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On the minimizing movement with the 1-Wasserstein distance

Martial AGUEH ^{*†} Guillaume CARLIER[‡] Nouredine IGBIDA [§]

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Abstract

We consider a class of doubly nonlinear constrained evolution equations which may be viewed as a nonlinear extension of the growing sandpile model of [15]. We prove existence of weak solutions for quite irregular sources by a semi-implicit scheme in the spirit of the seminal works of [13] and [14] but with the 1-Wasserstein distance instead of the quadratic one. We also prove an L^1 -contraction result when the source is L^1 and deduce uniqueness and stability in this case.

Keywords: 1-Wasserstein distance, minimizing movement, L^1 -contraction, growing sandpiles.

MS Classification: 35K55, 35D30, 49N15.

1 Introduction

Given a convex nonlinearity F , and Ω , an open bounded subset of $\mathbb{R}^d$, we are interested in (a suitable weak notion of solution for) the following evolution system:

$$\partial_t u - \operatorname{div}(a \nabla F'(u)) = f, \quad (t, x) \in (0, T) \times \Omega, \quad u|_{t=0} = u_0, \quad (1.1)$$

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[†]This work was completed after Martial passed away. We wish to dedicate this article to his memory.

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supplemented with the zero flux condition i.e. the requirement that $a\nabla F'(u)$ is tangential to $\partial\Omega$ and constrained by

$$a \geq 0, |\nabla F'(u)| \leq 1, a(1 - |\nabla F'(u)|) = 0. \quad (1.2)$$

At least formally, (1.1)-(1.2) can be viewed as the limit as $q \rightarrow \infty$ of the doubly nonlinear evolution equation:

$$\partial_t u - \Delta_q(F'(u)) = f, \quad u|_{t=0} = u_0 \quad (1.3)$$

where Δ_q is the q -Laplace operator, $\Delta_q v = \operatorname{div}(|\nabla v|^{q-2} \nabla v)$. In the linear case where $F'(u) = u$, this equation arises as a model for growing sandpiles introduced by Prigozhin [15] and very much studied since, see in particular [2], [11], [5], [6], [10], [9], [12] and the references therein.

We shall address existence of weak solutions to (1.1)-(1.2) by a simple constructive Euler scheme reminiscent of the seminal works of Jordan-Kinderlehrer and Otto [13] and Kinderlehrer and Walkington [14] but with the 1-Wasserstein distance and not the more traditional quadratic one. Thanks to this point of view, we will obtain weak solutions for irregular sources f , namely $f \in L^1((0, T), (C^{0,\alpha_0}(\overline{\Omega}))')$. If the source is in fact L^1 in t and x , then the flow of (1.1)-(1.2) defines a contraction in L^1 which implies uniqueness, stability as well as full convergence of the Euler scheme.

The paper is organized as follows. Section 2 is devoted to some preliminaries, the definition of weak solutions and a summary of our main results. Existence is proven via a variational scheme à la Jordan-Kinderlehrer and Otto [13] and Kinderlehrer and Walkington [14] in section 3. In section 4, an L^1 -contraction result that implies uniqueness and stability for an L^1 source f is proved. Finally, section 5 is devoted to some variants and concluding remarks.

2 The PDE and its weak formulation

2.1 Preliminaries

It is well-known that the constraints (1.2) are very much related to the 1-Wasserstein distance and the notion of Kantorovich potentials. In the following, we assume that Ω is an open bounded convex subset of $\mathbb{R}^d$ and $T > 0$. We denote by Lip_1 the set of 1-Lipschitz functions on Ω , and by $\operatorname{Lip} := W^{1,\infty}(\Omega)$ the set of Lipschitz functions on Ω , given a distribution of order one, $g \in \operatorname{Lip}'$ such that g is balanced i.e.

$$\langle g \rangle := \langle g, 1 \rangle = 0,$$

we denote by W_1 the dual semi norm of g :

$$W_1(g) := \sup_{\theta \in \text{Lip}_1} \langle g, \theta \rangle$$

which, when g is signed measure $g = g^+ - g^-$, with $g^\pm$ probability measures on $\overline{\Omega}$ is the well-know 1-Wasserstein distance between g^+ and g^- (see [17]). Define

$$\text{Lip}'_0 := \{g \in \text{Lip}' : \langle g \rangle = 0\}$$

a $\theta \in \text{Lip}_1$ for which $W_1(g) = \langle g, \theta \rangle$ is called a Kantorovich potential of g and we denote by $K(g)$ the set of Kantorovich potentials of g i.e.:

$$K(g) := \{\theta \in \text{Lip}_1 : \langle g, \theta \rangle \geq \langle g, \psi \rangle, \forall \psi \in \text{Lip}_1\}.$$

For an arbitrary $g \in \text{Lip}'_0$, it may be the case that $K(g)$ is empty, nevertheless, $K(g) \neq \emptyset$ as soon as $g \in \text{Lip}_0 \cap X'$ where X is a space of functions such that the embedding from Lip to X is compact (for instance $X = C^0, C^{0,\alpha}$ with $\alpha \in [0, 1), \dots$).

Using the Fenchel-Rockafellar duality theorem gives the following dual formula for $W_1(g)$:

$$W_1(g) = \inf_{\sigma \in (L^\infty(\Omega, \mathbb{R}^d))'} \left\{ \|\sigma\|_{(L^\infty)'} : -\text{div}(\sigma) = g \right\}. \quad (2.1)$$

where the equilibrium condition $-\text{div}(\sigma) = g$ has to be understood in the weak sense i.e.

$$\langle \sigma, \nabla \theta \rangle_{L^\infty', L^\infty} = \langle g, \theta \rangle_{\text{Lip}', \text{Lip}}, \forall \theta \in W^{1,\infty}(\Omega). \quad (2.2)$$

It actually also follows from the Fenchel-Rockafellar duality theorem that (2.1) admits solutions (in $(L^\infty)'$ and not in L^1 in general) whatever $g \in \text{Lip}'_0$ is, such solutions are called optimal flows. A Kantorovich potential $\theta \in K(g)$ is related to an optimal flow σ in (2.1) by the extremality relation

$$\langle \sigma, \nabla \theta \rangle_{(L^\infty)', L^\infty} = \|\sigma\|_{(L^\infty)'} \text{ with } \|\nabla \theta\|_{L^\infty} \leq 1$$

which, very informally, means that σ is concentrated on the set where $|\nabla \theta|$ equals 1 and is collinear to $\nabla \theta$. If by chance σ is L^1 , the previous relation expresses the fact that $\sigma = a \nabla \theta$ with $a \geq 0$ as well as the complementary slackness condition $a(1 - |\nabla \theta|) = 0$, note also that σ is in some weak sense tangential to $\partial\Omega$ because of (2.2).

For an arbitrary $g \in \text{Lip}'$ not necessarily balanced we define

$$\widetilde{W}_1(g) := W_1(g - \langle g \rangle) + |\langle g \rangle|$$

and observe that $\widetilde{W}_1$ is equivalent to the usual norm of Lip' .

With the previous considerations in mind it is natural to interpret the PDE (1.1) coupled with (1.2) as the inclusion

$$-F'(u) \in K(\partial_t u - f). \quad (2.3)$$

whose implicit in time discretization, given a time-step $\tau > 0$, reads as

$$-F'(u_{k+1}) \in K\left(u_{k+1} - u_k - \int_{k\tau}^{(k+1)\tau} f\right). \quad (2.4)$$

As we shall see later, these conditions appear as the Euler-Lagrange equations for the following Euler implicit scheme à la Jordan-Kinderlehrer-Otto (henceforth JKO) [13] to construct weak solutions but using W_1 instead of the more familiar 2-Wasserstein distance, W_2 (the idea to incorporate the source in an explicit way in the scheme was actually introduced by Kinderlehrer and Walkington [14]). Let $\tau > 0$ be a time step, let us construct inductively a sequence u_k^τ by setting $u_0^\tau = u_0$ and

$$u_{k+1}^\tau \in \operatorname{argmin}_u \left\{ W_1(u - (u_k^\tau + \tau f_k^\tau)) + \int_\Omega F(u) dx : \langle u - u_k^\tau - \tau f_k^\tau \rangle = 0 \right\} \quad (2.5)$$

where (extending f by 0 outside $[0, T]$ if necessary)

$$f_k^\tau := \frac{1}{\tau} \int_{k\tau}^{(k+1)\tau} f(s, \cdot) ds. \quad (2.6)$$

From now on, in addition to the assumption that Ω is convex and bounded, we suppose that there exists $\alpha_0 \in [0, 1)$ such that¹

$$f \in L^1((0, T), (C^{0, \alpha_0}(\overline{\Omega}))') \quad (2.7)$$

which in particular implies that

$$\int_0^T |\langle f(t, \cdot) \rangle| dt + \int_0^T W_1(f(t, \cdot) - \langle f(t, \cdot) \rangle) dt < +\infty. \quad (2.8)$$

and to make things as elementary as possible we take a power nonlinearity for F :

$$F(u) = \frac{1}{m} |u|^m, \text{ with } m > 1 \quad (2.9)$$

and

$$u_0 \in L^m(\Omega). \quad (2.10)$$

¹note that this allows a rough dependence in x , not even a measure for instance $f(t, \cdot) := \sum_n (\delta_{x_n(t)} - \delta_{y_n(t)})$ with $\int_0^T \sum_n |x_n(t) - y_n(t)|^{\alpha_0} dt < +\infty$.

It then follows directly from the fact that W_1 is lsc for the weak L^m topology as well as the strict convexity of F (and the convexity of W_1) that the sequence u_k^τ of the W_1 -JKO scheme (2.5) is uniquely well-defined. We define then two curves corresponding to linear and piecewise constant interpolation:

$$u^\tau(t) := u_k^\tau + \frac{t - k\tau}{\tau}(u_{k+1}^\tau - u_k^\tau), \quad \tilde{u}^\tau(t) := u_{k+1}^\tau, \quad t \in (k\tau, (k+1)\tau], \quad (2.11)$$

for $k = 0, \dots, [\tau^{-1}T]$. We also define the piecewise constant approximation of the source f :

$$f^\tau(t) := f_k^\tau, \quad t \in [k\tau, (k+1)\tau), \quad k = 0, \dots, [\tau^{-1}T]. \quad (2.12)$$

Note that by construction

$$\langle u_{k+1}^\tau \rangle = \langle u_k^\tau \rangle + \int_{k\tau}^{(k+1)\tau} \langle f(t, \cdot) \rangle dt$$

so that with (2.8) and (2.10), we have

$$\langle \tilde{u}^\tau(t, \cdot) \rangle, \langle u^\tau(t, \cdot) \rangle \in L^\infty((0, T)). \quad (2.13)$$

2.2 Weak solutions

The notion of weak solution of (1.1)-(1.2) we consider heavily relies on (2.3) and the following (slightly formal) observations. Recall that (2.3) means that $F'(u) \in \text{Lip}_1$ and for every $\xi \in \text{Lip}_1$, one has

$$\langle \partial_t u(t), F'(u(t)) - \xi \rangle \leq \langle f(t), F'(u(t)) - \xi \rangle.$$

Note that giving a pointwise in time sense to this condition would require that $\partial_t u \in L^1((0, T), \text{Lip}')$, which will not be guaranteed by the rather weak assumption (2.7).

Defining for every $k > 0$, the truncation map $T_k: \mathbb{R} \rightarrow \mathbb{R}$ by

$$T_k(u) := \begin{cases} k & \text{if } u \geq k \\ u & \text{if } u \in [-k, k] \\ -k & \text{if } u \leq -k. \end{cases}$$

and observing that for any $\theta \in \text{Lip}_1$, $\xi := F'(u) - T_k(F'(u) - \theta)$ belongs to Lip_1 too, we get

$$\begin{aligned} \langle \partial_t u, T_k(F'(u) - \theta) \rangle &= \frac{d}{dt} \int_{\Omega} \int_0^{u(t)} T_k(F'(s) - \theta) ds \, dx \\ &\leq \langle f, T_k(F'(u) - \theta) \rangle. \end{aligned}$$

These considerations lead to the following definition of weak solutions:

Definition 2.1. A weak solution of (1.1)-(1.2) is an $u \in L^\infty((0, T), C^0(\Omega))$ such that $F'(u) \in \text{Lip}_1$ for a.e. $t \in [0, T]$, and for every $\theta \in \text{Lip}_1$ and every $k > 0$,

$$\frac{d}{dt} \int_{\Omega} \int_0^{u(t)} T_k(F'(s) - \theta) ds \, dx \leq \langle f, T_k(F'(u) - \theta) \rangle \quad (2.14)$$

in the sense of distributions. In other words, $u \in L^\infty((0, T), C^0(\Omega))$ such that $F'(u) \in \text{Lip}_1$ for a.e. $t \in [0, T]$ is a weak solution of (1.1)-(1.2) if for every $\theta \in \text{Lip}_1$ and every $\phi \in C_c^1([0, T], \mathbb{R}_+)$ one has

$$\begin{aligned} - \int_0^T \int_{\Omega} \dot{\phi}(t) \int_0^{u(t)} T_k(F'(s) - \theta) ds \, dx \, dt &\leq \phi(0) \int_{\Omega} \int_0^{u_0} T_k(F'(s) - \theta) ds \, dx \\ &+ \int_0^T \phi(t) \langle f(t, \cdot), T_k(F'(u(t, \cdot)) - \theta) \rangle \, dt. \end{aligned}$$

Note that if u is a weak solution (letting $k \rightarrow \infty$), we have for every $\theta \in \text{Lip}_1$

$$\frac{d}{dt} \int_{\Omega} F(u(t, x)) \, dx \leq \frac{d}{dt} \langle \theta, u \rangle + \langle f, F'(u) - \theta \rangle$$

in the sense of distributions, which implies in particular that $t \mapsto \int_{\Omega} F(u(t, x)) \, dx$ as well as $t \mapsto \langle u(t, \cdot), \theta \rangle$ (with $\theta \in \text{Lip}$) are BV functions (but not necessarily absolutely continuous).

Remark 2.2. One can see here that the notion of solution we are using in Definition 2.1 is weaker than the standard one which consists in requiring that $u \in L^\infty((0, T), C^0(\Omega))$, with $\partial_t u \in L^1(0, T; \text{Lip}')$ and $F'(u(t)) \in \text{Lip}_1$ for a.e. $t \in [0, T]$, $u(0) = u_0$ and

$$-F'(u(t)) \in K(\partial_t u(t) - f(t)), \quad \text{for a.e. } t \in (0, T). \quad (2.15)$$

Indeed, by using the same arguments of Lemma 4 of [7], one can prove that if u is such a solution it is also a solution in the sense of Definition 2.1. Indeed, if $\partial_t u \in L^1(0, T; \text{Lip}')$ one can prove rigorously that (2.15) yields

$$\begin{aligned} \int_0^T \langle \partial_t u(t), T_k(F'(u(t)) - \theta) \rangle \phi(t) \, dt &= - \int_0^T \int_{\Omega} \dot{\phi}(t) \int_0^{u(t)} T_k(F'(s) - \theta) ds \, dx \, dt \\ &- \phi(0) \int_{\Omega} \int_0^{u_0} T_k(F'(s) - \theta) ds \, dx, \end{aligned}$$

for any $\theta \in \text{Lip}_1$ and $\phi \in C_c^1([0, T], \mathbb{R})$.

2.3 Main results

Our main results concerning the existence and uniqueness of weak solutions can be summarized as follows. First, existence will be obtained (proof will be detailed in section 3) by convergence of the JKO-scheme:

Theorem 2.3. *Assume (2.7)-(2.9)-(2.10) and define u^τ by (2.5)-(2.11), then there exists $u \in L^\infty((0, T), C^0(\Omega))$ such that $F'(u) \in \text{Lip}_1$ for a.e. $t \in [0, T]$, and a vanishing family of stepsizes $\tau_n \rightarrow 0$ such that u^{τ_n} converges strongly to u in $L^p((0, T), C^0(\overline{\Omega}))$ for every $p \in [1, \infty)$ and u is weak solution of (1.1)-(1.2).*

Uniqueness will be guaranteed by the following L^1 -contraction result (see section 4) which requires an L^1 assumption on the source:

Theorem 2.4. *Assume $f \in L^1((0, T), L^1(\Omega))$, (2.9), $u_0 \in L^m(\Omega)$, $v_0 \in L^m(\Omega)$ and let u and v be weak solutions associated respectively to the initial conditions u_0 and v_0 respectively, then*

$$\|u(t, \cdot) - v(t, \cdot)\|_{L^1(\Omega)} \leq \|u_0 - v_0\|_{L^1(\Omega)}, \text{ a.e. } t \in [0, T].$$

In particular (1.1)-(1.2) has a unique weak solution.

Combining theorems 2.3 and 2.4 and the a priori estimates for the JKO scheme of section 3, we deduce:

Corollary 2.5. *Assume that $f \in L^1((0, T), L^1(\Omega))$, (2.9) and (2.10) then the whole family u^τ defined by (2.5)-(2.11) converges in $L^p((0, T), C^0(\overline{\Omega}))$ for every $p \in [1, \infty)$ to the unique solution of (1.1)-(1.2).*

3 Convergence of the JKO-like W_1 -scheme

3.1 Euler-Lagrange equation for the discrete scheme

The fact that the Euler-Lagrange equation of the variational problem in (2.5) is very much linked to an implicit time discretization of (2.3) follows from:

Lemma 3.1. *Let $\alpha_0 \in [0, 1)$ and $v \in (C^{0, \alpha_0}(\overline{\Omega}))'$, if $u \in L^m$ solves*

$$\inf \left\{ W_1(u - v) + \int_{\Omega} F(u) : u \in L^m, \langle u - v \rangle = 0 \right\} \quad (3.1)$$

then

$$-F'(u) \in K(u - v). \quad (3.2)$$

Proof. We proceed by duality. Consider the convex minimization problem

$$\inf_{z \in \text{Lip}_1} \left\{ \int_{\Omega} F^*(-z) + \langle v, z \rangle \right\} \quad (3.3)$$

It is easy to see that it admits a (unique by strict convexity of F^*) solution, indeed if z_n is a minimizing sequence, it possesses a subsequence that converges strongly in $C^{0,\alpha_0}(\overline{\Omega})$ (and thus also in $L^{m'}$ with $m' = \frac{m}{m-1}$ the conjugate exponent of m) to some z which obviously solves (3.3) since $v \in (C^{0,\alpha_0}(\overline{\Omega}))'$. By Fenchel-Rockafellar Theorem, we also deduce that (3.3) (written as a convex minimization problem on $L^{m'}$ observing that Lip_1 is closed in $L^{m'}$) is dual to (3.1). Moreover, the solution $z \in \text{Lip}_1$ of (3.3) is related to the solution u of (3.1) by the extremality relation

$$\begin{aligned} 0 &= \int_{\Omega} F(u) + W_1(u - v) + \int_{\Omega} F^*(-z) + \langle v, z \rangle \\ &= \int_{\Omega} F(u) + \int_{\Omega} F^*(-z) + \langle u, z \rangle + W_1(u - v) - \langle u - v, z \rangle \end{aligned}$$

since both terms are nonnegative this gives $z = -F'(u)$ and $z \in K(u - v)$. □

Proposition 3.2. *If for $i = 1, 2$, $v_i \in (C^{0,\alpha_0}(\overline{\Omega}))'$ and $v_1 - v_2 \in L^1(\Omega)$, then the corresponding solutions u_i to (3.1) satisfy*

$$\int_{\Omega} |u_1 - u_2| \leq \int_{\Omega} |v_1 - v_2|. \quad (3.4)$$

Proof. It follows from Lemma 3.1, that for $i = 1, 2$, we have

$$\langle v_i - u_i, F'(u_i) - \xi \rangle \geq 0, \quad \text{for any } \xi \in \text{Lip}_1,$$

and since $F'(u_1) - T_k(F'(u_1) - F'(u_2)) \in \text{Lip}_1$, we have

$$\langle v_1 - u_1, T_k(F'(u_1) - F'(u_2)) \rangle \geq 0.$$

Similarly,

$$\langle v_2 - u_2, T_k(F'(u_2) - F'(u_1)) \rangle \geq 0.$$

hence

$$\langle u_1 - u_2, T_k(F'(u_2) - F'(u_1)) \rangle \leq \langle v_1 - v_2, T_k(F'(u_2) - F'(u_1)) \rangle.$$

Dividing both terms by k , using the fact that $v_1 - v_2 \in L^1$ and that $\frac{1}{k}|T_k(F'(u_1) - F'(u_2))| \leq 1$, we get

$$\frac{1}{k} \int_{\Omega} (u_1 - u_2) T_k(F'(u_1) - F'(u_2)) \leq \int_{\Omega} |v_1 - v_2|.$$

At last, letting $k \rightarrow 0$ and using Lebesgue's dominated convergence theorem and the fact that F' is increasing, we obtain (3.4). $\square$

As a consequence, we deduce that the discrete JKO scheme contracts the L^1 distance. Let us indeed consider the same JKO construction (2.5) as before but for two different initial conditions u_0 and v_0 , we denote by u_k^τ and v_k^τ the corresponding discrete in time sequences. We then have:

Corollary 3.3. *The discrete JKO scheme given by (2.5) satisfies the discrete flow equation (2.4) and contracts the L^1 distance (whatever the time step $\tau > 0$ is). In other words, the sequences u_k^τ and v_k^τ constructed by the scheme (2.5) corresponding to the initial conditions u_0 and v_0 satisfy*

$$\|u_{k+1}^\tau - v_{k+1}^\tau\|_{L^1} \leq \|u_k^\tau - v_k^\tau\|_{L^1}, \quad (3.5)$$

for any $k = 0, 1, \dots, [\tau^{-1}T] - 1$.

Now, in order to pass to the limit in (2.4) for the discrete JKO scheme, as $\tau \rightarrow 0$, we give in this paragraph the main a priori estimates on u^τ and $\tilde{u}^\tau$. Lemma 3.1 first gives the estimate

$$\|\nabla[F'(\tilde{u}^\tau)]\|_{L^\infty((0,T) \times \Omega)} \leq 1. \quad (3.6)$$

and since $[F']^{-1}$ is $C^{0,\beta}$ for the exponent $\beta := \min(1, \frac{1}{m-1})$, thanks to (2.13), we in fact have the Hölder bound:

$$\|\tilde{u}^\tau\|_{L^\infty((0,T), C^{0,\beta}(\Omega))} + \|u^\tau\|_{L^\infty((0,T), C^{0,\beta}(\Omega))} \leq C. \quad (3.7)$$

Using $u_k^\tau + \tau \langle f_k^\tau \rangle$ as a competitor to u_{k+1}^τ in (2.5), we first have:

$$\begin{aligned} W_1(u_{k+1}^\tau - (u_k^\tau + \tau f_k^\tau)) + \int_{\Omega} F(u_{k+1}^\tau) &\leq \int_{k\tau}^{(k+1)\tau} W_1(f(t, \cdot) - \langle f(t, \cdot) \rangle) dt \\ &+ \int_{\Omega} F(u_k^\tau + \tau \langle f_k^\tau \rangle). \end{aligned} \quad (3.8)$$

Thanks to (3.7) u_k^τ is bounded and thanks to (2.8)

$$\tau |\langle f_k^\tau \rangle| \leq \|\langle f \rangle\|_{L^1(k\tau, (k+1)\tau)} \leq \omega(\tau) \text{ with } \omega(\tau) \rightarrow 0 \text{ as } \tau \rightarrow 0. \quad (3.9)$$

The mean-value theorem therefore enables to write

$$F(u_k^\tau + \tau \langle f_k^\tau \rangle) \leq F(u_k^\tau) + C\tau |\langle f_k^\tau \rangle|,$$

which, together with (3.8), yields

$$\begin{aligned} W_1(u_{k+1}^\tau - (u_k^\tau + \tau f_k^\tau)) &\leq \int_{k\tau}^{(k+1)\tau} W_1(f(t, \cdot) - \langle f(t, \cdot) \rangle) dt \\ &\quad + \int_{\Omega} F(u_k^\tau) - \int_{\Omega} F(u_{k+1}^\tau) + C\tau |\langle f_k^\tau \rangle|. \end{aligned} \quad (3.10)$$

Now, the right-hand side of (3.10) contains a telescopic sum and terms on which we have L^1 bounds thanks to (2.8). Hence, since $F \geq 0$ and $u_0 \in L^m$ we get

$$\sum_{k=0}^{[\tau^{-1}T]} W_1(u_{k+1}^\tau - u_k^\tau - \tau f_k^\tau) \leq C. \quad (3.11)$$

Next we observe that

$$\begin{aligned} \|\partial_t u^\tau\|_{L^1((0,T),(\text{Lip}', \widetilde{W}_1))} &= \sum_{k=0}^{[\tau^{-1}T]} \widetilde{W}_1(u_{k+1}^\tau - u_k^\tau) \\ &\leq \int_0^T |\langle f(t, \cdot) \rangle| dt + \sum_{k=0}^{[\tau^{-1}T]} W_1(u_{k+1}^\tau - u_k^\tau - \tau \langle f_k^\tau \rangle) \\ &\leq C + \sum_{k=0}^{[\tau^{-1}T]} (W_1(u_{k+1}^\tau - u_k^\tau - \tau f_k^\tau) + \tau W_1(f_k^\tau - \langle f_k^\tau \rangle)) \end{aligned}$$

together with (3.11) and (2.8) we deduce

$$\|\partial_t u^\tau\|_{L^1((0,T),(\text{Lip}', \widetilde{W}_1))} \leq C. \quad (3.12)$$

The Euler-Lagrange equation of (2.5) from Lemma 3.1 reads

$$-F'(u_{k+1}^\tau) \in K(u_{k+1}^\tau - u_k^\tau - \tau f_k^\tau) = K\left(\frac{u_{k+1}^\tau - u_k^\tau}{\tau} - f_k^\tau\right) \quad (3.13)$$

Note that by the very construction of the interpolations u^τ and $\widetilde{u}^\tau$, (3.13) can be rewritten as for a.e. $t \in [0, T]$

$$-F'(\widetilde{u}^\tau(t)) \in K(\partial_t u^\tau(t) - f^\tau(t)). \quad (3.14)$$

i.e. $F'(\widetilde{u}^\tau) \in \text{Lip}_1$ and for every $\xi \in \text{Lip}_1$ and for a.e. time one has

$$\langle \partial_t u^\tau, F'(\widetilde{u}^\tau) - \xi \rangle \leq \langle f^\tau, F'(\widetilde{u}^\tau) - \xi \rangle. \quad (3.15)$$

As already observed, given $k > 0$ and $\theta \in \text{Lip}_1$, $\xi := F'(\tilde{u}^\tau) - T_k(F'(\tilde{u}^\tau) - \theta)$ belongs to Lip_1 , hence we have

$$\langle \partial_t u^\tau, T_k(F'(\tilde{u}^\tau) - \theta) \rangle \leq \langle f^\tau, T_k(F'(\tilde{u}^\tau) - \theta) \rangle. \quad (3.16)$$

Now we observe that if $t \in (k\tau, (k+1)\tau)$ by the strict convexity of F and the fact that T_k is nondecreasing, $T_k(F'(\tilde{u}^\tau(t)) - \theta) - T_k(F'(u^\tau(t)) - \theta)$ has the sign of $(\tilde{u}^\tau(t) - u^\tau(t))$ i.e. that of $\partial_t u^\tau(t)$, hence

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \int_0^{u^\tau} T_k(F'(s) - \theta) ds &= \langle \partial_t u^\tau, T_k(F'(u^\tau) - \theta) \rangle \\ &\leq \langle \partial_t u^\tau, T_k(F'(\tilde{u}^\tau) - \theta) \rangle. \end{aligned}$$

With (3.16), this yields

$$\frac{d}{dt} \int_{\Omega} \int_0^{u^\tau} T_k(F'(s) - \theta) ds \leq \langle f^\tau, T_k(F'(\tilde{u}^\tau) - \theta) \rangle. \quad (3.17)$$

Our aim now is of course to pass to the limit $\tau \rightarrow 0$ in (3.17). We first have:

Proposition 3.4. *There exist a vanishing sequence of time steps $\tau_n \rightarrow 0$ as $n \rightarrow \infty$ and $u \in L^\infty((0, T), C^0(\overline{\Omega}))$, such that setting $u_n := u^{\tau_n}$ and $\tilde{u}_n := \tilde{u}^{\tau_n}$ one has:*

$$u_n \rightarrow u \text{ in } L^p((0, T), C^0(\overline{\Omega})), \forall p \in [1, +\infty) \quad (3.18)$$

$$\tilde{u}_n \rightarrow u \text{ in } L^1((0, T), C^0(\overline{\Omega})) \quad (3.19)$$

$$F'(\tilde{u}_n) \rightarrow F'(u) \text{ in } L^p((0, T), C^{0,\alpha}(\overline{\Omega})), \forall p \in [1, +\infty), \forall \alpha \in [0, 1) \quad (3.20)$$

$$F'(u(t, \cdot)) \in \text{Lip}_1 \text{ for a.e. } t \quad (3.21)$$

Proof. Thanks to (3.7), u^τ is bounded in $L^1((0, T), C^{0,\beta})$ and $\partial_t u^\tau$ is bounded in $L^1((0, T), \text{Lip}')$, since the embedding $C^{0,\beta}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ is compact and the embedding $C^0(\overline{\Omega}) \hookrightarrow \text{Lip}(\Omega)'$ is continuous (it is actually compact as well...), it follows from the Aubin-Lions-Simon Theorem (see [3], [16]) that $\{u^\tau\}_\tau$ has a cluster point u in $L^1((0, T), C^0(\overline{\Omega}))$. For a suitable vanishing sequence of stepsizes we may thus assume that the corresponding sequence u_n converges to u in $L^1((0, T), C^0(\overline{\Omega}))$ but also (up to a further extraction) that $u_n(t, \cdot)$ converges to $u(t, \cdot)$ in $C^0(\overline{\Omega})$ for a.e. t . Thanks to the uniform bound (3.7) with Lebesgue's dominated convergence theorem, we deduce (3.18).

As for $\tilde{u}^\tau$, in addition to (3.7) and (3.12), we observe that

$$\begin{aligned} \|\tilde{u}^\tau - u^\tau\|_{L^1((0, T), (\text{Lip}', \widetilde{W}_1))} &= \sum_{k=0}^{[\tau^{-1}T]} \int_{k\tau}^{(k+1)\tau} \widetilde{W}_1(\tilde{u}^\tau(t) - u^\tau(t)) dt \\ &= \frac{\tau}{2} \sum_{k=0}^{[\tau^{-1}T]} \widetilde{W}_1(u_{k+1}^\tau - u_k^\tau) \leq C\tau \end{aligned}$$

where the last rightmost inequality follows from (3.12). Since u_n obviously converges to u in $L^1((0, T), \text{Lip}')$, we deduce from the latter inequality that $\tilde{u}^n$ is relatively compact in $L^1((0, T), \text{Lip}')$, together with (3.7), the fact that the embedding $C^{0,\beta} \hookrightarrow C^0$ is compact, that the embedding $C^0 \hookrightarrow \text{Lip}'$ is continuous and Lemma 9 in Simon [16], we can conclude that up to further extractions, $\tilde{u}_n$ converges to u in $L^1((0, T), C^0(\overline{\Omega}))$. Again, we may also assume as well that $\tilde{u}_n(t, \cdot)$ converges to $u(t, \cdot)$ in $C^0(\overline{\Omega})$ for a.e. t . This implies that $F'(\tilde{u}_n(t, \cdot))$ converges to $F'(u(t, \cdot))$ in $C^0(\overline{\Omega})$ for a.e. t which in particular implies (3.21). Thanks to (3.7), for every $\alpha \in [0, 1)$, $F'(\tilde{u}_n(t, \cdot))$ is relatively compact in $C^{0,\alpha}$ and thus converges to $F'(u(t, \cdot))$ in $C^{0,\alpha}$ for a.e. t , the $L^p((0, T), C^{0,\alpha}(\overline{\Omega}))$ convergence in (3.20) thus simply follows again from the uniform bound (3.7) and Lebesgue's dominated convergence theorem. $\square$

3.2 Proof of theorem 2.3

We are now ready to prove our main result which in particular implies existence of weak solution of (1.1)-(1.2) via convergence of the JKO scheme (2.5), namely

Proposition 3.5. *The limit function u from Proposition 3.4 is a weak solution of (1.1)-(1.2).*

Proof. Let $\theta \in \text{Lip}_1$ and $\phi \in C_c^1([0, T], \mathbb{R}_+)$, multiplying (3.17) by ϕ and integrating by parts in time (observing that $\int_{\Omega} \int_0^{u^\tau} T_k(F'(s) - \theta) ds$ is absolutely continuous) we first have:

$$\begin{aligned} - \int_0^T \int_{\Omega} \dot{\phi}(t) \int_0^{u^\tau} T_k(F'(s) - \theta) ds &\leq \phi(0) \int_0^{u_0} T_k(F'(s) - \theta) ds \\ &\quad + \int_0^T \phi(t) \langle f^\tau(t, \cdot), T_k(F'(\tilde{u}^\tau(t, \cdot)) - \theta) \rangle dt. \end{aligned}$$

For the last term in this inequality, we remark that it can be rewritten as

$$\begin{aligned} \int_0^T \phi(t) \langle f^\tau(t, \cdot), T_k(F'(\tilde{u}^\tau(t, \cdot)) - \theta) \rangle dt &= \langle f, \phi^\tau T_k(F'(\tilde{u}^\tau) - \theta) \rangle \\ &= \int_0^T \phi^\tau(t) \langle f(t, \cdot), T_k(F'(\tilde{u}^\tau(t, \cdot)) - \theta) \rangle dt \end{aligned}$$

with

$$\phi^\tau(t) := \frac{1}{\tau} \int_{k\tau}^{(k+1)\tau} \phi(s) ds, \text{ for } t \in [k\tau, (k+1)\tau).$$

It follows from proposition 3.4 that

$$T_k(F'(\tilde{u}_n) - \theta) \rightarrow T_k(F'(u) - \theta) \text{ in } C^{0,\alpha_0}(\overline{\Omega}) \text{ for a.e. } t.$$

Remarking then that $\|\phi^\tau - \phi\|_{L^\infty} \rightarrow 0$ as $\tau \rightarrow 0$, thanks to (2.7), and Lebesgue's dominated convergence theorem, we get

$$\lim_n \int_0^T \phi(t) \langle f^{\tau_n}(t, \cdot), F'(\tilde{u}_n(t, \cdot) - \theta) \rangle dt = \lim_n \int_0^T \phi(t) \langle f(t, \cdot), F'(u(t, \cdot)) - \theta \rangle dt$$

Taking $\tau = \tau_n$, letting $n \rightarrow \infty$ and using proposition 3.4, we thus easily deduce that u is a weak solution of (1.1)-(1.2). $\square$

4 L^1 -contraction, uniqueness and stability

Theorem 4.1. *Assume that $f \in L^1((0, T), L^1(\Omega))$. Let u_1 and u_2 be two weak solutions of (1.1)-(1.2), then*

$$\frac{d}{dt} \int_{\Omega} |u_1(t, x) - u_2(t, x)| dx \leq 0 \quad \text{in } \mathcal{D}'(0, T). \quad (4.1)$$

In particular, for any $u_0 \in L^1$ such that $F'(u_0) \in K$, and $f \in L^1((0, T), L^1(\Omega))$, the problem (1.1)-(1.2) has most one weak solution such that $u(0) = u_0$.

Proof. Let u_1 and u_2 be two weak solutions. Dividing (2.14) by k and letting k tend to 0^+ , for $i = 1, 2$, thanks to the fact that $f \in L^1((0, T), L^1(\Omega))$ and Lebesgue's dominated convergence theorem, we have², for any $\theta \in \text{Lip}_1$

$$\frac{d}{dt} \int_{\Omega} \int_0^{u_i(t, x)} \text{sign}(F'(r) - \theta(x)) dr dx \leq \langle f, \text{sign}(F'(u_i(t)) - \theta) \rangle, \quad \text{in } \mathcal{D}'(0, T).$$

In particular, if θ is of the form $\theta = F'(\eta)$, we get

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \int_0^{u_i(t, x)} \text{sign}(r - \eta(x)) dr dx &= \frac{d}{dt} \int_{\Omega} \int_{\eta(x)}^{u_i(t, x)} \text{sign}(r - \eta(x)) dr dx \\ &\leq \langle f, \text{sign}(F'(u_i(t)) - F'(\eta)) \rangle, \quad \text{in } \mathcal{D}'(0, T). \end{aligned}$$

Thus

$$\frac{d}{dt} \int_{\Omega} |u_i(t, x) - \eta(x)| dx \leq \langle f(t), \text{sign}(u_i(t) - \eta) \rangle, \quad \text{in } \mathcal{D}'(0, T). \quad (4.2)$$

²Here and in the sequel sg refers to the selection of the sign function which takes value 0 at 0.

Now, we use doubling and dedoubling variables techniques to prove the L^1 contraction principle. Let $\sigma \in \mathcal{D}((0, T)^2)$, $\sigma \geq 0$, if we take for $i = 1$, $\eta(x) = F'(u_2(s, x))$, (s fixed for the moment) in the inequality above, we get

$$\begin{aligned} & - \int_0^T \int_{\Omega} \partial_t \sigma(t, s) |u_1(t, x) - u_2(s, x)| dx dt \\ & \leq \int_0^T \int_{\Omega} f(t, x) \text{sign}(u_1(t, x) - u_2(s, x)) \sigma(t, s) dx dt \end{aligned}$$

In a similar way, using $\eta(x) = F'(u_1(t, x))$ (with t fixed) for u_2 , we have

$$\begin{aligned} & - \int_0^T \int_{\Omega} \partial_s \sigma(t, s) |u_1(t, x) - u_2(s, x)| dx ds \\ & \leq \int_0^T \int_{\Omega} f(s, x) \text{sign}(u_1(t, x) - u_2(s, x)) \sigma(t, s) dx ds \end{aligned}$$

Integrating the preceding equations with respect to s and t , respectively, and adding the resulting inequalities, we thus get

$$\begin{aligned} & - \int_0^T \int_0^T \int_{\Omega} |u_1(t, x) - u_2(s, x)| (\partial_t \sigma(t, s) + \partial_s \sigma(t, s)) dx dt ds \\ & \leq \int_0^T \int_0^T \int_{\Omega} |f(t, x) - f(s, x)| \sigma(t, s) dx dt ds. \end{aligned}$$

Now, to dedouble variables, given a small $\varepsilon > 0$ we take $\sigma = \sigma_{\varepsilon}$ of the form

$$\sigma_{\varepsilon}(t, s) = \frac{\phi(t)}{\varepsilon} \rho\left(\frac{s-t}{\varepsilon}\right)$$

with $\phi \in \mathcal{D}(0, T)$, $\phi \geq 0$ and $\rho \geq 0$, $\rho \in \mathcal{D}(-1, 1)$, even, and such that $\int_{\mathbb{R}} \rho = 1$. We then have

$$\partial_t \sigma_{\varepsilon}(t, s) + \partial_s \sigma_{\varepsilon}(t, s) = \frac{\dot{\phi}(t)}{\varepsilon} \rho\left(\frac{t-s}{\varepsilon}\right).$$

Performing the change of variables $(t, s) \mapsto (t, \frac{s-t}{\varepsilon})$, we thus arrive at

$$-I_{\varepsilon} \leq J_{\varepsilon} \tag{4.3}$$

where

$$I_{\varepsilon} := \int_{-1}^1 \int_0^T \int_{\Omega} |u_1(t, x) - u_2(t + \varepsilon \tau, x)| \rho(\tau) \dot{\phi}(t) dx dt d\tau$$

and

$$J_\varepsilon := \int_{-1}^1 \int_0^T \int_\Omega |f(t, x) - f(t + \varepsilon\tau, x)| \rho(\tau) \phi(t) dx dt d\tau.$$

Now using the fact that if $g \in L^1((0, T) \times \Omega)$, then for a and b such that $0 < a < b < T$, $\|g(\cdot + h, \cdot) - g(\cdot, \cdot)\|_{L^1((a, b) \times \Omega)}$ tends to 0 uniformly as $|h| \rightarrow 0$ (see for instance Chapter 4 of [4] for details), we easily obtain

$$\lim_{\varepsilon \rightarrow 0^+} I_\varepsilon = \int_0^T \int_\Omega |u_1(t, x) - u_2(t, x)| \dot{\phi}(t) dx dt$$

and, since $f \in L^1((0, T) \times \Omega)$, we similarly have

$$\lim_{\varepsilon \rightarrow 0^+} J_\varepsilon = 0.$$

Letting ε tend to 0^+ in (4.3), we deduce the desired contraction result (4.1). $\square$

Remark 4.2. Exactly the same proof as above, gives the following stability result for weak solutions u_1 and u_2 associated to different $(L^1((0, T) \times \Omega))$ sources, respectively f_1 and f_2 :

$$\frac{d}{dt} \|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^1(\Omega)} \leq \|f_1(t, \cdot) - f_2(t, \cdot)\|_{L^1(\Omega)} \quad (4.4)$$

and in particular

$$\|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^1(\Omega)} \leq \|u_1(0, \cdot) - u_2(0, \cdot)\|_{L^1(\Omega)} + \int_0^t \|f_1(s, \cdot) - f_2(s, \cdot)\|_{L^1(\Omega)} ds. \quad (4.5)$$

5 Variants and concluding remarks

We have proposed an elementary Euler scheme à la JKO with W_1 to deal with nonlinear evolution equations of the form (1.1)-(1.2) and addressed stability and uniqueness issues thanks to an L^1 -contraction argument. We presented the L^1 -contraction directly at the level of the PDE, but another approach, leading to the same conclusion, would have been to consider contraction at the discretized in time level (in a similar way as the estimate of Proposition 3.2) and conclude by the classical semi-group theory in Banach spaces of Crandall and Liggett [8]. Indeed, thanks to Proposition 3.2, it is possible to handle the evolution problem (1.1)-(1.2) by using the classical semi-group theory in the Banach space $L^1(\Omega)$, whenever the source term is L^1 in space. In particular, one sees that (4.2) is closely connected to the notion of integral solution in the sense of non linear semi-group theory in $L^1(\Omega)$. If the source term is regular enough, we believe that it is possible to prove the existence of a weak solution in the standard sense (cf. remark 2.2). Thanks to

remark 2.2, this solution coincides with ours and the uniqueness holds true if the source term remains in L^1 .

Let us stress the fact that the W_1 -JKO scheme is constructive. We indeed believe that since the scheme consists in a sequence of relatively simple convex minimization problems, it is well suited for numerical purposes but we leave this aspect for future research.

An easy extension of the W_1 -JKO approach concerns the case of a reaction term in the right-hand side i.e.

$$\partial_t u - \operatorname{div}(a \nabla F'(u)) = f(u), \quad (t, x) \in (0, T) \times \Omega, \quad u|_{t=0} = u_0, \quad (5.1)$$

again constrained by (1.2). If the reaction term $f(u)$ is sublinear, i.e. there exists a positive constant C such that

$$|f(u)| \leq C(1 + |u|), \quad \forall u \in \mathbb{R}, \quad (5.2)$$

and if we modify (2.5) in a straightforward way by

$$u_{k+1}^\tau \in \operatorname{argmin}_u \left\{ W_1(u - (u_k^\tau + \tau f(u_k^\tau))) + \int_\Omega F(u) dx : \langle u - u_k^\tau - \tau f(u_k^\tau) \rangle = 0 \right\}. \quad (5.3)$$

then, thanks to (5.2), one can obtain similar estimates as in section 3 to deduce convergence of the scheme (5.3) as $\tau \rightarrow 0^+$ to a solution of (5.1)-(1.2). If, in addition f is Lipschitz, then it follows directly from (4.5) and Gronwall's Lemma that we also have uniqueness and stability in L^1 . To make things simple we have considered a power convex nonlinearity for F , but this is not really essential, what is important is that F' is an homeomorphism. An interesting limit case, out of the scope of the present analysis, is when F' is a general monotone graph, possibly set-valued or empty-valued such as in the compression molding model of Aronsson and Evans [1].

We have also have left unanswered two questions that seem natural to us. The first one is what happens if the source term f is only $L^1((0, T), \operatorname{Lip}')$: can one expect convergence of the JKO scheme, and more generally, does there exist a weak solution to (1.1)-(1.2) in this case? The second one is the uniqueness of weak solutions to the Cauchy problem when the source is not $L^1((0, T) \times \Omega)$ but only $L^1((0, T), \operatorname{Lip}')'$ or $L^1((0, T), (C^{0, \alpha_0}(\overline{\Omega}))')$, we actually suspect that uniqueness is false in such irregular cases but have not found any counterexample.

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