Reassigning and synchrosqueezing the Stockwell Transform: Complementary proofs

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1 Relationship between the squared modulus of the S-transform and the Wigner-Ville distribution

$$|\mathrm{ST}_x(t,\omega)|^2 = \mathrm{ST}_x(t,\omega)\cdot\mathrm{ST}_x(t,\omega)^* \tag{1}$$
$$= \frac{\omega^2}{2\pi(\omega_0 T)^2}\iint_{\mathbb{R}^2} x(\tau')x(\tau'')^*\,\mathbf{e}^{-\frac{\omega^2(t-\tau')^2}{2(\omega_0 T)^2}}\,\mathbf{e}^{-\frac{\omega^2(t-\tau'')^2}{2(\omega_0 T)^2}}\,\mathbf{e}^{-j\omega(\tau'-\tau'')}\,d\tau'\,d\tau'' \tag{2}$$

As the Wigner-Ville distribution can be seen as the Fourier transform of the instantaneous autocorrelation function of a signal x

$$\mathrm{WV}_x(t,\omega) = \int_{\mathbb{R}} x(t+\frac{\tau}{2})x(t-\frac{\tau}{2})^*\,\mathbf{e}^{-j\omega\tau}\,d\tau. \tag{3}$$

Hence, using the inverse Fourier transform we obtain

$$x(t+\frac{\tau}{2})x(t-\frac{\tau}{2})^* = \int_{\mathbb{R}} \mathrm{WV}_x(t,\Omega)\,\mathbf{e}^{j\Omega\tau}\,\frac{d\Omega}{2\pi}. \tag{4}$$

By replacing $\tau' = t+\frac{\tau}{2}$ and $\tau'' = t-\frac{\tau}{2}$ we obtain $t = \frac{\tau'+\tau''}{2}$ and $\tau = \tau'-\tau''$ thus we have

$$x(\tau')x(\tau'')^* = \int_{\mathbb{R}} \mathrm{WV}_x\left(\frac{\tau'+\tau''}{2},\Omega\right)\,\mathbf{e}^{j\Omega(\tau'-\tau'')}\,\frac{d\Omega}{2\pi}. \tag{5}$$

Similarly with $h(\lambda) = \frac{1}{\sqrt{2\pi}T}\,\mathbf{e}^{-\frac{\lambda^2}{2T^2}}$ we have

$$\mathrm{WV}_h(\lambda,\omega) = \frac{1}{2\pi T^2}\int_{\mathbb{R}} \mathbf{e}^{-\frac{(\lambda+\frac{\tau}{2})^2}{2T^2}}\,\mathbf{e}^{-\frac{(\lambda-\frac{\tau}{2})^2}{2T^2}}\,\mathbf{e}^{-j\omega\tau}\,d\tau \tag{6}$$

($\mathrm{WV}_h(\lambda,\omega)$ is further expressed in Section 1.1) and

$$\mathbf{e}^{-\frac{(\lambda+\frac{\tau}{2})^2}{2T^2}}\,\mathbf{e}^{-\frac{(\lambda-\frac{\tau}{2})^2}{2T^2}} = 2\pi T^2\int_{\mathbb{R}} \mathrm{WV}_h(\lambda,\Omega')\,\mathbf{e}^{j\Omega'\tau}\,\frac{d\Omega'}{2\pi}. \tag{7}$$

If we define

$$(\lambda+\frac{\tau}{2})^2 = \frac{\omega^2}{\omega_0^2}(t-\tau')^2 \tag{8}$$
$$(\lambda-\frac{\tau}{2})^2 = \frac{\omega^2}{\omega_0^2}(t-\tau'')^2 \tag{9}$$

we obtain $\lambda = \frac{\omega}{\omega_0}(t-\frac{\tau'+\tau''}{2})$ and $\tau = \frac{\omega}{\omega_0}(\tau''-\tau')$ thus we have

$$\mathbf{e}^{-\frac{\omega^2(t-\tau')^2}{2(\omega_0 T)^2}}\,\mathbf{e}^{-\frac{\omega^2(t-\tau'')^2}{2(\omega_0 T)^2}} = 2\pi T^2\int_{\mathbb{R}} \mathrm{WV}_h\left(\frac{\omega}{\omega_0}\left(t-\frac{\tau'+\tau''}{2}\right),\Omega'\right)\,\mathbf{e}^{j\Omega'\frac{\omega}{\omega_0}(\tau''-\tau')}\,\frac{d\Omega'}{2\pi}. \tag{10}$$

So, (2) can be expressed using Eqs. (5) and (10) as

$$|\mathrm{ST}_x(t,\omega)|^2 = \frac{\omega^2}{\omega_0^2}\iiiint_{\mathbb{R}^4} \mathrm{WV}_x(\frac{\tau'+\tau''}{2},\Omega)\,\mathbf{e}^{j\Omega(\tau'-\tau'')}\mathrm{WV}_h(\frac{\omega}{\omega_0}(t-\frac{\tau'+\tau''}{2}),\Omega')\,\mathbf{e}^{j\frac{\omega\Omega'}{\omega_0}(\tau''-\tau')-j\omega(\tau'-\tau'')}\,\frac{d\Omega}{2\pi}\,\frac{d\Omega'}{2\pi}d\tau'd\tau''. \tag{11}$$

If we define $\tau_1 = \frac{\tau'+\tau''}{2}$ and $\tau_2 = \tau'-\tau''$, we obtain

$$\begin{pmatrix}\tau'\\ \tau''\end{pmatrix} = \Phi(\tau_1,\tau_2) = \begin{pmatrix}\tau_1+\frac{\tau_2}{2}\\ \tau_1-\frac{\tau_2}{2}\end{pmatrix}, \tag{12}$$

as a result for a multiple substitution integration we obtain

$$d\tau'\,d\tau'' = |\det J_\Phi|\,d\tau_1\,d\tau_2 = d\tau_1\,d\tau_2 \tag{13}$$

where $\det J_\Phi$ is the determinant of the Jacobian of matrix Φ. As a result, (11) can be expressed as

$$|\mathrm{ST}_x(t,\omega)|^2 = \frac{\omega^2}{\omega_0^2}\iiiint_{\mathbb{R}^4} \mathrm{WV}_x(\tau_1,\Omega)\mathrm{WV}_h(\frac{\omega}{\omega_0}(t-\tau_1),\Omega')\,\mathbf{e}^{j\tau_2(\Omega-\omega-\frac{\omega\Omega'}{\omega_0})}\,\frac{d\Omega}{2\pi}\,\frac{\Omega'}{2\pi}\,d\tau_1\,d\tau_2 \tag{14}$$
$$= \frac{\omega^2}{\omega_0^2}\iiint_{\mathbb{R}^3} \mathrm{WV}_x(\tau_1,\Omega)\mathrm{WV}_h(\frac{\omega}{\omega_0}(t-\tau_1),\Omega')\delta\left(\Omega-\omega-\frac{\omega\Omega'}{\omega_0}\right)\,\frac{d\Omega}{2\pi}\,d\Omega'\,d\tau_1 \tag{15}$$
$$= \frac{|\omega|}{\omega_0}\iiint_{\mathbb{R}^3} \mathrm{WV}_x(\tau_1,\Omega)\mathrm{WV}_h(\frac{\omega}{\omega_0}(t-\tau_1),\Omega')\delta\left(\frac{\omega_0\Omega}{\omega}-\omega_0-\Omega'\right)\,\frac{d\Omega}{2\pi}\,d\Omega'\,d\tau_1 \tag{16}$$
$$= \frac{|\omega|}{\omega_0}\iint_{\mathbb{R}^2} \mathrm{WV}_x(\tau_1,\Omega)\mathrm{WV}_h(\frac{\omega}{\omega_0}(t-\tau_1),\frac{\omega_0}{\omega}(\Omega-\omega))\frac{d\Omega}{2\pi}d\tau_1 \tag{17}$$

1.1 Wigner-Ville distribution of a Gaussian analysis window

If we define $h(t) = \frac{1}{\sqrt{2\pi}T}\,\mathbf{e}^{-\frac{t^2}{2T^2}}$, its Wigner-Ville distribution can be expressed as

$$\mathrm{WV}_h(t,\omega) = \int_{\mathbb{R}} h\left(t+\frac{\tau}{2}\right) h\left(t-\frac{\tau}{2}\right)^* \mathbf{e}^{-j\omega\tau}\,d\tau \tag{18}$$

$$= \frac{1}{2\pi T^2}\int_{\mathbb{R}} \mathbf{e}^{-\frac{(t+\frac{\tau}{2})^2}{2T^2}}\,\mathbf{e}^{-\frac{(t-\frac{\tau}{2})^2}{2T^2}}\,\mathbf{e}^{-j\omega\tau}\,d\tau \tag{19}$$

$$= \frac{1}{2\pi T^2}\int_{\mathbb{R}} \mathbf{e}^{-\frac{t^2+\frac{\tau^2}{4}}{T^2}}\,\mathbf{e}^{-j\omega\tau}\,d\tau \tag{20}$$

$$= \frac{1}{2\pi T^2}\,\mathbf{e}^{-\frac{t^2}{T^2}}\int_{\mathbb{R}} \mathbf{e}^{-\frac{\tau^2}{4T^2}}\,\mathbf{e}^{-j\omega\tau}\,d\tau \tag{21}$$

$$= \frac{1}{\sqrt{\pi}T}\,\mathbf{e}^{-\frac{t^2}{T^2}}\,\mathbf{e}^{-\omega^2T^2} \tag{22}$$

Thus, from (17) we finally obtain

$$|\mathrm{ST}_x(t,\omega)|^2 = \frac{|\omega|}{\sqrt{\pi}\omega_0 T}\iint_{\mathbb{R}^2}\mathrm{WV}_x(\tau,\Omega)\,\mathbf{e}^{-\frac{\omega^2(t-\tau)^2}{(\omega_0T)^2}}\,\mathbf{e}^{-\frac{(\omega_0T)^2(\Omega-\omega)^2}{\omega^2}}\,d\tau\frac{d\Omega}{2\pi} \tag{23}$$

2 Frequency domain expression of the S-transform

As a signal x can be expressed as the inverse of its Fourier transform

$$x(t) = \int_{-\infty}^{+\infty} F_x(\xi)\,\mathbf{e}^{+j\xi t}\,\frac{d\xi}{2\pi} \tag{24}$$

thus, $\mathrm{ST}_x(t,\omega)$ can now be expressed as

$$\mathrm{ST}_x(t,\omega) = \frac{|\omega|}{\sqrt{2\pi}\omega_0 T}\iint_{\mathbb{R}^2} F_x(\xi)\,\mathbf{e}^{-\frac{(t-\tau)^2\omega^2}{2(\omega_0T)^2}}\,\mathbf{e}^{-j(\omega-\xi)\tau}\,d\tau\,\frac{d\xi}{2\pi} \tag{25}$$

$$= \frac{|\omega|}{\omega_0}\iint_{\mathbb{R}^2} F_x(\xi) h\left(\frac{\omega}{\omega_0}(t-\tau)\right)\mathbf{e}^{-j(\omega-\xi)\tau}\,d\tau\,\frac{d\xi}{2\pi} \tag{26}$$

where $h(t) = \frac{1}{\sqrt{2\pi}T}\,\mathbf{e}^{-\frac{t^2}{2T^2}}$.

The Fourier transform of $g(t, \frac{\omega_0 T}{|\omega|}) = \frac{|\omega|}{\omega_0} h\left(\frac{\omega}{\omega_0}(t-\tau)\right)$ can be expressed as

$$F_g(\Omega) = \frac{|\omega|}{\omega_0}\int_{-\infty}^{+\infty} h\left(\frac{\omega}{\omega_0}(t-\tau)\right)\mathbf{e}^{-j\Omega t}\,dt \tag{27}$$

$$= \mathbf{e}^{-j\Omega\tau}\int_{-\infty}^{+\infty} h(\alpha)\,\mathbf{e}^{-j\frac{\omega_0}{\omega}\Omega\alpha}\,d\alpha = \mathbf{e}^{-j\Omega\tau}F_h(\frac{\omega_0}{\omega}\Omega) \tag{28}$$

Thus (26) can now be expressed as

$$\mathrm{ST}_x(t,\omega) = \iiint_{\mathbb{R}^3} F_x(\xi)F_h\left(\frac{\omega_0}{\omega}\Omega\right)\mathbf{e}^{j(\Omega(t-\tau)}\,\mathbf{e}^{j(\xi-\omega)\tau}\,d\tau\,\frac{d\xi}{2\pi}\frac{d\Omega}{2\pi} \tag{29}$$

$$= \iint_{\mathbb{R}^2} F_x(\xi)F_h\left(\frac{\omega_0}{\omega}\Omega\right)\mathbf{e}^{j\Omega t}\int_{-\infty}^{+\infty}\mathbf{e}^{j(\xi-\Omega-\omega)\tau}\,d\tau\,\frac{d\xi}{2\pi}\frac{d\Omega}{2\pi} \tag{30}$$

$$= \iint_{\mathbb{R}^2} F_x(\xi)F_h\left(\frac{\omega_0}{\omega}\Omega\right)\mathbf{e}^{j\Omega t}\delta(\xi-\Omega-\omega)\,d\xi\,\frac{d\Omega}{2\pi} \tag{31}$$

$$= \mathbf{e}^{-j\omega t}\int_{-\infty}^{+\infty} F_x(\xi)F_h\left(\frac{\omega_0}{\omega}(\xi-\omega)\right)\mathbf{e}^{j\xi t}\,\frac{d\xi}{2\pi} \tag{32}$$

$$= \int_{-\infty}^{+\infty} F_x(\Omega+\omega)F_h\left(\frac{\omega_0}{\omega}\Omega\right)\mathbf{e}^{j\Omega t}\,\frac{d\Omega}{2\pi} \tag{33}$$

3 S-transform of particular signals

3.1 S-transform of a time-delayed signal

If $y(t) = x(t - t_1)$, then we obtain

$$\mathrm{ST}_y(t,\omega) = \frac{|\omega|\,\mathbf{e}^{-j\omega t}}{\sqrt{2\pi}\omega_0 T}\int_{-\infty}^{+\infty} x(t+\tau-t_1)\,\mathbf{e}^{-\frac{\omega^2\tau^2}{2(\omega_0 T)^2}}\,\mathbf{e}^{-j\omega\tau}\,d\tau$$

and

$$\mathrm{ST}_x(t-t_1,\omega) = \frac{|\omega|\,\mathbf{e}^{-j\omega(t-t_1)}}{\sqrt{2\pi}\omega_0 T}\int_{-\infty}^{+\infty} x(t-t_1+\tau)\,\mathbf{e}^{-\frac{\omega^2\tau^2}{2(\omega_0 T)^2}}\,\mathbf{e}^{-j\omega\tau}\,d\tau \tag{34}$$

$$= \mathrm{ST}_y(t,\omega)\,\mathbf{e}^{-j\omega t_1}. \tag{35}$$

Thus, we finally deduce

$$\mathrm{ST}_y(t,\omega) = \mathrm{ST}_x(t-t_1,\omega)\,\mathbf{e}^{j\omega t_1}. \tag{36}$$

3.2 S-transform of a frequency-shifted signal

If we define $y(t) = x(t)\,\mathbf{e}^{j\omega_1 t}$, then using $F_y(\omega) = F_x(\omega-\omega_1)$ we obtain

$$\mathrm{ST}_y(t,\omega) = \int_{-\infty}^{+\infty} F_x(\omega-\omega_1+\Omega)F_h\left(\frac{\omega_0\Omega}{\omega}\right)\mathbf{e}^{j\Omega t}\,\frac{d\Omega}{2\pi} \tag{37}$$

As,

$$\mathrm{ST}_x(t,\omega-\omega_1) = \int_{-\infty}^{+\infty} F_x(\omega-\omega_1+\Omega)F_h\left(\frac{\omega_0\Omega}{\omega-\omega_1}\right)\mathbf{e}^{j\Omega t}\frac{d\Omega}{2\pi} \tag{38}$$

we finally deduce that

$$\mathrm{ST}_y(t,\omega) \neq \mathrm{ST}_x(t,\omega-\omega_1). \tag{39}$$

3.3 S-transform of a rescaled signal

If we define $z(t) = \frac{1}{\sqrt{s}}x\left(\frac{t}{s}\right)$, then we have

$$\mathrm{ST}_z(t,\omega) = \frac{|\omega|}{\sqrt{2\pi}\omega_0 T}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{s}}x\left(\frac{\tau}{s}\right)\mathbf{e}^{-\frac{\omega^2(t-\tau)^2}{2(\omega_0 T)^2}}\,\mathbf{e}^{-j\omega\tau}d\tau \tag{40}$$

$$= \frac{|\omega|}{\sqrt{2\pi}\omega_0 T}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{s}}x(\tau')\,\mathbf{e}^{-\frac{\omega^2(t-s\tau')^2}{2(\omega_0 T)^2}}\,\mathbf{e}^{-j\omega s\tau'}s d\tau' \tag{41}$$

$$= \frac{1}{\sqrt{s}}\frac{s|\omega|}{\sqrt{2\pi}\omega_0 T}\int_{-\infty}^{+\infty} x(\tau')\,\mathbf{e}^{-\frac{s^2\omega^2(\frac{t}{s}-\tau')^2}{2(\omega_0 T)^2}}\,\mathbf{e}^{-js\omega\tau'}d\tau' \tag{42}$$

$$= \frac{1}{\sqrt{s}}\mathrm{ST}_x\left(\frac{t}{s},s\omega\right). \tag{43}$$

3.4 S-transform of a sinusoid

If we consider a signal $x(t) = A\,\mathbf{e}^{j\omega_1 t}$, we have $F_x(\omega) = 2\pi A\delta(\omega-\omega_1)$. Hence, the S-transform of x can be expressed as

$$\mathrm{ST}_x(t,\omega) = A\int_{-\infty}^{+\infty}\delta(\Omega+\omega-\omega_1)F_h\left(\frac{\omega_0}{\omega}\Omega\right)\mathbf{e}^{j\Omega t}\,d\Omega \tag{44}$$

$$= AF_h\left(\frac{\omega_0}{\omega}(\omega_1-\omega)\right)\mathbf{e}^{j(\omega_1-\omega)t} \tag{45}$$

$$= A\,\mathbf{e}^{-\frac{(\omega_1-\omega)^2(\omega_0 T)^2}{2\omega^2}}\,\mathbf{e}^{j(\omega_1-\omega)t}. \tag{46}$$

Thus, we finally deduce

$$\mathrm{ST}_x(t,\omega_1) = A \tag{47}$$

3.5 Tuning $\omega_0 T$ to compute the S-transform of a sinusoid

Let's begin with the S-transform of a sinusoid expressed as $\mathrm{ST}_x(t,\omega) = A\,\mathbf{e}^{-\frac{(\omega_1-\omega)^2(\omega_0 T)^2}{2\omega^2}}\,\mathbf{e}^{j(\omega_1-\omega)t}$. If one wants the set $\{\omega, |\mathrm{ST}_x(t,\omega)| > \Gamma\}$ with $\Gamma < 1$, to have a width $\Delta\omega_1$ around ω_1, we have to find the two values ω and ω' that verifies $\Delta\omega_1 = \omega - \omega' > 0$ (with $\omega > \omega'$) obtained by solving the equation

$$\mathbf{e}^{-\frac{(\omega_1-\omega)^2(\omega_0 T)^2}{2\omega^2}} = \Gamma \tag{48}$$

$$-\frac{(\omega_1-\omega)^2(\omega_0 T)^2}{2\omega^2} = \log(\Gamma) \tag{49}$$

$$(\omega_1-\omega)^2 = \frac{2\log(1/\Gamma)}{(\omega_0 T)^2}\omega^2 \tag{50}$$

$$\left(1 - \frac{2\log(1/\Gamma)}{(\omega_0 T)^2}\right)\omega^2 - 2\omega_1\omega + \omega_1^2 = 0 \tag{51}$$

Thus we obtain the solutions

$$\omega = \frac{\omega_1}{1 - \frac{\sqrt{2\log(1/\Gamma)}}{\omega_0 T}} = \frac{\omega_1}{1-\alpha} \quad \text{with } \alpha = \frac{\sqrt{2\log(1/\Gamma)}}{\omega_0 T} \tag{52}$$

$$\omega' = \frac{\omega_1}{1 + \frac{\sqrt{2\log(1/\Gamma)}}{\omega_0 T}} = \frac{\omega_1}{1+\alpha}. \tag{53}$$

Thus we have,

$$\Delta\omega_1 = \omega - \omega' = \frac{\omega_1}{1-\alpha} - \frac{\omega_1}{1+\alpha} = \frac{2\alpha}{1-\alpha^2}\omega_1 = \frac{2\frac{\sqrt{2\log(1/\Gamma)}}{(\omega_0 T)^2}}{1 - \frac{2\log(1/\Gamma)}{\omega_0 T}}\omega_1. \tag{54}$$

that leads to

$$1 - \frac{2\log(1/\Gamma)}{(\omega_0 T)^2} = \frac{\omega_1}{\Delta\omega_1}2\frac{\sqrt{2log(1/\Gamma)}}{\omega_0 T} \tag{55}$$

$$(\omega_0 T)^2 - 2\log(1/\Gamma) = \frac{\omega_1}{\Delta\omega_1}2\sqrt{2\log(1/\Gamma)}\omega_0 T \tag{56}$$

$$(\omega_0 T)^2 - \frac{\omega_1}{\Delta\omega_1}2\sqrt{2\log(1/\Gamma)}\omega_0 T - 2\log(1/\Gamma) = 0 \tag{57}$$

So finally, as $\omega_0 T > 0$, we obtain a unique solution

$$\omega_0 T = \sqrt{2\log(1/\Gamma)}\left(\sqrt{1 + \frac{\omega_1^2}{\Delta\omega_1^2}} + \frac{\omega_1}{\Delta\omega_1}\right) \tag{58}$$

3.6 S-transform of an impulse signal

If $x(t) = \delta(t - t_1)$ then we obtain

$$\mathrm{ST}_x(t,\omega) = \frac{|\omega|}{\omega_0}\int_{-\infty}^{+\infty} \delta(\tau - t_1) h\left(\frac{\omega}{\omega_0}(t-\tau)\right)\mathbf{e}^{-j\omega\tau}\,d\tau \tag{59}$$

$$= \frac{|\omega|}{\omega_0} h\left(\frac{\omega}{\omega_0}(t - t_1)\right)\mathbf{e}^{-j\omega t_1} \tag{60}$$

$$= \frac{|\omega|}{\sqrt{2\pi}\omega_0 T}\,\mathbf{e}^{-\frac{\omega^2(t-t_1)^2}{2(\omega_0 T)^2}}\,\mathbf{e}^{-j\omega t_1} \tag{61}$$

Thus, we finally deduce

$$\mathrm{ST}_x(t_1,\omega) = \frac{|\omega|}{\sqrt{2\pi}\omega_0 T}\,\mathbf{e}^{-j\omega t_1} \tag{62}$$

3.7 Tuning $\omega_0 T$ to compute the S-transform of an impulse

Let's begin with the S-transform of an impulse located at instant t_1 expressed as $\mathrm{ST}_x(t,\omega) = \frac{|\omega|}{\sqrt{2\pi}\omega_0 T}\,\mathbf{e}^{-\frac{\omega^2(t-t_1)^2}{2(\omega_0 T)^2}}\,\mathbf{e}^{-j\omega t_1}$. If one wants the set $\{t, \frac{\mathrm{ST}_x(t,\omega)}{\mathrm{ST}_x(t_1,\omega)} > \Gamma\}$ with $\Gamma < 1$, to have a width Δt_1 around t_1, we have to find the two values t and t' that verifies $\Delta t_1 = t - t'$ obtained by solving the equation

$$\frac{\mathrm{ST}_x(t,\omega)}{\mathrm{ST}_x(t_1,\omega)} = \mathbf{e}^{-\frac{\omega^2(t-t_1)^2}{2(\omega_0 T)^2}} = \Gamma \tag{63}$$

that leads to

$$-\frac{\omega^2(t-t_1)^2}{2(\omega_0 T)^2} = \log(\Gamma) \tag{64}$$

$$(t-t_1)^2 = 2\log(1/\Gamma)\frac{(\omega_0 T)^2}{\omega^2} \tag{65}$$

$$t^2 - 2tt_1 + t_1^2 - 2\log(1/\Gamma)\frac{(\omega_0 T)^2}{\omega^2} = 0. \tag{66}$$

From the solutions

$$t = \frac{t+\alpha}{2} \quad \text{with } \alpha = 2\sqrt{2\log(1/\Gamma)}\frac{\omega_0 T}{\omega} \tag{67}$$

$$t' = \frac{t-\alpha}{2} \tag{68}$$

$$\tag{69}$$

one can deduce

$$\Delta t_1 = t - t' = 2\sqrt{2\log(1/\Gamma)}\frac{\omega_0 T}{\omega} \tag{70}$$

So finally we have

$$\omega_0 T = \frac{\omega \Delta t_1}{2\sqrt{2\log(1/\Gamma)}} \tag{71}$$

4 Marginalization of the S-transform over time

$$\int_{-\infty}^{+\infty} \mathrm{ST}(t,\omega)\,dt = \int_{-\infty}^{+\infty} x(\tau)\,\mathbf{e}^{-j\omega\tau}\underbrace{\left[\frac{|\omega|}{\sqrt{2\pi}\omega_0 T}\int_{-\infty}^{+\infty}\mathbf{e}^{-\frac{(t-\tau)^2\omega^2}{2(\omega_0 T)^2}}\,dt\right]}_{1} d\tau \tag{72}$$

$$= \int_{-\infty}^{+\infty} x(\tau)\,\mathbf{e}^{-j\omega\tau}\,d\tau = F_x(\omega) \tag{73}$$

5 S-transform simplified reconstruction formula

5.1 Proof

The S-transform simplified reconstruction formula leads to the following result

$$\int_{-\infty}^{+\infty} \mathrm{ST}_x(t,\omega)\,\mathbf{e}^{j\omega t}\frac{d\omega}{|\omega|} = \iint_{\mathbb{R}^2} F_x(\Omega+\omega)F_h\left(\frac{\omega_0}{\omega}\Omega\right)\mathbf{e}^{j\Omega t}\,\mathbf{e}^{j\omega t}\frac{d\Omega}{2\pi}\frac{d\omega}{|\omega|} \tag{74}$$

$$= \iint_{\mathbb{R}^2} F_x(\Omega')\,\mathbf{e}^{j\Omega' t}F_h\left(\frac{\omega_0}{\omega}(\Omega'-\omega)\right)\frac{d\Omega'}{2\pi}\frac{d\omega}{|\omega|} \tag{75}$$

Using the following variables substitution with the corresponding Jacobian matrix

$$\Phi(\xi,\Omega') = \begin{pmatrix}\frac{\omega_0\Omega'}{\xi}\\ \Omega'\end{pmatrix} \tag{76}$$

$$J_\Phi(\xi,\Omega') = \begin{pmatrix}-\frac{\omega_0\Omega'}{\xi^2} & \frac{\omega_0}{\xi}\\ 0 & 1\end{pmatrix} \tag{77}$$

we obtain

$$\int_{-\infty}^{+\infty} \mathrm{ST}_x(t,\omega)\,\mathbf{e}^{j\omega t}\,\frac{d\omega}{|\omega|} = \iint_{\mathbb{R}^2} F_x(\Omega')\,\mathbf{e}^{j\Omega' t} F_h\left(\xi-\omega_0\right)\left(\frac{|\omega_0\Omega'|}{\xi^2}\frac{|\xi|}{|\omega_0\Omega'|}\right)\frac{d\Omega'}{2\pi}d\xi \tag{78}$$

$$= \int_{-\infty}^{+\infty} F_x(\Omega')\,\mathbf{e}^{j\Omega' t}\,\frac{d\Omega'}{2\pi}\cdot\int_{-\infty}^{+\infty} F_h(\xi-\omega_0)\,\frac{d\xi}{|\xi|} \tag{79}$$

$$= x(t)\cdot C_h(\omega_0 T) \tag{80}$$

$$\text{with} \quad C_h(\omega_0 T) = \int_{-\infty}^{+\infty} \mathbf{e}^{-\frac{(\xi T-\omega_0 T)^2}{2}}\,\frac{d\xi}{|\xi|} = \int_{-\infty}^{+\infty} \mathbf{e}^{-\frac{(x-\omega_0 T)^2}{2}}\,\frac{dx}{|x|}. \tag{81}$$

Hence, x can be recovered from ST_x using the simplified S-transform inversion formula

$$x(t) = \frac{1}{C_h(\omega_0 T)}\int_{-\infty}^{+\infty} \mathrm{ST}_x(t,\omega)\,\mathbf{e}^{j\omega t}\,\frac{d\omega}{|\omega|}. \tag{82}$$

5.2 Relationship with the Morlet wavelet synthesis formula

For the wavelet transform of a signal x expressed as

$$W_x(t,s) = \frac{1}{\sqrt{|s|}}\int_{\mathbb{R}} x(\tau)\Psi(\frac{\tau-t}{s})^*\,d\tau \tag{83}$$

the simplified wavelet reconstruction formula is given by

$$\hat{x}(t) = \frac{1}{C_\Psi}\int_{\mathbb{R}} W_x(t,s)|s|^{-3/2}\,ds \tag{84}$$

$$= \frac{1}{\sqrt{\omega_0}C_\Psi}\int_{\mathbb{R}} \mathrm{CW}_x(t,\omega)\,\frac{d\omega}{\sqrt{|\omega|}} \tag{85}$$

with $s=\frac{\omega_0}{\omega}$ for $\omega_0>0$ and $\mathrm{CW}_x(t,\omega)=W_x(t,\frac{\omega_0}{\omega})$.

For the Morlet wavelet transform denoted $\mathrm{MW}_x(t,\omega)$ (*i.e.* $\mathrm{MW}_x(t,\omega)=\mathrm{CW}_x(t,\omega)$ when $\Psi(t)=\frac{\pi^{-1/4}}{\sqrt{T}}\,\mathbf{e}^{\frac{-t^2}{2T^2}}\,\mathbf{e}^{j\omega_0 t}$), thus using $F_\Psi(\omega)=\sqrt{2T}\pi^{1/4}\,\mathbf{e}^{-\frac{(\omega-\omega_0)^2T^2}{2}}$ (see Section 12) we have

$$C_\Psi = \sqrt{2T}\pi^{1/4}\int_{\mathbb{R}} \mathbf{e}^{-\frac{(\omega-\omega_0)^2T^2}{2}}\,\frac{d\omega}{|\omega|} \tag{86}$$

$$= \sqrt{2T}\pi^{1/4}\int_{\mathbb{R}} \mathbf{e}^{-\frac{(\omega T-\omega_0 T)^2}{2}}\,\frac{d\omega}{|\omega|} \tag{87}$$

$$= \sqrt{2T}\pi^{1/4}\int_{\mathbb{R}} \mathbf{e}^{-\frac{(x-\omega_0 T)^2}{2}}\,\frac{dx}{|x|} \tag{88}$$

$$= \sqrt{2T}\pi^{1/4}C_h(\omega_0 T) \tag{89}$$

Using the relation between the S-transform and the Morlet wavelet transform expressed as $\mathrm{MW}_x(t,\omega) = \sqrt{2\omega_0 T}\pi^{1/4}\frac{1}{\sqrt{|\omega|}}\,\mathbf{e}^{j\omega t}\mathrm{ST}_x(t,\omega)$ (see the main article) and using Eq. (89), we obtain from Eq. (85)

$$\hat{x}(t) = \frac{1}{\sqrt{2\omega_0 T}\pi^{1/4}C_h(\omega_0 T)}\int_{\mathbb{R}} \mathrm{MW}_x(t,\omega)\,\frac{d\omega}{\sqrt{|\omega|}} \tag{90}$$

$$= \frac{1}{C_h(\omega_0 T)}\int_{\mathbb{R}} \mathrm{ST}_x(t,\omega)\,\mathbf{e}^{j\omega t}\frac{d\omega}{|\omega|} \tag{91}$$

that is the simplified S-transform reconstruction formula given by Eq. (82).

As the Morlet wavelet is considered to be be approximately analytic ($F_\Psi(\omega)\approx 0$ for $\omega<0$ when $\omega_0 T\gg 0$), thus we have $|C_Psi|<\infty$. Using the relationship of $C_h(\omega_0 T)$ and $B_h(\omega_0 T)$ with C_Ψ, one can deduce that $|C_h(\omega_0 T)|<\infty$ and $|B_h(\omega_0 T)|<\infty$.

6 Energy conservation of the S-transform

Starting from the frequency domain expression of the S-transform given by Eq.(33)

$$\mathrm{ST}_x(t,\omega) = \int_{-\infty}^{+\infty} F_x(\omega+\Omega) F_h(\frac{\omega_0}{\omega}\Omega)\,\mathrm{e}^{j\Omega t}\frac{d\Omega}{2\pi}, \tag{92}$$

the Stockwellogram can be expressed as

$$|\mathrm{ST}_x(t,\omega)|^2 = \iint_{\mathbb{R}^2} F_x(\omega+\Omega_1)F_x(\omega+\Omega_2)^* F_h(\frac{\omega_0}{\omega}\Omega_1)F_h(\frac{\omega_0}{\omega}\Omega_2)^*\,\mathrm{e}^{-j(\Omega_2-\Omega_1)t}\,\frac{d\Omega_1}{2\pi}\frac{d\Omega_2}{2\pi} \tag{93}$$

Hence, when the Stockwellogram is marginalized over time, we obtain

$$\begin{aligned}
\int_{-\infty}^{+\infty}|\mathrm{ST}_x(t,\omega)|^2 dt &= \iint_{\mathbb{R}^2} F_x(\omega+\Omega_1)F_x(\omega+\Omega_2)^* F_h(\frac{\omega_0}{\omega}\Omega)F_h(\frac{\omega_0}{\omega}\Omega)^*\delta(\Omega_2-\Omega_1)\,\frac{d\Omega_1}{2\pi}d\Omega_2 && (94)\\
&= \int_{-\infty}^{+\infty} |F_x(\omega+\Omega')|^2|F_h(\frac{\omega_0}{\omega}\Omega')|^2\frac{d\Omega'}{2\pi} && (95)\\
&= \int_{-\infty}^{+\infty} |F_x(\Omega)|^2|F_h(\frac{\omega_0}{\omega}(\Omega-\omega))|^2\frac{d\Omega}{2\pi} && (96)\\
&= \int_{-\infty}^{+\infty} |F_x(\Omega)|^2\;\mathrm{e}^{-\frac{(\omega_0 T)^2(\Omega-\omega)^2}{\omega^2}}\;\frac{d\Omega}{2\pi} && (97)
\end{aligned}$$

that corresponds to a smoothed version of the energy spectral density of the analyzed signal x.

Now, one can calculate

$$\iint_{\mathbb{R}^2}|\mathrm{ST}_x(t,\omega)|^2\;dt\frac{d\omega}{|\omega|} = \iint_{\mathbb{R}^2}|F_x(\Omega)|^2\,|F_h(\frac{\omega_0\Omega}{\omega}-\omega_0)|^2\,\frac{d\Omega}{2\pi}\frac{d\omega}{|\omega|} \tag{98}$$

Using the following variables substitution with the corresponding Jacobian matrix

$$\Phi(\omega',\Omega') = \begin{pmatrix}\frac{\Omega_0\Omega}{\omega'}\\ \Omega'\end{pmatrix} \tag{99}$$

$$J_\Phi(\omega',\Omega') = \begin{pmatrix}\frac{-\omega_0\Omega}{\omega'^2} & \frac{\omega_0}{\omega'}\\ 0 & 1\end{pmatrix}, \tag{100}$$

we obtain

$$\begin{aligned}
\iint_{\mathbb{R}^2}|\mathrm{ST}_x(t,\omega)|^2\;dt\frac{d\omega}{|\omega|} &= \int_{-\infty}^{+\infty}|F_h(\omega'-\omega_0)|^2\;\frac{d\omega'}{|\omega'|}\int_{-\infty}^{+\infty}|F_x(\Omega)|^2\;\frac{d\Omega}{2\pi} && (101)\\
&= B_h(\omega_0 T)E_x && (102)
\end{aligned}$$

So finally, the Parseval's theorem leads to

$$E_x = \int_{-\infty}^{+\infty}|x(t)|^2\,dt = \int_{-\infty}^{+\infty}|F_x(\omega)|^2\;\frac{d\omega}{2\pi} = \frac{1}{B_h(\omega_0 T)}\iint_{\mathbb{R}^2}|\mathrm{ST}_x(t,\omega)|^2\;dt\frac{d\omega}{|\omega|} \tag{103}$$

where the proportionality factor $B_h(\omega_0 T)$ is given by

$$B_h(\omega_0T) = \int_{-\infty}^{+\infty}|F_h(\omega-\omega_0)|^2\;\frac{d\omega}{|\omega|} = \int_{-\infty}^{+\infty}\mathrm{e}^{-(\omega-\omega_0)^2T^2}\;\frac{d\omega}{|\omega|} = \int_{-\infty}^{+\infty}\mathrm{e}^{-\omega_0(x-1)^2T^2}\;\frac{dx}{|x|} = \int_{-\infty}^{+\infty}|F_h(\omega_0(x-1))|^2\;\frac{dx}{|x|} \tag{104}$$

7 Frequency derivative of the phase of the S-transform

Let define the S-transform as a particular case of the Short-Time Fourier Transform (STFT) using the window $g(t,\omega) = \frac{|\omega|}{\sqrt{2\pi}\omega_0 T}\,\mathrm{e}^{-\frac{\omega^2t^2}{2(\omega_0 T)^2}}$

$$\mathrm{ST}_x(t,\omega) = M_x(t,\omega)\,\mathrm{e}^{j\Phi_x(t,\omega)} = \int_{\mathbb{R}} x(\tau)g(t-\tau,\omega)\,\mathrm{e}^{-j\omega\tau}\,d\tau \tag{105}$$

When $\omega \neq 0$, we have

$$\frac{\partial \mathrm{ST}_x}{\partial \omega}(t,\omega) = \frac{\partial M_x}{\partial \omega}(t,\omega)\,\mathbf{e}^{j\Phi_x(t,\omega)} + j\frac{\partial \Phi_x}{\partial \omega}(t,\omega)\mathrm{ST}_x(t,\omega) \tag{106}$$

$$= \frac{\partial}{\partial \omega}\left[\mathbf{e}^{-j\omega t}\int_{\mathbb{R}} x(\tau)g(t-\tau,\frac{\omega_0}{|\omega|}T)\,\mathbf{e}^{j\omega(t-\tau)}\,d\tau\right] \tag{107}$$

$$= -jt\mathrm{ST}_x(t,\omega) + \mathbf{e}^{-j\omega t}\left[\int_{\mathbb{R}} x(\tau)\frac{\partial g}{\partial \omega}(t-\tau,\frac{\omega_0}{|\omega|}T)\,\mathbf{e}^{j\omega(t-\tau)}d\tau + j\int_{\mathbb{R}} x(\tau)(t-\tau)g(t-\tau,\frac{\omega_0}{|\omega|}T)\,\mathbf{e}^{j\omega(t-\tau)}d\tau\right] \tag{108}$$

$$= -jt\mathrm{ST}_x(t,\omega) + j\mathrm{ST}_x^{\mathcal{T}g}(t,\omega) + \frac{\omega}{|\omega|\sqrt{2\pi}\omega_0 T}\int_{\mathbb{R}} x(\tau)\,\mathbf{e}^{-\frac{\omega^2(t-\tau)^2}{2(\omega_0 T)^2}}\,\mathbf{e}^{-j\omega\tau}d\tau - \frac{|\omega|\omega}{\sqrt{2\pi}(\omega_0 T)^3}\int_{\mathbb{R}} x(\tau)(t-\tau)^2\,\mathbf{e}^{-\frac{\omega^2(t-\tau)^2}{2(\omega_0 T)^2}-j\omega\tau}d\tau \tag{109}$$

$$= -jt\mathrm{ST}_x(t,\omega) + j\mathrm{ST}_x^{\mathcal{T}g}(t,\omega) + \frac{\omega}{|\omega|^2}\mathrm{ST}_x(t,\omega) - \frac{\omega}{(\omega_0 T)^2}\mathrm{ST}_x^{\mathcal{T}^2 g}(t,\omega). \tag{110}$$

Thus, we have

$$\frac{\frac{\partial \mathrm{ST}_x}{\partial \omega}(t,\omega)}{\mathrm{ST}_x(t,\omega)} = \frac{1}{M_x(t,\omega)}\frac{\partial M_x}{\partial \omega}(t,\omega) + j\frac{\partial \Phi_x}{\partial \omega}(t,\omega) = -jt + j\frac{\mathrm{ST}_x^{\mathcal{T}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)} + \frac{\omega}{|\omega|^2} - \frac{\omega}{(\omega_0 T)^2}\frac{\mathrm{ST}_x^{\mathcal{T}^2 g}(t,\omega)}{\mathrm{ST}_x(t,\omega)} \tag{111}$$

so finally we deduce

$$\frac{\partial \Phi_x}{\partial \omega}(t,\omega) = \mathrm{Im}\left(\frac{\frac{\partial \mathrm{ST}_x}{\partial \omega}(t,\omega)}{\mathrm{ST}_x(t,\omega)}\right) = -t + \mathrm{Re}\left(\frac{\mathrm{ST}_x^{\mathcal{T}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)}\right) - \mathrm{Im}\left(\frac{\omega}{(\omega_0 T)^2}\frac{\mathrm{ST}_x^{\mathcal{T}^2 g}(t,\omega)}{\mathrm{ST}_x(t,\omega)}\right) \tag{112}$$

8 Computation of the reassignment operators using S-transforms

8.1 time coordinate

We consider first the definition of the reassigned time coordinates

$$\hat{t}(t,\omega) = \frac{\iint_{\mathbb{R}^2} \tau \mathrm{WV}_x(\tau,\Omega)\mathrm{WV}_h(\frac{\omega}{\omega_0}(t-\tau),\frac{\omega_0}{\omega}(\Omega-\omega))d\tau d\Omega}{\iint_{\mathbb{R}^2} \mathrm{WV}_x(\tau,\Omega)\mathrm{WV}_h(\frac{\omega}{\omega_0}(t-\tau),\frac{\omega_0}{\omega}(\Omega-\omega))d\tau d\Omega} \tag{113}$$

$$= \mathrm{Re}\left(\frac{\iint_{\mathbb{R}^2} \tau \mathrm{Ri}_x(\tau,\Omega)\mathrm{Ri}_h(\frac{\omega}{\omega_0}(t-\tau),\frac{\omega_0}{\omega}(\Omega-\omega))d\tau d\Omega}{\iint_{\mathbb{R}^2} \mathrm{Ri}_x(\tau,\Omega)\mathrm{Ri}_h(\frac{\omega}{\omega_0}(t-\tau),\frac{\omega_0}{\omega}(\Omega-\omega))d\tau d\Omega}\right) \tag{114}$$

where $\mathrm{Ri}(t,\omega) = x(t)F_x(\omega)^*\,\mathbf{e}^{-j\omega t}$ is the Rihaczeck disctribution. Thus we have

$$\hat{t}(t,\omega) = \frac{1}{\frac{\omega_0}{|\omega|}|\mathrm{ST}_x(t,\omega)|^2}\mathrm{Re}\left(\int_{\mathbb{R}} x(\tau)\tau h(\frac{\omega}{\omega_0}(t-\tau))\,\mathbf{e}^{-j\omega\tau}d\tau \cdot \int_{\mathbb{R}} F_x(\Omega)^* F_h(\frac{\omega_0}{\omega}(\Omega-\omega))^*\,\mathbf{e}^{-j(\Omega-\omega)t}\frac{d\Omega}{2\pi}\right) \tag{115}$$

$$= t - \mathrm{Re}\left(\frac{\int_{\mathbb{R}} x(\tau)(t-\tau)h(\frac{\omega}{\omega_0}(t-\tau))\,\mathbf{e}^{-j\omega\tau}d\tau\mathrm{ST}_x(t,\omega)^*}{\frac{\omega_0}{|\omega|}|\mathrm{ST}_x(t,\omega)|^2}\right) \tag{116}$$

$$= t - \mathrm{Re}\left(\frac{\mathrm{ST}_x^{\mathcal{T}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)}\right) \tag{117}$$

8.2 Frequency coordinate

We consider first the definition of the reassigned frequency coordinates

$$\hat{\omega}(t,\omega) = \frac{\iint_{\mathbb{R}^2} \Omega \mathrm{WV}_x(\tau,\Omega)\mathrm{WV}_h(\frac{\omega}{\omega_0}(t-\tau), \frac{\omega_0}{\omega}(\Omega-\omega))d\tau d\Omega}{\iint_{\mathbb{R}^2} \mathrm{WV}_x(\tau,\Omega)\mathrm{WV}_h(\frac{\omega}{\omega_0}(t-\tau), \frac{\omega_0}{\omega}(\Omega-\omega))d\tau d\Omega} \tag{118}$$

$$= \mathrm{Re}\left(\frac{\iint_{\mathbb{R}^2} \Omega \mathrm{Ri}_x(\tau,\Omega)\mathrm{Ri}_h(\frac{\omega}{\omega_0}(t-\tau), \frac{\omega_0}{\omega}(\Omega-\omega))d\tau d\Omega}{\iint_{\mathbb{R}^2} \mathrm{Ri}_x(\tau,\Omega)\mathrm{Ri}_h(\frac{\omega}{\omega_0}(t-\tau), \frac{\omega_0}{\omega}(\Omega-\omega))d\tau d\Omega} \right) \tag{119}$$

$$= \frac{1}{\frac{\omega_0}{|\omega|}|\mathrm{ST}_x(t,\omega)|^2} \mathrm{Re}\left(\int_{\mathbb{R}} x(\tau) h(\frac{\omega}{\omega_0}(t-\tau)\,\mathbf{e}^{-j\omega\tau} d\tau \cdot \int_{\mathbb{R}} \Omega F_x(\Omega)^* F_h(\frac{\omega_0}{\omega}(\Omega-\omega))\,\mathbf{e}^{-(\Omega-\omega)t} \frac{d\Omega}{2\pi} \right) \tag{120}$$

$$= \omega + \mathrm{Re}\left(\frac{\mathrm{ST}_x(t,\omega)}{|\mathrm{ST}_x(t,\omega)|^2} \cdot \int_{\mathbb{R}} (\Omega-\omega) F_x(\Omega)^* F_h(\frac{\omega_0}{\omega}(\Omega-\omega))^*\,\mathbf{e}^{-(\Omega-\omega)t} \frac{d\Omega}{2\pi} \right) \tag{121}$$

$$= \omega + \mathrm{Re}\left(\frac{\mathrm{ST}_x(t,\omega)}{|\mathrm{ST}_x(t,\omega)|^2} \cdot \int_{\mathbb{R}} (\Omega-\omega) F_x(\Omega)^* F_g(\Omega-\omega)^*\,\mathbf{e}^{-(\Omega-\omega)t} \frac{d\Omega}{2\pi} \right) \tag{122}$$

Using the following property

$$F_h\left(\frac{\omega_0}{\omega}(\Omega-\omega)\right) = \int_{\mathbb{R}} h(t)\,\mathbf{e}^{-j\frac{\omega_0}{\omega}(\Omega-\omega)t}\,dt = \frac{|\omega|}{\omega_0}\int_{\mathbb{R}} h(\frac{\omega}{\omega_0}\tau)\,\mathbf{e}^{-j(\Omega-\omega)\tau}\,d\tau \tag{123}$$

$$= \int_{\mathbb{R}} g(\tau,\sigma_t)\,\mathbf{e}^{-j(\Omega-\omega)\tau}\,d\tau = F_g(\Omega-\omega) \tag{124}$$

If we define $\mathcal{D}g(t) = \frac{dg}{dt}(t) = \frac{-t}{\sigma_t^2}g(t)$ and we have $F_{\mathcal{D}g}(\omega) = j\omega F_g(\omega)$

Thus using $F_g(\omega) = -j\frac{1}{\omega}F_{\mathcal{D}g}(\omega)$, we obtain

$$\hat{\omega}(t,\omega) = \omega - \mathrm{Re}\left(j\frac{\mathrm{ST}_x(t,\omega)}{|\mathrm{ST}_x(t,\omega)|^2} \cdot \int_{\mathbb{R}} F_x(\Omega)^* F_{\mathcal{D}g}(\Omega-\omega)^*\,\mathbf{e}^{-(\Omega-\omega)t} \frac{d\Omega}{2\pi} \right) \tag{125}$$

$$= \omega - \mathrm{Im}\left(\frac{\mathrm{ST}_x^{\mathcal{D}g}(t,\omega)^*}{\mathrm{ST}_x(t,\omega)^*} \right) \tag{126}$$

$$= \omega + \mathrm{Im}\left(\frac{\mathrm{ST}_x^{\mathcal{D}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)} \right) \tag{127}$$

9 Computation of the Levenberg-Marquardt reassignment operators using S-transforms

Starting from the Stockwellogram classical reassignment operators given by

$$\hat{t}(t,\omega) = t - \mathrm{Re}\left(\frac{\mathrm{ST}_x^{\mathcal{T}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)} \right) \tag{128}$$

$$\hat{\omega}(t,\omega) = \omega + \mathrm{Im}\left(\frac{\mathrm{ST}_x^{\mathcal{D}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)} \right) \tag{129}$$

and the definition of the Levenberg-Marquardt reassignment operators

$$\begin{pmatrix} \tilde{t}(t,\omega) \\ \tilde{\omega}(t,\omega) \end{pmatrix} = \begin{pmatrix} t \\ \omega \end{pmatrix} - \left(\nabla^t R_x(t,\omega) + \mu I_2\right)^{-1} R_x(t,\omega) \tag{130}$$

$$\text{with} \quad R_x(t,\omega) = \begin{pmatrix} t \\ \omega \end{pmatrix} - \begin{pmatrix} \hat{t}(t,\omega) \\ \hat{\omega}(t,\omega) \end{pmatrix} \tag{131}$$

$$\nabla^t R_x(t,\omega) = \begin{pmatrix} \frac{\partial R_x}{\partial t}(t,\omega) & \frac{\partial R_x}{\partial \omega}(t,\omega) \end{pmatrix} \tag{132}$$

we directly obtain

$$R_x(t,\omega) = \begin{pmatrix} \mathrm{Re}\left(\frac{\mathrm{ST}_x^{\mathcal{T}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)}\right) \\ -\mathrm{Im}\left(\frac{\mathrm{ST}_x^{\mathcal{D}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)}\right) \end{pmatrix} \tag{133}$$

$$\nabla^t R_x(t,\omega) = \begin{pmatrix} \frac{\partial}{\partial t}\left[\mathrm{Re}\left(\frac{\mathrm{ST}_x^{\mathcal{T}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)}\right)\right] & \frac{\partial}{\partial \omega}\left[\mathrm{Re}\left(\frac{\mathrm{ST}_x^{\mathcal{T}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)}\right)\right] \\ -\frac{\partial}{\partial t}\left[\mathrm{Im}\left(\frac{\mathrm{ST}_x^{\mathcal{D}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)}\right)\right] & -\frac{\partial}{\partial \omega}\left[\mathrm{Im}\left(\frac{\mathrm{ST}_x^{\mathcal{D}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)}\right)\right] \end{pmatrix} \tag{134}$$

The coefficients of matrix $\nabla^t R_x(t,\omega)$ can be detailed as below.

9.1 Computation of the partial time derivatives of $R_x(t,\omega)$

Using $\frac{\partial \mathrm{ST}_x}{\partial t} = \mathrm{ST}_x^{\mathcal{D}g}(t,\omega)$ and $\frac{\partial \mathrm{ST}_x^{\mathcal{T}g}}{\partial t} = \mathrm{ST}_x(t,\omega) + \mathrm{ST}_x^{\mathcal{T}\mathcal{D}g}(t,\omega)$ we obtain

$$\frac{\partial}{\partial t}\left[\mathrm{Re}\left(\frac{\mathrm{ST}_x^{\mathcal{T}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)}\right)\right] = \mathrm{Re}\left(\frac{\frac{\partial \mathrm{ST}_x^{\mathcal{T}g}}{\partial t}(t,\omega)\mathrm{ST}_x(t,\omega) - \mathrm{ST}_x^{\mathcal{T}g}(t,\omega)\frac{\partial \mathrm{ST}_x}{\partial t}(t,\omega)}{\mathrm{ST}_x(t,\omega)^2}\right) \tag{135}$$

$$= 1 + \mathrm{Re}\left(\frac{\mathrm{ST}_x^{\mathcal{T}\mathcal{D}g}(t,\omega)\mathrm{ST}_x(t,\omega) - \mathrm{ST}_x^{\mathcal{T}g}(t,\omega)\mathrm{ST}_x^{\mathcal{D}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)^2}\right) \tag{136}$$

$$= \boxed{1 + \mathrm{Re}\left(\frac{\mathrm{ST}_x^{\mathcal{T}\mathcal{D}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)} - \frac{\mathrm{ST}_x^{\mathcal{T}g}(t,\omega)\mathrm{ST}_x^{\mathcal{D}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)^2}\right)}. \tag{137}$$

Using $\frac{\partial \mathrm{ST}_x^{\mathcal{D}g}}{\partial t}(t,\omega) = \mathrm{ST}_x^{\mathcal{D}^2 g}(t,\omega)$ we obtain

$$-\frac{\partial}{\partial t}\left[\mathrm{Im}\left(\frac{\mathrm{ST}_x^{\mathcal{D}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)}\right)\right] = -\mathrm{Im}\left(\frac{\frac{\partial \mathrm{ST}_x^{\mathcal{D}g}}{\partial t}(t,\omega)\mathrm{ST}_x(t,\omega) - \mathrm{ST}_x^{\mathcal{D}g}(t,\omega)\frac{\partial \mathrm{ST}_x}{\partial t}(t,\omega)}{\mathrm{ST}_x(t,\omega)^2}\right) \tag{138}$$

$$= \boxed{-\mathrm{Im}\left(\frac{\mathrm{ST}_x^{\mathcal{D}^2 g}(t,\omega)}{\mathrm{ST}_x(t,\omega)} - \left(\frac{\mathrm{ST}_x^{\mathcal{D}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)}\right)^2\right)}. \tag{139}$$

9.2 Computation of the partial frequency derivatives of $R_x(t,\omega)$

Using the following equalities (computation details in Section 7)

$$\frac{\partial \mathrm{ST}_x}{\partial \omega}(t,\omega) = \frac{1}{\omega}\mathrm{ST}_x(t,\omega) - \frac{\omega}{(\omega_0 T)^2}\mathrm{ST}_x^{\mathcal{T}^2 g}(t,\omega) + j(\mathrm{ST}_x^{\mathcal{T}g}(t,\omega) - t\mathrm{ST}_x(t,\omega)) \tag{140}$$

$$\text{and} \quad \frac{\partial \mathrm{ST}_x^{\mathcal{T}g}}{\partial \omega}(t,\omega) = \frac{1}{\omega}\mathrm{ST}_x^{\mathcal{T}g}(t,\omega) - \frac{\omega}{(\omega_0 T)^2}\mathrm{ST}_x^{\mathcal{T}^3 g}(t,\omega) + j(\mathrm{ST}_x^{\mathcal{T}^2 g}(t,\omega) - t\mathrm{ST}_x^{\mathcal{T}g}(t,\omega)) \tag{141}$$

we obtain

$$\frac{\partial}{\partial \omega}\left[\mathrm{Re}\left(\frac{\mathrm{ST}_x^{\mathcal{T}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)}\right)\right]$$

$$= \mathrm{Re}\left(\frac{\frac{\partial \mathrm{ST}_x^{\mathcal{T}g}}{\partial \omega}(t,\omega)\mathrm{ST}_x(t,\omega) - \mathrm{ST}_x^{\mathcal{T}g}(t,\omega)\frac{\partial \mathrm{ST}_x}{\partial \omega}(t,\omega)}{\mathrm{ST}_x(t,\omega)^2}\right) \tag{142}$$

$$= \mathrm{Re}\left(j\frac{\mathrm{ST}_x^{\mathcal{T}^2 g}(t,\omega)\mathrm{ST}_x(t,\omega) - \mathrm{ST}_x^{\mathcal{T}g}(t,\omega)^2}{\mathrm{ST}_x(t,\omega)^2} + \frac{\omega}{(\omega_0 T)^2}\frac{\mathrm{ST}_x^{\mathcal{T}^2 g}(t,\omega)\mathrm{ST}_x^{\mathcal{T}g}(t,\omega) - \mathrm{ST}_x^{\mathcal{T}^3 g}(t,\omega)\mathrm{ST}_x(t,\omega)}{\mathrm{ST}_x(t,\omega)^2}\right) \tag{143}$$

$$= \boxed{-\mathrm{Im}\left(\frac{\mathrm{ST}_x^{\mathcal{T}^2 g}(t,\omega)}{\mathrm{ST}_x(t,\omega)} - \left(\frac{\mathrm{ST}_x^{\mathcal{T}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)}\right)^2\right) + \frac{\omega}{(\omega_0 T)^2}\mathrm{Re}\left(\frac{\mathrm{ST}_x^{\mathcal{T}^2 g}(t,\omega)\mathrm{ST}_x^{\mathcal{T}g}(t,\omega) - \mathrm{ST}_x^{\mathcal{T}^3 g}(t,\omega)\mathrm{ST}_x(t,\omega)}{\mathrm{ST}_x(t,\omega)^2}\right)} \tag{144}$$

Using $\mathrm{ST}_x^{\mathcal{D}g}(t,\omega) = -\frac{\omega^2}{(\omega_0 T)^2}\mathrm{ST}_x^{\mathcal{T}g}(t,\omega)$ we have

$$\frac{\partial \mathrm{ST}_x^{\mathcal{D}g}}{\partial \omega}(t,\omega) = -\frac{\partial}{\partial \omega}\left[\frac{\omega^2}{(\omega_0 T)^2}\mathrm{ST}_x^{\mathcal{T}g}\right] \tag{145}$$

$$= -\frac{2\omega}{(\omega_0 T)^2}\mathrm{ST}_x^{\mathcal{T}g}(t,\omega) - \frac{\omega^2}{(\omega_0 T)^2}\left(\frac{\partial \mathrm{ST}_x^{\mathcal{T}g}}{\partial \omega}(t,\omega)\right) \tag{146}$$

$$= -\frac{-3\omega}{(\omega_0 T)^2}\mathrm{ST}_x^{\mathcal{T}g}(t,\omega) + \frac{\omega^3}{(\omega_0 T)^4}\mathrm{ST}_x^{\mathcal{T}^3 g}(t,\omega) - j\frac{\omega^2}{(\omega_0 T)^2}\mathrm{ST}_x^{\mathcal{T}^2 g}(t,\omega) + jt\frac{\omega^2}{(\omega_0 T)^2}\mathrm{ST}_x^{\mathcal{T}g}(t,\omega) \tag{147}$$

we obtain

$$-\frac{\partial}{\partial \omega}\left[\mathrm{Im}\left(\frac{\mathrm{ST}_x^{\mathcal{D}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)}\right)\right] \tag{148}$$

$$= -\mathrm{Im}\left(\frac{\frac{\partial \mathrm{ST}_x^{\mathcal{D}g}}{\partial \omega}(t,\omega)\mathrm{ST}_x - \mathrm{ST}_x^{\mathcal{D}g}\frac{\partial \mathrm{ST}_x}{\partial \omega}(t,\omega)}{\mathrm{ST}_x(t,\omega)^2}\right) \tag{149}$$

$$= -\mathrm{Im}\left(\frac{\frac{-2\omega}{(\omega_0 T)^2}\mathrm{ST}_x^{\mathcal{T}g}(t,\omega)\mathrm{ST}_x(t,\omega)}{\mathrm{ST}_x^2} + \frac{\omega^3}{(\omega_0 T)^4}\frac{\mathrm{ST}_x^{\mathcal{T}^3 g}(t,\omega)\mathrm{ST}_x(t,\omega) - \mathrm{ST}_x^{\mathcal{T}^2 g}(t,\omega)\mathrm{ST}_x^{\mathcal{T}g}(t,\omega))}{\mathrm{ST}_x(t,\omega)^2}\right.$$

$$\left. -j\frac{\omega^2}{(\omega_0 T)^2}\frac{\mathrm{ST}_x^{\mathcal{T}^2 g}(t,\omega)\mathrm{ST}_x(t,\omega) - (\mathrm{ST}_x^{\mathcal{T}g})^2}{\mathrm{ST}_x(t,\omega)^2}\right) \tag{150}$$

$$= \boxed{-\mathrm{Re}\left(\frac{\mathrm{ST}_x^{\mathcal{T}\mathcal{D}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)} - \frac{\mathrm{ST}_x^{\mathcal{T}g}(t,\omega)\mathrm{ST}_x^{\mathcal{D}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)^2}\right) - \frac{\omega^3}{(\omega_0 T)^4}\mathrm{Im}\left(\frac{\mathrm{ST}_x^{\mathcal{T}^3 g}(t,\omega)}{\mathrm{ST}_x(t,\omega)} - \frac{\mathrm{ST}_x^{\mathcal{T}^2 g}(t,\omega)\mathrm{ST}_x^{\mathcal{T}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)^2}\right)}$$

$$\boxed{+\frac{2\omega}{(\omega_0 T)^2}\mathrm{Im}\left(\frac{\mathrm{ST}_x^{\mathcal{T}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)}\right)} \tag{151}$$

10 Definition of the S-transform local instantaneous modulation operator

Let be $x(t) = a(t)\,\mathbf{e}^{j\phi(t)}$ a Gaussian-modulated linear chirp (*i.e.* $\phi(t)$ and $\log(a(t))$ functions are quadratic. Thus, the frequency modulation $\hat{q}_x$ can be defined as

$$\hat{q}_x(t,\omega) = \frac{d^2\phi}{dt^2}(t). \tag{152}$$

The proof of this result follows. By definition, $x(t)$ is differentiable and its derivative is

$$\frac{dx}{dt}(t) = \left(\frac{da}{dt}(t) + j\frac{d\phi}{dt}(t)a(t)\right)\mathbf{e}^{j\phi(t)} \tag{153}$$

$$= \underbrace{\left(\frac{d}{dt}\left(\log(a(t))\right) + j\frac{d\phi}{dt}(t)\right)}_{l(t)} x(t) \tag{154}$$

where l is an affine complex-valued function since $\log(a(t))$ and $\phi(t)$ are quadratic. Thus, there exists complex numbers α and β such that $l(t) = \alpha t + \beta$ with

$$\alpha = \frac{dl}{dt}(t) = \frac{d^2}{dt}\left(\log(a(t))\right) + j\frac{d^2\phi}{dt^2}(t). \tag{155}$$

Using $\mathrm{ST}_x(t,\omega) = \mathbf{e}^{-j\omega t}\int_{\mathbb{R}} x(t+\tau)g(-\tau,\sigma_t)\,\mathbf{e}^{-j\omega\tau}\,d\tau$ as the expression of the S-transform of a signal x, we obtain

$$\frac{\partial \mathrm{ST}_x}{\partial t}(t,\omega) = -j\omega\,\mathbf{e}^{-j\omega t}\int_{\mathbb{R}} x(t+\tau)g(-\tau,\sigma_t)\,\mathbf{e}^{-j\omega\tau}\,d\tau + \mathbf{e}^{-j\omega t}\int_{\mathbb{R}} \frac{dx}{dt}(t+\tau)g(-\tau,\sigma_t)\,\mathbf{e}^{-j\omega\tau}\,d\tau \tag{156}$$

$$= -j\omega \mathrm{ST}_x(t,\omega) + \mathbf{e}^{-j\omega t}\int_{\mathbb{R}} l(t+\tau)x(t+\tau)g(-\tau,\sigma_t)\,\mathbf{e}^{-j\omega\tau}\,d\tau \tag{157}$$

$$= -j\omega \mathrm{ST}_x(t,\omega) + \mathbf{e}^{-j\omega t}\alpha\int_{\mathbb{R}} x(t+\tau)(t+\tau)g(-\tau,\sigma_t)\,\mathbf{e}^{-j\omega\tau}\,d\tau + \beta\mathrm{ST}_x(t,\omega) \tag{158}$$

$$= -j\omega \mathrm{ST}_x(t,\omega) - \alpha\int_{\mathbb{R}} x(\tau)((t-\tau)-t)g(t-\tau,\sigma_t)\,\mathbf{e}^{-j\omega\tau}\,d\tau + \beta\mathrm{ST}_x(t,\omega) \tag{159}$$

$$= -j\omega \mathrm{ST}_x(t,\omega) - \alpha\left(\mathrm{ST}_x^{\mathcal{T}g}(t,\omega) - t\mathrm{ST}_x(t,\omega)\right) + \beta\mathrm{ST}_x(t,\omega). \tag{160}$$

Dividing by $\mathrm{ST}_x(t,\omega)$ we obtain

$$\frac{\frac{\partial \mathrm{ST}_x}{\partial t}(t,\omega)}{\mathrm{ST}_x(t,\omega)} = \frac{\mathrm{ST}_x^{\mathcal{D}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)} = -j\omega - \alpha\left(\frac{\mathrm{ST}_x^{\mathcal{T}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)} - t\right) + \beta. \tag{161}$$

Differentiating with respect to t, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\mathrm{ST}_x^{\mathcal{D}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)}\right) = -\alpha\left(\frac{\partial}{\partial t}\left(\frac{\mathrm{ST}_x^{\mathcal{T}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)}\right) - 1\right) \tag{162}$$

Thus, we have

$$\alpha = \frac{d^2}{dt}\left(\log(a(t))\right) + j\frac{d^2\phi}{dt^2}(t) = -\frac{\frac{\partial}{\partial t}\left(\frac{\mathrm{ST}_x^{\mathcal{D}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)}\right)}{\frac{\partial}{\partial t}\left(\frac{\mathrm{ST}_x^{\mathcal{T}g}(t,\omega)}{\mathrm{ST}_x(t,\omega)}\right) - 1} \tag{163}$$

that leads to

$$\hat{q}_x(t,\omega) = -\mathrm{Im}\left(\frac{\mathrm{ST}_x^{\mathcal{D}^2g}(t,\omega)\mathrm{ST}_x(t,\omega) - \mathrm{ST}_x^{\mathcal{D}g}(t,\omega)^2}{\mathrm{ST}_x^{\mathcal{T}\mathcal{D}g}(t,\omega)\mathrm{ST}_x(t,\omega) - \mathrm{ST}_x^{\mathcal{T}g}(t,\omega)\mathrm{ST}_x^{\mathcal{D}g}(t,\omega)}\right). \tag{164}$$

11 Implementation of the reassigned Gabor spectrogram and of the synchrosqueezed STFT

The discrete-time Gabor transform of a signal x can be approximated by $F_x^g(nT_s, \frac{2\pi m}{MT_s}) \approx F_x^g[n,m]$ with $n \in \mathcal{Z}$ and $m \in \mathcal{M}$ as

$$F_x^g[n,m] = \mathbf{e}^{-j\frac{2\pi mn}{M}}\sum_{k=-K/2}^{K/2} x[n+k]g[-k,L]\,\mathbf{e}^{-j\frac{2j\pi mk}{M}} \tag{165}$$

$$= \frac{\mathbf{e}^{-j\frac{2\pi mn}{M}}}{\sqrt{2\pi}L}\sum_{k=-K/2}^{K/2} x[n+k]\,\mathbf{e}^{-\frac{k^2}{2L^2}}\,\mathbf{e}^{-j\frac{2j\pi mk}{M}} \tag{166}$$

with $g[k,L] = T_s g(kT_s, \frac{T}{T_s}) = \frac{1}{\sqrt{2\pi}L}\,\mathbf{e}^{-\frac{n^2}{2L^2}}$, where $K = 2L\sqrt{2\log(1/\Gamma)}$ is obtained by setting a threshold Γ thus we have

$$\mathbf{e}^{-\frac{(K/2)^2}{2L^2}} \leq \Gamma. \tag{167}$$

The signal x can be recovered using the STFT synthesis formula as

$$\hat{x}[n] = \frac{1}{g(0,T)}\sum_{m\in\mathcal{M}} F_x^g[n,m]\,\mathbf{e}^{\frac{2j\pi mn}{M}}\Delta\omega \text{ with } \Delta\omega = \frac{2\pi}{MT_s} \tag{168}$$

$$= \frac{\sqrt{2\pi}}{ML}\sum_{m\in\mathcal{M}} F_x^g[n,m]\,\mathbf{e}^{\frac{2j\pi mn}{M}}. \tag{169}$$

11.1 Computation of the spectrogram and the reassigned spectrogram

The spectrogram is simply computed as $|F_x^g[n,m]|^2$ and the reassigned spectrogram is computed as

$$\mathrm{RV}_x[n,m] = \sum_{n\in\mathcal{Z}} \sum_{m\in\mathcal{Z}} |F_x^g[n,m]|^2 \delta[n - \hat{n}[n,m]]\delta[m - \hat{m}[n,m]] \tag{170}$$

where $\hat{n}$ and $\hat{m}$ correspond to the reassigned time-frequency coordinates computed as

$$\hat{n}[n,m] = n - \left[\!\!\left[\mathrm{Re}\left(\frac{T_s^{-1} F_x^{\mathcal{T}g}[n,m]}{F_x^g[n,m]} \right) \right]\!\!\right] \tag{171}$$

$$\hat{m}[n,m] = m + \left[\!\!\left[\mathrm{Im}\left(\frac{T_s F_x^{\mathcal{D}g}[n,m]}{F_x^g[n,m]} \right) \right]\!\!\right] \tag{172}$$

$$\tag{173}$$

where $\mathcal{T}g(t,T) = tg(t,T)$ and $\mathcal{D}g(t,T) = \frac{dg}{dt}(t,T) = \frac{-1}{T^2}\mathcal{T}g(t,T)$.

The Levenberg-Marquardt reassigned time-frequency coordinates are computed as

$$\begin{pmatrix} \tilde{n}[n,m] \\ \tilde{m}[n,m] \end{pmatrix} = \begin{pmatrix} n \\ m \end{pmatrix} - \left[\!\!\left[\left(\nabla^t R_x[n,m] + \mu I_2 \right)^{-1} R_x[n,m] \right]\!\!\right] \tag{174}$$

$$\text{with} \quad R_x[n,m] = \begin{pmatrix} n - \tilde{n}[n,m] \\ m - \tilde{m}[n,m] \end{pmatrix} \tag{175}$$

$$\nabla^t R_x[n,m] = \begin{pmatrix} 1 + \mathrm{Re}\left(\frac{F_x^{\mathcal{T}\mathcal{D}g}[n,m]}{F_x^g[n,m]} - \frac{F_x^{\mathcal{T}g}[n,m] F_x^{\mathcal{D}g}[n,m]}{F_x^g[n,m]^2} \right) & -\mathrm{Im}\left(\frac{T_s^{-2} F_x^{\mathcal{T}^2 g}[n,m]}{F_x^g[n,m]} - \left(\frac{T_s^{-1} F_x^{\mathcal{T}g}[n,m]}{F_x^g[n,m]} \right)^2 \right) \\ -\mathrm{Im}\left(\frac{T_s^2 F_x^{\mathcal{D}^2 g}[n,m]}{F_x^g[n,m]} - \left(\frac{T_s F_x^{\mathcal{D}g}[n,m]}{F_x[n,m]} \right)^2 \right) & -\mathrm{Re}\left(\frac{F_x^{\mathcal{T}\mathcal{D}g}[n,m]}{F_x^g[n,m]} - \frac{F_x^{\mathcal{T}g}[n,m] F_x^{\mathcal{D}g}[n,m]}{F_x^g[n,m]^2} \right) \end{pmatrix} \tag{176}$$

where $\mathcal{T}\mathcal{D}g(t,T) = t\frac{dg}{dt}(t,T)$, $\mathcal{T}^2 g(t,T) = t^2 g(t,T)$ and $\mathcal{D}^2 g(t,T) = \frac{d^2 g}{dt^2}(t,T)$. Thus, the Levenberg-Marquardt spectrogram is simply obtained by replacing $(\hat{n},\hat{m})$ by $(\tilde{n},\tilde{m})$ in Eq. (170).

11.2 Computation of the synchrosqueezed STFT

The discrete-time synchrosqueezed STFT can be defined as

$$T_x[n,m] = \frac{1}{2\pi} \sum_{m'\in\mathcal{M}} F_x^g[n,m']\, \mathbf{e}^{\frac{2j\pi m' n}{M}}\, \delta[m' - \hat{m}[n,m']] \tag{177}$$

where $(\hat{n},\hat{m})$ can be replaced by $(\tilde{n},\tilde{m})$ to obtain the Levenberg-Marquardt synchrosqueezed STFT.

The signal x can be recovered from the synchrosqueezed STFT as

$$\hat{x}[n] = \frac{1}{g(0,T)} \sum_{m'\in\mathcal{M}} T_x[n,m]\Delta\omega \tag{178}$$

$$= \frac{(2\pi)^{3/2} L}{M} \sum_{m'\in\mathcal{M}} T_x[n,m]. \tag{179}$$

11.3 Computation of the vertical synchrosqueezed STFT

Let be a signal $x(t) = A(t)\,\mathbf{e}^{j\phi(t)}$ with $\phi(t) = \phi_0 + \omega t + \frac{qt^2}{2}$. Thus we have

$$\frac{dx}{dt}(t) = \underbrace{\left(\frac{\frac{dA}{dt}(t)}{A(t)} + j\frac{d\phi}{dt}(t) \right)}_{l(t)} x(t) \tag{180}$$

thus, there exists $(\alpha,\beta) \in \mathbb{C}^2$ thus we have $l(t) = \alpha t + \beta$, $\frac{dl}{dt}(t) = \frac{d^2}{dt^2}\left(\log\left(A(t)\right)\right) + j\frac{d^2\phi}{dt^2}(t)$, hence $q = \mathrm{Im}(\alpha)$.

Thus, as $F_x^g(t,\omega) = \mathbf{e}^{-j\omega t} \int_{\mathbb{R}} x(t+\tau)g(-\tau,T)\,\mathbf{e}^{-j\omega\tau}$ we have

$$\frac{\partial}{\partial t}\left(F_x^g(t,\omega)\right) = -j\omega F_x^g(t,\omega) + \mathbf{e}^{-j\omega t}\int_{\mathbb{R}} \frac{dx}{dt}(t+\tau)g(-\tau,T)\,\mathbf{e}^{-j\omega\tau}d\tau \tag{181}$$

$$= -j\omega F_x^g(t,\omega) + \mathbf{e}^{-j\omega t}\int_{\mathbb{R}} (\alpha(t+\tau)+\beta)x(t+\tau)g(-\tau,T)\,\mathbf{e}^{-j\omega\tau}d\tau \tag{182}$$

$$= (\beta - j\omega + \alpha t)\,F_x^g(t,\omega) - \alpha\,\mathbf{e}^{-j\omega t}\int_{\mathbb{R}} x(t+\tau)(-\tau)g(-\tau,T)\,\mathbf{e}^{-j\omega\tau}d\tau \tag{183}$$

$$= (\beta - j\omega + \alpha t)\,F_x^g(t,\omega) - \alpha F_x^{\mathcal{T}g}(t,\omega) \tag{184}$$

then,

$$\frac{\frac{\partial}{\partial t}\left(F_x^g(t,\omega)\right)}{F_x^g(t,\omega)} = (\beta - j\omega + \alpha t) - \alpha\frac{F_x^{\mathcal{T}g}(t,\omega)}{F_x^g(t,\omega)} \tag{185}$$

then,

$$\frac{\partial}{\partial t}\left(\frac{\frac{\partial}{\partial t}\left(F_x^g(t,\omega)\right)}{F_x^g(t,\omega)}\right) = \alpha\left(1 - \frac{\partial}{\partial t}\left(\frac{F_x^{\mathcal{T}g}(t,\omega)}{F_x^g(t,\omega)}\right)\right) \tag{186}$$

$$\frac{F_x^{\mathcal{D}^2g}(t,\omega)F_x^g(t,\omega) - F_x^{\mathcal{D}g}(t,\omega)}{F_x^g(t,\omega)^2} = \alpha\left(1 - \frac{(F_x^g(t,\omega) + F_x^{\mathcal{T}\mathcal{D}g}(t,\omega))F_x^g(t,\omega) - F_x^{\mathcal{T}g}(t,\omega)F_x^{\mathcal{D}g}(t,\omega)}{F_x^g(t,\omega)^2}\right). \tag{187}$$

Hence we have

$$\alpha = \frac{F_x^{\mathcal{D}^2g}(t,\omega)F_x^g(t,\omega) - F_x^{\mathcal{D}g}(t,\omega)^2}{F_x^{\mathcal{T}g}(t,\omega)F_x^{\mathcal{D}g}(t,\omega) - F_x^{\mathcal{T}\mathcal{D}g}(t,\omega)F_x^g(t,\omega)}. \tag{188}$$

As a result, the discrete-time local frequency modulation can be estimated as

$$\hat{q}_x[n,m] = T_s^{-2}\mathrm{Im}\left(\frac{T_s^2F_x^{\mathcal{D}^2g}[n,m]F_x^g[n,m] - (T_sF_x^{\mathcal{D}g}[n,m])^2}{F_x^{\mathcal{T}g}[n,m]F_x^{\mathcal{D}g}[n,m] - F_x^{\mathcal{T}\mathcal{D}g}[n,m]F_x^g[n,m]}\right). \tag{189}$$

thus an enhanced frequency estimator is given by

$$\bar{m}_q[n,m] = \hat{m}[n,m] + \frac{M}{2\pi}T_s^2\hat{q}_x[n,m](n - \hat{n}[n,m]). \tag{190}$$

So finally, the vertical synchrosqueezed STFT is obtained by replacing $\hat{m}[n,m]$ by $\bar{m}_q[n,m]$ in Eq. (177).

12 Fourier transform of the Morlet Wavelet

According to the definition of the Morlet mother wavelet, we have

$$F_\Psi(\omega) = \int_{-\infty}^{+\infty} \Psi(t)\,\mathbf{e}^{-j\omega t}\,dt \tag{191}$$

$$= \frac{\pi^{-1/4}}{\sqrt{T}}\int_{-\infty}^{+\infty} \mathbf{e}^{-\frac{t^2}{2T^2}+j(\omega_0-\omega)t}\,dt \tag{192}$$

$$= \sqrt{2T}\pi^{1/4}\,\mathbf{e}^{-\frac{(\omega-\omega_0)^2T^2}{2}} \tag{193}$$