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A MINI-COURSE ON MORAVA STABILIZER GROUPS AND THEIR COHOMOLOGY

HANS-WERNER HENN

1. INTRODUCTION

The Morava stabilizer groups play a dominating role in chromatic stable homotopy theory. In fact, for suitable spectra X , for example all finite spectra, the chromatic homotopy type of X at chromatic level $n > 0$ and a given prime p is largely controlled by the continuous cohomology of a certain p -adic Lie group \mathbb{G}_n , in stable homotopy theory known under the name of Morava stabilizer group of level n at p , with coefficients in the corresponding Morava module $(E_n)_*X$.

These notes are slightly edited notes of a mini-course of 4 lectures delivered at the Vietnam Institute for Advanced Study in Mathematics in August 2013. The aim of the course was to introduce participants to joint work of the author with Goerss, Karamanov, Mahowald and Rezk which uses group cohomology in a crucial way to give a new approach to previous work by Miller, Ravenel, Wilson, and by Shimomura and his collaborators. This new approach has lead to a better understanding of old results as well as to substantial new results.

The notes are structured as follows. In section 2 and section 3 we give a short survey on certain aspects of chromatic stable homotopy theory. In section 2 we recall Bousfield localization and the chromatic set up. In section 3 we discuss the problem of finding finite resolutions of the trivial \mathbb{G}_n -module \mathbb{Z}_p and associated resolutions of the $K(n)$ -local sphere and we describe known resolutions. The form of these resolutions depend on cohomological properties of the groups \mathbb{G}_n and the remaining sections concentrate on those properties. Section 4 contains an essentially self contained discussion of some basic group theoretical properties of these groups. Section 5 discusses the (co)homology of these groups with trivial coefficients; this is self contained except for the discussion of Poincaré duality and the discussion of the case $n = 2$ and $p = 3$ which is only outlined. Section 6 concentrates mostly on the continuous cohomology $H^*(\mathbb{G}_1, (E_1)_*)$ and gives a fairly detailed account on how the short resolutions of the \mathbb{G}_1 -module \mathbb{Z}_p can be used to understand the homotopy of $L_{K(1)}S^0$. This homotopy is closely related to the image of the J-homomorphism studied in the 1960's by Adams, Mahowald, Quillen, Sullivan, Toda and others. Section 6 also contains some brief comments on how the algebraic resolutions surveyed in section 3 can be used to analyze $H^*(\mathbb{G}_2, (E_2)_*)$, at least for odd primes.

2. BOUSFIELD LOCALIZATION AND THE CHROMATIC SET UP

This section is a very brief introduction to the chromatic set up. More details with more references can be found in the introduction of [7].

2.1. Bousfield localization. Let E_* be a generalized homology theory. Bousfield localization with respect to E_* is a functor L_E from spectra to spectra together with a natural transformation $\lambda : X \rightarrow L_E X$ which is *terminal* among all E_* -equivalences. L_E exists for all homology theories E_* [2]. Bousfield-localization makes precise the idea to ignore spectra which are trivial to the eyes of E_* -homology.

Example Let MG be a Moore spectrum for an abelian group G . Then $L_{M\mathbb{Z}_{(p)}}$ resp. $L_{M\mathbb{Q}}$ are the homotopy theoretic versions of arithmetic localization with respect to $\mathbb{Z}_{(p)}$ resp. \mathbb{Q} (e.g. homology groups and homotopy groups of a spectrum get localized by these functors).

2.2. Morava K -theories. Fix a prime p . We are interested in the localization functors $L_{K(n)}$ with respect to Morava K -theory $K(n)$. We recall that $K(n)$ is a multiplicative periodic cohomology theory with coefficient ring $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$, where v_n is of degree $2(p^n - 1)$ if $n > 0$. In case $n = 0$ the convention is that $K(0) = M\mathbb{Q}$, independant of p . Furthermore $K(n)$ admits a theory of characteristic classes and the associated formal group law Γ_n is the Honda formal group law of height n .

The functors $L_{K(n)}$ are elementary “building blocks” of the stable homotopy category of finite p -local complexes in the following sense.

a) The localization functor $L_{K(n)}$ is “simple” in the sense that the category of $K(n)$ -local spectra contains no nontrivial localizing subcategory, i.e. no non-trivial thick subcategory which is closed under arbitrary coproducts [14]

b) There is a tower of localization functors

$$\dots \rightarrow L_n \rightarrow L_{n-1} \rightarrow \dots$$

(with $L_n = L_{K(0) \vee \dots \vee K(n)}$) together with natural transformations $id \rightarrow L_n$ such that

$$X \simeq \text{holim}_n L_n X$$

for every finite p -local spectrum X . Furthermore, for each n and p there is a homotopy pullback diagram (a “chromatic square”)

$$\begin{array}{ccc} L_n X & \longrightarrow & L_{K(n)} X \\ \downarrow & & \downarrow \\ L_{n-1} X & \longrightarrow & L_{n-1} L_{K(n)} X \end{array}$$

i.e. L_n is determined by $L_{K(n)}$ and L_{n-1} .

The functors $L_{K(n)}$ do not commute with smash products. Therefore the appropriate smash product of $K(n)$ -local spectra X and Y is given by $X \wedge_{K(n)} Y := L_{K(n)}(X \wedge Y)$.

2.3. $L_{K(n)}S^0$ as homotopy fixed point spectrum. The functors $L_{K(n)}$ are controlled by cohomological properties of the Morava stabilizer group \mathbb{S}_n resp. \mathbb{G}_n where \mathbb{S}_n is the group of automorphisms of the formal group law Γ_n (extended to the finite field \mathbb{F}_q with $q = p^n$). The Galois group $\text{Gal}(\mathbb{F}_q : \mathbb{F}_p)$ acts on \mathbb{S}_n and \mathbb{G}_n is defined as semidirect product $\mathbb{G}_n = \mathbb{S}_n \rtimes \text{Gal}(\mathbb{F}_q : \mathbb{F}_p)$. This group acts on the Lubin-Tate ring which classifies deformations of Γ_n (in the sense of Lubin-Tate). The Lubin-Tate spectrum E_n is a complex oriented 2-periodic cohomology theory whose associated formal group law is a universal deformation of Γ_n ; its homotopy groups are given as $(E_n)_* = \pi_*(E_n) = \pi_0(E_n)[u^{\pm 1}]$ with $u \in \pi_{-2}(E)$ and $\pi_0(E_n) \cong \mathbb{W}_{\mathbb{F}_q}[[u_1, \dots, u_{n-1}]]$, the ring of power series on $n-1$ generators over the ring of Witt vectors of \mathbb{F}_q . The group \mathbb{G}_n acts on deformations and hence on $(E_n)_*$, and by the Hopkins-Miller-Goerss theorem [10] this action can be lifted to E_∞ -ring spectra, i.e. \mathbb{G}_n acts on E_n through E_∞ -maps.

By Devinatz-Hopkins [4] the “homotopy fixed point spectrum” $E_n^{h\mathbb{G}_n}$ can be identified with $L_{K(n)}S^0$ and its Adams-Novikov spectral sequence can be identified with the associated homotopy fixed point spectral sequence

$$(2.1) \quad E_2^{s,t} \cong H_{cts}^s(\mathbb{G}_n, (E_n)_t) \implies \pi_{t-s}L_{K(n)}S^0.$$

Therefore methods of group theory and group cohomology can be used to study the $K(n)$ -local sphere and more generally the $K(n)$ -local category.

Warning: The “homotopy fixed point spectrum” is taken with respect to the action of a profinite group. We will not try to explain how this is done in detail but we insist that in [4] there is a construction such that there is an associated homotopy fixed point spectral sequence with an E_2 -term which is given in terms of continuous group cohomology as in (2.1).

3. RESOLUTIONS OF $K(n)$ -LOCAL SPHERES

The case $n = 0$ is both exceptional and trivial: $K(0) = M\mathbb{Q} = H\mathbb{Q}$ (with $H\mathbb{Q}$ the Eilenberg-MacLane spectrum for the rationals) and $L_{K(0)}$ is rationalization. From now on we will assume $n > 1$.

3.1. The example $n = 1$ and $p > 2$. The case $n = 1$ is well understood. In this case we have $E_1 = K\mathbb{Z}_p$ (p -adic complex K -theory). The formal group law Γ is the multiplicative group law given by $1 + (x +_\Gamma y) = (1 + x)(1 + y)$. The endomorphism ring of Γ over \mathbb{F}_p is isomorphic to \mathbb{Z}_p : in fact, the element $p \in \mathbb{Z}_p$ corresponds to the endomorphism $[p]_\Gamma(x) = (1 + x)^p - 1 \equiv x^p \pmod{p}$ and the canonical homomorphism $\mathbb{Z} \rightarrow \text{End}(\Gamma), n \mapsto [n](x)$ extends to an continuous isomorphism $\mathbb{Z}_p \rightarrow \text{End}(\Gamma)$. Therefore the group $\mathbb{G}_1 = \mathbb{S}_1$ can be identified with \mathbb{Z}_p^\times , the units in the p -adic integers. The group acts on $K\mathbb{Z}_p$ by Adams operations, and the action on its homotopy $\pi_*(K) = \mathbb{Z}_p[u^{\pm 1}]$ is via graded ring automorphisms determined by $(\lambda, u) \mapsto \lambda u$. If p is odd then $\mathbb{Z}_p^\times \cong C_{p-1} \times \mathbb{Z}_p$, and the homotopy fixed points with respect to \mathbb{Z}_p^\times can be formed in two steps, first with respect to the cyclic group C_{p-1} and then with respect to \mathbb{Z}_p . Taking homotopy fixed points with respect to C_{p-1} is quite simple; on homotopy groups it amounts to taking invariants with respect

to the action of C_{p-1} . Hence we get

$$\pi_*(K\mathbb{Z}_p^{hC_{p-1}}) \cong \mathbb{Z}_p[u^{\pm(p-1)}] .$$

In fact, $K\mathbb{Z}_p^{hC_{p-1}}$ is the Adams summand of $K\mathbb{Z}_p$. The Adams operation ψ^{p+1} still acts on $K\mathbb{Z}_p^{hC_{p-1}}$, taking homotopy fixed points with respect to \mathbb{Z}_p amounts to taking the fibre of $\psi^{p+1} - id$ and we get a fibration

$$(3.1) \quad L_{K(1)}S^0 \rightarrow K\mathbb{Z}_p^{hC_{p-1}} \xrightarrow{\psi^{p+1} - id} K\mathbb{Z}_p^{hC_{p-1}} .$$

We will get back to this in section 6.1.1.

3.2. The case that $p-1$ does not divide n . The fibration (3.1) can be considered as an example of a $K\mathbb{Z}_p$ -resolution in the sense of Miller [19].

Following Miller we say that a $K(n)$ -local spectrum I is E_n -*injective* if the canonical map $I \rightarrow L_{K(n)}(E_n \wedge I)$ splits, i.e. it has a left inverse in the homotopy category. A sequence of maps $X_1 \rightarrow X_2 \rightarrow X_3$ is said to be E_n -*exact* if the composition of the two maps is nullhomotopic and if $[-, I]_*$ transforms $X_1 \rightarrow X_2 \rightarrow X_3$ into an exact sequence of abelian groups for each E_n -injective spectrum I . An E_n -*resolution* of a spectrum X is a sequence

$$I_\bullet : * \rightarrow X \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$$

such that the sequence is E_n -exact and each I^s is E_n -injective. If there exists an integer $k \geq 0$ such that k is minimal with the property that I^s is contractible for all $s > k$ then we say that the E_n -resolution is of length k .

The spectrum E_n is E_n -injective because E_n is $K(n)$ -local and a ring spectrum.

The following result is in essence due to Morava.

Theorem 3.1. [12] *If n is neither divisible by $p-1$ nor by p then $L_{K(n)}S^0$ admits an E_n -resolution of length n^2 in which each I^s is a summand in a finite wedge of E_n 's.*

Remarks a) Suppose $G = \lim_\alpha G_\alpha$ is a profinite group and suppose p is a prime. We write $\mathbb{Z}_p[[G]] = \lim_\alpha \mathbb{Z}_p[G_\alpha]$ for the profinitely completed group algebra over \mathbb{Z}_p . Likewise, for a profinite set $S = \lim_\alpha S_\alpha$ we write $\mathbb{Z}_p[[S]]$ for $\lim_\alpha \mathbb{Z}_p[S_\alpha]$. The theorem is derived from the existence of a finite projective resolution of length n^2

$$P_\bullet : 0 \rightarrow P_{n^2} \rightarrow \dots \rightarrow P_0 \rightarrow \mathbb{Z}_p \rightarrow 0$$

of the trivial \mathbb{G}_n -module \mathbb{Z}_p in the category of profinite $\mathbb{Z}_p[[\mathbb{G}_n]]$ -modules. A more precise form of the theorem is that the E_n -resolution “realizes the projective resolution” in the sense that there is an isomorphism of chain complexes

$$(3.2) \quad \mathrm{Hom}_{cts}(P_\bullet, (E_n)_*) \cong E_*(I_\bullet)$$

where here and elsewhere in these notes we adopt the convention that $(E_n)_*X$ for $K(n)$ -local X means $\pi_*(E_n \wedge_{K(n)} X)$.

b) The assumption that n is divisible by p (but not by $p-1$) is not a very serious restriction. There is still a useful variation of this theorem which holds. However, the assumption that n is not divisible by $p-1$ is quite crucial.

3.3. The example $n = 2$ and $p > 3$. In the case $n = 2$ and $p > 3$ (even if $p = 3$) the group \mathbb{G}_2 can be decomposed as a product $\mathbb{G}_2 \cong \mathbb{G}_2^1 \times \mathbb{Z}_p$ (cf. section 5.4). The following two results are analogues of results of Ravenel (cf. chapter 6 of [22]).

Theorem 3.2. [12] *There is an exact complex of projective $\mathbb{Z}_p[[\mathbb{G}_2^1]]$ -modules*

$$0 \rightarrow C_3 \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow \mathbb{Z}_p \rightarrow 0$$

with $C_0 = C_3 = \mathbb{Z}_3[[\mathbb{G}_2/F_{2(p^2-1)}]]$ and $C_1 = C_2 = \mathbb{Z}_3[[\mathbb{G}_2^1]] \otimes_{\mathbb{Z}_p[F_{2(p^2-1)}]} \lambda_{1-p}$ where λ_{1-p} is a certain projective $\mathbb{Z}_p[F_{2(p^2-1)}]$ -module of \mathbb{Z}_p -rank 2 and $F_{2(p^2-1)}$ is a maximal finite subgroup of order $2(p^2-1)$ of \mathbb{G}_n^1 .

Theorem 3.3. [12] *There exists a fibration*

$$L_{K(2)}S^0 \rightarrow E_2^{h\mathbb{G}_2^1} \rightarrow E_2^{h\mathbb{G}_2^1}$$

and an E_2 -resolution

$$* \rightarrow E_2^{h\mathbb{G}_2^1} \rightarrow X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow *$$

with $X_0 = X_3 = E_2^{hF_{2(p^2-1)}}$ and $X_1 = X_2 = \Sigma^{2(p-1)}E_2^{hF_{2(p^2-1)}} \vee \Sigma^{2(1-p)}E_2^{hF_{2(p^2-1)}}$.

3.4. The example $n = 1$ and $p = 2$. This case is again well understood. The isomorphism $\mathbb{G}_1 = \mathbb{Z}_2^\times \cong C_2 \times \mathbb{Z}_2$ allows, as before, to form the homotopy fixed points in two stages and we obtain the following fibration

$$(3.3) \quad I_\bullet : L_{K(1)}S^0 \rightarrow K\mathbb{Z}_2^{hC_2} \xrightarrow{\psi^3 - id} K\mathbb{Z}_2^{hC_2}.$$

The homotopy fixed points $K\mathbb{Z}_2^{hC_2}$ can be identified with 2-adic real K -theory $KO\mathbb{Z}_2$. Note that this is not an example of Theorem 3.1, in fact a finite length E_n -resolution as in Theorem 3.1 cannot exist in this case because \mathbb{G}_1 contains an element of order 2, and hence $H_{cts}^*(\mathbb{G}_1, \mathbb{F}_2)$ is nontrivial in arbitrarily high cohomological degrees. Nevertheless this is a very useful substitute. We will get back to this in section 6.1.2.

3.5. The general case $p-1$ divides n . The natural question arises whether there are generalizations of the fibre sequence (3.3) for higher n and p such that $p-1$ divides n . What could they look like? In other words, can we explain the appearance of $K\mathbb{Z}_2^{hC_2}$ in (3.3) so that it fits into a more general framework?

A good point of view is provided by *group cohomology* as follows:

Applying the functor $K\mathbb{Z}_{2*}$ to (3.3) gives a short exact sequence

$$0 \rightarrow K\mathbb{Z}_{2*} \rightarrow K\mathbb{Z}_{2*}(K\mathbb{Z}_2^{hC_2}) \rightarrow K\mathbb{Z}_{2*}(K\mathbb{Z}_2^{hC_2}) \rightarrow 0$$

in which $K\mathbb{Z}_{2*}(K\mathbb{Z}_2^{hC_2})$ can be identified with the group of continuous homomorphisms from the permutation module $\mathbb{Z}_2[[\mathbb{Z}_2^\times/C_2]]$ to $(K\mathbb{Z}_2)_*$. The fibre sequence

(3.3) can therefore be considered as a homotopy theoretic realization of the exact sequence of profinite $\mathbb{Z}_2[[\mathbb{Z}_2^\times]]$ -modules (cf. section 6.1)

$$(3.4) \quad P_\bullet : 0 \rightarrow \mathbb{Z}_2[[\mathbb{Z}_2^\times/C_2]] \rightarrow \mathbb{Z}_2[[\mathbb{Z}_2^\times/C_2]] \rightarrow \mathbb{Z}_2 \rightarrow 0 .$$

in the sense that $K\mathbb{Z}_{2*}(I_\bullet) \cong \text{Hom}_{cts}(P_\bullet, K\mathbb{Z}_{2*})$ where I_\bullet is the fibration of (3.3) and P_\bullet the exact sequence of (3.4). However, in this case I_\bullet is not a $K\mathbb{Z}_2$ -resolution in the sense of section 3.2 and P_\bullet is not a free (neither a projective) resolution but rather a resolution by permutation modules.

This suggests that we should look for a resolution of the trivial \mathbb{G}_n -module \mathbb{Z}_p in terms of permutation modules $\mathbb{Z}_p[[\mathbb{G}_n/F]]$ with F running through finite subgroups (or summands thereof) and try to realize those in the sense of (3.2). In fact, if F is any finite subgroup of \mathbb{G}_n there is a canonical isomorphism

$$(3.5) \quad (E_n)_* E_n^{hF} \cong \text{Hom}_{cts}(\mathbb{Z}_p[[\mathbb{G}_n/F]], E_{n*}) .$$

This leads to the following questions?

Questions: 1) Are there resolutions of finite length and finite type of the trivial $\mathbb{Z}_p[[\mathbb{G}_n]]$ -module \mathbb{Z}_p by (direct summands of) permutation modules of the form $\mathbb{Z}_p[[\mathbb{G}_n/F]]$ for finite subgroups $F \subset \mathbb{G}_n$?

2) Can these resolutions be realized by resolutions of spectra where the resolving spectra are the corresponding homotopy fixed point spectra with respect to these finite subgroups?

3) If the answers to (a) and (b) are yes, how unique are these resolutions?

Here we call a sequence of spectra

$$* \rightarrow X = X_{-1} \rightarrow X_0 \rightarrow X_1 \rightarrow \dots$$

a *resolution* of X if the composite of any two consecutive maps is nullhomotopic and if each of the maps $X_i \rightarrow X_{i+1}$, $i \geq 0$, can be factored as $X_i \rightarrow C_i \rightarrow X_{i+1}$ such that $C_{i-1} \rightarrow X_i \rightarrow C_i$ is a cofibration for every $i \geq 0$ (with $C_{-1} := X_{-1}$). We say that the resolution is of length n if $C_n \simeq X_n$ and $X_i \simeq *$ if $i > n$.

Remark The group \mathbb{S}_n is of finite virtual mod- p cohomological dimension (vcd_p) equal to n^2 , i.e. there is a finite index subgroup whose continuous mod- p cohomology vanishes in degrees $> n^2$. In the case of a discrete group G of finite vcd_p there is a geometric source for resolutions of the trivial module \mathbb{Z}_p by permutation modules of the form $\mathbb{Z}_p[G/F]$ with F running through finite subgroups. In fact, they can be obtained as the cellular chains of a contractible finite dimensional G -CW-complex on which G acts with finite stabilizers. Such spaces always exist (if $vcd_p(G) < \infty$) and hence such resolutions always exist. In our case such spaces are not known to exist and we have to manufacture our resolutions by hand.

3.6. The example $n = 2$ and $p = 3$. This is the first new case.

Theorem 3.4. [7] *There is an exact complex of $\mathbb{Z}_3[[\mathbb{G}_2^1]]$ -modules*

$$0 \rightarrow C_3 \rightarrow C_2 \rightarrow C_1 \rightarrow C_1 \rightarrow \mathbb{Z}_3 \rightarrow 0$$

with $C_0 = C_3 = \mathbb{Z}_3[[\mathbb{G}_2/G_{24}]]$ and $C_1 = C_2 = \mathbb{Z}_3[[\mathbb{G}_2^1]] \otimes_{\mathbb{Z}_3[SD_{16}]} \chi$ where SD_{16} is a maximal finite subgroup of \mathbb{G}_2 which is isomorphic to the semidihedral group of order 16, χ is a suitable character χ of SD_{16} defined over \mathbb{Z}_3 , and G_{24} is another maximal finite subgroup of order 24 of \mathbb{G}_2 .

Theorem 3.5. [7] *There exists a fibration*

$$L_{K(2)}S^0 \rightarrow E_2^{h\mathbb{G}_2^1} \longrightarrow E_2^{h\mathbb{G}_2^1}$$

and a resolution of $E_2^{h\mathbb{G}_2^1}$ of length 3

$$* \rightarrow E_2^{h\mathbb{G}_2^1} \rightarrow X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow *$$

with $X_0 = E_2^{hG_{24}}$, $X_1 = \Sigma^8 E_2^{hSD_{16}} \simeq X_2 = \Sigma^{40} E_2^{hSD_{16}}$ and $X_3 = \Sigma^{48} E_2^{hG_{24}}$.

Remarks a) The homotopy fixed point spectrum $E_2^{hSD_{16}}$ is 16-periodic and the suspensions $\Sigma^8 E_2^{hSD_{16}}$ and $\Sigma^{40} E_2^{hSD_{16}}$ are due to the presence of the character χ in the previous theorem. The $(E_2)_*$ -homology of $E_2^{hG_{24}}$ is 24-periodic and this resolution realizes the one of the previous theorem in the same sense as before, i.e. there is an isomorphism of complexes $(E_2)_*(X_\bullet) \cong \text{Hom}_{cts}(C_\bullet, (E_2)_*)$. However, the spectrum $E_2^{hG_{24}}$ itself is only 72-periodic and the 48-fold suspension appearing with X_3 is a homotopy theoretic subtlety which is not explained by the algebra.

b) The spectrum $E_2^{hG_{24}}$ is a version of the Hopkins-Miller higher real K -theory spectrum EO_2 . It is equivalent to $L_{K(2)}tmf$, the $K(2)$ -localization of the spectrum tmf of topological modular forms at $p = 3$.

There is a second resolution which can be described as follows: we choose an 8-th primitive root of unity in $\mathbb{W}_{\mathbb{F}_9}$. This defines a one-dimensional faithful representation of C_8 over $\mathbb{W}_{\mathbb{F}_9}$ which we denote it by λ_1 , and its k -th tensor power by λ_k . Then the λ_k are naturally $\mathbb{Z}_3[SD_{16}]$ -modules and λ_4 splits as $\lambda_{4,+} \oplus \lambda_{4,-}$. Furthermore $\lambda_{4,-}$ is the representation χ of 3.4.

The following results are implicit in [6].

Theorem 3.6. [12] *There is an exact complex of $\mathbb{Z}_3[[\mathbb{G}_2^1]]$ -modules*

$$0 \rightarrow \mathbb{Z}_3[[\mathbb{G}_2^1/SD_{16}]] \rightarrow \mathbb{Z}_3[[\mathbb{G}_2^1]] \otimes_{\mathbb{Z}_3[SD_{16}]} \lambda_2 \rightarrow (\mathbb{Z}_3[[\mathbb{G}_2^1]] \otimes_{\mathbb{Z}_3[G_{24}]} \tilde{\chi}) \oplus (\mathbb{Z}_3[[\mathbb{G}_2^1]] \otimes_{\mathbb{Z}_3[SD_{16}]} \lambda_{4,-}) \rightarrow \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]] \rightarrow \mathbb{Z}_3 \rightarrow 0$$

where $\tilde{\chi}$ is a suitable nontrivial one-dimensional character of G_{24} defined over \mathbb{Z}_3 .

Theorem 3.7. [12] *There exists a resolution of $E_2^{h\mathbb{G}_2^1}$ of length 3*

$$* \rightarrow E_2^{h\mathbb{G}_2^1} \rightarrow E_2^{hG_{24}} \rightarrow \Sigma^{36} E_2^{hG_{24}} \vee \Sigma^8 E_2^{hSD_{16}} \rightarrow \Sigma^{48} E_2^{hSD_{16}} \vee \Sigma^{12} E_2^{hSD_{16}} \rightarrow E_2^{hSD_{16}} \rightarrow *.$$

Remark As in Theorem 3.5 the suspensions are due to the presence of the characters in the previous theorem.

3.7. Permutation resolutions and realizations.

Proposition 3.8. [12] *Let p be an odd prime and $n = k(p - 1)$ with $k \not\equiv 0 \pmod{p}$. Then the trivial $\mathbb{Z}_p[[\mathbb{G}_n]]$ -module \mathbb{Z}_p admits a resolution of finite length in which all modules are finite direct sums of modules which are of the form $\mathbb{Z}_p[[\mathbb{G}_n/F]]$ with F a finite subgroup of \mathbb{G}_n .*

In the case of general profinite groups G work of Symonds [26] suggests that such resolutions exist under suitable finiteness assumptions on G . In the case of the stabilizer group [12] provides a more direct approach to their construction.

Theorem 3.9. [12] *For p odd and $n = p - 1$ there is a resolution of $L_{K(n)}S^0$ of finite length in which all spectra are summands in finite wedges of spectra of the form E_n^{hF} and F is a finite subgroup of \mathbb{G}_n .*

3.8. Applications and work in progress. The pioneering work of Shimomura and collaborators on calculating the homotopy groups $\pi_*(L_{K(2)}X)$ for $X = S^0$ [24] resp. the Moore spectrum $V(0)$ [23] at the prime 3 and of $\pi_*(L_{K(2)}S^0)$ for primes $p > 3$ [25] have been poorly understood by the community. Therefore an alternative approach (using group cohomology in a systematic way) is useful. Accomplished respectively ongoing projects include the following:

3.8.1. The exact complex of Theorem 3.4 has been made into an efficient calculational tool in the thesis of Nasko Karamanov [15]. This has lead to calculations at $p = 3$ of $\pi_*(L_{K(2)}X)$ for $X = V(1)$, the cofibre of the Adams self map of $V(0)$ [6], as well as for $V(0)$ [13]. The results in [13] refine Shimomura's results of [23] and correct some errors. The case of S^0 is a joint project with Goerss, Karamanov and Mahowald. Details should appear in the near future.

The main result of [7] together with partial information from [13] have lead to major structural results on the homotopy category of $K(2)$ -local spectra at the prime 3: the rational homotopy of $L_{K(2)}S^0$ has been calculated and the chromatic splitting conjecture for $n = 2$ and $p = 3$ has been established in [8], the Picard group of smash-invertible $K(2)$ -local spectra has been calculated in [16] and [9] and the Brown-Comenetz dual of the sphere has been determined in [5].

3.8.2. The exact complex of Theorem 3.2 has been turned into an efficient calculational tool in the thesis of O. Lader [17]. Among other things he has recovered Shimomura's calculation of $\pi_*L_{K(2)}V(0)$ and Hopkins unpublished calculation of the Picard group Pic_2 , both for primes $p > 3$.

3.8.3. Resolutions for $n = p = 2$ which resemble those of section 3.5 were announced in [12] although the precise form of X_3 in the analogue of Theorem 3.5 remained unclear at the time. These resolutions have since been constructed in the recent Northwestern theses of Agnès Beaudry and Irina Bobkova. Beaudry has used this to disprove the chromatic splitting conjecture at $n = p = 2$ [1]. The resolutions can be expected to lead to further progress in $K(2)$ -local homotopy at the prime 2 similar to the case of the prime 3 mentioned in subsection 3.8.1 above.

In the remaining sections 4-6 of these notes we will explain some of the algebraic aspects of this story in more detail, in particular group theoretical and cohomological properties of \mathbb{G}_n . The homotopy theoretic aspects will mostly remain in the background.

4. THE MORAVA STABILIZER GROUPS. FIRST PROPERTIES

There are different ways to discuss these groups. They arise in stable homotopy theory as automorphism groups of certain p -typical formal group laws Γ_n defined over \mathbb{F}_p . For our purposes it seems best to introduce them as follows.

Definition 4.1. Let p be a prime and let \mathcal{O}_n be the non-commutative algebra over $\mathbb{W}(\mathbb{F}_{p^n})$, the ring of Witt vectors for the field \mathbb{F}_{p^n} , generated by an element S subject to the relations $S^n = p$ and $Sw = w^\sigma S$ for each $w \in \mathbb{W}(\mathbb{F}_{p^n})$ where w^σ is the result of applying the lift of Frobenius on w . In other words

$$(4.1) \quad \mathcal{O}_n = \mathbb{W}(\mathbb{F}_{p^n})\langle S \rangle / (S^n = p, Sw = w^\sigma S) .$$

Remarks (on Witt vectors) a) The ring of Witt vectors $\mathbb{W}(\mathbb{F}_{p^n})$ is a \mathbb{Z}_p -algebra which is a complete local ring with maximal ideal (p) . It is an integral domain which is free of rank n as \mathbb{Z}_p -module. As the notation suggests \mathbb{W} is a functor, say from the category of finite field extensions of \mathbb{F}_p to the category of integral domains which are unramified \mathbb{Z}_p -algebras.

b) Because of functoriality the Frobenius automorphism of \mathbb{F}_{p^n} lifts to a \mathbb{Z}_p -algebra automorphism.

c) By Hensel's lemma each root of unity in $\mathbb{F}_{p^n}^\times$ lifts uniquely to a root of unity in $\mathbb{W}(\mathbb{F}_{p^n})$.

d) Each element of $w \in \mathbb{W}(\mathbb{F}_{p^n})$ can be uniquely written as $\sum_{i \geq 0} w_i p^i$ where all $w_i \in \mathbb{W}(\mathbb{F}_{p^n})$ satisfy $w_i^{p^n} = w_i$. (Already for $n = 1$ this is a non-trivial statement).

e) A concrete construction (which, however, does not immediately reveal the functoriality of the construction) can be given as follows. Over $\mathbb{F}_p[X]$ the polynomial $X^{p^n} - X$ can be factored as product of irreducible polynomials whose degrees divide n . For each divisor d of n there is at least one factor p_d of degree d . Then $\mathbb{F}_{p^n} \cong \mathbb{F}_p[X]/(p_n)$ and $\mathbb{W}(\mathbb{F}_{p^n}) \cong \mathbb{Z}_p[X]/(\tilde{p}_n)$ where \tilde{p}_n is any lift of p_n to a polynomial $\tilde{p}_n \in \mathbb{Z}_p[X]$.

Remarks (on \mathcal{O}_n) a) The left $\mathbb{W}(\mathbb{F}_{p^n})$ -submodule of \mathcal{O}_n generated by S is a two sided ideal with quotient $\mathcal{O}_n/(S) \cong \mathbb{F}_{p^n}$ and \mathcal{O}_n is complete with respect to the filtration given by the powers of the ideal (S) . In fact, \mathcal{O}_n is a non-commutative complete discrete valuation ring. The valuation v is normalized such that $v(p) = 1$, i.e. $v(S) = \frac{1}{n}$.

b) \mathcal{O}_n is a free $\mathbb{W}(\mathbb{F}_{p^n})$ -module of rank n . A basis is given by the elements $1, S, \dots, S^{n-1}$ and every element in $x \in \mathcal{O}_n$ can be uniquely written as

$$x = \sum_{i=0}^{n-1} a_i S^i$$

with $a_i \in \mathbb{W}(\mathbb{F}_{p^n})$, and thus as

$$x = \sum_{j=0}^{\infty} x_j S^j$$

with all $x_j \in \mathbb{W}(\mathbb{F}_{p^n})$ satisfying $x_j^{p^n} = x_j$. In fact, if $a_i = \sum_{j=0}^{\infty} a_{i,j} p^j$ then $x_{i+jn} = a_{i,j}$.

c) Inverting p makes \mathcal{O}_n into a division algebra \mathbb{D}_n which is central over \mathbb{Q}_p and free of rank n^2 as a vector space over \mathbb{Q}_p . In fact, \mathcal{O}_n is a domain and if $x = \sum_{j \geq k} x_j S^j$ with $x_k \neq 0$, then $x = S^k x'$ and x' is invertible in \mathcal{O}_n . Inverting p also inverts S and thus every nontrivial element admits an inverse.

d) The Galois group $\text{Gal}(\mathbb{F}_p^n : \mathbb{F}_p)$ of the extension $\mathbb{F}_p \subset \mathbb{F}_{p^n}$ acts on \mathcal{O}_n by algebra automorphisms via $(\sigma, \sum_{i=0}^{n-1} x_i S^i) \mapsto \sum_{i=0}^{n-1} x_j^\sigma S^j$ where as before x_j^σ is the result of applying the lift of Frobenius to x_j . We note that by the relation in (4.1) this action of Frobenius can be realized by conjugation by S inside \mathbb{D}_n^\times .

Definition 4.2. The n -th Morava stabilizer group at p is defined as the group of units in \mathcal{O}_n . It is denoted \mathbb{S}_n , i.e. $\mathbb{S}_n = \mathcal{O}_n^\times$. The extended n -th Morava stabilizer group at p is the semidirect product $\mathbb{G}_n := \mathbb{S}_n \rtimes \text{Gal}(\mathbb{F}_p^n : \mathbb{F}_p)$.

Remarks a) Because \mathcal{O}_n is a complete (non-commutative) discrete valuation ring, an element $x \in \mathcal{O}_n$ is invertible in \mathcal{O}_n if and only if $v(x) = 0$.

b) It can be shown that \mathbb{S}_n is the group of automorphisms of a suitable formal group law Γ_n (associated to the complex oriented cohomology theory given by Morava K -theory $K(n)$). The group law Γ_n is already defined over \mathbb{F}_p but \mathbb{S}_n is its automorphism group considered as a formal group law over the field \mathbb{F}_{p^n} .

4.1. The Morava stabilizer group as a profinite group. The filtration of \mathcal{O}_n by powers of (S) leads to a very useful filtration of \mathbb{S}_n . For $i = \frac{k}{n}$ with $k \in \mathbb{N}$ we let

$$F_i := F_i \mathbb{S}_n := \{x \in \mathbb{S}_n \mid x \equiv 1 \pmod{(S^{in})}\}$$

Then we get a decreasing filtration

$$(4.2) \quad \mathbb{S}_n = F_0 \supset F_{\frac{1}{n}} \supset F_{\frac{2}{n}} \supset$$

by normal subgroups and \mathbb{S}_n is complete and separated with respect to this filtration, i.e. the canonical map $\mathbb{S}_n \rightarrow \varprojlim_i \mathbb{S}_n / F_i \mathbb{S}_n$ is an isomorphism. In particular \mathbb{S}_n is a *profinite group*. Furthermore $F_{\frac{1}{n}} \mathbb{S}_n$ is the kernel of the reduction homomorphism

$$\mathbb{S}_n = \mathcal{O}_n^\times \rightarrow \mathbb{F}_{p^n}^\times.$$

This group is also denoted by S_n and is often called the *strict Morava stabilizer group*. Furthermore for each $i = \frac{k}{n} > 0$ there are canonical isomorphism

$$(4.3) \quad F_{\frac{k}{n}}/F_{\frac{k+1}{n}} \rightarrow \mathbb{F}_{p^n}, \quad x = 1 + aS^k \mapsto \bar{a}$$

if $a \in \mathcal{O}_n$ and if \bar{a} denotes the residue class of a in $\mathcal{O}_n/(S) \cong \mathbb{F}_{p^n}$. In particular S_n/F_i is a finite p -group for each $i > 0$ and S_n is a *profinite p -group*. As S_n is also normal in \mathbb{S}_n , S_n is the p -Sylow subgroup of the profinite group \mathbb{S}_n . Furthermore the exact sequence

$$1 \rightarrow S_n \rightarrow \mathbb{S}_n \rightarrow \mathbb{F}_{p^n}^\times \rightarrow 1$$

splits, i.e. $\mathbb{S}_n \cong S_n \rtimes \mathbb{F}_{p^n}^\times$ is a semidirect product. In fact, the splitting is given by Remark c on Witt vectors above.

4.2. The associated mixed Lie algebra of S_n . The associated graded object $gr S_n$ with respect to the above filtration with

$$gr_i S_n := F_{\frac{k}{n}} S_n / F_{\frac{k+1}{n}} S_n$$

for $i = \frac{k}{n}$ becomes a graded Lie algebra with Lie bracket $[\bar{a}, \bar{b}]$ induced by the commutator $[x, y] := xyx^{-1}y^{-1}$ in S_n . Furthermore, if we define a function φ from $\{\frac{k}{n} | k = 1, 2, \dots\}$ to itself by $\varphi(i) := \min\{i + 1, pi\}$ then the p -th power map on S_n induces maps

$$P : gr_i S_n \longrightarrow gr_{\varphi(i)} S_n$$

which define on $gr S_n$ the structure of a mixed Lie algebra in the sense of Lazard [18]. If we identify the filtration quotients with \mathbb{F}_{p^n} as above then the Lie bracket and the map P are explicitly given as follows.

Proposition 4.3. [11] *Let $\bar{a} \in gr_i S_n$, $\bar{b} \in gr_j S_n$. With respect to the isomorphism (4.3) the mixed Lie algebra structure maps are given by*

a)

$$[\bar{a}, \bar{b}] = \bar{a}\bar{b}^{p^{ni}} - \bar{b}\bar{a}^{p^{nj}} \in gr_{i+j} S_n$$

b)

$$P\bar{a} = \begin{cases} \bar{a}^{\frac{p^{pni}-1}{p^{ni}-1}} & i < (p-1)^{-1} \\ \bar{a} + \bar{a}^{\frac{p^{pni}-1}{p^{ni}-1}} & i = (p-1)^{-1} \\ \bar{a} & i > (p-1)^{-1} \end{cases}.$$

Proof. a) Write $i = \frac{k}{n}$, $j = \frac{l}{n}$ and choose representatives $x = 1 + aS^k \in F_i S_n$, $y = 1 + bS^l \in F_j S_n$. Then $x^{-1} = 1 - aS^k \pmod{S^{k+1}}$, $y^{-1} = 1 - bS^l \pmod{S^{l+1}}$ and the formula

$$xyx^{-1}y^{-1} = 1 + ((x-1)(y-1) - (y-1)(x-1))x^{-1}y^{-1}$$

shows

$$xyx^{-1}y^{-1} = 1 + (aS^k bS^l - bS^l aS^k) \pmod{S^{k+l+1}}.$$

Because $\mathcal{O}_n/(S) \cong \mathbb{W}(\mathbb{F}_{p^n})/(p)$ we can choose a and b from $\mathbb{W}(\mathbb{F}_{p^n})$. Then $Sw = w^\sigma S$ and $w^\sigma \equiv w^p \pmod{(p)}$ give the stated formula.

b) Again we write $i = \frac{k}{n}$ and we choose a representative $x = 1 + aS^k$ with $a \in \mathbb{W}_{\mathbb{F}_{p^n}}$. Consider the expression $x^p = \sum_r \binom{p}{r} (aS^k)^r$. Because $\binom{p}{r}$ is divisible by p for $0 < r < p$ and because $S^n = p$ we get

$$x^p \equiv 1 + aS^{n+k} + \dots + (aS^k)^p \pmod{S^{2k+n}}.$$

Furthermore, modulo S^{kp+1} we get

$$(aS^k)^p = aa^{\sigma^k} \dots a^{\sigma^{(p-1)k}} S^{pk} \equiv aa^{p^k} \dots a^{p^{(p-1)k}} S^{pk} \equiv a^{1+p^k+\dots+p^{(p-1)k}} S^{pk}.$$

Now we only have to determine whether pk is smaller resp. equal resp. larger than $n+k$. i.e. whether pi is smaller resp. equal resp. larger than $1+i$. These cases are equivalent to $i < (p-1)^{-1}$ resp. $i = (p-1)^{-1}$ resp. $i > (p-1)^{-1}$ and hence we are done. \square

4.3. Torsion in the Morava stabilizer groups. As an immediate consequence of Proposition 4.3 we obtain the following result.

Corollary 4.4.

a) If $g \in F_i$ has finite order and $i > (p-1)^{-1}$ then $g = 1$.

b) S_n is torsionfree if n is not divisible by $p-1$. \square

Examples a) In particular, if $n = 1$ and $p > 2$ and $n = 2$ and $p > 3$ then the groups S_n are torsionfree.

b) For $n = 1$ we have $\mathcal{O}_n = \mathbb{Z}_p$, $S_1 = \{x \in \mathbb{Z}_p^\times \mid x \equiv 1 \pmod{(p)}\}$. Furthermore, it is well known that

$$\mathbb{Z}_p^\times \cong \begin{cases} F_1 \times \mathbb{F}_{p^n}^\times & p > 2 \\ F_2 \times \{\pm 1\} & p = 2 \end{cases}$$

and F_1 is isomorphic to the additive group \mathbb{Z}_p if p is odd. For $p = 2$ it is F_2 which is isomorphic to the additive group \mathbb{Z}_2 .

c) For $n = 2$ the group S_2 is nonabelian and its structure is complicated. Non-trivial torsion elements can exist only if $p = 2$ or $p = 3$.

For $p = 3$ a non-trivial torsion element must be nontrivial in $F_{\frac{1}{2}}/F_1$. An easy calculation shows that if ω is a fixed chosen primitive 8-th root of unity in $\mathbb{W}_{\mathbb{F}_9}$ then the element

$$(4.4) \quad a = -\frac{1}{2}(1 + \omega S)$$

satisfies $a^3 = 1$. (It is clearly in $F_{\frac{1}{2}}$ and its image in $F_{\frac{1}{2}}/F_1$ is \overline{a} .)

For $p = 2$ there is always, i.e. for each n , the element $-1 = 1 - S^n$ which is in F_1 and is a nontrivial element of order 2. If $n = 2$ there are elements of order 4 which must be nontrivial in $F_{\frac{1}{2}}/F_1$.

d) If $n = 4$ and $p = 2$ there is a chance for the existence of elements of order 8 which are nontrivial in $F_{\frac{1}{4}}/F_{\frac{1}{2}}$. In fact, such elements exist and they are in the background of the recent solution of the Kervaire invariant one problem by Hill, Hopkins and Ravenel.

5. ON THE COHOMOLOGY OF THE STABILIZER GROUPS WITH TRIVIAL COEFFICIENTS

The stabilizer groups are examples of p -adic Lie groups. For such groups the category of profinite modules over $\mathbb{Z}_p[[G]]$ has enough projectives and one can define continuous cohomology with coefficients in a profinite $\mathbb{Z}_p[[G]]$ -module M simply as $H_{cts}^s(G, M) = \text{Ext}_{\mathbb{Z}_p[[G]]}^s(\mathbb{Z}_p, M)$. Likewise one can define continuous homology with coefficients in a profinite $\mathbb{Z}_p[[G]]$ -module N simply as $H_{cts}^s(G, M) = \text{Tor}_{\mathbb{Z}_p[[G]]}^s(\mathbb{Z}_p, N)$. In the sequel cohomology resp homology will always be continuous cohomology resp continuous homology and we will simply write it as $H^*(G, M)$ resp. $H_*(G, M)$.

5.1. H_1 . The stabilizer group made abelian. The commutator formula in Proposition 4.3 can be used to calculate the abelianization of the groups S_n . In this profinite setting it is the quotient $S_n / \overline{[S_n, S_n]}$ which identifies with the homology $H_1(S_n, \mathbb{Z}_p)$. (Here \overline{E} denotes the closure of a given subset $E \subset S_n$). Likewise $H_1(S_n, \mathbb{Z}/p)$ identifies with the quotient $S_n / \langle [S_n, S_n], S_n^p \rangle$.

Here is the crucial lemma on commutators.

Lemma 5.1. *Let p be any prime and let k and l be integers > 0 .*

- a) *If $\frac{k+1}{n}$ is not an integer then the commutator map $gr_{\frac{k}{n}} S_n \otimes gr_{\frac{1}{n}} S_n \rightarrow gr_{\frac{k+1}{n}} S_n$ is onto.*
- b) *If $\frac{k+1}{n}$ is an integer then the image of the commutator map $gr_{\frac{k}{n}} S_n \otimes gr_{\frac{1}{n}} S_n \rightarrow gr_{\frac{k+1}{n}} S_n$ is equal to the kernel of the trace $tr : \mathbb{F}_{p^n} \rightarrow \mathbb{F}_p$*
- c) *If $\frac{k+l}{n}$ is an integer then the image of the commutator map $gr_{\frac{k}{n}} S_n \otimes gr_{\frac{l}{n}} S_n \rightarrow gr_{\frac{k+l}{n}} S_n$ is contained in the kernel of the trace $tr : \mathbb{F}_{p^n} \rightarrow \mathbb{F}_p$.*

Proof. a) By Proposition 4.3 the commutator map is given by the formula

$$[\bar{a}, \bar{b}] = \bar{a}\bar{b}^{p^k} - \bar{b}\bar{a}^p$$

By taking $b = 1$ one sees that all elements of the form $\bar{a} - \bar{a}^p$ belong to the image. This is an \mathbb{F}_p -linear subspace of \mathbb{F}_{p^n} of \mathbb{F}_p -codimension 1 which is contained in and therefore equal to the kernel of the trace. Furthermore, if $\frac{k+1}{n}$ is not an integer, it is enough to exhibit a couple (\bar{a}, \bar{b}) such that

$$tr(\bar{a}\bar{b}^{p^k} - \bar{b}\bar{a}^p) = tr(\bar{a}^p(\bar{b}^{p^{k+1}} - \bar{b})) \neq 0.$$

Now, if $k+1$ is not divisible by n there exists \bar{b} such that $c := \bar{b}^{p^{k+1}} - \bar{b} \neq 0$. Because the trace is a nontrivial linear form and because

$$\mathbb{F}_{p^n} \rightarrow \mathbb{F}_{p^n}, \bar{a} \rightarrow \bar{a}^p c$$

is bijective we are done.

b) If $\frac{k+1}{n}$ is an integer, i.e. $k+1$ divisible by n , then $\bar{b}^{p^{k+1}} - \bar{b} = 0$ for all \bar{b} and therefore

$$\text{tr}(\bar{a}\bar{b}^{p^k} - \bar{b}\bar{a}^{p^k}) = \text{tr}(\bar{a}^{p^k}(\bar{b}^{p^{k+1}} - \bar{b})) = 0.$$

On the other hand we have already seen in the proof of (a) that the kernel of the trace is in the image of the commutator map.

c) In general the commutator map $gr_{\frac{k}{n}}S_n \otimes gr_{\frac{l}{n}}S_n \rightarrow gr_{\frac{k+l}{n}}S_n$ is given by

$$[\bar{a}, \bar{b}] = \bar{a}\bar{b}^{p^k} - \bar{b}\bar{a}^{p^l}$$

and hence

$$\text{tr}(\bar{a}\bar{b}^{p^k} - \bar{b}\bar{a}^{p^l}) = \text{tr}(\bar{a}^{p^l}(\bar{b}^{p^{k+l}} - \bar{b})).$$

If $\frac{k+l}{n}$ is an integer then $k+l$ is divisible by n and hence $\bar{b}^{p^{k+l}} - \bar{b} = 0$. \square

Proposition 5.2. *Let p be an odd prime and $n > 1$. Then*

$$H_1(S_n, \mathbb{Z}_p) \cong \mathbb{Z}_p \oplus (\mathbb{Z}/p)^n.$$

As topological generator of \mathbb{Z}_p one can choose $1 + cS^n = 1 + 2c$ where c is in $\mathbb{W}_{\mathbb{F}_{p^n}}$ of valuation 0 with $\text{tr}(\bar{c}) \neq 0$ and as generators of the n summands \mathbb{Z}/p one can choose the elements $1 + \omega^{p^j}S$, $j = 0, \dots, n-1$ of order p where ω is a fixed primitive root of unity of order $p^n - 1$.

Proof. The filtration on S_n introduced in (4.2) induces one on $S_n/\overline{[S_n, S_n]}$ and Lemma 5.1 shows that $gr_i(S_n/\overline{[S_n, S_n]})$ is isomorphic to $gr_i S_n \cong \mathbb{F}_q$ if $i = \frac{1}{n}$, isomorphic to the image of $\text{tr} : \mathbb{F}_q \rightarrow \mathbb{F}_p$ if i is an integer, and zero otherwise. By Proposition 4.3 the induced p -th power map sends $gr_i(S_n/\overline{[S_n, S_n]})$ isomorphically to $gr_{i+1}(S_n/\overline{[S_n, S_n]})$ if i is an integer, and it is clearly trivial on $gr_{\frac{1}{n}}(S_n/\overline{[S_n, S_n]})$ except possibly if $n = p$. Furthermore, if $n = p$ we get $\text{tr}(P(\bar{a})) = \text{tr}(\bar{a}^{1+p+\dots+p^{(p-1)}}) = 0$ because $\bar{a}^{1+p+\dots+p^{(p-1)}}$ is fixed by Frobenius and thus the trace is p times this element, hence trivial modulo p . Now Lemma 5.1 implies that the induced p -th power map is always trivial on $gr_{\frac{1}{n}}(S_n/\overline{[S_n, S_n]})$ and this implies the result. \square

Proposition 5.3. *Let $p = 2$ and $n > 1$. Then*

$$H_1(S_n, \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus (\mathbb{Z}/2)^{n+1}.$$

As topological generator of \mathbb{Z}_2 one can choose $1 + cS^{2^n} = 1 + 4c$ and as generators of the $n+1$ summands $\mathbb{Z}/2$ one can choose the elements $1 + cS^{2^n}$, $1 + \omega^{2^k}S$, $k = 1, \dots, n$, where c is in $\mathbb{W}_{\mathbb{F}_{2^{2^n}}}$ of valuation 0 and $\text{tr}(\bar{c}) \neq 0$, and ω is a fixed primitive root of unity of order $2^n - 1$.

Proof. Again the filtration on S_n introduced in (4.2) induces one on $S_n/\overline{[S_n, S_n]}$ and the previous lemma shows that $gr_i(S_n/\overline{[S_n, S_n]})$ is isomorphic to $gr_i S_n \cong \mathbb{F}_q$ if $i = \frac{1}{n}$ and isomorphic to the image of $\text{tr} : \mathbb{F}_q \rightarrow \mathbb{F}_p$ if i is an integer, and zero otherwise. By Proposition 4.3 the induced p -th power map on $gr_i(S_n/\overline{[S_n, S_n]})$ sends $gr_i(S_n/\overline{[S_n, S_n]})$ isomorphically to $gr_{i+1}(S_n/\overline{[S_n, S_n]})$ if i is an integer > 1 , and it is clearly trivial on gr_i except possibly if $i = \frac{1}{2}$ or $i = 1$. The same argument

as in the previous proof shows that the induced p -th power map is trivial on $gr_{\frac{1}{2}}$. For $i = 1$ Proposition 4.3 gives

$$P(\bar{a}) = \bar{a} + \bar{a}^{\frac{2^{2n}-1}{2^n-1}} = \bar{a} + \bar{a}^{2^n+1} = \bar{a} + \bar{a}^2 .$$

The trace of this element is again trivial and the result follows once again by Lemma 5.1. \square

Corollary 5.4. *Let p be a prime and $n > 1$. Then*

$$H_1(S_n, \mathbb{Z}/p) \cong \begin{cases} (\mathbb{Z}/p)^{n+1} & p > 2 \\ (\mathbb{Z}/2)^{n+2} & p = 2 . \end{cases} \quad \square$$

5.2. The cohomology of S_1 . This case is fairly easy.

Proposition 5.5.

a) *If p is odd then*

$$H^*(S_1, \mathbb{Z}_p) \cong \begin{cases} \mathbb{Z}_p & n = 0, 1 \\ 0 & \text{else} . \end{cases}$$

b) *If $p = 2$ then*

$$H^*(S_1, \mathbb{Z}_p) \cong \begin{cases} \mathbb{Z}_2 & n = 0, 1 \\ \mathbb{Z}/2 & n \geq 2 . \end{cases}$$

Proof. We have $S_1 = \mathbb{Z}_p$ if $p > 2$ and $S_1 = \mathbb{Z}_2^\times \cong \mathbb{Z}/2 \times \mathbb{Z}_2$ if $p = 2$. The result follows therefore as soon as we know that $H^n(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p$ if $n = 0, 1$ and trivial otherwise. (For $p = 2$ we use the Kuenneth theorem). Now cohomology is calculated from a resolution of the trivial module \mathbb{Z}_p by projective modules over the completed group ring $\mathbb{Z}_p[[\mathbb{Z}_p]]$. There is an obvious algebra homomorphism from the polynomial algebra $\mathbb{Z}_p[T]$ to the group algebra $\mathbb{Z}_p[\mathbb{Z}_p]$ which sends T to $t - e$ where t is a topological generator of the group \mathbb{Z}_p . This map extends to a continuous homomorphism from the power series ring $\mathbb{Z}_p[[T]]$ to $\mathbb{Z}_p[[\mathbb{Z}_p]]$ which can be checked to be an isomorphism. In fact, this isomorphism is the starting point for Iwasawa theory in number theory (cf. [20]). Now it is obvious that the trivial $\mathbb{Z}_p[[T]]$ -module \mathbb{Z}_p admits a projective resolution

$$(5.1) \quad 0 \rightarrow \mathbb{Z}_p[[T]] \xrightarrow{T} \mathbb{Z}_p[[T]] \rightarrow \mathbb{Z}_p$$

and the result follows. \square

5.3. Structural properties of $H^*(S_n, \mathbb{Z}/p)$. Proposition 5.5 and its proof yield immediately the additive structure of $H^*(S_1, \mathbb{Z}/p)$ resp. of $H^*(\mathbb{Z}_p, \mathbb{Z}_p)$. In fact, there is a cup product structure which is uniquely determined by the additive result.

Proposition 5.6. *Let p be any prime. Then*

$$H^*(\mathbb{Z}_p, \mathbb{Z}/p) \cong \Lambda_{\mathbb{Z}/p}(H^1(S_1, \mathbb{Z}/p)) \cong \Lambda(e)$$

with $e \in H^1$ given by the reduction homomorphism $\mathbb{Z}_p \rightarrow \mathbb{Z}/p$ considered as an element in $H^1(\mathbb{Z}_p, \mathbb{Z}/p) = \text{Hom}(\mathbb{Z}_p, \mathbb{Z}/p)$ and $\Lambda_{\mathbb{Z}/p}$ denotes the exterior algebra over \mathbb{Z}/p . \square

Via the Kuenneth theorem we get the following corollary.

Corollary 5.7. *Let p be any prime. Then*

$$H^*(\mathbb{Z}_p^n, \mathbb{Z}/p) \cong \Lambda_{\mathbb{Z}/p}(H^1(\mathbb{Z}_p^n, \mathbb{Z}/p)) \cong \Lambda_{\mathbb{Z}/p}(e_1, \dots, e_n) .$$

with $e_i \in H^1$ for $i = 1, \dots, i = n$, a dual basis of \mathbb{Z}_p^n/p and $\Lambda_{\mathbb{Z}/p}$ denoting the exterior algebra over \mathbb{Z}/p . \square

An interesting feature of the stabilizer groups is that although they do not contain abelian subgroups of rank $> n$ (i.e. free \mathbb{Z}_p submodules of rank $> n$) they do contain finite index subgroups which look abelian of rank n^2 from the point of view of mod- p cohomology. The following result follows from [18].

Proposition 5.8.

a) *Let $p > 2$ and let $i = \frac{k}{n} \geq 1$. Then*

$$H^*(F_i, \mathbb{Z}/p) \cong \Lambda_{\mathbb{Z}/p}(H^1(F_i, \mathbb{Z}/p)) \cong \Lambda_{\mathbb{Z}/p}(e_{i,j})$$

where $0 \leq i, j \leq n-1$ and $e_{i,j}$ is dual to $1 + \omega^i S^{k+j}$.

b) *For $p = 2$ the same result holds if $i = \frac{k}{n} > 1$.*

Corollary 5.9. *The mod- p cohomology ring of S_n is a noetherian algebra over \mathbb{Z}/p .*

Proof. This follows from Proposition 5.8 by analyzing the spectral sequence of the group extension $1 \rightarrow F_i \rightarrow S_n \rightarrow S_n/F_i \rightarrow 1$. \square

Definition 5.10. Let p be any prime. A profinite p -group is called a *Poincaré duality group of dimension d* if

- $H^s(G, \mathbb{Z}/p)$ is finite dimensional for each $s \geq 0$
- $H^d(G, \mathbb{Z}/p) \cong \mathbb{Z}/p$
- The cup product $H^s(G, \mathbb{Z}/p) \times H^{d-s}(G, \mathbb{Z}/p) \rightarrow H^d(G, \mathbb{Z}/p)$ is a nondegenerate bilinear form for each $s > 0$.

Examples a) \mathbb{Z}_p^d is a Poincaré duality group of dimension d .

b) $F_i S_n$ is a Poincaré duality group of dimension n^2 whenever $i = \frac{k}{n} \geq 1$ if $p > 2$, and whenever $i > 1$ if $p = 2$.

Theorem 5.11. [18] *Suppose that G is a profinite p -group without torsion which contains a finite index subgroup which is a Poincaré duality group of dimension n . Then G is itself a Poincaré duality group of dimension n .*

5.4. The reduced norm and a decomposition of S_n . If $n = 2$ and $p > 3$ then S_2 is torsionfree and hence it is a Poincaré duality group of dimension $2^2 = 4$. In fact, we can even reduce to the case of a Poincaré duality group 3 as follows.

In the case of general n and p we consider \mathcal{O}_n as a left $\mathbb{W}(\mathbb{F}_{p^n})$ -module of rank n . Multiplying on the right gives a multiplicative homomorphism

$$\mathcal{O}_n \rightarrow M_n(\mathbb{W}(\mathbb{F}_{p^n}))$$

and hence

$$\mathbb{S}_n \rightarrow GL_n(\mathbb{W}(\mathbb{F}_{p^n})) .$$

Following this by the determinant gives a homomorphism $\mathbb{S}_n \rightarrow (\mathbb{W}(\mathbb{F}_{p^n}))^\times$ which is invariant with respect to the natural actions of $Gal(\mathbb{F}_{p^n} : \mathbb{F}_p)$. On the other hand we have noted in the remark preceeding Definition 4.2 that the Galois action on \mathbb{S}_n is induced by conjugation by the element S in D_n^\times . It follows that the determinant restricted to \mathbb{S}_n takes its values in the Galois invariant part \mathbb{Z}_p^\times of $\mathbb{W}(\mathbb{F}_{p^2})^\times$.

If $n = 2$ this can also be seen by an easy calculation as follows. If we choose 1 and S as $\mathbb{W}(\mathbb{F}_{p^2})$ basis for \mathcal{O}_2 then right multiplication defines

$$\mathcal{O}_2 \rightarrow M_2(\mathbb{W}(\mathbb{F}_{p^2})), \quad a + bS \mapsto \begin{pmatrix} a & pb^\sigma \\ b & a^\sigma \end{pmatrix}$$

with obviously Galois-invariant determinant.

The resulting homomorphism $\mathbb{S}_n \rightarrow \mathbb{Z}_p^\times$ is often called the *reduced norm*. Restricted to the central \mathbb{Z}_p^\times in \mathbb{S}_n the reduced norm is given by the n -th power map. By restricting to the p -Sylow subgroup and assuming that p does not divide n we get a splitting of the sequence

$$1 \rightarrow S_n^1 \rightarrow S_n \rightarrow P(\mathbb{Z}_p^\times) \rightarrow 1$$

where $P(\mathbb{Z}_p^\times)$ is the p -Sylow subgroup of \mathbb{Z}_p^\times .

Proposition 5.12. *Suppose p does not divide n . Then the group S_n is isomorphic to the direct product of its subgroups S_n^1 and $P(\mathbb{Z}_p^\times)$, i.e.*

$$S_n \cong \begin{cases} S_n^1 \times \{g \in \mathbb{Z}_p^\times \mid g \equiv 1 \pmod{(p)}\} & p > 2 \\ S_n^1 \times \mathbb{Z}_2^\times & p = 2 . \end{cases} \quad \square$$

5.5. Cohomology in case $n = 2$ and $p > 2$.

5.5.1. The case $p > 3$. In this case we have

$$S_2 \cong S_2^1 \times \{g \in \mathbb{Z}_p^\times \mid g \equiv 1 \pmod{(p)}\} \cong S_2^1 \times \mathbb{Z}_p$$

The group S_2 is a Poincaré duality group of dimension 4, hence S_2^1 is a Poincaré duality group of dimension 3. Calculating its mod p -cohomology is therefore easy.

By Poincaré duality it is enough to calculate $H^1(S_2^1, \mathbb{F}_p)$. From Corollary 5.4 we obtain the following result.

Theorem 5.13. [12] *Let $p > 3$. Then*

$$H^*(S_2^1, \mathbb{Z}/p) \cong \begin{cases} \mathbb{Z}/p & * = 0, 3 \\ (\mathbb{Z}/p)^2 & * = 1, 2 \\ 0 & * > 3 \end{cases} . \quad \square$$

5.5.2. *The case $p = 3$.* The cases $n = 2$ and $p = 2, 3$ are considerably more complicated. In this case the groups S_n do contain p -torsion and they are no longer Poincaré duality groups. In fact, their vcd_p is infinite. We will be content to discuss the case $p = 3$. For $p = 3$ we still have the decomposition

$$S_2 \cong S_2^1 \times \{g \in \mathbb{Z}_p^\times \mid g \equiv 1 \pmod{(p)}\} \cong S_2^1 \times \mathbb{Z}_p$$

and the problem is again reduced to the case of S_2^1 . Even though the group S_2^1 is not a Poincaré duality group it contains one of index 9, namely the group $F_1 S_2^1 = F_1 S_2 \cap S_2^1$. In fact, it even contains one of index 3. In order to see this we consider the formula for the 3-rd power map

$$P : F_{\frac{1}{2}} \rightarrow F_1, \quad \bar{a} \mapsto \bar{a} + \bar{a}^{1+3+9} .$$

This shows that if there is an element $g \in S_2^1$ of order 3 then it has the form $g = 1 + aS \pmod{F_1}$ with $\bar{a}^4 = -1$. Thus if we define K to be the kernel of the homomorphism $S_2^1 \rightarrow S_2^1/F_1 S_2^1 \cong \mathbb{F}_9 \rightarrow \mathbb{F}_9/\mathbb{F}_3$ then K is torsion-free and by Theorem 5.11 it is a Poincaré duality group of dimension 3.

Proposition 5.14. [11]

$$H^*(K, \mathbb{Z}/3) \cong \begin{cases} \mathbb{Z}/3 & * = 0, 3 \\ (\mathbb{Z}/3)^2 & * = 1, 2 \\ 0 & * > 3 \end{cases} .$$

Proof. Because K is without torsion Theorem 5.11 implies that it is a Poincaré duality group (of dimension 3). So it is enough to calculate $H^1(K, \mathbb{Z}/3) \cong H_1(K, \mathbb{Z}/3)$. For this we consider the filtration on K given by $F_i K := K \cap F_i S_2$. It is not hard to check that $H_1(K, \mathbb{Z}_3) \cong \mathbb{Z}/9 \oplus \mathbb{Z}/3$ generated by $b := [a, \omega]$ and $c = [a, b]$ where as before

$$a = -\frac{1}{2}(1 + \omega S)$$

is the element of order 3 of (4.4) and ω is a primitive 8th root of unity in $\mathbb{W}(\mathbb{F}_9)$. This implies the desired result. \square

The cohomology of S_2^1 can now be calculated by using the (non-central) split exact sequence

$$1 \rightarrow K \rightarrow S_2^1 \rightarrow \mathbb{Z}/3 \rightarrow 1 .$$

The quotient map $S_2^1 \rightarrow \mathbb{Z}/3$ makes $H^*(S_2^1, \mathbb{Z}/3)$ into a module over the polynomial algebra generated by $y \in H^2(\mathbb{Z}/3, \mathbb{Z}/3)$. It is true (but far from obvious) that this spectral sequence degenerates at E_2 . In fact, it is equivalent to knowing that

$H^*(S_2^1, \mathbb{Z}/3)$ is a free module over the polynomial algebra $\mathbb{Z}/3[y]$. Using that we obtain the following result.

Theorem 5.15. [11] *Let $p = 3$. Then $H^*(S_2^1, \mathbb{Z}/3)$ is a free module over $\mathbb{Z}/3[y]$ on 8 generators in degrees 0, 1, 1, 2, 2, 3, 3, 4.*

The cup product structure is also known. It can be approached as follows. Up to conjugacy there are two subgroups of order 3 in S_2^1 , namely the subgroup $\langle a \rangle$ generated by a and the subgroup $\langle \omega a \omega^{-1} \rangle$. The centralizers of these elements are isomorphic and $C_{S_2^1}(a) \cong \langle a \rangle \times \mathbb{Z}_3$. The cup product structure is determined by the following result.

Theorem 5.16. [11] *a) The restriction homomorphisms induce a monomorphism*

$$H^*(S_2^1, \mathbb{Z}/3) \rightarrow H^*(C_{S_2^1}(a), \mathbb{Z}/3) \times H^*(C_{S_2^1}(\omega a \omega^{-1}), \mathbb{Z}/3)$$

whose target is isomorphic to $\prod_{i=1}^2 \mathbb{Z}/3[y_i] \otimes \Lambda_{\mathbb{Z}/3}(x_i, a_i)$ where the elements y_i are of degree 2 and x_i and a_i are of degree 1.

b) This map is an isomorphism in degrees > 2 . Its image in degree 0 is the diagonal, in degree 1 it is the subspace generated by x_1 and x_2 and in degree 2 the subspace generated by y_1, y_2 and $x_1 a_1 - x_2 a_2$.

c) The image is a free module over $\mathbb{Z}/3[y_1 + y_2]$ on the following 8 generators: 1, $x_1, x_2, y_1 - y_2, x_1 a_1 - x_2 a_2, y_1 a_1, y_2 a_2, y_1 x_1 a_1 + y_2 x_2 a_2$.

5.5.3. *The case $p = 2$.* The case of the prime 2 is even more complicated but it is also understood (cf. [12] and the recent Northwestern theses of Beaudry and Bobkova).

6. COHOMOLOGY WITH NON-TRIVIAL COEFFICIENTS AND RESOLUTIONS

For homotopy theoretic applications we will be interested in calculating cohomology with certain non-trivial coefficients, in particular $H^*(\mathbb{G}_n, (E_n)_*)$. For this we use explicit resolutions of the trivial module. In this section we will discuss the classical case $n = 1$ in fair detail and briefly comment on the case $n = 2$.

6.1. **The case $n = 1$.** In the case $n = 1$ we have already seen such resolutions for the group S_1 , at least if $p > 2$. More precisely we have seen in (5.1) seen that there is a free resolution

$$0 \rightarrow P_1 \xrightarrow{t-\xi} P_0 \rightarrow \mathbb{Z}_p \rightarrow 0$$

of the trivial $\mathbb{Z}_p[[\mathbb{Z}_p]]$ -module with $P_0 = P_1 = \mathbb{Z}_p[[\mathbb{Z}_p]]$ and t a topological generator of \mathbb{Z}_p . In the case of $\mathbb{S}_1 = \mathbb{G}_1$ we can use the same resolutions but enriched as resolutions by $\mathbb{Z}_p[[\mathbb{G}_1]]$ -modules. In fact because of the product decomposition $\mathbb{G}_1 = \mathbb{Z}_p^\times \cong \mathbb{Z}_p \times F$ where $F = \{\pm 1\}$ if $p = 2$ resp. $F = \mathbb{Z}/p^\times$ if $p > 2$, every $\mathbb{Z}_p[[\mathbb{Z}_p]]$ -module resp. every $\mathbb{Z}_p[[\mathbb{Z}_p]]$ -module homomorphism can be considered as a $\mathbb{Z}_p[[\mathbb{G}_1]]$ -module resp. $\mathbb{Z}_p[[\mathbb{G}_1]]$ -module homomorphism via the projection map $\mathbb{G}_1 \rightarrow \mathbb{Z}_p$. Of course, in this case the modules are no longer free modules. However, if p is odd they

are still projective and in case $p = 2$ they are at least permutation modules. Indeed as $\mathbb{Z}_p[[\mathbb{G}_1]]$ -modules we always have $P_0 = P_1 = \mathbb{Z}_p[[\mathbb{G}_1/F]] = \mathbb{Z}_p[[\mathbb{G}_1]] \otimes_{\mathbb{Z}_p[F]} \mathbb{Z}_p$ and the trivial $\mathbb{Z}_p[F]$ -module \mathbb{Z}_p is projective if $p > 2$ because the order of F is prime to p in this case. But even in the case $p = 2$ this permutation resolution is useful for calculating group cohomology. In fact, it gives rise to a long exact sequence (with $R = \mathbb{Z}_p[[\mathbb{G}_1]]$ and ψ denoting a topological generator of $\mathbb{Z}_p^\times/F \cong \mathbb{Z}_p$)

$$\dots \rightarrow \text{Ext}_R^s(\mathbb{Z}_2, M) \rightarrow \text{Ext}_R^s(P_0, M) \xrightarrow{\psi - id} \text{Ext}_R^s(P_1, M) \rightarrow \text{Ext}_R^{s+1}(\mathbb{Z}_2, M) \rightarrow \dots$$

which can be identified by definition of H^* and by using Shapiro's Lemma with

$$(6.1) \quad \dots \rightarrow H^s(\mathbb{G}_1, M) \rightarrow H^s(F, M) \xrightarrow{\psi - id} H^s(F, M) \rightarrow H^{s+1}(\mathbb{G}_1, M) \rightarrow \dots$$

6.1.1. *The case $p > 2$.* If $p > 2$ the groups in the middle of (6.1) are trivial unless $s = 0$. Now we consider the graded module $M = (E_1)_* = \mathbb{Z}_p[u^{\pm 1}]$ with $|u| = -2$. The action of $\mathbb{G}_1 = \mathbb{Z}_p^\times$ on this graded algebra is by algebra homomorphisms and is thus specified by the action on the polynomial generator u . It is the tautological action $(g, u) \mapsto g.u$. Then we get

$$H^*(F, \mathbb{Z}_p[u^{\pm 1}]) = \begin{cases} \mathbb{Z}_p[u^{\pm(p-1)}] & s = 0 \\ 0 & s \neq 0 \end{cases}.$$

For ψ we can take the element $p + 1 \in \mathbb{Z}_p^\times$. Then

$$(\psi - id)_*(u^{t(p-1)}) = ((p + 1)^{t(p-1)} - 1)u^{t(p-1)} = cp^{\nu_p(t)+1}u^{t(p-1)}$$

where $\nu_p(t)$ is the p -adic valuation of the integer t and c is a unit modulo p . This proves the following result.

Theorem 6.1. *Let p be an odd prime. Then*

$$H^s(\mathbb{S}_1, (E_1)_t) = \begin{cases} \mathbb{Z}_p & t = 0, s = 0, 1 \\ \mathbb{Z}/p^{\nu_p(t')+1} & t = 2(p-1)t', s = 1 \\ 0 & \text{else} \quad \square \end{cases}$$

Then the homotopy fixed point spectral sequence (2.1)

$$E_2^{s,t} = H^s(\mathbb{S}_1, (E_1)_t) \implies \pi_{t-s}(L_{K(1)}S^0)$$

collapses by sparseness, and we get the following result which is essentially equivalent to Ravenel's calculation of $\pi_*L_1S^0$ (cf. [21]).

Theorem 6.2. *Let p be an odd prime. Then*

$$\pi_n(L_{K(1)}S^0) \cong \begin{cases} \mathbb{Z}_p & n = 0, -1 \\ \mathbb{Z}/p^{\nu_p(t')+1} & n = 2(p-1)t' - 1 \end{cases} \quad \square$$

6.1.2. *The case $p = 2$.* For $p = 2$ we get

$$(6.2) \quad H^s(C_2, \mathbb{Z}_2[u^{\pm 1}]) = \begin{cases} \mathbb{Z}_2[u^{\pm 2}] & s = 0 \\ \mathbb{Z}/2[u^{\pm 2}]\{y^{s'}\} & s = 2s' \\ \mathbb{Z}/2[u^{\pm 2}]\{y^{s'}x\} & s = 2s' + 1 \end{cases}$$

with $y \in H^2(C_2, \mathbb{Z}_2)$, $x \in H^1(C_2, \mathbb{Z}_2\{u\})$. We note that this group cohomology is the E_2 -term for the homotopy fixed point spectral sequence converging to $\pi_* KO\mathbb{Z}_2 = \pi_* KU^{hC_2}$. The bidegree of u , y and x are $|u| = (0, -2)$, $|y| = (2, 0)$ and $|x| = (1, -2)$. The full multiplicative structure is determined by the relation $x^2 = yu^2$. One can thus rewrite this E_2 -term as

$$(6.3) \quad E_2^{*,*} = \mathbb{Z}/2[u^{\pm 2}, \eta]/(2\eta)$$

with $\eta = xu^{-2}$. The notation is chosen so as to agree with usual notation in homotopy theory, i.e. $\eta \in E_2^{1,2}$ is a permanent cycle which represents the image of $\eta \in \pi_1^S$ in $\pi_1(KO\mathbb{Z}_2)$. In the homotopy fixed point spectral sequence there is a single differential. In fact, in the sphere we have $\eta^4 = 0$ so η^4 has to be hit by a differential. There is only one way how this can happen, namely via

$$(6.4) \quad d_3(u^{-2}) = \eta^3.$$

The spectral sequence is multiplicative. Therefore we get

$$E_4^{s,t} = \mathbb{Z}_2[u^{\pm 4}]\{2u^2, \eta, \eta^2\}/(2\eta),$$

the spectral sequence degenerates at E_4 and we find the well known homotopy groups of $\pi_*(KO\mathbb{Z}_2)$ given as

$$\pi_s(KO\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & s \equiv 0, 4 \pmod{8} \\ \mathbb{Z}/2 & s \equiv 1, 2 \pmod{8} \\ 0 & s \equiv 3, 5, 6, 7 \pmod{8} \end{cases}.$$

Independently of this homotopy theoretic calculation we can use (6.2) and the long exact sequence (6.1) in order to calculate $H^s(\mathbb{G}_1, (E_1)_*)$. The induced homomorphism in H^s is trivial if $s > 0$ because ψ necessarily acts trivially on $H^s(C_2, (E_1)_t)$ if $s > 0$ (because the group is either trivial or $\mathbb{Z}/2$). For $s = 0$ we have a similar phenomenon as before, namely we take for ψ the element $3 = 1 + 2 \in \mathbb{Z}_2^\times$. Because of

$$(\psi - id)_*(u^{2t}) = ((1 + 2)^{2t} - 1)u^{2t} = ((1 + 8)^t - 1)u^{2t} = c2^{\nu_2(t)+3}u^{2t}$$

with c a unit modulo 2 we obtain the following result.

Theorem 6.3. *Let $p = 2$. Then*

$$H^s(\mathbb{S}_1, (E_1)_t) \cong \begin{cases} \mathbb{Z}_2 & t = 0, s = 0, 1 \\ \mathbb{Z}/(2^{\nu_2(t')+3}) & s = 1, t = 4t' \neq 0 \\ \mathbb{Z}/2 & s = 1, t = 4t' + 2 \\ \mathbb{Z}/2 & s \geq 2, t \text{ even} \\ 0 & \text{else} . \quad \square \end{cases}$$

In cohomological degrees $s \geq 2$ the E_2 -term of the homotopy fixed point spectral sequence (2.1)

$$E_2^{s,t} = H^s(\mathbb{S}_1, (E_1)_t) \implies \pi_{t-s}(L_{K(1)}S^0)$$

agrees with the algebra $\mathbb{Z}_2[u^{\pm 2}, \eta, \zeta]/(2\eta, \zeta^2)$ with $\zeta \in H^1(\mathbb{S}_1, \mathbb{Z}_2[u^{\pm 1}]_0)$. In cohomological degree $s = 0$ the E_2 -term is isomorphic to \mathbb{Z}_2 concentrated in internal degree $t = 0$. In bidegrees $(1, 4t' + 2)$ it is isomorphic to $\mathbb{Z}/2$ generated by $\eta u^{-2t'}$ and in bidegrees $(1, 4t')$ it is isomorphic to $\mathbb{Z}/(2^{\nu_2(t') + 3})$ resp. \mathbb{Z}_2 generated by $\zeta u^{-2t'}$ if $t' \neq 0$ resp. by ζ if $t' = 0$.

To get at the homotopy of $L_{K(1)}S^0$ we still need to understand the differentials in this spectral sequence. They are determined via naturality and via the geometric boundary theorem [3] by those for the homotopy fixed point spectral sequence for $KO\mathbb{Z}_2$, i.e. by (6.4). The d_3 -differential is linear with respect to the permanent cycles η and ζ , and it is determined by

$$d_3(\eta u^{-2t}) = \begin{cases} \eta^4 u^{-2t+2} & t \equiv 1 \pmod{2} \\ 0 & t \equiv 0 \pmod{2} \end{cases}$$

and

$$d_3(\zeta u^{-2t}) = \begin{cases} \zeta \eta^3 u^{-2t+2} & t \equiv 1 \pmod{2} \\ 0 & t \equiv 0 \pmod{2} \end{cases}.$$

By sparseness all higher differentials are trivial. In dimension $\equiv 1, 3 \pmod{8}$ there are still extension problems to be solved. This can be done, for example, by comparing with the calculation for the mod-2 Moore space. In dimension $\equiv 1 \pmod{8}$ the extensions turn out to be trivial while in dimensions $\equiv 3 \pmod{8}$ they are non-trivial. The final result reads as follows and is essentially once again equivalent to Ravenel's calculation of $\pi_* L_1 S^0$ in [21].

Theorem 6.4. *Let $p = 2$. Then*

$$\pi_n(L_{K(1)}S^0) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}/2 & i = 0 \\ \mathbb{Z}/2 & i \equiv 0 \pmod{8}, i \neq 0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & i \equiv 1 \pmod{8} \\ \mathbb{Z}/2 & i \equiv 2 \pmod{8} \\ \mathbb{Z}/8 & i \equiv 3 \pmod{8} \\ \mathbb{Z}_2 & i = -1 \\ \mathbb{Z}/(2^{\nu_2(s)+4}) & i = 8s - 1, i \neq -1 \\ 0 & \text{otherwise} . \quad \square \end{cases}$$

6.2. Some comments on the case $n = 2$. As a first step one needs to establish the resolutions of the trivial module described in sections 3.3 and 3.6 and make them into effective calculational tools.

For $p > 3$ these resolutions are projective minimal resolutions which are constructed from the calculations of $H^*(S_2^1, \mathbb{Z}/p)$ discussed in section 5.5. For example, one can construct a resolution of the trivial S_2^1 -module \mathbb{Z}_p of the form

$$0 \rightarrow P_3 \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z}_p \rightarrow 0$$

such that $P_0 = P_3 = \mathbb{Z}_p[[S_2^1]]$ and $P_1 = P_2 = \mathbb{Z}_p[[S_2^1]]^{\oplus 2}$. As in the case $n = 1$ this projective resolution can be promoted to one of \mathbb{G}_2^1 . However, it is not true that

S_2^1 is a quotient of \mathbb{G}_2^1 so the details of promoting this resolution to one of \mathbb{G}_2^1 are not as straightforward.

Nevertheless the existence of the resolution is fairly formal and knowledge of the existence of such resolutions is already quite useful. However, mere existence is not good enough to carry out actual calculations like that of $H^*(\mathbb{G}_2^1, (E_2)_*)$. A great deal of work is necessary in order to describe the homomorphism in the resolution explicitly, or at least closely enough such that actual calculations can be carried out. Another problem is that of getting sufficient control of the action of \mathbb{G}_2 on $(E_2)_*$. At least modulo p all these problems have been resolved in the thesis of O. Lader [17] (cf. section 3.8.2).

In the case of the prime 3 the resolution has a similar form except that the modules are no longer projective (just as in the case $p = 2$ and $n = 1$ before). Nevertheless in the thesis of Karamanov [15] and in [13] this resolution has been made into a very effective computational tool (cf. section 3.8.2).

As noted in section 3.8.3 the case of the prime 2 is currently actively developed.

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