}_2[c_1, c_2] \otimes E(q_1, q_3) \tag{3.3}$$

via two ring homomorphisms

$$\pi : \mathbb{Z}[\frac{1}{2}, i] \to \mathbb{F}_5\ , \quad \pi' : \mathbb{Z}[\frac{1}{2}, i] \to \mathbb{F}_5\ . \tag{3.4}$$

We choose π such that i is sent to 3 and π' such that i is sent to 2.

Then consider the two commutative diagrams (with horizontal arrows induced by inclusion and vertical arrows induced by π resp. π')

$$\begin{array}{ccc} D_2(\mathbb{Z}[\frac{1}{2}]) & \to & GL_2(\mathbb{Z}[\frac{1}{2}]) \\ \downarrow & & \downarrow \\ D_2(\mathbb{F}_5) & \to & GL_2(\mathbb{F}_5)\ . \end{array} \tag{3.5}$$

By abuse of notation we can write

$$H^*(D_2(\mathbb{F}_5); \mathbb{F}_2) \cong H^*(\mathbb{F}_5^\times \times \mathbb{F}_5^\times; \mathbb{F}_2) \cong \mathbb{F}_2[y_1, y_2] \otimes E(x_1, x_2) \tag{3.6}$$

with $y_1 \in H^2(\mathbb{F}_5^\times; \mathbb{F}_2)$ and $x_1 \in H^2(\mathbb{F}_5^\times; \mathbb{F}_2)$ coming from the first factor and likewise with y_2 and x_2 coming from the second factor. Then these ring homomorphisms induce two homomorphisms

$$\pi^*, \pi'^* : H^*(D_2(\mathbb{F}_5); \mathbb{F}_2)) \to H^*(D_2(\mathbb{Z}[\frac{1}{2}]); \mathbb{F}_2)$$

which in term of the isomorphisms (3.6) and (3.1) are explicitly given by

$$\pi^*(y_i) = y_i = \pi'^*(y_i), \quad \pi^*(x_i) = x_i, \quad \pi'^*(x_i) = x_i + x_i' \quad \text{for} \quad i = 1, 2\ . \tag{3.7}$$

The cohomology of $GL_2(\mathbb{F}_5)$ is detected by restriction to the cohomology of diagonal matrices and restriction is given explicitly as follows:

$$c_1 \mapsto y_1 + y_2, \quad c_2 \mapsto y_1 y_2, \quad q_1 \mapsto x_1 + x_2 \quad q_3 \mapsto y_1 x_2 + y_2 x_1\ . \tag{3.8}$$

Then e_1, e_1', e_3, e_3' are defined via

$$e_1 = \pi^*(q_1), \quad e_3 = \pi^*(q_3), \quad e_1' = \pi'^*(q_1), \quad e_3' = \pi'^*(q_3)\ . \tag{3.9}$$

If c_1 and c_2 are the Chern classes of the tautological 2-dimensional complex representation of $GL_2(\mathbb{Z}[\frac{1}{2}], i)$, then the restriction homomorphism from $H^*(GL_2(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$ to the cohomology of the subgroup of diagonal matrices is injective and by using (3.5) and (3.8) we see that it is explicitly given by

$$\begin{array}{llll} c_1 \mapsto & y_1 + y_2 & c_2 \mapsto & y_1 y_2 \\ e_1 \mapsto & x_1 + x_2 & e_3 \mapsto & y_1 x_2 + y_2 x_1 \\ e_1' \mapsto & x_1 + x_1' + x_2 + x_2' & e_3' \mapsto & y_1(x_2 + x_2') + y_2(x_1 + x_1')\ . \end{array} \tag{3.10}$$

3.2.3. *Functoriality.* We note that together with the isomorphisms (3.1) and (3.2) the restriction (3.10) also describes the map

$$\alpha_* : H^*(C_\Gamma(E_1); \mathbb{F}_2) \to H^*(C_\Gamma(E_2); \mathbb{F}_2)$$

induced from the standard inclusion of E_1 into E_2.

To finish the description of $H^*(C_\Gamma(-); \mathbb{F}_2)$ as a functor on $\mathcal{A}$ it remains to describe the action of the symetric group $Aut_{\mathcal{A}}(E_2) \cong \mathfrak{S}_3$ of rank 3 on $H^*(C_\Gamma(E_2); \mathbb{F}_2) \cong \mathbb{F}_2[y_1, y_2] \otimes \Lambda(x_1, x_1', x_2, x_2')$ and because of the multiplicative structure we need it only on the generators.

If $\tau \in \mathrm{Aut}_{\mathcal{A}}(E_2)$ corresponds to permuting the factors in $C_\Gamma(E_2) \cong GL_1(\mathbb{Z}[\frac{1}{2}, i]) \times GL_1(\mathbb{Z}[\frac{1}{2}, i])$ then

$$\begin{array}{lll} \tau_*(y_1) = y_2 & \tau_*(x_1) = x_2 & \tau_*(x_1') = x_2' \\ \tau_*(y_2) = y_1 & \tau_*(x_2) = x_1 & \tau_*(x_2') = x_1' \end{array} \tag{3.11}$$

and if $\sigma \in \mathrm{Aut}_{\mathcal{A}}(E_2)$ corresponds to the cyclic permutation of the diagonal entries (in suitable order) then

$$\begin{array}{lll} \sigma_*(y_1) = y_2 & \sigma_*(x_1) = x_2 & \sigma_*(x_1') = x_2' \\ \sigma_*(y_2) = y_1 + y_2 & \sigma_*(x_2) = x_1 + x_2 & \sigma_*(x_2') = x_1' + x_2'\ . \end{array} \tag{3.12}$$

3.3. **Calculating the limit and its derived functors.** In Proposition 4.3 of [H1] we showed that for any functor F from $\mathcal{A}$ to $\mathbb{Z}_{(2)}$-modules there is an exact sequence

$$0 \to \lim_{\mathcal{A}} F \to F(E_1) \xrightarrow{\varphi} \mathrm{Hom}_{\mathbb{Z}[\mathfrak{S}_3]}(St_{\mathbb{Z}}, F(E_2)) \to \lim^1_{\mathcal{A}} F \to 0 \tag{3.13}$$

where $St_{\mathbb{Z}}$ is the $\mathbb{Z}[\mathfrak{S}_3]$ module given by the kernel of the augmentation $\mathbb{Z}[\mathfrak{S}_3/\mathfrak{S}_2] \to \mathbb{Z}$, and if a and b are chosen to give an integral basis of $St_{\mathbb{Z}}$ on which τ and σ act via

$$\begin{array}{ll} \tau_*(a) = b & \tau_*(b) = a \\ \sigma_*(a) = -b & \sigma_*(b) = a - b \end{array} \tag{3.14}$$

then $\varphi(x)(a) = \alpha_*(x) - (\sigma_*)^2\alpha_*(x)$ and $\varphi(x)(b) = \alpha_*(x) - \sigma_*\alpha_*(x)$ if $x \in F(E_1)$.

Because in our case the functor takes values in $\mathbb{F}_2$-vector spaces we can replace $\mathrm{Hom}_{\mathbb{Z}[\mathfrak{S}_3]}$ by $\mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}$ and $St_{\mathbb{Z}}$ by its mod-2 reduction. The following elementary lemma is needed in the analysis of the third term in the exact sequence (3.13).

Lemma 3.1.

a) Let St be the $\mathbb{F}_2[\mathfrak{S}_3]$-module given as the kernel of the augmentation $\mathbb{F}_2[\mathfrak{S}_3/\mathfrak{S}_2] \to \mathbb{F}_2$. The tensor product $St \otimes St$ decomposes as $\mathbb{F}_2[\mathfrak{S}_3]$-module canonically as

$$St \otimes St \cong \mathbb{F}_2[\mathfrak{S}_3/A_3] \oplus St$$

where A_3 denotes the alternating group on three letters. In fact, the decomposition is given by

$$St \otimes St \cong Im(id + \sigma_* + \sigma_*^2) \oplus Ker(id + \sigma_* + \sigma_*^2)$$

and the first summand is isomorphic to $\mathbb{F}_2[\mathfrak{S}_3/A_3]$ *while the second summand is isomorphic to* St.

b) The tensor product $\mathbb{F}_2[\mathfrak{S}_3/A_3]\otimes St$ *is isomorphic to* $St\oplus St$.

Proof. a) It is well known that St is a projective $\mathbb{F}_2[\mathfrak{S}_3]$-module, hence $St\otimes St$ is also projective. It is also well known that every projective indecomposable $\mathbb{F}_2[\mathfrak{S}_3]$-module is isomorphic to either St or $\mathbb{F}_2[\mathfrak{S}_3/A_3]$. Both modules can be distinguished by the fact that $e := id+\sigma_*+\sigma_*$ acts trivially on St and as the identity on $\mathbb{F}_2[\mathfrak{S}_3/A_3]$.

Furthemore e is a central idempotent in $\mathbb{F}_2[\mathfrak{S}_3]$ and hence each $\mathbb{F}_2[\mathfrak{S}_3]$-module M decomposes as direct sum of $\mathbb{F}_2[\mathfrak{S}_3]$-modules

$$M\cong \mathrm{Im}(e:M\to M)\oplus \mathrm{Ker}(e:M\to M)\ .$$

An easy calculation shows that in the case of $St\otimes St$ both submodules are non-trivial and this together with the fact these submodules must be projective proves the claim.

b) Again each of the factors in the tensor product is a projective $\mathbb{F}_2[\mathfrak{S}_3]$-module, hence the tensor product is a projective $\mathbb{F}_2[\mathfrak{S}_3]$-module. Because σ acts as the identity on $\mathbb{F}_2[\mathfrak{S}_3/A_3]$ we see that the idempotent e acts trivially on the tensor product and this forces the tensor product to be isomorphic to $St\oplus St$. □

Lemma 3.2. *The Poincaré series* χ_2 *of* $\mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(St,\mathbb{F}_2[y_1,y_2]\otimes E(x_1,x_1',x_2,x_2'))$ *is given by*

$$\chi_2=\frac{2t^2(1+3t^2+3t^4+t^6)+2t(1+2t^2+2t^4+2t^6+t^8)}{(1-t^4)(1-t^6)}\ .$$

Proof. The isomorphism of (3.1) is an isomorphism of $\mathbb{F}_2[\mathfrak{S}_3]$-modules where the action of $\mathfrak{S}_3$ is given by (3.11) and (3.12). In particular we see that $H^1(GL_1(\mathbb{Z}[\frac{1}{2},i])\times GL_1(\mathbb{Z}[\frac{1}{2},i]);\mathbb{F}_2)$ is isomorphic to $St\oplus St$ generated by x_1,x_1',x_2,x_2'. The exterior powers of H^1 are given as

$$E^k(x_1,x_2,x_1',x_2')\cong E^k(St\oplus St)\cong\bigoplus_{j=0}^{k}E^jSt\otimes E^{k-j}St$$

and, because $E^k(St)$ is isomorphic to $\Sigma^k\mathbb{F}_2$ if $k=0,2$, isomorphic to ΣSt if $k=1$, and trivially otherwise, we obtain

$$E^k(x_1,x_2,x_1',x_2')\cong\begin{cases}\Sigma^k\mathbb{F}_2 & k=0,4\\ \Sigma^k(St\oplus St) & k=1,3\\ \Sigma^2\mathbb{F}_2\oplus\Sigma^2(St\otimes St)\oplus\Sigma^2\mathbb{F}_2 & k=2\\ 0 & k\neq 0,1,2,3,4\end{cases}$$

where $\mathbb{F}_2$ denotes the trivial $\mathbb{F}_2[\mathfrak{S}_3]$-module whose additive structure is that of $\mathbb{F}_2$.

Therefore the Poincaré series χ_2 of $\mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(St,H^*(C_G(E_2);\mathbb{F}_2))$ decomposes according to the decomposition of $\Lambda(x_1,x_2',x_1',x_2')$ as sum

(3.15) $$\chi_2:=(1+2t^2+t^4)\chi_{2,0}+t^2\chi_{2,1}+2(t+t^3)\chi_{2,2}$$

where $\chi_{2,0}$ is the Poincaré series of $\mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(St,\mathbb{F}_2[y_1,y_2])$, $\chi_{2,1}$ is the Poincaré series of $\mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(St,St\otimes St\otimes\mathbb{F}_2[y_1,y_2])$ and $\chi_{2,2}$ is that of $\mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(St,St\otimes\mathbb{F}_2[y_1,y_2])$.

Furthermore it is well known (and elementary to verify) that there is an isomorphism of $\mathbb{F}_2[\mathfrak{S}_3]$-modules $St \oplus St \oplus \mathbb{F}_2[\mathfrak{S}_3/A_3] \cong \mathbb{F}_2[\mathfrak{S}_3]$ and therefore an isomorphism

$$\begin{aligned} \mathbb{F}_2[y_1,y_2] &\cong \mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(St \oplus St \oplus \mathbb{F}_2[\mathfrak{S}_3/A_3], \mathbb{F}_2[y_1,y_2]) \\ &\cong \mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(St, \mathbb{F}_2[y_1,y_2])^{\oplus 2} \oplus \mathbb{F}_2[y_1,y_2]^{A_3} \ . \end{aligned}$$

Together with the elementary fact that the A_3-invariants $\mathbb{F}_2[y_1,y_2]^{A_3}$ form a free module over $\mathbb{F}_2[y_1,y_2]^{\mathfrak{S}}_3 \cong \mathbb{F}_2[c_2,c_3]$ on two generators 1 and $y_1^3 + y_1y_2^2 + y_2^3$ of degree 0 resp. 6 this implies

$$2\chi_{2,0} + \frac{1+t^6}{(1-t^4)(1-t^6)} = \frac{1}{(1-t^2)^2}$$

and hence

$$\chi_{2,0} = \frac{t^2}{(1-t^2)(1-t^6)} \ . \tag{3.16}$$

It is elementary to check that St and $\mathbb{F}_2[\mathfrak{S}_3/A_3]$ are both self-dual $\mathbb{F}_2[\mathfrak{S}_3]$-modules and hence Lemma 3.1 gives

$$St \otimes St^* \cong \mathbb{F}_2[\mathfrak{S}_3/A_3] \oplus St$$

and

$$\begin{aligned} St \otimes St^* \otimes St^* &\cong (\mathbb{F}_2[\mathfrak{S}_3/A_3] \oplus St) \otimes St^* \\ &\cong (\mathbb{F}_2[\mathfrak{S}_3/A_3] \otimes St) \oplus (St \otimes St) \\ &\cong St \oplus St \oplus St \oplus \mathbb{F}_2[\mathfrak{S}_3/A_3] \ . \end{aligned}$$

Therefore, if $\chi_{\mathbb{F}_2[y_1,y_2]^{A_3}}$ denotes the Poincaré series of the A_3-invariants then

$$\chi_{2,1} = 3\chi_{2,0} + \chi_{\mathbb{F}_2[y_1,y_2]^{A_3}} = \frac{3t^2}{(1-t^2)(1-t^6)} + \frac{1+t^6}{(1-t^4)(1-t^6)} = \frac{1+3t^2+3t^4+t^6}{(1-t^4)(1-t^6)} \tag{3.17}$$

$$\chi_{2,2} = \chi_{2,0} + \chi_{\mathbb{F}_2[y_1,y_2]^{A_3}} = \frac{t^2}{(1-t^2)(1-t^6)} + \frac{1+t^6}{(1-t^4)(1-t^6)} = \frac{1+t^2+t^4+t^6}{(1-t^4)(1-t^6)} \ . \tag{3.18}$$

Finally (3.15), (3.16), (3.17) and (3.18) give

$$\begin{aligned} \chi_2 &= \frac{(1+2t^2+t^4)t^2(1+t^2) + t^2(1+3t^2+3t^4+t^6) + 2(t+t^3)(1+t^2+t^4+t^6)}{(1-t^4)(1-t^6)} \\ &= \frac{2t^2(1+3t^2+3t^4+t^6) + 2t(1+2t^2+2t^4+2t^6+t^8)}{(1-t^4)(1-t^6)} \ , \end{aligned}$$

and this finishes the proof. □

Theorem 1.1 is now an immediate consequence of Theorem 2.1 and the following result.

Proposition 3.3. *Let $p = 2$ and $\Gamma = SL_3(\mathbb{Z}[\frac{1}{2}, i])$.*

a) There is an isomorphism of graded $\mathbb{F}_2$-algebras

$$\lim_{\mathcal{A}_*(\Gamma)} H^*(C_\Gamma(E); \mathbb{F}_2) \cong \mathbb{F}_2[b_2,b_3] \otimes E(d_3,d_3',d_5,d_5') \ .$$

Furthermore, if we identify this limit with a subalgebra of $H^*(C_\Gamma(E_1);\mathbb{F}_2) \cong \mathbb{F}_2[c_1,c_2] \otimes E(e_1,e_1',e_3,e_3')$ then

$$\begin{aligned} b_2 &= c_1^2 + c_2 \qquad & b_3 &= c_1c_2 \\ d_3 &= e_3 & d_5 &= c_1e_3 + c_2e_1 \\ d_3' &= e_3' & d_5' &= c_1e_3' + c_2e_1' \ . \end{aligned}$$

b) There is an isomorphism of graded $\mathbb{F}_2$-vector spaces

$$\lim^1_{\mathcal{A}_*(\Gamma)} H^*(C_\Gamma(E);\mathbb{F}_2) \cong \Sigma^3\mathbb{F}_2 \oplus \Sigma^3\mathbb{F}_2 \oplus \Sigma^6\mathbb{F}_2 \ .$$

c) For any $s > 1$

$$\lim^s_{\mathcal{A}_*(\Gamma)} H^*(C_\Gamma(E);\mathbb{F}_2) = 0 \ .$$

Proof. a) It is straightforward to check that the subalgebra of $\mathbb{F}_2[c_1, c_2] \otimes E(e_1, e_1', e_3, e_3')$ generated by the elements $c_1^2 + c_2$, c_1c_2, $e_3, e_3', c_1e_3 + c_2e_1, c_1e_3' + c_2e_1'$ is isomorphic to the tensor product of a polynomial algebra on two generators b_2 and b_3 of degree 4 and 6 and an exterior algebra on 4 generators d_3, d_3', d_5 and d_5' of degree 3, 3, 5 and 5. In fact, it is clear that $c_1^2+c_2$ and c_1c_2 are algebraically independant and the elements $e_3, e_3', c_1e_3+c_2e_1, c_1e_3'+c_2e_1'$ are exterior classes; their product is given as $c_2^2e_3e_3'e_1e_1' \neq 0$, and this implies easily that the exterior monomials in these elements are linearly independant over the polynomial algebra generated by $c_1^2 + c_2$, c_1c_2. From now on we identify b_2, b_3, d_3, d_3', d_5 and d_5' with $c_1^2 + c_2$, c_1c_2, e_3, e_3', $c_1e_3 + c_2e_1$ and $c_1e_3' + c_2e_1'$.

Now we use the exact sequence (3.13) and the description of φ to determine the inverse limit. Because α_* is injective, we see that if we identify $H^*(C_\Gamma(E_1);\mathbb{F}_2)$ with its image in $H^*(C_\Gamma(E_2);\mathbb{F}_2)$ then the inverse limit can be identified with the intersection of the image of α_* with the invariants in $\mathbb{F}_2[y_1, y_2] \otimes E(x_1, x_1', x_2, x_2')$ with respect to the action of the cyclic group of order 3 of $\mathrm{Aut}_{\mathcal{A}}(E_2) \cong \mathfrak{S}_3$ generated by σ. This action has been described in (3.12) and with these formulas it is straightfoward to check that the elements

$$\begin{array}{rcl} b_2 &=& y_1^2 + y_1y_2 + y_2^2 \\ b_3 &=& y_1y_2(y_1 + y_2) \\ d_3 &=& y_1x_2 + y_2x_1 \\ d_5 &=& (y_1 + y_2)(y_1x_2 + y_2x_1) + y_1y_2(x_1 + x_2) = y_1^2x_2 + y_2^2x_1 \\ d_3' &=& y_1(x_2 + x_2') + y_2(x_1 + x_1') \\ d_5' &=& (y_1 + y_2)(y_1(x_2 + x_2') + y_2(x_1 + x_1')) + y_1y_2(x_1 + x_1' + x_2 + x_2') \\ &=& y_1^2(x_2 + x_2') + y_2^2(x_1 + x_1') \ . \end{array} \tag{3.19}$$

all belong to the inverse limit.

Now consider the following Poincaré series

$$\chi_0 := \sum_{n\geq 0} \dim_{\mathbb{F}_2}(\mathbb{F}_2[b_2, b_3] \otimes E(e_3, e_3', e_5, e_5')^n)t^n = \frac{(1+t^3)^2(1+t^5)^2}{(1-t^4)(1-t^6)}$$

$$\chi_1 := \sum_{n\geq 0} \dim_{\mathbb{F}_2} H^n(C_\Gamma(E_1);\mathbb{F}_2)t^n = \frac{(1+t)^2(1+t^3)^2}{(1-t^2)(1-t^4)}$$

$$\chi_2 := \frac{2t^2(1+3t^2+3t^4+t^6)+2t(1+2t^2+2t^4+2t^6+t^8)}{(1-t^4)(1-t^6)} \ .$$

Then we have the following identity

$$\chi_0 + \chi_2 - \chi_1 = \frac{p}{(1 - t^4)(1 - t^6)}$$

with

$$\begin{aligned} p &= (1 + t^3)^2(1 + t^5)^2 + 2t^2(1 + 3t^2 + 3t^4 + t^6) \\ &\quad + 2t(1 + 2t^2 + 2t^4 + 2t^6 + t^8) - (1 + t)^2(1 + t^3)^2(1 + t^2 + t^4) \\ &= 2t^3 + t^6 - 2t^7 - 2t^9 - t^{10} - t^{12} + 2t^{13} + t^{16} = (2t^3 + t^6)(1 - t^4)(1 - t^6) \end{aligned}$$

and therefore

$$\chi_0 + \chi_2 = \chi_1 + (2t^3 + t^6) \ . \tag{3.20}$$

This together with the fact that $\lim_{\mathcal{A}_*(\Gamma)} H^*(C_\Gamma(E);\mathbb{F}_2)$ contains a subalgebra which is isomorphic to $\mathbb{F}_2[b_2,b_3]\otimes\Lambda(d_3,d_3',d_5,d_5')$ already implies that the sequence

$$0\to\mathbb{F}_2[b_2,b_3]\otimes E(d_3,d_3',d_5,d_5')\to H^*(C_\Gamma(E_1);\mathbb{F}_2)\xrightarrow{\varphi}\mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(St,H^*(C_\Gamma(E_1);\mathbb{F}_2))\to 0$$

in which the left hand arrow is given by inclusion is exact except possibly in dimensions 3 and 6.

In order to complete the proof of a) it is now enough to verify that in degrees 3 and 6 the inverse limit is not bigger than $\mathbb{F}_2[b_2,b_3]\otimes E(d_3,d_3',d_5,d_5')$. We leave this straightforward verification to the reader.

Then b) follows immediately from (a) together with (3.20) and the exact sequence (3.13), and (c) follows from Theorem 2.1 and the fact that $r_2(G)=2$. □

We can now give the proof of Theorem 1.2.

Proof. a) The exact sequence of Theorem 1.1 is obtained from the exact sequence (2.2) via Proposition 3.3. Therefore the epimorphism of Theorem 1.1 is the edge homomorphism of the centralizer spectral sequence. The result then follows from (2.1) by observing that we have identified the target of the edge homomorphism with the subalgebra $\mathbb{F}_2[b_2,b_3]\otimes E(d_3,d_3',d_5,d_5')$ of $H^*(C_\Gamma(E_1);\mathbb{F}_2)$ and by recalling that $C_\Gamma(E_1)$ is equal to $SD_3(\mathbb{Z}[\frac{1}{2},i])$.

b) The two ring homomorphisms $\pi,\pi':\mathbb{Z}[\frac{1}{2},i]\to\mathbb{F}_5$ of (3.4) determine homomorphisms $SL_3(\mathbb{Z}[\frac{1}{2},i])\subset GL_3(\mathbb{Z}[\frac{1}{2},i])\to GL_3(\mathbb{F}_5)$. By [Q2] we have

$$H^*GL_3(\mathbb{F}_5);\mathbb{F}_2)\cong\mathbb{F}_3[c_1,c_2,c_3]\otimes E(q_1,q_3,q_5)\ .$$

We get a well defined homomorphism of $\mathbb{F}_2$-graded algebras

$$\varphi:\mathbb{F}_2[c_2,c_3]\otimes E(e_3,e_3',e_5,e_5')\to H^*(\Gamma;\mathbb{F}_2)$$

by sending c_i to the i-th Chern class of the tautological 3-dimensional representation of Γ and by declaring $\varphi(e_i)=\pi^*(q_i)$ and $\varphi(e_i')=\pi'^*(q_i')$ for $i=3,5$. The classes q_1 resp. q_3 resp. q_5 are the symmetrisations of x_1 resp. y_1x_2 resp. $y_1y_2x_3$ with respect to the natural action of $\mathfrak{S}_3$ on $H^*(GL_3(\mathbb{F}_5);\mathbb{F}_2)\cong\mathbb{F}_2[y_1,y_2,y_3]\otimes E(x_1,x_2,x_3)$ (cf. (5.1) below).

Next we determine the composition $\psi\phi$. The universal Chern classes c_i are the elementary symmetric polynomials in variables, say y_i, and the inclusion $GL_2(\mathbb{C})\subset SL_3(\mathbb{C})\subset GL_3(\mathbb{C})$ imposes the relation $y_1+y_2+y_3=0$. This implies that the behaviour of ψ on Chern classes is given by

$$c_1\mapsto 0,\quad c_2\mapsto c_1^2+c_2=y_1^2+y_1y_2+y_2^2=b_2,\quad c_3\mapsto c_1c_2=y_1y_2(y_1+y_2)=b_3\ .$$

In these equations we have identified $H^*(GL_2(\mathbb{Z}[\frac{1}{2},i];\mathbb{F}_2)$, as in the proof of Proposition 3.3, via restriction with a subalgebra of $\mathbb{F}_2[y_1,y_2]\otimes E(x_1,x_1',x_3,x_3')$.

In order to determine the composition $\psi\varphi$ on the classes e_3, e_3', e_5 and e_5' we calculate at the level of $\mathbb{F}_5$ and use naturality with respect to the homomorphisms induced by π and π'. In fact, the inclusion

$$j:GL_2(\mathbb{F}_5)\subset SL_3(\mathbb{F}_5)\subset GL_3(\mathbb{F}_5)$$

induce in cohomology a map

$$\mathbb{F}_3[c_1,c_2,c_3]\otimes E(q_1,q_3,q_5)\to\mathbb{F}_2[c_1,c_2]\otimes E(e_1,e_3)\subset\mathbb{F}_2[y_1,y_2]\otimes E(q_1,q_3)$$

which is easily determined from (5.1) below by imposing the relations $y_1+y_2+y_3=0$ and $x_1+x_2+x_3=0$ on the symmetrisation of the classes y_1x_2 resp. $y_1y_2x_3$ with respect to the natural action of $\mathfrak{S}_3$ on the cohomology of diagonal matrices $H^*(D_3(\mathbb{F}_5);\mathbb{F}_2) \cong \mathbb{F}_2[y_1,y_2,y_3] \otimes E(x_1,x_2,x_3)$. Explicitly we get

$$c_1 \mapsto 0, \quad c_2 \mapsto y_1^2 + y_1y_2 + y_2^2, \quad c_3 \mapsto y_1y_2(y_1+y_2)$$
$$q_1 \mapsto 0, \quad q_3 \mapsto y_1x_2 + y_2x_1, \quad q_5 \mapsto y_1^2x_2 + y_2^2x_1$$

and if i denotes the inclusion

$$GL_2(\mathbb{Z}[\frac{1}{2},i]) \subset SL_3(\mathbb{Z}[\frac{1}{2},i]) \subset GL_3(\mathbb{Z}[\frac{1}{2},i])$$

then (3.7) and (3.19) imply

$$\begin{array}{lllllllll} \psi(\varphi(e_3)) & = & i^*(\pi^*(q_3)) & = & \pi^*j^*(q_3) & = & \pi^*(y_1x_2+y_2x_1) & = & d_3 \\ \psi(\varphi(e_5)) & = & i^*(\pi^*(q_5)) & = & \pi^*j^*(q_5) & = & \pi^*(y_1^2x_2+y_2^2x_1) & = & d_5 \\ \psi(\varphi(e_3')) & = & i^*(\pi'^*(q_3)) & = & \pi'^*j^*(q_3) & = & \pi'^*(y_1x_2+y_2x_1) & = & d_3' \\ \psi(\varphi(e_5')) & = & i^*(\pi'^*(q_5)) & = & \pi'^*j^*(q_5) & = & \pi'^*(y_1^2x_2+y_2^2x_1) & = & d_5' \end{array}$$

where we have identified the target of ψ with a subalgebra of $H^*(GL_2(\mathbb{Z}[\frac{1}{2},i];\mathbb{F}_2)$ and the latter via restriction with a subalgebra of $\mathbb{F}_2[y_1,y_2] \otimes E(x_1x_1',x_3,x_3')$.

c) The space X can be taken to be the product of symmetric space $X_\infty := SL_3(\mathbb{C})/SU(2)$ and the Bruhat-Tits building X_2 for $SL_3(\mathbb{Q}_2[i])$. Now $SL_3(\mathbb{Q}_2[i])\backslash X_2$ is a 2-simplex (cf. [B]) and the projection map $X \to X_2$ induces a map

$$SL_3(\mathbb{Q}_2[i])\backslash X \to SL_3(\mathbb{Q}_2[i])\backslash X_2$$

whose fibres have the homotopy type of a 6-dimensional $SL_3(\mathbb{Z}[\frac{1}{2},i])$-invariant deformation retract (cf. section 4). Therefore we get $H^n_G(X,X_s;\mathbb{F}_2) = 0$ if $n > 8$ and the inclusion $X_s \subset X$ induces an isomorphism $H^n_G(X;\mathbb{F}_2) \cong H^n_G(X_s;\mathbb{F}_2)$ if $n > 8$. Then part c) simply follows from a) except for the finiteness statement for the kernel for which we refer to (4.1) and (4.2) below. $\square$

4. Comments on step 2

The situation for $p = 2$ and $G = SL_3(\mathbb{Z}[\frac{1}{2},i])$ is analogous to the situation for $p = 2$ and $G = SL_3(\mathbb{Z}[\frac{1}{2}])$ for which step 2 was carried out in [H2] via a detailed study of the relative cohomology $H^*_G(X,X_s;\mathbb{F}_2)$ for X equal to the product of the symmetric space $X_\infty := SL_3(\mathbb{R})/SO(3)$ with the Bruhat-Tits building X_2 for $SL_3(\mathbb{Q}_2)$; the spaces involved had a few hundred cells and the calculation was painful. In the case of $SL_3(\mathbb{Z}[\frac{1}{2},i])$ with X the product of $SL_3(\mathbb{C})/SU(3)$ with the Bruhat-Tits building for $SL_3(\mathbb{Q}_2[i])$ the calculational complexity of the second step is much more involved and an explicit calculation by hand does not look feasible. However, in recent years there have been a lot of machine aided calculations of the cohomology of various arithmetic groups (for example [GG], [BRW]) and a machine aided calculation seems to be within reach.

The natural strategy for undertaking this second step is to follow the same path as in [H2]. The equivariant cohomology $H^*_\Gamma(X,X_s;\mathbb{F}_2)$ can be studied via the spectral sequence of the projection map

$$p : X = X_\infty \times X_2 \to X_2 \ .$$

This gives a spectral sequence with

$$E_1^{s,t} \cong \bigoplus_{\sigma\in\Lambda_s} H^t_{\Gamma_\sigma}(X_\infty, X_{\infty,s};\mathbb{F}_2) \Longrightarrow H^{s+t}_\Gamma(X,X_s;\mathbb{F}_2) \ . \tag{4.1}$$

Here Λ_s indexes the s-dimensional cells in the orbit space of X_2 with respect to the action of Γ. The orbit space is a 2-simplex, i.e. Λ_0 and Λ_1 contain 3 elements and Λ_2 is a singleton. Furthermore Γ_σ is the isotropy group of a chosen representative in X_2 of the cell σ in the quotient space. For fixed s all s-dimensional cells have isomorphic isotropy groups because the Γ-action on the Bruhat-Tits building is the restriction of a natural action of $GL_3(\mathbb{Z}[\frac{1}{2}, i])$ on X_2 and this action is transitive on the set of s-dimensional cells (cf. [B]).

Therefore all isotropy subgroups for the action on X_2 are, up to isomorphism, subgroups of $SL_3(\mathbb{Z}[i])$ which itself appears as isotropy group of a 0-dimensional cell in X_2. The isotropy groups of 1-dimensional and 2-dimensional cells are isomorphic to well-known congruence subgroups of $SL_3(\mathbb{Z}[i])$. By the Soulé-Lannes method the fibre X_∞ of the projection map p admits a 6-dimensional $SL_3(\mathbb{Z}[i])$-equivariant deformation retract (the space of "well-rounded hermitean forms" modulo arithmetic equivalence) with compact quotient (cf. [Ash]) and therefore we have

(4.2) $$E_1^{s,t} = 0 \text{ unless } s = 0, 1, 2,\ 0 \le t \le 6, \text{ and } \dim_{\mathbb{F}_2} E_1^{s,t} < \infty \text{ for all } (s, t)\ .$$

The E_1-term of this spectral sequence should be accessible to machine calculation. The spectral sequence will necessarily degenerate at E_3 and the calculation of the d_1-differential and, if necessary the d_2-differential, is likely to need human intervention, as it was necessary in the case of $SL(3, \mathbb{Z}[\frac{1}{2}])$ (cf. section 3.4 of [H2]). Likewise the calculation of the connecting homomorphism for the mod-2 Borel cohomology of the pair (X, X_s) is likely to require human intervention.

5. Relation to Quillen's conjecture

The next result gives gives 2 reformulations of Quillen's conjecture which we had briefly discussed in the introduction. The classes e_{2k-1}, e'_{2k-1} figuring in part c) will be introduced in (5.1) below.

Theorem 5.1. *Suppose $n \ge 2$. Then the following statements are equivalent.*

a) $C(n, \mathbb{Z}[\frac{1}{2}, i], 2)$ holds, i.e. $H^*(GL_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$ is a free module over $\mathbb{Z}/2[c_1, \ldots, c_n]$ where the c_i are the mod-2 Chern classes of the tautological n-dimensional complex representation of $GL_n(\mathbb{Z}[\frac{1}{2}, i])$.

b) The restriction homomorphism $H^*(GL_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2) \to H^*(D_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$ is injective where $D_n(\mathbb{Z}[\frac{1}{2}, i])$ denotes the subgroup of diagonal matrices in $GL_n(\mathbb{Z}[\frac{1}{2}])$.

c) There are isomorphisms

$$H^*(GL_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2) \cong \mathbb{F}_2[c_1, \ldots, c_n] \otimes E(e_1, e'_1, \ldots, e_{2n-1}, e'_{2n-1})$$

where the classes c_k are the Chern classes of the tautological n-dimensional complex representation of $GL_n(\mathbb{Z}[\frac{1}{2}, i])$ and the classes e_{2k-1}, e'_{2k-1} are of cohomological degree $2k-1$ for $k = 1, \ldots, n$.

Proof. It is trivial that (c) implies (a).

In order to show that (a) implies (b) we observe that $D_n(\mathbb{Z}[\frac{1}{2}, i])$ is the centralizer of the unique, up to conjugacy, maximal elementary abelian 2-subgroup E_n of $GL_n(\mathbb{Z}[\frac{1}{2}, i])$ given by the subgroup of diagonal matrices of order 2. Now consider the top Dickson invariant ω in $H^*(BGL_n(\mathbb{C}); \mathbb{F}_2)$, i.e. the class whose restriction to $H^*B(\prod_{i=1}^n GL_1(\mathbb{C})); \mathbb{F}_2)$ is the product of all non-trivial classes of degree 2. The image of ω in $H^*(GL_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$ restricts trivially to the cohomology of all elementary abelian 2-subgroups E of $GL_n(\mathbb{Z}[\frac{1}{2}, i])$ of rank less than n. If (a) holds then the image of ω is not a zero divisor in $H^*(GL_n(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$ and hence Corollary I.5.8 of [HLS] implies that the restriction to the centralizer of E_n is injective.

The implication $(b) \Rightarrow (c)$ follows from Proposition 5.3 below. □

Before we go on we introduce the classes e_{2k-1} and e'_{2k-1}. As in the case of GL_2 they are obtained from Quillen's classes $q_{2k-1} \in H^{2k-1}(GL_n(\mathbb{F}_5);\mathbb{F}_2)$ [Q2] which restrict in the cohomology of diagonal matrices in $\mathbb{F}_5$ to the symmetrization of the class $y_1 \dots y_{k-1}x_k$ where y_k is of cohomological degree 2 corresponding to the k-th factor in the product $\prod_{k=1}^n \mathbb{F}_5^\times$ and x_k is of cohomological degree 1 of the same factor. Then we define

$$e_{2k-1} := \pi^*(q_{2k-1}), \quad e'_{2k-1} := \pi'^*(q_{2k-1}) \tag{5.1}$$

where π, π' are the two ring homomorphisms $\mathbb{Z}[\frac{1}{2},i] \to \mathbb{F}_5$ with π sending i to 3 and π' sending i to 2 which we considered earlier in section 3. If we identify the mod-2 cohomology $H^*(D_n(\mathbb{Z}[\frac{1}{2},i]);\mathbb{F}_2)$ with $\mathbb{F}_2[y_1,\dots y_n] \otimes E(x_1,x'_1\dots,x_n,x'_n)$ with y_k, $k=1,\dots,n$ of degree 2 and x_k, x'_k, $k=1,\dots,n$ of degree 1 where as before we choose x_k and x'_k to be the basis which is dual to the basis of the k-th factor in

$$D_n(\mathbb{Z}[\frac{1}{2},i])/D_n(\mathbb{Z}[\frac{1}{2},i])^2 \cong \Big(\mathbb{Z}[\frac{1}{2},i]^\times/(\mathbb{Z}[\frac{1}{2},i]^\times)^2\Big)^n$$

given by the classes of i and $1+i$ then we get the following lemma which generalizes (3.10) and whose straighforward proof we leave to the reader.

Lemma 5.2. *The class e_{2k-1} restricts in the cohomology of the subgroup of diagonal matrices $H^*(D_n(\mathbb{Z}[\frac{1}{2},i];\mathbb{F}_2))$ to the symmetrization of $y_1\dots y_{k-1}x_k$ and the class e'_{2k-1} restricts to the symmetrization of $y_1\dots y_{k-1}(x_k+x'_k)$.* □

The following result determines the image of the restriction homomorphism and shows that (b) implies (c) in Theorem 5.1. It resembles results of Mitchell [M] for $GL_n(\mathbb{Z}[\frac{1}{2}])$ for $p=2$ and of Anton [An1] for $GL_n(\mathbb{Z}[\frac{1}{3},\zeta_3])$ for $p=3$.

Proposition 5.3. *Let $n \geq 1$ be an integer. The image of the restriction map*

$$i^*: H^*(GL_n(\mathbb{Z}[\frac{1}{2},i]);\mathbb{F}_2) \to H^*(D_n(\mathbb{Z}[\frac{1}{2},i]);\mathbb{F}_2) \cong \mathbb{F}_2[y_1,\dots y_n] \otimes E(x_1,x'_1\dots,x_n,x'_n)$$

is isomorphic to

$$\mathbb{F}_2[c_1,\dots c_n] \otimes E(e_1,e'_1,\dots,e_{2n-1},e'_{2n-1}) \ .$$

Here we have identified the Chern classes c_i and the classes e_{2i-1} and e'_{2i-1} with their image via i^*. The images of the elements c_i are, of course, the elementary symmetric polynomials in the y_i and the images of the classes e_{2i-1} and e'_{2i-1} have been determined in Lemma 5.2. We remark that even though i^* need not be injective, it is injective on the subalgebra of $H^*(GL_n(\mathbb{Z}[\frac{1}{2},i]);\mathbb{F}_2)$ generated by the classes c_i, e_{2i-1} and e'_{2i-1}, $1 \leq i \leq n$.

This proposition is an analogue of Proposition 3.6 of [An2]. Its proof uses crucially condition (5.3) below, which also plays a central role in [An2].

Proof. In this proof we denote the subalgebra

$$\mathbb{F}_2[c_1,\dots c_n] \otimes E(e_1,e'_1,\dots,e_{2n-1},e'_{2n-1}) \ .$$

of $H^*(D_n(\mathbb{Z}[\frac{1}{2},i]);\mathbb{F}_2)$ by C_n and the image of the restriction map by B_n. We need to show that $B_n = C_n$. This is trivial if $n=1$ and for $n=2$ this follows from Theorem 1 of [W] (cf. (3.2) and (3.8) and Lemma 5.2).

The classes $c_1,\dots,c_n$ are in B_n as images of the Chern classes with the same name and the classes $e_1,\dots e_{2n-1}$, $e'_1,\dots e'_{2n-1}$ are in B_n by Lemma 5.2. Therefore we have $C_n \subset B_n$. We will show $B_n \subset C_n$ for $n \geq 2$ by induction on n. This will be done in three steps.

1. From the inclusions

$$GL_{n-2}(\mathbb{Z}[\tfrac{1}{2},i]) \times GL_2(\mathbb{Z}[\tfrac{1}{2},i]) \quad \subset \quad GL_n(\mathbb{Z}[\tfrac{1}{2},i])$$

$$GL_{n-1}(\mathbb{Z}[\tfrac{1}{2},i]) \times GL_1(\mathbb{Z}[\tfrac{1}{2},i]) \quad \subset \quad GL_n(\mathbb{Z}[\tfrac{1}{2},i])$$

given by matrix block sum and the identifications of $D_{n-2}(\mathbb{Z}[\frac{1}{2},i])\times D_2(\mathbb{Z}[\frac{1}{2},i])$ with $D_n(\mathbb{Z}[\frac{1}{2},i])$ and of $D_{n-1}(\mathbb{Z}[\frac{1}{2},i]) \times D_1(\mathbb{Z}[\frac{1}{2},i])$ with $D_n(\mathbb{Z}[\frac{1}{2},i])$ we see that

$$B_n \subset B_{n-1} \otimes B_1 \cap B_{n-2} \otimes B_2$$

and by induction hypothesis the latter subalgebra is equal to

$$C_{n-1} \otimes C_1 \cap C_{n-2} \otimes C_2 \ ,$$

in particular we have

$$B_n \subset C_{n-1} \otimes C_1 \cap C_{n-2} \otimes C_2 \ . \tag{5.2}$$

2. The monomial basis in

$$H^*(D_n(\mathbb{Z}[\frac{1}{2},i]);\mathbb{F}_2) \cong \mathbb{F}_2[y_1,\ldots,y_n] \otimes E(x_1,\ldots,x_n,x'_1,\ldots,x'_n)$$

is in bijection with the set $S(n)$ of sequences

$$I = (a_1,\varepsilon_{1,1},\varepsilon_{2,1},\ldots,a_n,\varepsilon_{1,n},\varepsilon_{2,n})$$

where the a_i are integers ≥ 0 and $\varepsilon_{i,j} \in \{0,1\}$ for $i = 1,2$ and $1 \leq j \leq n$. More precisely to I we associate the monomial

$$y^I := y_1^{a_1}\ldots y_n^{a_n} x_1^{\varepsilon_{1,1}} \ldots x_n^{\varepsilon_{1,n}} {x'_1}^{\varepsilon_{2,1}} \ldots {x'_n}^{\varepsilon_{2,n}} \ .$$

We equip $S(n)$ with the lexicographical order and denote it by $<_n$. This order has the property that for each $1 \leq k < n$ it agrees with the lexicographical order on $S(k)\times S(n-k)$ if $S(k)$ and $S(n-k)$ are equipped with the orders $<_k$ and $<_{n-k}$ and $S(n)$ is identified with $S(k) \times S(n-k)$ via concetanation of sequences.

In the sequel we replace the symmetrizations of the elements $y_1 \ldots y_{i-1}(x_i + x'_i)$, $i = 1,\ldots,n$, by the symmetrization of $y_1 \ldots y_{i-1}x'_i$ and by abuse of notation we continue to denote them by e'_{2i-1}. This does not change the subalgebra C_n. This subalgebra

$$\mathbb{F}_2[c_1,\ldots c_n] \otimes E(e_1,e'_1,\ldots,e_{2n-1},e'_{2n-1}) \subset \mathbb{F}_2[y_1,\ldots,y_n] \otimes E(x_1,\ldots,x_n,x'_1,\ldots,x'_n)$$

has a monomial basis which is in bijection with the set $T(n)$ of sequences

$$K = (k_1,\ldots,k_n;\phi_{1,1},\ldots,\phi_{1,n};\phi_{2,1}\ldots,\phi_{2,n})$$

where the k_i are integers ≥ 0 and $\phi_{i,j} \in \{0,1\}$ for $i = 1,2$ and $1 \leq j \leq n$. More precisely to K we associate the monomial

$$c^K := c_1^{k_1}\ldots c_n^{k_n} e_1^{\phi_{1,1}} \ldots e_n^{\phi_{1,n}} e_1^{\phi_{2,1}} \ldots e_n^{\phi_{2,n}} \ .$$

We define a map

$$\alpha : T(n) \to S(n)$$

by associating to $K \in T(n)$ the largest monomial in $S(n)$ which occurs in the decomposition of c^K as linear combination of elements x^I with $I \in S(n)$. The proof of the following result is elementary and is left to the reader.

Lemma 5.4. *The map α is explicitly given by*

$$\alpha((k_1,\ldots,k_n;\phi_{1,1},\ldots,\phi_{1,n};\phi_{2,1}\ldots,\phi_{2,n})) = (a_1,\varepsilon_{1,1},\varepsilon_{2,1},\ldots,a_n,\varepsilon_{1,n},\varepsilon_{2,n})$$

with

$$\begin{aligned}
a_1 &= k_1 + \dots k_n + \sum_{i=1}^{2} (\phi_{i,2} + \dots \phi_{i,n}) \\
a_2 &= k_2 + \dots k_n + \sum_{i=1}^{2} (\phi_{i,3} + \dots \phi_{i,n}) \\
&\dots \quad \dots \\
a_j &= k_j + \dots k_n + \sum_{i=1}^{2} (\phi_{i,j+1} + \dots \phi_{i,n}) \\
&\dots \quad \dots \\
a_n &= k_n \\
\varepsilon_{i,j} &= \phi_{i,j}, \quad 1 \leq j \leq n, \ i = 1, 2 \ . \qquad \square
\end{aligned}$$

From this lemma it is obvious that α is injective and a sequence

$$I = (a_1, \varepsilon_{1,1}, \varepsilon_{2,1}, \dots, a_n, \varepsilon_{1,n}, \varepsilon_{2,n}) \in S(n)$$

is in the image of α if and only if we have

$$a_j - a_{j+1} \geq \varepsilon_{1,j+1} + \varepsilon_{2,j+1} \ \text{ for all } \ 1 \leq j < n \ . \tag{5.3}$$

In particular, if an element x is in C_n then the maximal sequence which appears in the decomposition of x as a linear combination of the monomials x^I with $I \in S(n)$ satisfies (5.3) for all $1 \leq j < n$. Likewise, if x is in $C_i \otimes C_{n-i}$ then this maximal sequence is equal to the maximal sequence which appears in the decomposition of x as a linear combination of the monomials x^I with $I \in S(k) \times S(n-k)$ and hence it satisfies (5.3) for all $1 \leq j < i$ and $i + 1 \leq j < n$.

3. Now let x be a homogeneous element of B_n and let I_0 be the maximal sequence in $S(n)$ appearing in the decomposition of x as a linear combination of the monomials x^I with $I \in S(n)$. By (5.2) we have $x \in C_{n-1} \otimes C_1$ and $x \in C_{n-2} \otimes C_2$, and I_0 remains the maximal sequence in $S(n-1) \times S(1)$ resp. $S(n-2) \times S(2)$ appearing in the decomposition of x as a linear combination of the monomials x^I with $I \in S(n-1) \times S(1)$ resp. $I \in S(n-2) \times S(2)$. Hence I_0 satisfies conditions (5.3) for $1 \leq j < n-1$ resp. $1 \leq j < n-2$ and $j = n-1$. In particular condition condition (5.3) holds for all $1 \leq j < n$ and therefore there exists $K_0 \in T(n)$ such that $\alpha(K_0) = I_0$. Then $x - c^{K_0}$ is still in B_n and the maximal sequence appearing in the decomposition of $x - c^{K_0}$ is smaller than that of x. By iterating this procedure we see that x belongs to C_n. $\square$

Finally we relate $C(3, \mathbb{Z}[\frac{1}{2}, i], 2)$ to the behaviour of the restriction homomorphism

$$H^*(\Gamma; \mathbb{F}_2) \to H^*(C_\Gamma(E_2); \mathbb{F}_2) \ .$$

For this we observe that the subgroups $\Gamma = SL_3(\mathbb{Z}[\frac{1}{2}, i])$ and the center $Z \cong \mathbb{Z}[\frac{1}{2}, i]^\times$ of $GL_3(\mathbb{Z}[\frac{1}{2}, i])$ have trivial intersection and their product is the kernel of the homomorphism

$$GL_3(\mathbb{Z}[\frac{1}{2}, i]) \to (\mathbb{Z}[\frac{1}{2}, i])^\times \to (\mathbb{Z}[\frac{1}{2}, i])^\times / ((\mathbb{Z}[\frac{1}{2}, i])^\times)^3 \cong \mathbb{Z}/3$$

given as the composition of the determinant with the natural quotient map. Therefore the spectral sequence of the extension

$$1 \to SL_3(\mathbb{Z}[\frac{1}{2}, i]) \times Z \to GL_3(\mathbb{Z}[\frac{1}{2}, i]) \to \mathbb{Z}/3 \to 1$$

gives an isomorphism

$$H^*(GL_3(\mathbb{Z}[\frac{1}{2},i]);\mathbb{F}_2) \cong \big(H^*(SL_3(\mathbb{Z}[\frac{1}{2},i]);\mathbb{F}_2)\otimes H^*(Z;\mathbb{F}_2)\big)^{\mathbb{Z}/3}\ . \tag{5.4}$$

Proposition 5.5. *The conjecture $C(3,\mathbb{Z}[\frac{1}{2},i],2)$ holds if and and only if either*

a) $H^*(SL_3(\mathbb{Z}[\frac{1}{2},i]);\mathbb{F}_2)\cong \mathbb{F}_2[b_2,b_3]\otimes E(d_3,d_3',d_5,d_5')$ or

b) the kernel of the map ψ of Theorem 1.2 is a finite dimensional vector space for which the action of $\mathbb{Z}/3\cong(\mathbb{Z}[\frac{1}{2},i])^\times/(\mathbb{Z}[\frac{1}{2},i])^\times)^3$ has trivial invariants.

Proof. The quotient $\mathbb{Z}/3\cong(\mathbb{Z}[\frac{1}{2},i])^\times/(\mathbb{Z}[\frac{1}{2},i])^\times)^3$ acts clearly trivially on $H^*(Z;\mathbb{F}_2)$ and on the image of the homomorphism φ of Theorem 1.2. Hence, the corollary follows immediately from (5.4) and Theorem 1.2. □

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