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Topological classification of Morse-Smale diffeomorphisms without heteroclinic curves on 3-manifolds

Ch. Bonatti V. Grines F. Laudenbach O. Pochinka *

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Abstract

We show that, up to topological conjugation, the equivalence class of a Morse-Smale diffeomorphism without heteroclinic curves on 3-manifold is completely defined by an embedding of two-dimensional stable and unstable heteroclinic laminations to a characteristic space.

Key words: Morse-Smale diffeomorphism, topological classification, heteroclinic lamination
MSC: 37C05, 37C15, 37C29, 37D15.

1 Introduction and formulation of the result

In 1937 A. Andronov and L. Pontryagin [2] introduced the notion of *rough system* of differential equation given in a bounded part of the plane, that is a system which keeps its qualitative properties after small changes in the right-hand side. They proved that the flows generated by such systems are exactly the flows having the next properties:

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- the set of fixed points and periodic orbits is finite and all its elements are hyperbolic;
- there are no separatrices going from one saddle to itself or to another one;
- all ω - and α -limit sets are contained in the union of fixed points and periodic orbits (limit cycles).

The above description characterizes the rough flows on the two-dimensional sphere also. Similar result for flows with closed section and without equilibrium states on two-dimension tori follows from paper of A. Mayer [16] in 1939. A. Andronov and L. Pontryagin have shown also in [2] that the set of the rough flows is dense in the space of C^1 -flows¹. Similar criterion and density were proved by M. Peixoto [22], [23] in 1962 for *structural stable flows* on orientable surfaces of genus greater than zero. Direct generalization of the properties of rough flows on surfaces leads to the following class of dynamical systems (continuous and discrete — flows and diffeomorphisms (cascades)).

Definition 1.1 *A smooth dynamical system given on an n -dimensional manifold ($n \geq 1$) M^n is called Morse-Smale if: 1) its non-wandering set consists of a finite number of fixed points and periodic orbits where each of them is hyperbolic; 2) the stable and unstable manifolds W_p^s , W_q^u of any pair of non-wandering points p and q intersect transversely².*

Let M be a given closed 3-dimensional manifold and $f : M \rightarrow M$ be a Morse-Smale diffeomorphism.

For $q = 0, 1, 2, 3$ denote by Ω_q the set of all periodic points of f with q -dimensional unstable manifold. Let Ω_f be the union of all periodic points. Let us represent the dynamics of f in the form “source-sink” in the following way. Set

$$A_f = W_{\Omega_0 \cup \Omega_1}^u, \quad R_f = W_{\Omega_2 \cup \Omega_3}^s, \quad V_f = M \setminus (A_f \cup R_f).$$

Due to Theorem 1.1 in [11] the set A_f (resp. R_f) is an attractor (resp. a repeller)³ of f whose topological dimension is equal to 0 or 1. Due to Theorem 1.2 in [11] the set V_f is a connected 3-manifold and $V_f = W_{A_f \cap \Omega_f}^s \setminus A_f = W_{R_f \cap \Omega_f}^u \setminus R_f$. Moreover, the quotient $\hat{V}_f = V_f/f$ is a closed connected 3-manifold and when \hat{V}_f is orientable, then it is either irreducible or diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$. The natural projection $p_f : V_f \rightarrow \hat{V}_f$ is an infinite cyclic covering. Therefore, there is a natural epimorphism from the the first homology group of \hat{V}_f to \mathbb{Z} ,

$$\eta_f : H_1(\hat{V}_f; \mathbb{Z}) \rightarrow \mathbb{Z},$$

¹This statement was not explicitly formulated in [2] and was mentioned first time in papers of E. Leontovich [15] and M. Peixoto [21]. G. Baggis [3] in 1955 made explicit some details of the proofs in [2], which were not published.

²Two smooth submanifolds X_1, X_2 of an n -manifold X intersect transversely if either $X_1 \cap X_2 = \emptyset$ or $T_x X_1 + T_x X_2 = T_x X$ for each point $x \in (X_1 \cap X_2)$ (here $T_x A$ stands for tangent space to the manifold A at the point x).

³A compact set $A \subset M$ is an attractor of a diffeomorphism $f : M \rightarrow M$ if there is a neighborhood U of the set A such that $f(U) \subset \text{int } U$ and $A = \bigcap_{n \in \mathbb{N}} f^n(U)$. A set $R \subset M$ is called a repeller of f if it is an attractor of f^{-1} .

defined as follows: if γ is a path in V_f joining x to $f^n(x)$, $n \in \mathbb{Z}$, then η_f maps the homology class of the cycle $p_f \circ \gamma$ to n . Set:

$$\begin{aligned}\Gamma_f^s &:= W_{\Omega_1}^s \setminus A_f \text{ and } \hat{\Gamma}_f^s := p_f(\Gamma_f^s); \\ \Gamma_f^u &:= W_{\Omega_2}^u \setminus R_f \text{ and } \hat{\Gamma}_f^u := p_f(\Gamma_f^u).\end{aligned}$$

Definition 1.2 *The sets $\hat{\Gamma}_f^s$ and $\hat{\Gamma}_f^u$ are called the two-dimensional stable and unstable laminations associated with the diffeomorphism f .*

A precise definition of what is a lamination will be given in Definition 2.1.

Definition 1.3 *The collection $S_f = (\hat{V}_f, \eta_f, \hat{\Gamma}_f^s, \hat{\Gamma}_f^u)$ is called the scheme of the diffeomorphism f .*

Definition 1.4 *The schemes S_f and $S_{f'}$ of two Morse-Smale diffeomorphisms $f, f' : M \rightarrow M$ are said to be equivalent if there is a homeomorphism $\hat{\varphi} : \hat{V}_f \rightarrow \hat{V}_{f'}$ with following properties:*

- 1) $\eta_f = \eta_{f'} \hat{\varphi}_*$;
- 2) $\hat{\varphi}(\hat{\Gamma}_f^s) = \hat{\Gamma}_{f'}^s$ and $\hat{\varphi}(\hat{\Gamma}_f^u) = \hat{\Gamma}_{f'}^u$.

Using the notion of the scheme above in a series of papers by Ch. Bonatti, V. Grines, V. Medvedev, E. Pecou, O. Pochinka [5], [7], [8], [9], the problem of topological classification⁴ of Morse-Smale diffeomorphisms on 3-manifolds was solved in different particular cases.

In the present article, we give the topological classification of the Morse-Smale diffeomorphisms belonging to the subset $G(M)$ of the Morse-Smale diffeomorphisms $f : M \rightarrow M$ which have no heteroclinic curves (see Section 2). According to [6], when the ambient manifold is orientable, then it is either sphere \mathbb{S}^3 or the connected sum of a finite number copies of $\mathbb{S}^2 \times \mathbb{S}^1$.

Theorem 1 *Two Morse-Smale diffeomorphisms in $G(M)$ are topologically conjugate if and only if their schemes are equivalent.*

2 Dynamics of diffeomorphisms in the class $G(M)$

In this section we introduce some notions connected with Morse-Smale diffeomorphisms on 3-manifold M . More detailed information on Morse-Smale diffeomorphisms is contained in [13] for example.

Let $f : M \rightarrow M$ be a Morse-Smale diffeomorphism. If x is a periodic point its *Morse index* is the dimension of its unstable manifold W_x^u ; the point x is called a *saddle point* when its two invariant manifolds have positive dimension, that is, its Morse index is not extremely. *Sink point* has Morse index 0 and *source point* has Morse index 3. The following notions are key concepts for describing the structure of intersection of stable and unstable manifolds of saddle periodic points. If x, y are distinct periodic saddle points of f and $W_x^u \cap W_y^s \neq \emptyset$, then:

⁴Recall that two diffeomorphisms f, f' given on M are said to be *topologically conjugate* if there is a homeomorphism $h : M \rightarrow M$ which satisfies $f'h = hf$.

- if $\dim W_x^s < \dim W_y^s$, any connected component of $W_x^u \cap W_y^s$ is 1-dimensional and called a *heteroclinic curve* (see figure 1);
- if $\dim W_x^s = \dim W_y^s$, the set $W_x^u \cap W_y^s$ is countable; each of its points is called a *heteroclinic point*; the orbit of a heteroclinic point is called a *heteroclinic orbit*.

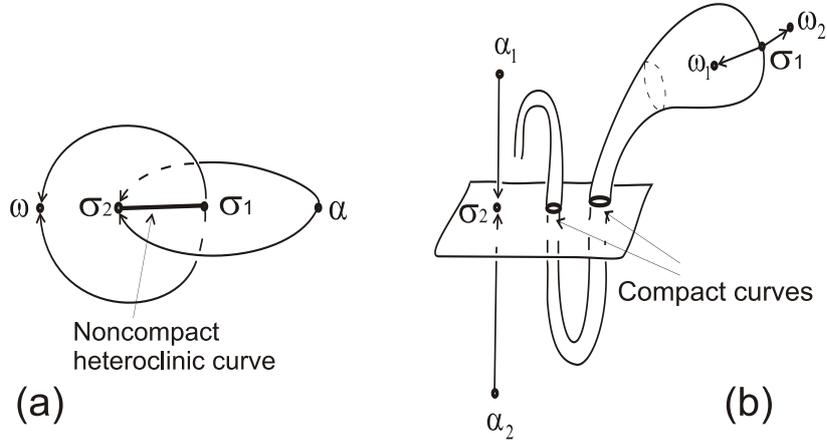


Figure 1: Heteroclinic curves

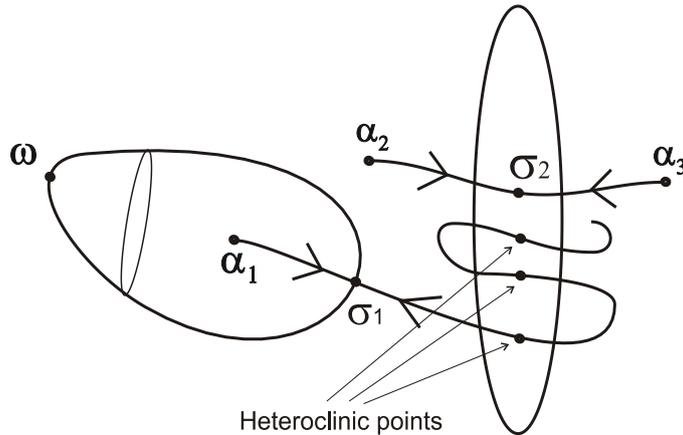


Figure 2: Heteroclinic points

According to S. Smale [24], it is possible to define a partial order in the set of saddle points of a given Morse-Smale diffeomorphism f as follows: for different periodic orbits $p \neq q$, one sets $p \prec q$ if and only if $W_q^u \cap W_p^s \neq \emptyset$. Smale proved that this relation is a partial order. In that case, it follows from lemma 1.5 of [18] that there is a sequence of different periodic orbits p_0, \dots, p_n satisfying the following conditions: $p_0 = p$, $p_n = q$ and $p_i \prec p_{i+1}$. The sequence p_0, \dots, p_n is said to be an *n-chain connecting p to q*. The length of the longest chain connecting p to q is denoted by $beh(q|p)$. If $W_q^u \cap W_p^s = \emptyset$, we pose $beh(q|p) = 0$. For a subset P of the

periodic orbits let us set $beh(q|P) = \max_{p \in P} \{beh(q|p)\}$. The present paper is devoted to studying Morse-Smale diffeomorphisms in dimension 3 which have no heteroclinic curve. We recall from the introduction that this class of diffeomorphisms is denoted by $G(M)$. Let $f \in G(M)$. It follows from [11] that if the set Ω_2 is empty then R_f consists of unique source. If $\Omega_2 \neq \emptyset$, denote by n the length of the longest chain connecting two points of Ω_2 . Divide the set Ω_2 into f -invariant disjoint parts $\Sigma_0, \Sigma_1, \dots, \Sigma_n$ using the rule: $beh(q|(\Omega_2 \setminus q)) = 0$ for each orbit $q \in \Sigma_0$ and $beh(q|\Sigma_i) = 1$ for each orbit $q \in \Sigma_{i+1}$, $i \in \{0, \dots, n-1\}$. Since Ω_1 for f is Ω_2 for f^{-1} , then it is possible to divide the periodic orbits of the set Ω_1 into parts in a similar way. The absence of heteroclinic curves means that there are no chains connecting a saddle from Ω_2 with a saddle from Ω_1 . Thus we explain all material for Ω_2 and say that all is similar for Ω_1 .

Set $W_i^u := W_{\Sigma_i}^u$, $W_i^s := W_{\Sigma_i}^s$. Then, $R_f := \bigcup_{i=0}^n cl(W_i^s)$, where $cl(\cdot)$ stands for the closure of (\cdot) . We now specify what a lamination is and which sort of regularity it may have.

Definition 2.1 *Let X be a n -dimensional and $Y \subset X$ be a closed subset. Let q be an integer $0 < q < n$. A codimension- q lamination with support Y is a decomposition $Y = \bigcup_{j \in J} L_j$ into pairwise disjoint smooth $(n-q)$ -dimensional connected manifolds L_j , which are called the leaves. The family $L = \{L_j, j \in J\}$ is said to be a $C^{1,0}$ -lamination⁵ if for every point $x \in Y$ the following conditions hold:*

- 1) *There are an open neighborhood $U_x \subset X$ of x and a homeomorphism $\psi : U_x \rightarrow \mathbb{R}^n$ such that ψ maps every plaque, that is a connected component of $U_x \cap L_j$, into a codimension- q subspace $\{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^{n-q+1} = c^{n-q+1}, \dots, x^n = c^n\}$. If $Y = X$ one says that \mathcal{L} is foliation.*
- 2) *The tangent plane field $TY := \bigcup_{j \in J} TL_j$ exists on Y and is continuous.*

By abuse, the lamination and its support are generally denoted in the same way. We recall the λ -Lemma in the strong form which is proved in [19, Remarks p. 85].

Lemma 2.2 (λ -lemma.) *Let $f : X \rightarrow X$ be a diffeomorphism of an n -manifold, and let p be a fixed point of f . Denote W_p^u and W_p^s be the unstable and stable manifold respectively; say $\dim W_p^u = m$, $0 < m < n$. Let B^s be a compact subset of W_p^s (containing p or not) and let $F : B^s \rightarrow C^1(\mathbb{D}^m, X)$ be a continuous family of embedded closed m -disks of class C^1 transverse to W_p^s ; set $F(x) := D_x^u$. Let $D^u \subset W_p^u$ be a compact m -disk and let $V \subset X$ be a compact n -ball such that D^u is a connected component of $W_p^u \cap V$. Then, when k goes to $+\infty$, the sequence $f^k(D_x^u) \cap V$ converges to D^u in the C^1 topology uniformly for $x \in B^s$.*

Notice that it is important for applications that B^s may not contained the point p . Going back to our setting, a first application of the λ -lemma is that we have $W_i^u \subset cl(W_{i+1}^u)$ and the closure of $cl(W_0^u) = W_0^u \cup \Omega_0$. Moreover, $cl(W_n^u) \cap (M \setminus \Omega_0)$ is a $C^{1,0}$ -lamination of codimension one. From this one derives that \hat{L}_f^u is also a $C^{1,0}$ -lamination. Here is a typical example.

⁵There are different possible notations. Here, we use the one which is given in [10, Definition 1.1.22].

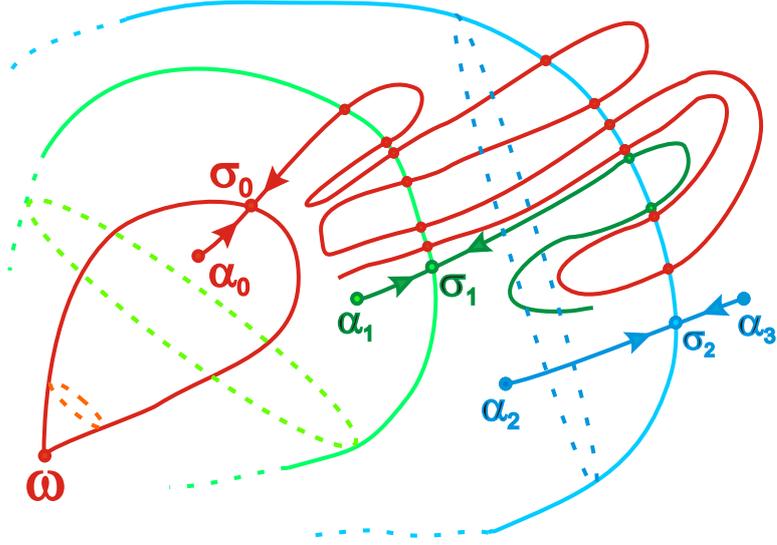


Figure 3: A phase portrait of a diffeomorphism from the class $G(M)$.

On Figure 3 there is a phase portrait of a diffeomorphism $f \in G(M)$ whose non-wandering set Ω_f consists of fixed points: one sink ω , three saddle points $\Sigma_0 = \sigma_0, \Sigma_1 = \sigma_1, \Sigma_2 = \sigma_2$ with two-dimensional unstable manifolds and four sources $\alpha_0, \alpha_1, \alpha_2, \alpha_3$. All further proofs we will illustrate on this diffeomorphism. For this case $V_f := W_\omega^s \setminus \{\omega\}$. As the restriction of f to the basin W_ω^s of ω is topologically conjugate to any homothety, \hat{V}_f is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$. As $f|_{W_i^u}$ is topologically conjugate to a homothetic then $(W_i^u \setminus \Sigma_i)/f$ is diffeomorphic to the 2-torus; but this torus does not embed to \hat{V}_f , except when $i = 0$. On Figure 4 there is the lamination associated with the diffeomorphism $f \in G(M)$ whose phase portrait is on Figure 3. On the left, the lamination is embedded in $\mathbb{S}^2 \times \mathbb{S}^1$. We represent the manifold $\mathbb{S}^2 \times \mathbb{S}^1$ as 3-dimensional annulus, suggesting that its two boundary components are *radially* identified. On the right, the lamination is embedded in \mathbb{R}^3 . We are going to show that the topological classification of diffeomorphisms in the class $G(M)$ reduces to classifying some appropriate laminations $\hat{\Gamma}_f^u$ and $\hat{\Gamma}_f^s$. The technical key to the proof consists of constructing special foliations in some neighborhoods of the laminations.

3 Compatible foliations

Let $f \in G(M)$. Recall that we divided the set Ω_2 into the f -invariant parts $\Sigma_0, \dots, \Sigma_n$. Using this partition, we explain how to construct compatible foliations (see definition 3.3) around $W_{\Omega_2}^u \cup W_{\Omega_2}^s$. Similarly, it is possible to construct compatible foliations around $W_{\Omega_1}^s \cup W_{\Omega_1}^u$. In what follows, we give ourselves four models of direct hyperbolic linear isomorphisms $a_{\mu,\nu} \in$

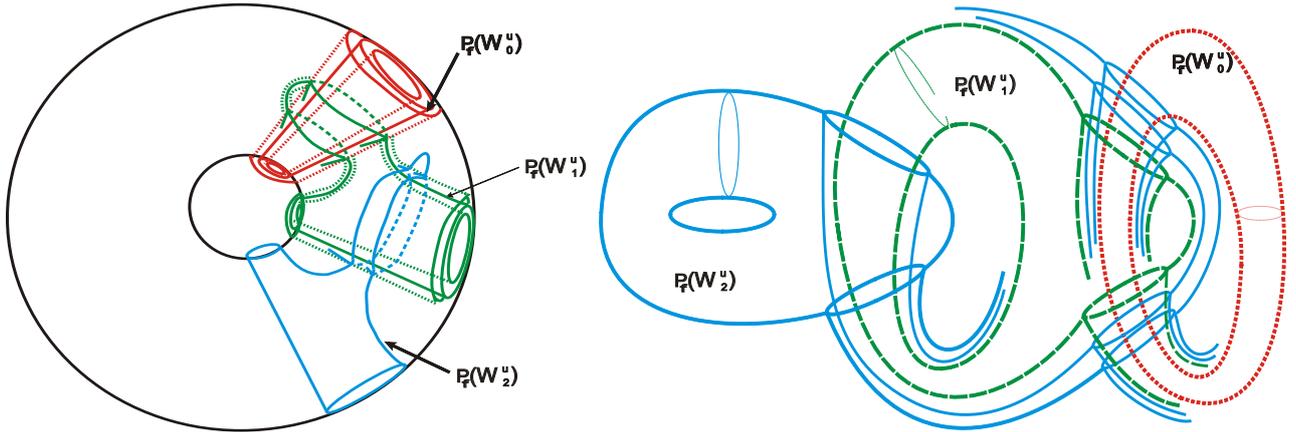


Figure 4: A lamination associated with the diffeomorphism $f \in G(M)$ whose phase portrait is pictured in Figure 3.

$GL(\mathbb{R}^3)$, $\mu, \nu \in \{-, +\}$ given by the following formula:

$$a_{\mu, \nu}(x_1, x_2, x_3) = (\mu 2x_1, 2x_2, \nu \frac{x_3}{4}).$$

The origin O is the unique fixed point which is a saddle point with unstable manifold $W_O^u = Ox_1x_2$ and stable manifold $W_O^s = Ox_3$. If $\mu = +$ (resp. $-$), the orientation of the unstable manifold is preserved (resp. reversed), and similarly for the orientation of the stable manifold with respect to ν . We refer to each of them as the *canonical diffeomorphism*, noted by a without taking \pm into account. For $p \in \Omega_2$, let $per(p)$ denote the period of f at p .

Definition 3.1 A neighborhood N_p of a saddle point $p \in \Omega_2$ is called linearizable if there is a homeomorphism $\mu_p : N_p \rightarrow \mathcal{N}$ which conjugates the diffeomorphism $f^{per(p)}|_{N_p}$ to the canonical diffeomorphism $a|_{\mathcal{N}}$.

According to the local topological classification of hyperbolic fixed point [19, Theorem 5.5], every $p \in \Omega_2$ has a linearizable neighbourhood N_p . For $t \in (0, 1)$, set $\mathcal{N}^t := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid -t < (x_1^2 + x_2^2)x_3 < t\}$ and $\mathcal{N} := \mathcal{N}^1$. The set \mathcal{N}^t is invariant by the canonical diffeomorphism a . By [24], W_p^s and W_p^u are smooth submanifolds of M . The *boundary* of \mathcal{N} is the surface in \mathbb{R}^3 defined by the equations $(x_1^2 + x_2^2)x_3 = \pm 1$. The open manifold N_p has a similar boundary in M denoted by ∂N_p . This boundary is formed by points which are not in N_p but limit points of arcs in N_p ; it is distinct from its closure as a subset of M . Clearly, the linearizing homeomorphism μ_p extends to ∂N_p . For each $i \in \{0, \dots, n\}$, choose some $p \in \Sigma_i$ and μ_p conjugating $f^{per(p)}$ to $a|_{\mathcal{N}}$. Then, for $k \in \{1, \dots, per(p) - 1\}$ define $\mu_{f^k(p)}$ so that the next formula holds for every $x \in N_{f^{k-1}(p)}$:

$$\mu_{f^k(p)}(f(x)) = \mu_{f^{k-1}(p)}(x).$$

We define a pair of transverse foliations $(\mathcal{F}^u, \mathcal{F}^s)$ in \mathcal{N} in the following way: – the leaves of \mathcal{F}^u are the fibres in \mathcal{N} of the projection $(x_1, x_2, x_3) \mapsto x_3$; – the leaves of \mathcal{F}^s are the fibres in \mathcal{N} of the projection $(x_1, x_2, x_3) \mapsto (x_1, x_2)$. By construction, W_O^u and W_O^s are leaves of \mathcal{F}^u and \mathcal{F}^s respectively. Let N_i denote the union $\bigcup_{p \in \Sigma_i} N_p$. This is an f -invariant neighbourhood of Σ_i .

Let $\mu_i : N_i \rightarrow \mathcal{N}$ be the map whose restriction to N_p is μ_p . Thus, taking the pullback of them by μ_i gives a pair of f -invariant foliations (F_i^u, F_i^s) on N_i which are said to be *linearizable*. By construction, W_i^u and W_i^s are made of leaves of F_i^u and F_i^s respectively. Sometimes we want to deform the linearizable neighborhood N_p by *shrinking*. Observe that the homotheties of ratio $\rho \in (0, 1)$ act on \mathcal{N} preserving \mathcal{F}^u and \mathcal{F}^s and map \mathcal{N} to \mathcal{N}^{ρ^3} . By conjugation, similar contractions c_ρ are available in N_p for every $p \in \Omega_2$. The neighborhood $c_\rho(N_p)$ is said to be obtained from N_p by *shrinking*.

Lemma 3.2 *For every $\rho \in (0, 1)$, the shrunk neighbourhood $c_\rho(N_p)$ is linearizable. Generically, the boundary of $c_\rho(N_p)$ does not contain any heteroclinic point.*

Proof: For a given μ_p , we define μ_p^ρ as follows: its domain is $\mu_p^{-1}(\mathcal{N}^{\rho^3})$ and, on this domain, it is defined by $\mu_p^\rho = c_\rho^{-1} \circ \mu_p$. Its range is \mathcal{N} . Since the heteroclinic points form a countable set, for almost every $\rho \in (0, 1)$ the boundary of the domain of μ_p^ρ avoids the heteroclinic points. \diamond

Observe that the canonical diffeomorphism and the contraction c_ρ keep both foliations invariant. Recall the f -invariant partition $\Omega_2 = \Sigma_0 \sqcup \Sigma_1 \sqcup \dots \sqcup \Sigma_n$. Let us introduce the following notations:

- for any $t \in (0, 1)$, set $N_p^t := \mu_p^{-1}(\mathcal{N}^t)$ and $N_i^t := \bigcup_{p \in \Sigma_i} N_p^t$;
- for any point $x \in N_i$, denote $F_{i,x}^u$ (resp. $F_{i,x}^s$) the leaf of the foliation F_i^u (resp. F_i^s) passing through x ;
- for each point $x \in N_i$, set $x_i^u = W_i^u \cap F_{i,x}^s$ and $x_i^s = W_i^s \cap F_{i,x}^u$. Thus, we have $x = (x_i^u, x_i^s)$ in the coordinates defined by μ_i .

We also introduce the *radial functions* $r_i^u, r_i^s : N_i \rightarrow [0, +\infty)$ defined by:

$$r_i^u(x) = \|\mu_i(x_i^u)\|^2 \quad \text{and} \quad r_i^s(x) = |\mu_i(x_i^s)|.$$

With this definition at hand, the neighborhood N_i^t of Σ_i is defined by the inequality

$$r_i^u(x) \cdot r_i^s(x) < t.$$

Observe that the radial function r_i^s endows each stable separatrix of $p \in \Sigma_p$ with a natural order which will be used later in the proof of Theorem 1.

Definition 3.3 *The linearizable neighborhoods N_0, \dots, N_n are called compatible if, for any $0 \leq i < j \leq n$ and $x \in N_i \cap N_j$, the following holds:*

$$F_{j,x}^s \cap N_i \subset F_{i,x}^s \quad \text{and} \quad F_{i,x}^u \cap N_j \subset F_{j,x}^u.$$

If linearizable neighbourhoods are compatible, they remain so after some of them are shrunk.

Remark 3.4 The notion of compatible foliations is a modification of the admissible systems of tubular families introduced by J. Palis and S. Smale in [18] and [20].

We introduce the following notations:

- For $i \in \{0, \dots, n\}$, set $A_i := A_f \cup \bigcup_{j=0}^i W_j^u$,
 $V_i := W_{A_i \cap \Omega_f}^s \setminus A_i$, $\hat{V}_i := V_i/f$. Observe that f acts freely on V_i and denote the natural projection by $p_i : V_i \rightarrow \hat{V}_i$.
- For $j, k \in \{0, \dots, n\}$ and $t \in (0, 1)$, set $\hat{W}_{j,k}^s = p_k(W_j^s \cap V_k)$, $\hat{W}_{j,k}^u = p_k(W_j^u \cap V_k)$,
 $\hat{N}_{j,k}^t = p_k(N_j^t \cap V_k)$.
- $L^u := \bigcup_{i=0}^n W_i^u$, $L^s := \bigcup_{i=0}^n W_i^s$, $L_i^u := L^u \cap V_i$, $L_i^s := L^s \cap V_i$, $\hat{L}_i^u := p_i(L_i^u)$, $\hat{L}_i^s := p_i(L_i^s)$.

Theorem 2 For each diffeomorphism $f \in G(M)$ there exist compatible linearizable neighborhoods of all saddle points whose Morse index is 2.

Proof: The proof consists of three steps.

Step 1. Here, we prove the following claim.

Lemma 3.5 There exist f -invariant neighborhoods U_0^s, \dots, U_n^s of the sets $\Sigma_0, \dots, \Sigma_n$ respectively, equipped with two-dimensional f -invariant foliations F_0^u, \dots, F_n^u of class $C^{1,0}$ such that the following properties hold for each $i \in \{0, \dots, n\}$:

- (i) the unstable manifolds W_i^u are leaves of the foliation F_i^u and each leaf of the foliation F_i^u is transverse to L_i^s ;
- (ii) for any $0 \leq i < k \leq n$ and $x \in U_i^s \cap U_k^s$, we have the inclusion $F_{k,x}^u \cap U_i^s \subset F_{i,x}^u$.

Proof: Let us prove this by a decreasing induction on i from $i = n$ to $i = 0$. For $i = n$, it follows from the definition of V_n that $(W_n^s \setminus \Sigma_n) \subset V_n$. Since f acts freely and properly on W_n^s , the quotient $\hat{W}_{n,n}^s$ is a smooth submanifold of \hat{V}_n ; it consists of finitely many circles. The lamination \hat{L}_n^s accumulates on $\hat{W}_{n,n}^s$. Choose an open tubular neighborhood \hat{N}_n^s of $\hat{W}_{n,n}^s$ in \hat{V}_n ; denote its projection by $\pi_n^u : \hat{N}_n^s \rightarrow \hat{W}_{n,n}^s$. Its fibers form a 2-disc foliation $\{d_{n,x}^u \mid x \in \hat{W}_{n,n}^s\}$ transverse to $\hat{W}_{n,n}^s$. Since \hat{L}_n^s is a $C^{1,0}$ -lamination containing $\hat{W}_{n,n}^s$, each plaque of $\hat{W}_{n,n}^s$ is the C^1 -limit of any sequence of plaques approaching it C^0 . Therefore, if the tube \hat{N}_n^s is small enough, its fibers are transverse to \hat{L}_n^s .

Set $U_n^s := p_n^{-1}(\hat{N}_n^s) \cup W_n^u$. This is an open set of M which carries a foliation F_n^u defined by taking the preimage of the fibers of π_n^u and by adding W_n^u as extra leaves. This is the wanted foliation satisfying (i) and (ii) for $i = n$. Notice that the plaques of F_n^u are smooth and by the λ -lemma, for any compact disc B in W_n^u there is $\varepsilon > 0$ such that every plaque of F_n^u which is

ε -close to B in topology C^0 is also ε -close to B in topology C^1 . Hence, F_n^u is a $C^{1,0}$ -foliation. For the induction, we assume the construction is done for every $j > i$ and we have to construct an f -invariant neighborhood U_i^s of the saddle points in Σ_i carrying an f -invariant foliation F_i^u satisfying (i) and (ii). Moreover, by genericity the boundary ∂U_j^s , $j > i$, is assumed to avoid all heteroclinic points. For $j > i$, let $\hat{U}_{j,i}^s := p_i(U_j^s \cap V_i)$ and $\hat{F}_{j,i}^u := p_i(F_j^u \cap V_i)$. For the same reason as in the case $i = n$, the set $\hat{W}_{i,i}^s$ is a smooth submanifold of \hat{V}_i consisting of circles. Choose a tubular neighborhood \hat{N}_i^s of $\hat{W}_{i,i}^s$ with a projection $\pi_i^s : \hat{N}_i^s \rightarrow \hat{W}_{i,i}^s$ whose fibers are 2-discs. Similarly, $\hat{W}_{i+1,i}^u$ is a compact submanifold, consisting of finitely many tori or Klein bottles. The set \hat{L}_i^u is a compact lamination and its intersection with $\hat{W}_{i,i}^s$ consists of a countable set of points which are the projections of the heteroclinic points belonging to the stable manifolds W_i^s . Actually, there is a hierarchy in $\hat{L}_i^u \cap \hat{W}_{i,i}^s$ which we are going to describe in more details.

Set $H_k := \hat{W}_{i+k,i}^u \cap \hat{W}_{i,i}^s$ for $k > 0$. Since $\hat{W}_{i+1,i}^u$ is compact, H_1 is a finite set: $H_1 = \{h_1^1, \dots, h_{t(1)}^1\}$. We are given neighborhoods, called *boxes*, B_ℓ^1 , $\ell = 1, \dots, t(1)$, about these points, namely, the connected components of $\hat{U}_{i+1,i}^s \cap \hat{N}_i^s$. Due to the fact that $\partial \hat{U}_{i+1,i}^s$ contains no heteroclinic point, $\partial \hat{U}_{i+1,i}^s \cap \hat{W}_{i,i}^s$ is isolated from \hat{L}_i^u . Therefore, if the tube \hat{N}_i^s is small enough, \hat{L}_i^u does not intersect $\partial \hat{U}_{i+1,i}^s \cap \hat{N}_i^s$. Then, by shrinking U_j^s , $j > i + 1$ (in the sense of Lemma 3.2) if necessary, we may guarantee that $\hat{U}_{j,i}^s \cap \hat{N}_i^s$ is disjoint from $\partial \hat{U}_{i+1,i}^s \cap \hat{N}_i^s$.

Since $\hat{W}_{i+2,i}^u$ accumulates on $\hat{W}_{i+1,i}^u$, there are only finitely many points of H_2 outside of all boxes B_ℓ^1 , $\ell = 1, \dots, t(1)$. Let $\bar{H}_2 := \{h_1^2, \dots, h_{t(2)}^2\}$ be this finite set. The open set $\hat{U}_{i+2,i}^s$ is a neighborhood of \bar{H}_2 . The connected components of $\hat{U}_{i+2,i}^s \cap \hat{N}_i^s$ which contain points of \bar{H}_2 will be the box B_ℓ^2 for $\ell = 1, \dots, t(2)$. We argue with B_ℓ^2 with respect to \hat{L}_i^u and the neighborhoods $\hat{U}_{j,i}$, $j > i + 1$, in a similar manner as we do with B_ℓ^1 . And so on, until \bar{H}_n .

Due to the induction hypothesis, each above-mentioned box is foliated. Namely, B_ℓ^1 is foliated by $\hat{F}_{i+1,i}^u$; the box B_ℓ^2 is foliated by $\hat{F}_{i+2,i}^u$, and so on. But the leaves are not contained in fibres of \hat{N}_i^s ; even more, not every leaf intersects $\hat{W}_{i,i}^s$. We have to correct this situation in order to construct the foliation F_i^u satisfying the wanted conditions (i) and (ii). For every $j > i$, the foliation F_j^u may be extended to the boundary ∂U_j^s and a bit beyond. Once this is done, if \hat{N}_i^s is enough shrunk, each leaf of $\hat{F}_{i+k,i}^u$ through $x \in B_\ell^k$ intersects $\hat{W}_{i,i}^s$ (it is understood that the boxes are intersected with the shrunk tube without changing their names). Thus, we have a projection along the leaves $\pi_{k,\ell} : B_\ell^k \rightarrow \hat{W}_{i,i}^s$; but, the image of $\pi_{k,\ell}$ is larger than $B_\ell^k \cap \hat{W}_{i,i}^s$. Then, we choose a small enlargement $B_\ell^{k'}$ of B_ℓ^k such that $B_\ell^{k'} \setminus B_\ell^k$ is foliated by $\hat{F}_{i+k,i}^u$ and avoids the lamination \hat{L}_i^u . On $B_\ell^{k'} \setminus B_\ell^k$ we have two projections: one is $\hat{\pi}_i^u$ and the other one is $\pi_{k,\ell}$. We are going to interpolate between both using a partition of unity (we do it for B_ℓ^k but it is understood that it is done for all boxes). Let $\phi : \hat{N}_i^s \rightarrow [0, 1]$ be a smooth function which equals 1 near B_ℓ^k and whose support is contained in $B_\ell^{k'}$. Define a global C^1 retraction $\hat{q} : \hat{N}_i^s \rightarrow \hat{W}_{i,i}^s$ by the formula

$$\hat{q}(x) = (1 - \phi(x))\hat{\pi}_i^u(x) + \phi(x)(\pi_{k,\ell}(x)).$$

Here, we use an affine manifold structure on each component of $\hat{W}_{i,i}^s$ by identifying it with the 1-torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. So, any positively weighted barycentric combination makes sense for a pair

of points sufficiently close. When $x \in \hat{W}_{i,i}^s$, we have $\hat{q}(x) = x$. Then, by shrinking the tube \hat{N}_i^s once more if necessary we make \hat{q} be a fibration whose fibres are transverse to the lamination \hat{L}_i^s and we make each leaf of $\hat{F}_{j,i}^u$, $j > i$, in every box B_k^ℓ be contained in a fibre of q . Henceforth, taking the preimage of that tube (and its fibration) by p_i and adding the unstable manifold W_i^u provide the wanted U_i^s and its foliation F_i^u satisfying the required properties. Thus, the induction is proved. \diamond

We also have the following statement.

Lemma 3.6 *There exist f -invariant neighborhoods U_0^u, \dots, U_n^u of the sets $\Sigma_0, \dots, \Sigma_n$ respectively, equipped with one-dimensional f -invariant foliations F_0^s, \dots, F_n^s of class $C^{1,0}$ such that the following properties hold for each $i \in \{0, \dots, n\}$: (iii) the stable manifold W_i^s is a leaf of the foliation F_i^s and each leaf of the foliation F_i^s is transverse to L_i^u ;*

(iv) *for any $0 \leq j < i$ and $x \in U_i^u \cap U_j^u$, we have the inclusion $(F_{j,x}^s \cap U_i^u) \subset F_{i,x}^s$.*

Proof: The proof is done by an increasing induction from $i = 0$; it is skipped due to similarity to the previous one. \diamond

Step 2. We prove the following statement for each $i = 0, \dots, n$.

Lemma 3.7 (v) *There exists an f -invariant neighborhood \tilde{N}_i of the set Σ_i contained in $U_i^s \cap U_i^u$ and such that the restrictions of the foliations F_i^u and F_i^s to \tilde{N}_i are transverse.*

Proof: For this aim, let us choose a fundamental domain⁶ K_i^s of the restriction of f to $W_i^s \setminus \Sigma_i$ and take a tubular neighborhood $N(K_i^s)$ of K_i^s whose disc fibres are contained in leaves of F_i^u . By construction, F_i^u is transverse to W_i^s and, according to the Lemma 3.6, F_i^s is a $C^{1,0}$ -foliation. Therefore, if the tube $N(K_i^s)$ is small enough, F_i^u is transverse to F_i^s in $N(K_i^s)$. Set

$$\tilde{N}_i := W_i^u \bigcup_{k \in \mathbb{Z}} f^k(N(K_i^s)).$$

This is a neighborhood of Σ_i ; it satisfies condition (v) and the previous properties (i)–(iv) still hold. A priori the boundary of \tilde{N}_i is only piecewise smooth; but, by choosing $N(K_i^s)$ correctly at its corners we may arrange that $\partial\tilde{N}_i$ be smooth. \diamond

Step 3. For proving Theorem 2 it remains to show the existence of linearizable neighborhoods $N_i \subset \tilde{N}_i$, $i = 0, \dots, n$, for which the required foliations are the restriction to N_i of the foliations F_i^u and F_i^s . For each orbit of f in Σ_i , choose one p . Let \tilde{N}_p be a connected component of \tilde{N}_i containing p . There is a homeomorphism $\varphi_p^u : W_p^u \rightarrow W_O^u$ (resp. $\varphi_p^s : W_p^s \rightarrow W_O^s$) conjugating the diffeomorphisms $f^{per(p)}|_{W_p^u}$ and $a|_{W_O^u}$ (resp. $f^{per(p)}|_{W_p^s}$ and $a|_{W_O^s}$). In addition, for any point $z \in \tilde{N}_p$ there is unique pair of points $z_s \in W_p^s$, $z_u \in W_p^u$

⁶Let us recall that a *fundamental domain of action $g : X \rightarrow X$ on X* is a closed set $D_g \subset X$ such that there is a set \tilde{D}_g with the following properties:

- 1) $cl(\tilde{D}_g) = D_g$;
- 2) $g^k(\tilde{D}_g) \cap \tilde{D}_g = \emptyset$ for all $k \in (\mathbb{Z} \setminus \{0\})$;
- 3) $\bigcup_{k \in \mathbb{Z}} g^k(\tilde{D}_g) = X$.

such that $z = F_{i,z_u}^s \cap F_{i,z_s}^u$. We define a topological embedding $\tilde{\mu}_p : \tilde{N}_p \rightarrow \mathbb{R}^3$ by the formula $\tilde{\mu}_p(z) = (x_1, x_2, x_3)$ where $(x_1, x_2) = \varphi_p^u(z_u)$ and $x_3 = \varphi_p^s(z_s)$. Since the foliations F_i^u and F_i^s are f -invariant, this definition makes $\tilde{\mu}_p$ conjugate the restriction $f^{per(p)}|_{\tilde{N}_p}$ to $a^{per(p)}$. For $k = 1, \dots, per(p) - 1$, set $\tilde{N}_{f^k(p)} := f^k(\tilde{N}_p)$ and define $\tilde{\mu}_{f^k(p)}$ so that the equivariance formula holds: $\tilde{\mu}_{f^k(p)}(f^k(x)) = a^k \tilde{\mu}_p(x)$ for every $x \in \tilde{N}_p$. Choose $t_0 \in (0, 1]$ such that $\mathcal{N}^{t_0} \subset \tilde{\mu}_p(\tilde{N}_p)$ for every $p \in \Sigma_i$. Observe that $a|_{\mathcal{N}^{t_0}}$ is conjugate to $a|_{\mathcal{N}}$ by the suitable homothety h . Set $N_p = \tilde{\mu}_p^{-1}(\mathcal{N}^{t_0})$ and $\mu_p = h\tilde{\mu}_p : N_p \rightarrow \mathcal{N}$. Then, N_p is the wanted neighborhood with its linearizing homeomorphism μ_p . This finishes the proof of Theorem 2. \diamond

4 Proof of the classification theorem

Let us prove that the diffeomorphisms f and f' in $G(M)$ are topologically conjugate if and only if there is a homeomorphism $\hat{\varphi} : \hat{V}_f \rightarrow \hat{V}_{f'}$ such that 1) $\eta_f = \eta_{f'} \hat{\varphi}_*$; 2) $\hat{\varphi}(\hat{\Gamma}_f^s) = \hat{\Gamma}_{f'}^s$ and $\hat{\varphi}(\hat{\Gamma}_f^u) = \hat{\Gamma}_{f'}^u$.

4.1 Necessity

Let $f : M \rightarrow M$ and $f' : M \rightarrow M$ be two elements in $G(M)$ which are topologically conjugated by some homeomorphism $h : M \rightarrow M$. Then h conjugates the invariant manifolds of periodic points of f and f' . More precisely, if p is a periodic point of f , then $h(p)$ is a periodic point of f' and $h(W^u(p)) = W^u(h(p))$, $h(W^s(p)) = W^s(h(p))$. In particular, h maps V_f to $V_{f'}$ by a homeomorphism noted φ . Moreover, if x is any points of V_f , for every $n \in \mathbb{Z}$ the following holds:

$$\varphi(f^n(x)) = f'^n(\varphi(x)).$$

This formula says exactly that φ is the lift of a map $\hat{\varphi} : \hat{V}_f \rightarrow \hat{V}_{f'}$. By construction of η_f , the same formula says that $\eta_f = \eta_{f'} \circ \hat{\varphi}_*$, where $\hat{\varphi}_* : H_1(\hat{V}_f; \mathbb{Z}) \rightarrow H_1(\hat{V}_{f'}; \mathbb{Z})$ denotes the map induced in homology. By definition of the quotient topology, $\hat{\varphi}$ is continuous. Since the same holds for φ^{-1} , one checks that $\hat{\varphi}$ is a homeomorphism. As φ conjugates the laminations Γ_f^s (resp. Γ_f^u) to $\Gamma_{f'}^s$ (resp. $\Gamma_{f'}^u$), the same holds for $\hat{\varphi}$ in the quotient spaces with respect the projections of the laminations.

4.2 Sufficiency

For proving the sufficiency of the conditions in Theorem 1, let us consider a homeomorphism $\hat{\varphi} : \hat{V}_f \rightarrow \hat{V}_{f'}$ such that:

- (1) $\eta_f = \eta_{f'} \hat{\varphi}_*$;
- (2) $\hat{\varphi}(\hat{\Gamma}_f^s) = \hat{\Gamma}_{f'}^s$ and $\hat{\varphi}(\hat{\Gamma}_f^u) = \hat{\Gamma}_{f'}^u$.

From now on, the dynamical objects attached to f' will be denoted by $L'^u, L'^s, \Sigma'_i, \dots$ with the same meaning as $L^u, L^s, \Sigma_i, \dots$ have with respect to f . Due to property (1), $\hat{\varphi}$ lifts to an *equivariant*⁷ homeomorphism $\varphi : V_f \rightarrow V_{f'}$, that is: $f'|_{V_{f'}} = \varphi f \varphi^{-1}|_{V_f}$. Due to property (2),

⁷For brevity, equivariance stands for (f, f') -equivariance.

φ maps Γ_f^u to $\Gamma_{f'}^u$, and Γ_f^s to $\Gamma_{f'}^s$. Thanks to Theorem 2 we may use compatible linearizable neighbourhoods of the saddle points of f (resp. f').

An idea of the proof is the following: we modify the homeomorphism φ in a neighborhood of Γ_f^u such that the final homeomorphism preserves the compatible foliations, then we do similar modification near Γ_f^s . So we get a homeomorphism $h : M \setminus (\Omega_0 \cup \Omega_3) \rightarrow M \setminus (\Omega'_0 \cup \Omega'_3)$ conjugating $f|_{M \setminus (\Omega_0 \cup \Omega_3)}$ with $f'|_{M \setminus (\Omega'_0 \cup \Omega'_3)}$. Notice that $M \setminus (W_{\Omega_1}^s \cup W_{\Omega_2}^s \cup \Omega_3) = W_{\Omega_0}^s$ and $M \setminus (W_{\Omega'_1}^s \cup W_{\Omega'_2}^s \cup \Omega'_3) = W_{\Omega'_0}^s$. Since $h(W_{\Omega_1}^s) = W_{\Omega'_1}^s$ and $h(W_{\Omega_2}^s) = W_{\Omega'_2}^s$ then $h(W_{\Omega_0}^s \setminus \Omega_0) = W_{\Omega'_0}^s \setminus \Omega'_0$. Thus for each connected component A of $W_{\Omega_0}^s \setminus \Omega_0$ there is a sink $\omega \in \Omega_0$ such that $A = W_{\omega}^s \setminus \omega$. Similarly $h(A)$ is a connected component of $W_{\Omega'_0}^s \setminus \Omega'_0$ such that $h(A) = W_{\omega'}^s \setminus \omega'$ for a sink $\omega' \in \Omega'_0$. Then we can continuously extend h to Ω_0 assuming $h(\omega) = \omega'$ for every $\omega \in \Omega_0$. A similar extension of h to Ω_3 finishes the proof. Thus below in a sequence of lemmas we explain only how to modify the homeomorphism φ in a neighborhood of Γ_f^u such that the final homeomorphism preserves the compatible foliations.

Recall the partition $\Sigma_0 \sqcup \dots \sqcup \Sigma_n$ associated with the Smale order on the periodic points of index 2.

Lemma 4.1 *For every $i = 0, \dots, n$ the following equality holds $\varphi(W_i^u \cap V_f) = W_i'^u \cap V_{f'}$ and there is a unique continuous extension of $\varphi|_{W_i^u \cap V_f}$ to Σ_i which is equivariant and bijective from Σ_i to Σ_i' .*

Proof: Let $p \in \Sigma_0$. Denote its orbit by $orb_f(p)$. The punctured unstable manifold $W^u(p) \setminus \{p\}$ projects by p_f to one compact leaf $\ell(p)$. Both sides of the next equality are f -invariant and project to the same leaf, thus:

$$p_f^{-1}(\ell(p)) = W^u(orb(p)) \setminus \{orb(p)\}.$$

Then, the number of connected components of $p_f^{-1}(\ell(p))$ is $per(p)$, the period of p . The image $\hat{\varphi}(\ell(p))$ is a compact leaf of $\hat{\Gamma}_{f'}^u$. By the previous argument, it is $\ell'(p')$ for some $p' \in \Sigma'_0$. Since $\hat{\varphi}$ lifts to φ , then $\varphi[p_f^{-1}(\ell(p))] = p_{f'}^{-1}(\ell'(p'))$ which implies the equality of the number of connected components. Thus $per(p) = per(p')$. From this, we can deduce that, up to replacing p' with $f'^k(p')$ for some integer k , we have $\varphi(W^u(p) \setminus \{p\}) = W^u(p') \setminus \{p'\}$. Using the property $p = \lim_{n \rightarrow -\infty} f^n(x)$ for every $x \in W^u(p)$ and the similar property for p' in addition to the equivariance of φ , one extends continuously $\varphi|_{W_p^u}$ by defining $\varphi(p) = p'$. Doing the same for every orbit of Σ_0 , we get a continuous extension of $\varphi|_{W_0^u}$ to Σ_0 which is still equivariant. One easily checks that this extension is continuous, unique, and hence equivariant. Then, arguing similarly with $\hat{\varphi}^{-1}$, we derive that the extension of φ maps Σ_0 bijectively onto Σ'_0 .

Denote $\ell_0 := \bigcup_{p \in \Sigma_0} \ell(p)$. We have $\hat{\varphi}(\ell_0) = \ell'_0$. Let now $p \in \Sigma_1$. The closure in \hat{V}_f of $\ell(p) := p_f(W^u(p) \setminus \{p\})$ is contained in $\ell(p) \cup \ell_0$. We deduce that $\hat{\varphi}(\ell(p))$ is a leaf of $\hat{\Gamma}_{f'}^u$ of the form $\ell'(p')$ for some $p' \in \Sigma'_1$ and we can continue inductively. Thus, there is a continuous extension of $\varphi|_{W_i^u}$ to every Σ_i for $i = 0, 1, \dots, n$ which is a bijection $\Sigma_i \rightarrow \Sigma'_i$. Arguing with $\hat{\varphi}^{-1}$, we derive that $n' = n$. \diamond

Recall the radial functions $r_i^u, r_i^s : N_i \rightarrow [0, +\infty)$ which are introduced above Definition 3.3; recall also the order which is defined by r_i^s on each stable separatrix γ_p of $p \in \Sigma_i$. Similar functions are associated with the dynamics of f' .

Lemma 4.2 *There is a unique continuous extension of $\varphi|_{\Gamma_f^u}$*

$$\varphi^{us} : \Gamma_f^u \cup (L^u \cap L^s) \longrightarrow \Gamma_{f'}^u \cup (L'^u \cap L'^s)$$

such that the following holds:

- 1) If $x \in W_j^u \cap W_i^s$, $j > i$, then $\varphi^{us}(x) \in W_j'^u \cap W_i'^s$.
- 2) If x and y lie in $\gamma_p \cap L^u$ with $r_p^s(x) < r_p^s(y)$, then $\varphi^{us}(x)$ and $\varphi^{us}(y)$ lie in $\gamma'_{\varphi(p)} \cap L'^u$ with $r_{\varphi(p)}'^s(\varphi^{us}(x)) < r_{\varphi(p)}'^s(\varphi^{us}(y))$.

Notice that $\varphi|_{\Gamma_f^u}$ being equivariant its continuous extension is also equivariant.

Proof: This statement is proved by induction on i . We recall that $V_i \setminus L_i^s = V_f \setminus cl(W_i^u)$ is a dense open set in V_f (and similarly with $'$), and according to Lemma 4.1, φ maps $V_i \setminus L_i^s$ to $V_i' \setminus L_i'^s$ homeomorphically and conjugates f to f' . Thus, for every $i = 0, \dots, n$, we have an equivariant homeomorphism $\varphi_i : V_i \setminus L_i^s \rightarrow V_i' \setminus L_i'^s$ which maps $W_j^u \setminus L_i^s$ to $W_j'^u \setminus L_i'^s$ for every $j > i$, again as a consequence of Lemma 4.1.

First, take $i = 0$. The manifold \hat{V}_0 is closed and three-dimensional. We have $\hat{L}_0^s = \hat{W}_{0,0}^s$, which consists of finite number disjoint smooth circles, and similarly with $'$.

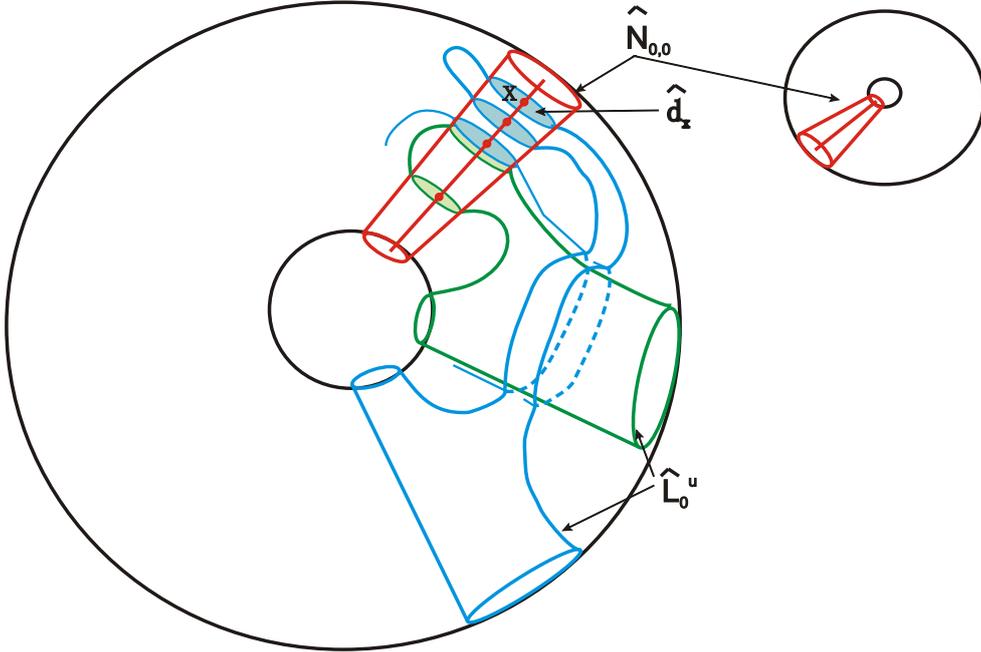


Figure 5: Case $i = 0$ in proof of Lemma 4.2 for the diffeomorphism from Figure 3.

We look for an extension $\hat{\varphi}_0^{us}$ of $\hat{\varphi}_0|_{\hat{L}_0^u}$ to $\hat{L}_0^s \cap \hat{L}_0^u$. If N_0 is the neighborhood of Σ_0 extracted from a compatible system given by Theorem 2 and if $\hat{N}_{0,0}$ denotes the corresponding tubular neighborhood of \hat{L}_0^s in \hat{V}_0 , the trace of \hat{L}_0^u in that tube is a lamination by disks:

$$\hat{L}_0^u \cap \hat{N}_{0,0} = \{\hat{d}_x \mid x \in \hat{L}_0^u \cap \hat{L}_0^s\},$$

where \hat{d}_x denotes the fiber of the tube over $x \in \hat{L}_0^s$ (see figure 5).

The complement in \hat{V}_0' of the interior of $\hat{N}_{0,0}'$ is a compact set contained in $\hat{V}_0' \setminus \hat{L}_0'^s$. Then its preimage K by the homeomorphism $\hat{\varphi}_0$ is a compact set contained in $\hat{V}_0 \setminus \hat{L}_0^s$. When $t = 0$, we have $\hat{N}_{0,0}^t = \hat{L}_0^s$ and hence disjoint from K . Then, if t is small enough, $\hat{\varphi}_0(\hat{N}_{0,0}^t \setminus \hat{L}_0^s) \subset \hat{N}_{0,0}' \setminus \hat{L}_0'^s$. Finally, the map $\hat{\varphi}_0$ (which is not defined on \hat{L}_0^s) possesses the two following properties:

1. If N_0 is shrunk enough, we have $(\hat{\varphi}_0(\hat{N}_{0,0}) \cap \hat{L}_0^u) \subset (\hat{N}_{0,0}' \cap \hat{L}_0'^u)$, where $\hat{N}_{0,0}'$ denotes the tube associated with the chosen linearizable neighborhood N_0' of Σ_0' .
2. If \hat{d}_x is a plaque of $\hat{L}_0^u \cap \hat{N}_{0,0}$, the image $\hat{\varphi}_0(\hat{d}_x \setminus \{x\})$ is contained in some fiber $\hat{d}_{x'}$, with $x' \in \hat{L}_0'^u \cap \hat{L}_0'^s$.

As a consequence, the wanted extension may be defined by $\hat{\varphi}_0^{us}(x) = x'$. As the considered plaques are arcwise connected, the construction lifts to the cover and yields a continuous map $\varphi_0^{us} : \Gamma_f^u \cup (L_0^s \cap L_0^u) \rightarrow \Gamma_{f'}^u \cup (L_0'^s \cap L_0'^u)$ which is a continuous equivariant extension of $\varphi|_{\Gamma_f^u}$.

We are left to prove that φ_0^{us} is increasing on its domain in each separatrix of Σ_0 . For this aim, consider a point $p \in \Sigma_0$, one of its separatrices γ_p and a connected component N_{γ_p} of $N_p \setminus W_p^u$ containing γ_p . Take an infinite proper arc C in $N_{\gamma_p} \setminus W_p^s$ which crosses transversely each leaf of foliation F_0^u and whose one end is p . It is ordered as its projection onto γ_p . Its image through φ_0 is a proper arc C' contained in $N_{\varphi(p)}' \setminus W_{\varphi(p)}'^u$. Moreover, $\varphi(p)$ is one end of C' . The property $r_{\varphi(p)}'^s(\varphi_0^{us}(x)) < r_{\varphi(p)}'^s(\varphi_0^{us}(y))$ if $r_p^s(x) < r_p^s(y)$ for $x, y \in \gamma_p \cap L^u$ follows if we are sure that C' intersects each plaque of $L_0'^u \cap N_{\varphi(p)}'$ in one point at most. This is true since φ_0 is a homeomorphism on its image from $N_0 \setminus W_0^s$ to $N_0' \setminus W_0'^s$ mapping L_0^u into $L_0'^u$.

For the induction, let $i \in \{1, \dots, n\}$ and let us assume that there is a continuous extension

$$\varphi_{i-1}^{us} : \Gamma_f^u \cup \bigcup_{j=0}^{i-1} (L_j^s \cap L_j^u) \rightarrow \Gamma_{f'}^u \cup \bigcup_{j=0}^{i-1} (L_j'^s \cap L_j'^u),$$

which is monotone on each separatrix of Σ_j , $j < i$. The image $\hat{W}_{i,i}^s$ of W_i^s by the projection $p_i : V_i \rightarrow \hat{V}_i$ is made of finitely many disjoint circles which are the images of the stable separatrices of Σ_i . About $\hat{W}_{i,i}^s$, there is a tube $\hat{N}_{i,i}$ which is the projection by p_i of a neighborhood N_i of Σ_i extracted from a compatible system given by Theorem 2 (see Figure 6). The trace of \hat{L}_i^u in that tube is a lamination by disks:

$$\hat{L}_i^u \cap \hat{N}_{i,i} = \{\hat{d}_x \mid x \in \hat{L}_i^u \cap \hat{W}_{i,i}^s\},$$

where \hat{d}_x denotes the fiber of the tube over $x \in \hat{W}_{i,i}^s$.

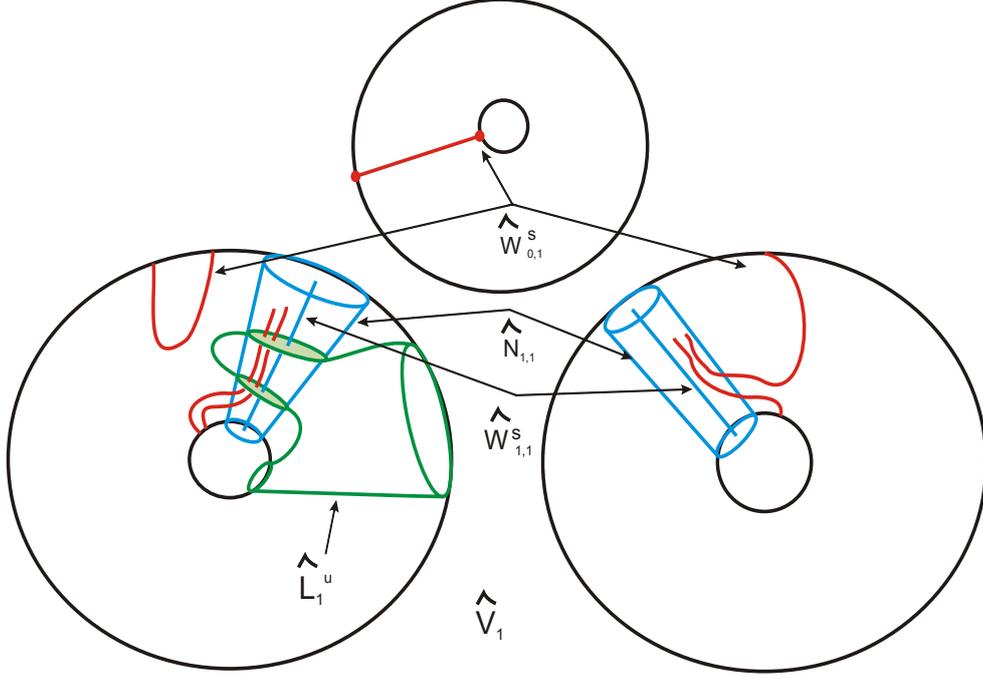


Figure 6: Illustration of induction in Lemma 4.2 for the diffeomorphism from Figure 3.

In V_i , there are two laminations L_i^u and L_i^s (and the corresponding objects with $'$). The map φ_i , not defined on L_i^s , sends $L_i^u \setminus L_i^s$ homeomorphically onto $L_i'^u \setminus L_i'^s$. Thanks to the induction hypothesis, the restriction $\varphi_i|_{(L_i^u \setminus L_i^s)}$ extends continuously to $L_i^u \setminus W_i^s$; this extension, automatically equivariant, is denoted ψ_i . This induces on the factor space \hat{V}_i a homeomorphism

$$\hat{\psi}_i : \hat{L}_i^u \setminus \hat{W}_{i,i}^s \rightarrow \hat{L}_i'^u \setminus \hat{W}_{i,i}'^s.$$

In order to extend $\hat{\psi}_i$ to $\hat{L}_i^u \cap \hat{L}_i^s$, we use the fact that \hat{L}_i^u is compact for arguing as in the case $i = 0$. Consider the tube $\hat{N}_{i,i}^t$ depending on $t \in (0, 1)$ and look at its compact lamination by disks $\hat{L}_i^u \cap \hat{N}_{i,i}^t$. After removing $\hat{W}_{i,i}^s$ which marks one puncture on each leaf, it is leaf-wise mapped by $\hat{\psi}_i$ into $\hat{V}_i' \setminus \hat{W}_{i,i}'^s$. The mentioned compactness allows us, as in case $i = 0$, to conclude that there exists some $t \in (0, 1)$ such that $\hat{L}_i^u \cap (\hat{N}_{i,i}^t \setminus \hat{W}_{i,i}^s)$ is mapped into $\hat{N}_{i,i}'$ where $\hat{N}_{i,i}'$ denotes the tube associated with the chosen linearizable neighborhood N_i' of Σ_i' . Finally, the map $\hat{\psi}_i$ possesses the two following properties:

1. If N_i is shrunk enough, we have $(\hat{\psi}_i(\hat{N}_{i,i}) \cap \hat{L}_i^u) \subset (\hat{N}_{i,i}' \cap \hat{L}_i'^u)$.
2. If \hat{d}_x is a plaque of $\hat{L}_i^u \cap \hat{N}_{i,i}$, the image $\hat{\psi}_i(\hat{d}_x \setminus \{x\})$ is contained in some fiber $\hat{d}_{x'}$, with $x' \in \hat{L}_i'^u \cap \hat{W}_{i,i}'^s$.

Now, the extension $\hat{\varphi}_i^{us}$ of $\hat{\psi}_i$ is defined by $x \mapsto x'$. One checks it is a continuous extension. The wanted φ_i^{us} is the lift of $\hat{\varphi}_i^{us}$ to V_i . It has the required properties allowing us to finish the induction. \diamond

Remark 4.3 Due to Lemma 3.2 we may assume that in all lemmas below the chosen values $t = \beta_i, a_i, \dots$ are such that the boundary of the linearizable neighbourhood N_i^t does not contain any heteroclinic point.

Lemma 4.4 *There are numbers $\beta_0, \dots, \beta_n \in (0, 1)$ such that, for every $i \in \{0, \dots, n\}$, for every point $p \in \Sigma_i$ and $x \in N_p^{\beta_i} \cap L^u$, the next inequality holds:*

$$r_i^{u'}(\varphi^{us}(x_i^u)) \cdot r_i^{s'}(\varphi^{us}(x_i^s)) < \frac{1}{2}.$$

Proof: As $N_n \cap L^u = W_n^u$ and $\varphi^{us}(W_n^u) = W_n^{u'}$, it is possible to choose any $\beta_n \in (0, 1)$.

Indeed, for $p \in \Sigma_n$ and $x \in W_p^u$, we have $r_{\varphi(p)}^{s'}(\varphi^{us}(x_i^s)) = 0$. For $i \in \{0, \dots, n-1\}$ and $p \in \Sigma_i$, choose some heteroclinic point $y \in W_p^s \cap L^u$ arbitrarily. Set:

$$\lambda_p^{u'}(t) = \sup_x \{r_{\varphi(p)}^{u'}(\varphi^{us}(x_i^u)) \mid x \in N_p^t \cap F_{i,y}^u\} \quad \text{and} \quad \lambda_p^{s'} = r_p^{s'}(\varphi^{us}(y)).$$

When t goes to 0, the arc $N_p^t \cap F_{i,y}^u$ shrinks to the point y . Then, according to Lemma 4.2, $\lambda_p^{u'}(t)$ also goes to 0. Therefore, there exists some $\beta_p \in (0, 1)$ such that $\lambda_p^{u'}(\beta_p) \cdot \lambda_p^{s'} < \frac{1}{8}$. Denote by Q_p the compact subset of M bounded by $\partial N_p^{\beta_p}$, $F_{i,y}^u$ and $f^{per(p)}(F_{i,y}^u)$. Notice that Q_p is a fundamental domain for the restriction of $f^{per(p)}$ to the connected component of $N_p^{\beta_p} \setminus W_p^u$ containing y . For every $x \in Q_p$, we have $r_{\varphi(p)}^{u'}(\varphi^{us}(x_i^u)) \leq 4\lambda_p^{u'}(\beta_p)$ and $r_{\varphi(p)}^{s'}(\varphi^{us}(x_i^s)) \leq \lambda_p^{s'}$. Then, for every $x \in Q_p \cap L^u$ we have:

$$r_p^{u'}(\varphi^{us}(x_i^u)) \cdot r_p^{s'}(\varphi^{us}(x_i^s)) \leq 4\lambda_p^{u'}(\beta_p) \cdot \lambda_p^{s'} < \frac{1}{2}.$$

Set $\beta_i = \min_{p \in \Sigma_i} \{\beta_p\}$. Hence, β_i is the required number. ◊

Lemma 4.5 *When $n > 0$, there exist real numbers $a_j \in (0, \beta_j]$ fulfilling the following property: for every $j = 1, \dots, n$ and every integer $i < j$, each connected component of $\hat{W}_{i,i}^{s'} \cap \hat{N}_{j,i}^{a_j}$ is an open interval which is either disjoint from $A_j^i := \bigcup_{k=i+1}^{j-1} \hat{N}_{k,i}^{a_k}$ or included in A_j^i . Moreover, only finitely many of these intervals are not covered by A_j^i .*

Proof: The proof is done by induction on j from 1 to n . For $j = 1$, one is allowed to take $a_1 = \beta_1$. Indeed, $\hat{W}_{0,0}^s$ is a smooth curve and $\hat{W}_{1,0}^u$ is a smooth closed surface which is transverse to $\hat{W}_{0,0}^s$. Therefore, there are finitely many intersection points. By the choice of β_1 , the projection in \hat{V}_0 of $N_1^{a_1}$ is a tubular neighborhood of $\hat{W}_{1,0}^u$. Moreover, each component of $\hat{W}_{0,0}^s \cap \hat{N}_{1,0}^{a_1}$ is a fiber of this tube.

For the induction, assume the numbers a_1, \dots, a_{j-1} are given with the required properties and let us find a_j . In particular, the subset A_j^i is assumed to be defined. According to Remark 4.3, the boundary of A_j^i contains no heteroclinic point.

First, fix $i < j$. Consider the projection $\hat{W}_{j,i}^u$ of W_j^u in \hat{V}_i . This is a union of leaves in the lamination \hat{L}_i^u . The following is a well-known fact (see, for example, Statement 1.1 in [12]): if x is a point from \hat{L}_i^u which is accumulated by a sequence of plaques from $\hat{W}_{j,i}^u$, then x does not lie in $\hat{W}_{j,i}^u$ but belongs to some $\hat{W}_{k,i}^u$ with $k < j$. Then the part of $\hat{W}_{j,i}^u$ which is covered by A_j^i contains every intersection points $\hat{W}_{i,i}^s \cap \hat{W}_{j,i}^u$ except finitely many of them. From this finiteness and the fact that $A_j^i \cap \hat{W}_{i,i}^s \cap \hat{W}_{j,i}^u$ is actually contained in A_j^i , an easy compactness argument⁸ allows us to find a positive number a_j^i such that the collection of disjoint intervals made by $\hat{W}_{i,i}^s \cap \hat{N}_j^{a_j^i}$ fulfills the wanted property with respect to the considered i . By defining $a_j := \inf\{a_j^0, \dots, a_j^{j-1}\}$, we are sure that $\hat{N}_j^{a_j}$ satisfies all the wanted properties. \diamond

The corollary below follows from Lemma 4.5 immediately.

Corollary 4.6 *For each $i \in \{0, \dots, n-1\}$ the intersection $\hat{W}_{i,i}^s \cap (\bigcup_{j=i+1}^n \hat{N}_{j,i}^{a_j^i})$ consists of finitely many open arcs $\hat{I}_1^i, \dots, \hat{I}_{r_i}^i$ such that, for each $l = 1, \dots, r_i$, the arc \hat{I}_l^i is a connected component of $\hat{W}_{i,i}^s \cap \hat{N}_{j,i}^{a_j^i}$ for some $j > i$.*

For brevity, for $i = 0, \dots, n$, we denote by φ_i^u the restriction $\varphi^{us}|_{W_i^u}$ in the rest of the proof of Theorem 1. Let $\psi_i^s : W_i^s \rightarrow W_i^{s'}$ be any equivariant homeomorphism which extends $\varphi^{us}|_{W_i^s \cap L^u}$ and let $t_i \in (0, 1)$ be a small enough number so that, for every $x \in N_i^{t_i}$, the next inequality holds:

$$(*)_i \quad r^{s'}(\varphi_i^u(x_i^u)) \cdot r^{u'}(\psi_i^s(x_i^s)) < 1.$$

In this setting, one derives an equivariant embedding $\phi_{\varphi_i^u, \psi_i^s} : N_i^{t_i} \rightarrow N_i^{t_i'}$ which is defined by sending $x \in N_i^{t_i}$ to $(\varphi_i^u(x_i^u), \psi_i^s(x_i^s))$.

Lemma 4.7 *There is an equivariant homeomorphism $\psi^s : L^s \rightarrow L^{s'}$ consisting of conjugating homeomorphisms $\psi_0^s : W_0^s \rightarrow W_0^{s'}, \dots, \psi_n^s : W_n^s \rightarrow W_n^{s'}$ such that for each $i \in \{0, \dots, n\}$:*

- 1) $\psi_i^s|_{W_i^s \cap L^u} = \varphi_i^u|_{W_i^s \cap L^u}$;
- 2) the topological embedding $\phi_{\varphi_i^u, \psi_i^s}$ is well-defined on $N_i^{a_i}$;
- 3) if $x \in (W_i^s \cap N_j^{a_j^i})$, $j > i$, then $\psi_i^s(x) = \phi_{\varphi_j^u, \psi_j^s}(x)$.

Proof: We are going to construct ψ_i^s by a decreasing induction on i from $i = n$ to $i = 0$. The stable manifolds of the saddles in Σ_n have no heteroclinic points. Therefore, the only constraints on ψ_n^s imposed by the first item is its value on Σ_n . In particular, we are allowed to change ψ_n^s to $f^{lk} \circ \psi_n^s$ if k is *admissible* in the sense that k is a multiple of all periods $per(p)$, $p \in \Sigma_n$.

This remark is used in the following way. One starts with any equivariant homeomorphism ψ_n^s such that for any $p \in \Sigma_n$ the stable manifold W_p^s is mapped to the stable manifold of $\varphi_n^u(p)$; hence, item 1 is fulfilled. Choose a fundamental domain I of $f|_{W_n^s \setminus \Sigma_n}$. Consider the fundamental

⁸Let $B(t)$ be the closure of $\partial A_j^i \cap \hat{W}_{i,i}^s \cap \hat{N}_j^t$. The intersection $\bigcap_{k \in \mathbb{N}} B(\frac{1}{k})$ is empty. Then $B(t)$ is empty when t is small enough.

domain of $f|_{N_n^{an} \setminus W_n^u}$ defined by $N_I := \{x \in N_n^{an} \mid x_n^s \in I\}$; set $\lambda_n'^u := \sup\{r'^u(\varphi_n^u(x_n^u)) \mid x \in N_I\}$ and $\lambda_n'^s := \sup\{r'^s(\psi_n^s(x_n^s)) \mid x \in N_I\}$. If the product $\lambda_n'^u \lambda_n'^s$ is less than 1, the inequality $(*)_n$ is fulfilled by the pair (φ_n^u, ψ_n^s) and hence, the embedding $\phi_{\varphi_n^u, \psi_n^s}$ is well-defined on N_n^{an} .

If not, we replace ψ_n^s with $f^{rk} \circ \psi_n^s$ with k admissible and large enough. Indeed, the effect of this change is to multiply $\lambda_n'^s$ by some positive factor bounded by $(\frac{1}{4})^k$ while $\lambda_n'^u$ is kept fixed and hence, $(*)_n$ becomes fulfilled when k is large enough. Since the third item is empty for $i = n$, we have built some ψ_n^s as desired.

For the induction, let us build ψ_i^s , $i < n$, with the required properties assuming that the homeomorphisms $\psi_n^s, \dots, \psi_{i+1}^s$ have been already built. The stable manifolds of saddles in Σ_i have heteroclinic intersections with unstable manifolds of saddles in Σ_j with $j > i$ only. The image $\hat{W}_{i,i}^s$ of W_i^s through $p_i : V_i \rightarrow \hat{V}_i$ is a closed smooth 1-dimensional submanifold. According to Corollary 4.6, the intersection $\hat{W}_{i,i}^s \cap (\bigcup_{j=i+1}^n \hat{N}_j^{aj})$ consists of finitely many open arcs $\hat{I}_1^i, \dots, \hat{I}_{r_i}^i$ such that \hat{I}_l^i for each $l = 1, \dots, r_i$ is a connected component of $\hat{W}_{i,i}^s \cap \hat{N}_{j,i}^{aj}$ for some $j > i$.

In order to satisfy the third item of the statement, ψ_i^s is defined on $p_i^{-1}(\hat{I}_l^i)$ in an equivariant way. Denote by $\psi_{i,l}^s$ this partial definition of ψ_i^s ; its image is contained in $W_{i,i}^s$.

More precisely, if $I_{l,\alpha}^i$ is a connected component of $p_i^{-1}(\hat{I}_l^i)$ it is a proper arc in some N_i^{aj} and it intersects W_j^u in a unique point $x_{l,\alpha}^i$. Set $x_{l,\alpha}^i = \varphi^{us}(x_{l,\alpha}^i)$ and denote $I_{l,\alpha}^i$ the connected component of $W_i^s \cap N_j^u$ passing through the point $x_{l,\alpha}^i$. Then, the restriction of $\psi_{i,l}^s$ to the arc $I_{l,\alpha}^i$ reads:

$$\psi_{i,l,\alpha}^s = \phi_{\varphi_j^u, \psi_j^s}|_{I_{l,\alpha}^i} : I_{l,\alpha}^i \rightarrow I_{l,\alpha}^i.$$

Due to lemma 4.2, the map φ^{us} sends $W_i^s \cap L^u$ to $W_i^s \cap L'^u$ by preserving the order on each separatrix of $W_i^s \setminus \Sigma_i$ and $W_i^s \setminus \Sigma'_i$. On the other hand, $\psi_{i,l,\alpha}^s$ is also order preserving. Both together imply that $\psi_{i,l}^s$ is order preserving since we know that it is an injective map. And even, the union of all $\psi_{i,l}^s$ – which makes sense as their respective domains are mutually disjoint – are order preserving. Therefore, there is an equivariant homeomorphism $\psi_i^s : W_i^s \setminus \Sigma_i \rightarrow W_i^s \setminus \Sigma_i$ which extends all $\psi_{i,l}^s$.

Since φ^{us} is continuous, the above homeomorphism extends continuously to $\psi_i^s : W_i^s \rightarrow W_i^s$. At this point of the construction items 1 and 3 of the statement are satisfied. The condition of item 2 follows from Lemma 4.4 for stable separatrices which contain heteroclinic points. If some stable separatrix has no heteroclinic points, one changes ψ_i^s to $f^{rk} \circ \psi_i^s$ on the separatrix where k is a large common multiple of the period of the separatrix, like to the construction made in the case $i = n$. \diamond

PROOF OF THEOREM 1 CONTINUED. Let us recall that we denoted by $a : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ the canonical linear diffeomorphism with the unique fixed point $O = (0,0,0)$ which is a saddle point with the plane Ox_1x_2 as the unstable manifold and the axis Ox_3 as the stable manifold; for simplicity, we assume that a has a sign $\nu = +$ (see the beginning of Section 3). Let

$$N = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \leq (x_1^2 + x_2^2)x_3 \leq 1\}.$$

Let $\rho > 0$, $\delta \in (0, \frac{\rho}{4})$ and

$$d = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq \rho^2, x_3 = 0\},$$

$$\begin{aligned}
U &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (\rho - \delta)^2 \leq x_1^2 + x_2^2 \leq \rho^2, x_3 = 0\}, \\
c &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = \rho^2, x_3 = 0\}, \\
c^0 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = (\rho - \frac{\delta}{2})^2, x_3 = 0\}, \\
c^1 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = (\rho - \delta)^2, x_3 = 0\}.
\end{aligned}$$

Let $K = d \setminus \text{int } a^{-1}(d)$, $V = (K \cup a(K)) \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq 0, x_3 = 0\}$ and $\beta = U \cap Ox_1^+$, where $Ox_1^+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 0, x_1 > 0\}$.

Choose a point $Z^0 = (0, 0, z^0) \in Ox_3^+$ such that $\rho^2 \cdot z^0 < \frac{1}{4}$ (see Figure 7). Then, choose a point $Z^1 = (0, 0, z^1)$ in Ox_3^+ so that $z^0 > z^1 > \frac{z^0}{4}$. Let $\Pi(z) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = z\}$. For every set $A \subset Ox_1x_2$, let $\tilde{A} = A \times [0, z^0]$. Denote by \mathcal{W} the 3-ball bounded by the surface \tilde{c} and the two planes $\Pi(z^0)$ and $\Pi(\frac{z^0}{4})$. Let Δ be a closed 3-ball bounded by the surface \tilde{c}^1 and the two planes Ox_1x_2 and $\Pi(z^1)$. Let

$$\mathcal{T} = \bigcup_{k \in \mathbb{Z}} a^k(\tilde{d}) \quad \text{and} \quad \mathcal{H} = \bigcup_{k \in \mathbb{Z}} a^k(\Delta).$$

Notice that the construction yields $\mathcal{H} \subset \text{int } \mathcal{T}$ and makes \mathcal{W} a fundamental domain for the action of a on \mathcal{T} .

Now, we come back to f . Let $n \geq 1$, $i \in \{0, \dots, n-1\}$ and let G_i be the union of all stable separatrices of saddle points in Σ_i which contains heteroclinic points. Let $\check{G}_i \subset G_i$ be the union of separatrices in G_i such that $G_i = \bigcup_{\gamma \in \check{G}_i} \text{orb}(\gamma)$ and, for every pair (γ_1, γ_2) of

distinct separatrices in \check{G}_i and every $k \in \mathbb{Z}$, one has $\gamma_2 \neq f^k(\gamma_1)$. For $\gamma \in G_i$ with the end point $p \in \Sigma_i$ and a point $q \in \Sigma_j$, $j > i$, let us consider a sequence of different periodic orbits $p = p_0 \prec p_1 \prec \dots \prec p_k = q$ such that $\gamma \cap W_{p_1}^u \neq \emptyset$, the length of the longest such chain is denoted by $\text{beh}(q|\gamma)$.

Let $\gamma \in \check{G}_i$ be a separatrix of $p \in \Sigma_i$ and let N_γ^t be the connected component of $N_p^t \setminus W_p^u$ which contains γ . We endow with the index γ (resp. p) the preimages in M (through the linearizing map μ_p) of all objects from the linear model \mathcal{N} associated with the separatrix γ (resp. p); for being precise we decide that $\mu_p(\gamma) = Ox_3^+$. For a separatrix γ in \check{G}_i , let us fix a saddle point q_γ such that $\text{beh}(q_\gamma|\gamma) = 1$. Notice that the intersection $\gamma \cap W_{q_\gamma}^u$ consists of a finite number of heteroclinic orbits.

Lemma 4.8 *Let $n \geq 1$, $i \in \{0, \dots, n-1\}$. For every $\gamma \in \check{G}_i$ there are positive numbers ρ , δ and ε (depending on γ) such that for every heteroclinic point $Z_\gamma^0 \in (\gamma \cap W_{q_\gamma}^u)$ with $z^0 < \varepsilon$ the following properties hold:*

- (1) U_p avoids all heteroclinic points;
- (2) $\varphi(\tilde{d}_p) \subset \phi_{\varphi_i^u, \psi_i^s}(N_i^{a_i})$;
- (3) $\varphi(\tilde{c}_p) \cap \phi_{\varphi_i^u, \psi_i^s}(\tilde{c}_p^0) = \emptyset$, $\varphi(\tilde{c}_p^1) \cap \phi_{\varphi_i^u, \psi_i^s}(\tilde{c}_p^0) = \emptyset$ and $\varphi(\tilde{\beta}_\gamma) \subset \phi_{\varphi_i^u, \psi_i^s}(\tilde{V}_\gamma)$.

Proof: Let $\gamma \in \check{G}_i$, $i \in \{0, \dots, n-1\}$. Due to Lemma 3.2, there is a generic $\rho > 0$ such that the curve c_γ avoids all heteroclinic points. Since W_l^s accumulates on W_k^s for every $l < k$,

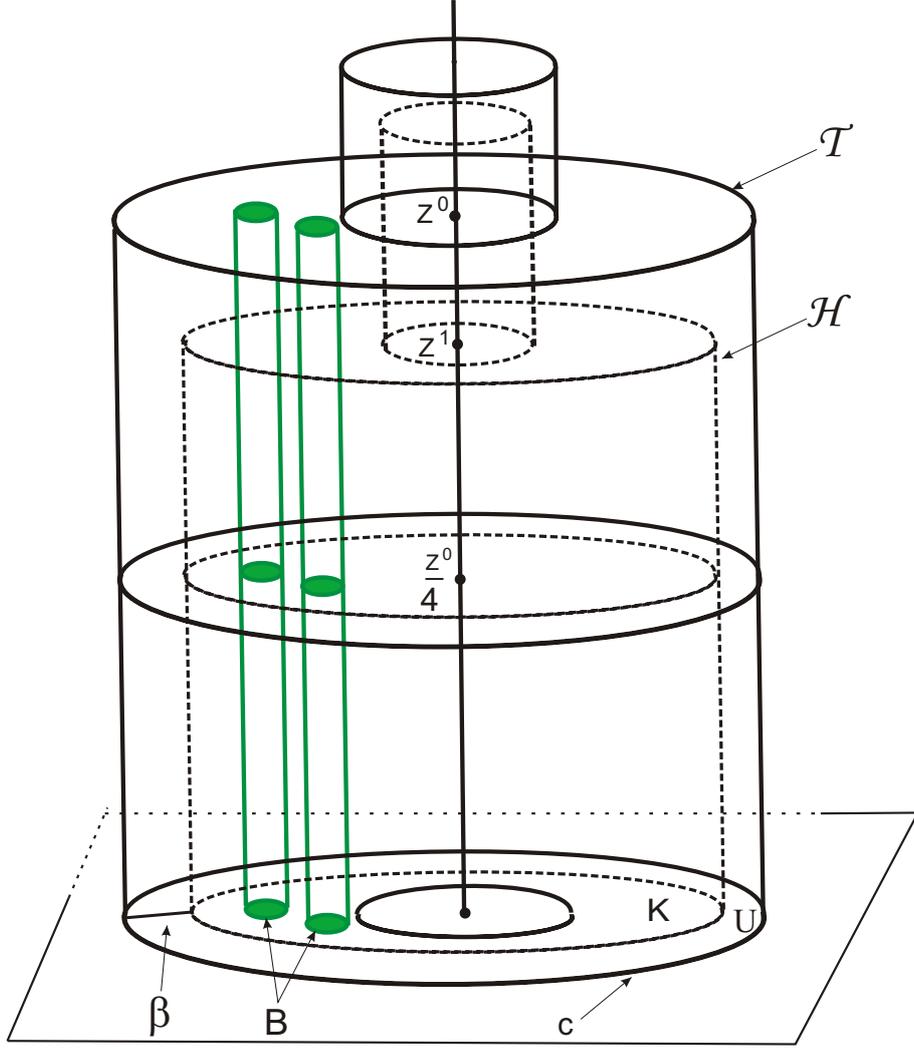


Figure 7: A linear model

then $K_p \cap W_{i-1}^s$ is made of a finite number of heteroclinic points y_1, \dots, y_r which we can cover by closed 2-discs $b_1, \dots, b_r \subset \text{int } K_p$. In $K_p \setminus \text{int}(b_1 \cup \dots \cup b_r)$ there is a finite number of heteroclinic points from W_{i-2}^s which we cover by the union of a finite number of closed 2-discs, and so on. Thus we get that all heteroclinic points in K_p belong to the union of finitely many closed 2-discs avoiding ∂K_p . Therefore, there is $\delta \in (0, \frac{\rho}{4})$ such that U_p avoids heteroclinic points. This proves item (1).

By assumption of Theorem 1, φ is defined on the complement of the stable manifolds and, by definition, $\phi_{\varphi_i^u, \psi_i^s}$ coincides with φ on $W_i^u \setminus L^s$, and hence on U_p . As φ and $\phi_{\varphi_i^u, \psi_i^s}$ are continuous, we can choose $\varepsilon > 0$ sufficiently small so that, if Z_γ^0 is any heteroclinic point in the intersection $\gamma \cap W_{q_\gamma}^u$ with $z^0 < \varepsilon$, the requirements of (2) and (3) are fulfilled. \diamond

Let us fix U_p satisfying item (1) of Lemma 4.8 and let us define

$$U_i = \bigcup_{p \in \Sigma_i} \left(\bigcup_{k=0}^{per(p)-1} f^k(U_p) \right), \quad K_i = \bigcup_{p \in \Sigma_i} \left(\bigcup_{k=0}^{per(p)-1} f^k(K_p) \right).$$

Lemma 4.9 *Let $n \geq 2$. For every $i \in \{0, \dots, n-2\}$ and $\gamma \in \check{G}_i$, there is a heteroclinic point $Z_\gamma^0 \in \gamma$ satisfying the conditions of Lemma 4.8 and in addition:*

$$\mathcal{T}_\gamma \cap \tilde{U}_j = \emptyset \quad \text{for } j \in \{i+1, \dots, n-1\}.$$

In this statement, it is meant that \tilde{U}_{n-1} is associated with the points $Z_{\gamma'}, \gamma' \in \check{G}_{n-1}$ given by Lemma 4.8 and \tilde{U}_j is associated with the points $Z_{\gamma''}, \gamma'' \in \check{G}_j$ given by Lemma 4.9 for every $j > i$. Therefore, it makes sense to prove that lemma by decreasing induction on i from $i = n-2$ to 0. That is what is done below. It is also worth noticing that nothing is required with respect to Σ_n ; the reason why is that the stable separatrices of Σ_n have no heteroclinic points.

Proof: Let us first prove the lemma for $i = n-2$. Let $\gamma \in \check{G}_{n-2}$ and let p be the saddle end point of γ . Notice that the intersection $\gamma \cap K_{n-1}$ consists of a finite number points a_1, \dots, a_l avoiding U_{n-1} . Let $d_1, \dots, d_l \subset K_{n-1}$ be compact discs with centres a_1, \dots, a_l and radius r_* (in linear coordinates of N_p) avoiding U_{n-1} . Let us choose a number $n^* \in \mathbb{N}$ such that $\frac{\rho}{2^{n^*}} < r_*$. Let $Z_\gamma^* \subset \gamma$ be a point such that the segment $[p, Z_\gamma^*]$ of γ avoids \tilde{K}_{n-1} and $\mu_p(Z_\gamma^*) = Z^* = (0, 0, z^*)$ where $z^* < \varepsilon$. Then every heteroclinic point z_γ^0 so that $z^0 < \frac{z^*}{2^{n^*}}$ possesses the property: $\mathcal{T}_\gamma \cap \tilde{K}_{n-1}$ avoids \tilde{U}_{n-1} .

For the induction, let us assume now that the construction of the desired heteroclinic points is done for $i+1, i+2, \dots, n-2$. Let us do it for i . Let $\gamma \in \check{G}_i$. By assumption of the induction $(\bigcup_{k=i+1}^{j-1} \mathcal{T}_k) \cap \tilde{U}_j = \emptyset$ for $j \in \{i+2, \dots, n-1\}$. Since W_{k-1}^s accumulates on W_k^s for every $k \in \{0, \dots, n\}$, then $(\bigcup_{k=i+1}^{j-1} \mathcal{T}_k) \cap K_j$ is a compact subset of K_j and the intersection $(\gamma \setminus (\bigcup_{k=i+1}^{j-1} \mathcal{T}_k)) \cap K_j$ consists of a finite number points a_1, \dots, a_l avoiding U_j . Let $d_1, \dots, d_l \subset K_j$ be compact discs with centres a_1, \dots, a_l and radius r_* (in linear coordinates of N_p) avoiding U_j and such that r_* is less than the distance between $\partial(K_j \setminus U_j)$ and $(\bigcup_{k=i+1}^{j-1} \mathcal{T}_k) \cap K_j$. Similar to the case $i = n-2$ it is possible to choose a heteroclinic point Z_γ^0 sufficiently close to the saddle p where γ ends such that the set $(\mathcal{T}_\gamma \setminus (\bigcup_{k=i+1}^{j-1} \mathcal{T}_k)) \cap \tilde{K}_j$ avoids \tilde{U}_j . \diamond

Everywhere below, we assume that for every $\gamma \subset \check{G}_i$ the neighborhood \mathcal{T}_γ satisfies to Lemmas 4.8 and 4.9. Let

$$\mathcal{T}_i = \bigcup_{\gamma \subset \check{G}_i} \left(\bigcup_{k=0}^{per(\gamma)-1} f^k(\mathcal{T}_\gamma) \right).$$

For $\gamma \subset \check{G}_i$, $j > i$, let us denote by $\mathcal{J}_{\gamma,j}$ the union of all connected components of $W_j^u \cap \mathcal{T}_\gamma$ which do not lie in $\text{int } \mathcal{T}_k$ with $i < k < j$. Let $\mathcal{J}_\gamma = \bigcup_{j=i+1}^n \mathcal{J}_{\gamma,j}$ and

$$\mathcal{J}_i = \bigcup_{\gamma \subset \check{G}_i} \left(\bigcup_{k=0}^{\text{per}(\gamma)-1} f^k(\mathcal{J}_\gamma) \right).$$

Let \mathcal{W}_γ be the fundamental domain of $f^{\text{per}(\gamma)}|_{\mathcal{T}_\gamma \setminus W_p^u}$ limited by the plaques of the two heteroclinic points Z_γ^0 and $f^{\text{per}(\gamma)}Z_\gamma^0$. Notice that $\gamma \cap \mathcal{W}_\gamma$ is a fundamental domain of $f^{\text{per}(\gamma)}|_\gamma$. Since W_k^u accumulates on W_l^u only when $l < k$, then the set $\mathcal{J}_{\gamma,j} \cap \mathcal{W}_\gamma$ consists of a finite number of closed 2-discs. Hence, the set $\mathcal{J}_\gamma \cap \gamma \cap \mathcal{W}_\gamma$ consists of a finite number of heteroclinic points; denote them $Z_\gamma^2, \dots, Z_\gamma^m$ (m depends on γ). Finally, choose an arbitrary point $Z_\gamma^1 \in \gamma$ so that the arc $(z_\gamma^0, z_\gamma^1) \subset \gamma$ does not contain heteroclinic points from \mathcal{J}_γ . Let us construct \mathcal{H}_γ using the point $Z^1 = \mu_p(Z_\gamma^1)$. Without loss of generality we will assume that $\mu_p(Z_\gamma^i) = Z^i = (0, 0, z^i)$ for $z^0 > z^1 > \dots > z^m > \frac{z^0}{4}$. For $i = 0, \dots, n-1$ let

$$\mathcal{H}_i = \bigcup_{\gamma \subset \check{G}_i} \left(\bigcup_{k=0}^{\text{per}(\gamma)-1} f^k(\mathcal{H}_\gamma) \right) \quad \text{and} \quad \mathcal{M}_i = V_f \bigcup_{k=0}^i (G_k \cup \Sigma_k).$$

Lemma 4.10 *There is an equivariant topological embedding $\varphi_0 : \mathcal{M}_0 \rightarrow M'$ with following properties:*

- 1) φ_0 coincides with φ out of \mathcal{T}_0 ;
- 2) $\varphi_0|_{\mathcal{H}_0} = \phi_{\psi_0^u, \psi_0^s}|_{\mathcal{H}_0}$, where $\psi_0^u = \varphi|_{W_0^u}$;
- 3) $\varphi_0(W_1^u) = W_1^{lu}$ and $\varphi_0(W_k^u \setminus \bigcup_{j=1}^{k-1} \text{int } \mathcal{T}_j) \subset W_k^{lu}$ for every $k \in \{2, \dots, n\}$.

Proof: The desired φ_0 should be an interpolation between $\varphi : V_f \setminus \mathcal{T}_0 \rightarrow M'$ and $\phi_{\varphi_0^u, \psi_0^s}|_{\mathcal{H}_0}$. Due to Lemma 4.8 (2) and the equivariance of the considered maps, the embedding

$$\xi_0 = \phi_{\psi_0^u, \psi_0^s}^{-1} \varphi : \mathcal{T}_0 \setminus W_0^s \rightarrow M$$

is well-defined. Let $\gamma \subset \check{G}_0$ be a separatrix ending at $p \in \Sigma_0$ and $\xi_\gamma = \xi_0|_{\mathcal{T}_\gamma}$. By construction, the topological embedding $\xi = \mu_p \xi_\gamma \mu_p^{-1} : \mathcal{T} \rightarrow N$ has the following properties:

- 1) $\xi a = a \xi$ (as $a \mu_p = \mu_p f^{\text{per}(\gamma)}$ and $\xi_\gamma f^{\text{per}(\gamma)} = f^{\text{per}(\gamma)} \xi_\gamma$);
- 2) ξ is the identity on Ox_1x_2 (as $\phi_{\psi_0^u, \psi_0^s}|_{W_0^u} = \varphi|_{W_0^u}$);
- 3) $\xi(\Pi(z^0) \cap \mathcal{T}) \subset \Pi(z^0)$ and $\xi(\Pi(z^i) \cap \partial \mathcal{T}) \subset \Pi(z^i)$, $i \in \{2, \dots, m\}$ (as $\xi_\gamma(L^u \cap \mathcal{T}_\gamma \setminus \gamma) \subset L^u$);
- 4) $\xi(\tilde{c}) \cap \tilde{c}^0 = \emptyset$, $\xi(\tilde{c}^1) \cap \tilde{c}^0 = \emptyset$ and $\xi(\tilde{\beta}) \subset \tilde{V}$ (due to Lemma 4.8 (3)).

Thus, ξ satisfies all conditions of Proposition 5.1 and, hence, there is an embedding $\zeta : \mathcal{T} \rightarrow N$ such that:

- 1) $\zeta a = a \zeta$;
- 2) ζ is the identity on \mathcal{H} ;

- 3) $\zeta(\Pi(z^i) \cap \mathcal{T}) \subset \Pi(z^i)$, $i \in \{0, 2, \dots, m\}$
- 4) ζ is ξ on $\partial\mathcal{T}$.

Then the embedding $\zeta_\gamma = \mu_p^{-1}\zeta\mu_p : \mathcal{T}_\gamma \rightarrow N_\gamma$ satisfies the properties:

- 1) $\zeta_\gamma f^{per(\gamma)} = f^{per(\gamma)}\zeta_\gamma$;
- 2) ζ_γ is the identity on \mathcal{H}_γ ;
- 3) $\zeta(\mathcal{J}_\gamma) \subset L^u$
- 4) ζ_γ is ξ_γ on $\partial\mathcal{T}_\gamma$.

Independently, one does the same for every separatrix $\gamma \in \check{G}_0$. Then, it is extended to all separatrices in G_0 by equivariance. As a result, we get a homeomorphism ζ_0 of \mathcal{T}_0 onto $\xi_0(\mathcal{T}_0)$ which coincides with ξ_0 on $\partial\mathcal{T}_0$. Now, define the embedding $\varphi_0 : \mathcal{M}_0 \rightarrow M'$ to be equal to $\phi_{\psi_0^u, \psi_0^s}\zeta_0$ on \mathcal{T}_0 and to φ on $\mathcal{M}_0 \setminus \mathcal{T}_0$. One checks the next properties:

- 1) φ_0 coincides with φ out of \mathcal{T}_0 ;
- 2) $\varphi_0|_{\mathcal{H}_0} = \phi_{\psi_0^u, \psi_0^s}|_{\mathcal{H}_0}$;
- 3) $\varphi_0(\mathcal{J}_0) \subset L^u$.

The last property and the definition of the set \mathcal{J}_γ imply that $\varphi_0(W_1^u) = W_1^u$ and $\varphi_0(W_k^u \setminus \bigcup_{j=1}^{k-1} \text{int } \mathcal{T}_j) \subset W_k^u$ for every $k \in \{2, \dots, n\}$. Thus φ_0 satisfies all required conditions of the lemma. \diamond

Lemma 4.11 *Assume $n \geq 2$, $i \in \{0, \dots, n-2\}$, and assume there is an equivariant topological embedding $\varphi_i : \mathcal{M}_i \rightarrow M'$ with following properties:*

- 1) φ_i coincides with φ_{i-1} out of \mathcal{T}_i ;
- 2) $\varphi_i|_{\mathcal{H}_i} = \phi_{\psi_i^u, \psi_i^s}$, where $\psi_i^u = \varphi_{i-1}|_{W_i^u}$ and $\varphi_{-1} = \varphi$;
- 3) there is an f -invariant union of tubes $\mathcal{B}_i \subset (\mathcal{T}_i \cap \bigcup_{j=0}^{i-1} \mathcal{H}_j)$ containing $(\mathcal{T}_i \cap (\bigcup_{j=0}^{i-1} W_j^s))$ where φ_i coincides with φ_{i-1} (we assume $\mathcal{B}_0 = \emptyset$);

- 4) $\varphi_i(W_{i+1}^u) = W_{i+1}^u$ and $\varphi_i(W_k^u \setminus \bigcup_{j=i+1}^{k-1} \text{int } \mathcal{T}_j) \subset W_k^u$ for every $k \in \{i+2, \dots, n\}$.

Then there is a homeomorphism φ_{i+1} with the same properties 1)-4)

Proof: The desired φ_{i+1} should be an interpolation between $\varphi_i : \mathcal{M}_{i+1} \setminus \mathcal{T}_{i+1} \rightarrow M'$ and $\phi_{\psi_{i+1}^u, \psi_{i+1}^s}|_{\mathcal{H}_{i+1}}$ where $\psi_{i+1}^u = \varphi_i|_{W_{i+1}^u}$. Let $\gamma \in \check{G}_{i+1}$ be a separatrix ending at $p \in \Sigma_{i+1}$. It follows from the definition of the set \mathcal{J}_i and the choice of the point q_γ that $(W_{q_\gamma}^u \cap \mathcal{T}_i) \subset \mathcal{J}_i$. Then, due to condition 4) for φ_i we have $\varphi_i(W_{q_\gamma}^u \cap \mathcal{T}_i) \subset W_{q_\gamma}^u$. By the property 1) of the homeomorphism φ_i and the properties of \mathcal{T}_{i+1} from Lemmas 4.8 (1) and 4.9, we get that $\varphi_i|_{\tilde{U}_p} = \varphi|_{\tilde{U}_p}$. Then $\phi_{\varphi_{i+1}^u, \psi_{i+1}^s}|_{\tilde{U}_p} = \phi_{\psi_{i+1}^u, \psi_{i+1}^s}|_{\tilde{U}_p}$. Thus it follows from the property (2) in Lemma 4.8 that the following embedding is well-defined: $\xi_\gamma = \phi_{\psi_{i+1}^u, \psi_{i+1}^s}^{-1}\varphi_i : \mathcal{T}_\gamma \setminus (\gamma \cup p) \rightarrow M'$.

By construction, the topological embedding $\xi = \mu_p \xi_\gamma \mu_p^{-1}$ satisfies to all conditions of Proposition 5.1. Let ζ be the embedding which is yielded by that proposition. Define $\zeta_\gamma = \mu_p^{-1}\zeta\mu_p$. Notice that by the property 3) of the homeomorphism ψ^s in Lemma 4.7 and by the properties $\psi_{i+1}^u = \varphi_i|_{W_i^u}$, we have that ζ_γ is the identity on a neighborhood $\tilde{B}_\gamma \subset (\mathcal{T}_\gamma \cap \bigcup_{j=0}^i \mathcal{H}_j)$ of

$\mathcal{T}_\gamma \cap (\bigcup_{j=0}^i W_j^s)$. Independently, one does the same for every separatrix $\gamma \in \check{G}_{i+1}$. Assuming that

$\zeta_{f(\gamma)} = f' \zeta_\gamma f^{-1}$ and $\tilde{B}_{i+1} = \bigcup_{\gamma \in \check{G}_{i+1}} \left(\bigcup_{k=0}^{per(\gamma)-1} f^k(\tilde{B}_\gamma) \right)$ we get a homeomorphism ζ_{i+1} on \mathcal{T}_{i+1} .

Thus the required homeomorphism coincides with $\phi_{\psi_{i+1}^u, \psi_{i+1}^s}$ on \mathcal{H}_{i+1} and with φ_i out of \mathcal{T}_{i+1} . \diamond

Let G be the union of all stable one-dimensional separatrices which do not contain heteroclinic points, $N_G^t = \bigcup_{\gamma \in G} N_\gamma^t$ and

$$\mathcal{M} = \mathcal{M}_{n-1} \cup G.$$

Lemma 4.12 *There are numbers $0 < \tau_1 < \tau_2 < 1$ and an equivariant embedding $h_{\mathcal{M}} : \mathcal{M} \rightarrow M'$ with the following properties:*

- 1) $h_{\mathcal{M}}$ coincides with φ_{n-1} out of $N_G^{\tau_2}$;
- 2) $h_{\mathcal{M}}$ coincides with $\phi_{\varphi_{n-1}, \psi^s}$ on $|_{N_G^{\tau_1}}$, where $\psi^s : L^s \rightarrow L^s$ is yielded by Lemma 4.7;
- 3) there is an f -invariant neighborhood of the set $N_G \cap (G_0 \cup \dots \cup G_{n-1})$ where $h_{\mathcal{M}}$ coincides with φ_{n-1} .

Proof: Let $\check{G} \subset G$ be a union of separatrices from G such that $\gamma_2 \neq f^k(\gamma_1)$ for every $\gamma_1, \gamma_2 \in \check{G}$, $k \in \mathbb{Z} \setminus \{0\}$ and $G = \bigcup_{\gamma \in \check{G}} orb(\gamma)$. Let $i \in \{0, \dots, n\}$, $p \in \Sigma_i$ and $\gamma \in G$.

Notice that $(N_\gamma \setminus (\gamma \cup p)) / f^{per(\gamma)}$ is homeomorphic to $X \times [0, 1]$ where X is 2-torus and the natural projection $\pi_\gamma : N_\gamma \setminus (\gamma \cup p) \rightarrow X \times [0, 1]$ sends ∂N_γ^t to $X \times \{t\}$ for each $t \in (0, 1)$ and sends $W_p^u \setminus p$ to $X \times \{0\}$. Let $\xi_\gamma = \phi_{\varphi_{n-1}|_{W_i^u, \psi_i^s}}^{-1} \varphi_{n-1}|_{N_\gamma^i \setminus (\gamma \cup p)}$ and $\hat{\xi}_\gamma = \pi_\gamma \xi_\gamma \pi_\gamma^{-1}|_{X \times [0, a_i]}$. Due to item 3) of Lemma 4.11, the homeomorphism $\hat{\xi}_\gamma$ coincides with the identity in some neighborhood of $\pi_\gamma(N_\gamma^{a_i} \cap (G_0 \cup \dots \cup G_{n-1}))$. Let us choose this neighborhood of the form $B_\gamma \times [0, a_i]$. Let us choose numbers $0 < \tau_{1,\gamma} < \tau_{2,\gamma} < a_i$ such that $\hat{\xi}_\gamma(X \times [0, \tau_{2,\gamma}]) \subset X \times [0, \tau_{1,\gamma}]$. By construction, $\hat{\xi}_\gamma : X \times [0, \tau_{2,\gamma}] \rightarrow X \times [0, 1]$ is a topological embedding which is the identity on $X \times \{0\}$ and $\hat{\xi}_\gamma|_{B_\gamma \times [0, \tau_{2,\gamma}]} = id|_{B_\gamma \times [0, \tau_{2,\gamma}]}$. Then, due to Proposition 5.2,

1. there is a homeomorphism $\hat{\zeta}_\gamma : X \times [0, \tau_{2,\gamma}] \rightarrow \hat{\xi}_\gamma(X \times [0, \tau_{2,\gamma}])$ such that $\hat{\zeta}_\gamma$ is identity on $X \times [0, \tau_{1,\gamma}]$ and is $\hat{\xi}_\gamma$ on $X \times \{\tau_{2,\gamma}\}$.

2. $\hat{\zeta}_\gamma|_{B_\gamma \times [0, \tau_{2,\gamma}]} = id|_{B_\gamma \times [0, \tau_{2,\gamma}]}$.

Let ζ_γ be a lift of $\hat{\zeta}_\gamma$ on $N_\gamma^{\tau_{2,\gamma}}$ which ξ_γ on $\partial N_\gamma^{\tau_{2,\gamma}}$. Thus $\varphi_\gamma = \phi_{\varphi_{n-1}|_{W_i^u, \psi_i^s}} \zeta_\gamma$ is the desired extension of φ_{n-1} to N_γ . Doing the same for every separatrix $\gamma \in \check{G}$ and extending it to the other separatrices from G by equivariance, we get the required homeomorphism $h_{\mathcal{M}}$ for $\tau_1 = \min_{\gamma \in \check{G}} \{\tau_{1,\gamma}\}$ and $\tau_2 = \min_{\gamma \in \check{G}} \{\tau_{2,\gamma}\}$. \diamond

So far in this section, we have modified the homeomorphism φ in a union of suitable linearizable neighborhoods of Ω_2 (with their 1-dimensional separatrices removed) so that the modified homeomorphism extends equivariantly to $W^s(\Omega_2)$. At the same time, we can perform a similar modification about Ω_1 since the involved linearizable neighborhoods of Ω_2 and Ω_1 are

mutually disjoint. Thus, we get a homeomorphism $h : M \setminus (\Omega_0 \cup \Omega_3) \rightarrow M \setminus (\Omega'_0 \cup \Omega'_3)$ conjugating $f|_{M \setminus (\Omega_0 \cup \Omega_3)}$ with $f'|_{M \setminus (\Omega'_0 \cup \Omega'_3)}$. Notice that $M \setminus (W_{\Omega_1}^s \cup W_{\Omega_2}^s \cup \Omega_3) = W_{\Omega_0}^s$ and $M \setminus (W_{\Omega'_1}^s \cup W_{\Omega'_2}^s \cup \Omega'_3) = W_{\Omega'_0}^s$. Since $h(W_{\Omega_1}^s) = W_{\Omega'_1}^s$ and $h(W_{\Omega_2}^s) = W_{\Omega'_2}^s$, then $h(W_{\Omega_0}^s \setminus \Omega_0) = W_{\Omega'_0}^s \setminus \Omega'_0$. Thus for each connected component A of $W_{\Omega_0}^s \setminus \Omega_0$, there is a sink $\omega \in \Omega_0$ such that $A = W_{\omega}^s \setminus \omega$. Similarly, $h(A)$ is a connected component of $W_{\Omega'_0}^s \setminus \Omega'_0$ such that $h(A) = W_{\omega'}^s \setminus \omega'$ for a sink $\omega' \in \Omega'_0$. Then we can continuously extend h to Ω_0 by defining $h(\omega) = \omega'$ for every $\omega \in \Omega_0$. A similar extension of h to Ω_3 finishes the proof of Theorem 1.

5 Topological background

We use below the notations which have been introduced before Lemma 4.8.

Proposition 5.1 *Let $z^0 > z^1 > \dots > z^m > \frac{z^0}{4} > 0$ and $\xi : \mathcal{T} \setminus Ox_3 \rightarrow N$ be a topological embedding with the following properties:*

- (i) $\xi a = a\xi$;
- (ii) ξ is the identity on Ox_1x_2 ;
- (iii) $\xi(\Pi(z^0 \cap \mathcal{T})) = \Pi(z^0)$ and $\xi(\Pi(z^i) \cap \partial\mathcal{T}) \subset \Pi(z^i)$, $i \in \{2, \dots, m\}$;
- (iv) $\xi(\tilde{c}) \cap \tilde{c}^0 = \emptyset$, $\xi(\tilde{c}^1) \cap \tilde{c}^0 = \emptyset$ and $\xi(\tilde{\beta}) \subset \tilde{V}$.

Then there is a homeomorphism $\zeta : \mathcal{T} \rightarrow N$ such that

- 1) $\zeta a = a\zeta$;
- 2) ζ is the identity on \mathcal{H} – and consequentially on Ox_1x_2 ;
- 3) $\zeta(\Pi(z^i) \cap \mathcal{T}) \subset \Pi(z^i)$, $i \in \{0, 2, \dots, m\}$
- 4) ζ is ξ on $\partial\mathcal{T}$.

Moreover, if ξ is identity on \tilde{B} for a set $B \subset (K \setminus U)$ then ζ is also identity on \tilde{B} .

Proof: For $j = 0, \dots, m$, we denote by E_j the domain of \mathbb{R}^3 located between the horizontal planes $\Pi(z_j)$ and $\Pi(z_{j+1})$, with $z_{m+1} = \frac{z_0}{4}$. Since the wanted ζ has to be equivariant with respect to a , it is useful to describe a fundamental domain \mathcal{V} for the action of a on the closure of $\mathcal{T} \setminus \mathcal{H}$; the natural one is

$$\mathcal{V} = cl(\mathcal{T} \setminus \mathcal{H}) \cap \left(\bigcup_{j=0}^m E_j \right),$$

where $cl(-)$ stands for *closure of* $(-)$. The domain \mathcal{V} is sliced by the horizontal planes $\Pi(z_j)$, $j = 2, \dots, m$, and the vertical cylinders $a^{-1}(\tilde{c})$ and \tilde{c}^1 , yielding the decomposition $\mathcal{V} = R_0 \cup R_1 \cup Q_0 \cup Q_2 \cup \dots \cup Q_m$ into solid tori whose interiors are mutually disjoint. Notice that the plane $\Pi(z_1)$ is not used in this decomposition.

More precisely, $R_0 \subset E_0$ is limited by the cylinders $a^{-1}(\tilde{c}^1)$ and $a^{-1}(\tilde{c})$; then, $R_1 \subset E_0$ is limited by the cylinders $a^{-1}(\tilde{c})$ and \tilde{c}^1 . The others of the list are obtained from \tilde{U} by slicing \mathcal{V} with horizontal planes. The first of the latter, namely Q_0 , is special as it is bounded by $\Pi(z_0)$ and $\Pi(z_2)$; then, Q_j is bounded by $\Pi(z_j)$ and $\Pi(z_{j+1})$ for $j = 2, \dots, m$. The vertical parts in the boundaries of the above-mentioned solid tori are provided by the vertical slices or the vertical parts of $\partial\mathcal{T} \cup \partial\mathcal{H}$.

For $j = 0, 2, \dots, m$, let $U(z_j) := \tilde{U} \cap \Pi(z_j)$. By construction, the top face of R_0 is $U'(z_0) := a^{-1}(U(z_{m+1})) = \Pi(z_0) \cap a^{-1}(\tilde{U})$; its bottom is $U'(z_1) := \Pi(z_1) \cap a^{-1}(\tilde{U})$. Similarly, the top of R_1 is $U''(z_0) := \Pi(z_0) \cap \tilde{K}$ and its bottom is $U''(z_1) := \Pi(z_1) \cap \tilde{K}$.

Very important is that each horizontal or vertical annulus Ann from the previous list is marked with a special arc noted $\beta(Ann)$ linking the two boundary components of Ann and defined as follows:

$$\beta(Ann) = Ann \cap \{x_1 > 0, x_2 = 0\}.$$

According to assumption (iv), all these arcs (except when $Ann = U''(z_0)$ or $U''(z_1)$) fulfill the next condition, referred to as the β -condition, namely: they are mapped by ξ into $\{x_1 > 0\}$.

First of all, we define $\zeta|_{R_1}$ by rescaling $\zeta|_{\tilde{K}}$ in the next way. There is a homeomorphism $\kappa : \tilde{K} \rightarrow R_1$ of the form: $(x_1, x_2, x_3) \mapsto (x_1, x_2, \rho(x_3))$ where $\rho : [0, z_0] \rightarrow [z_1, z_0]$ is any increasing continuous function. Then, we define $\zeta|_{R_1} = \kappa \xi|_{\tilde{K}} \kappa^{-1}$. Observe that ζ equals ξ on $U''(z_0)$ and coincide with the identity on $U''(z_1)$. As a consequence, the complement part of the statement follows directly. Indeed, if B lies in K and $\xi|_{\tilde{B}} = Id$ then its conjugate by κ is the identity on $\tilde{B} \cap R_1$.

We continue by defining ζ on the other horizontal annuli from the previous list. As required, ζ is the identity when this annulus lies in \mathcal{H} . For the others, that is, $U'(z_0)$ and $U(z_j)$, $j = 0, 2, \dots, m$, Lemma 5.4 is applicable as it is explained right below.

Each of these annuli is bounded by two curves; one of the two lies in the frontier of \mathcal{T} on which ζ has to coincide with ξ and is mapped in the respective plane $\Pi(z_j)$ – according to (iii); and the other lies in \mathcal{H} where ζ has to coincide with $Id|_{\mathcal{H}}$. In order to satisfy the equivariance property 3), we choose

$$(*) \quad \zeta|_{U(z_{m+1})} = a \zeta|_{U'(z_0)} a^{-1}.$$

Moreover, due to the β -condition, Lemma 5.4 holds and yields ζ on each of the listed horizontal annuli.

We continue by defining ζ on the vertical annuli in the above splitting of \mathcal{V} . When such an annulus lies in ∂H (resp. $\partial \mathcal{T}$), we must take $\zeta = Id$ (resp. $\zeta = \xi$) over there. The last two annuli are $R_0 \cap R_1$ and $R_1 \cap Q_0$ on which ζ is already defined by conjugating by κ . Notice that the β -condition holds for these two annuli because conjugating by κ preserves the β -condition.

Let us look more precisely to ∂Q_0 . It is made of the following: two horizontal annuli $U(z_0)$ and $U(z_2)$, and three vertical ones $R_1 \cap Q_0$, $\tilde{c}^1 \cap E_1$ (lying in \mathcal{H}) and $\tilde{c} \cap (E_0 \cup E_1)$. The β -condition holds for each of these latter annuli.

For finishing the proof, it remains to extend the ζ which we have defined right above on the tori ∂R_0 , ∂Q_0 and ∂Q_j , $j = 2, \dots, m$ to embeddings of the solid tori $R_0, Q_0, Q_2, \dots, Q_m$ with values in $E_0, E_0 \cup E_1, E_2, \dots, E_m$ respectively. The boundary data consists of annuli where ζ fulfils the β -condition. Therefore, the assumptions of Proposition 5.5 are fulfilled; then the conclusion holds true and yields the desired extension of ζ to the listed solid tori.

According to (*), we can extend the ζ which is built above on a fundamental domain to $\mathcal{T} \setminus \mathcal{H}$ equivariantly. Since this extension coincides with the identity on \mathcal{H} , it extends by $Id|_{Ox_1x_2}$. This is a continuous extension because any point of the plane Ox_1x_2 adheres only to $\mathcal{H} \setminus Ox_1x_2$ when considering the closure of $\mathcal{T} \setminus Ox_1x_2$.

◇

Proposition 5.2 *Let X be a compact topological space, $0 < \tau_1 < \tau_2 < 1$ and $\hat{\xi} : X \times [0, \tau_2] \rightarrow X \times [0, 1]$ be a topological embedding which is the identity on $X \times \{0\}$, $X \times [0, \tau_1] \subset \hat{\xi}(X \times [0, \tau_2])$. Then*

1. *there is a homeomorphism $\hat{\zeta} : X \times [0, \tau_2] \rightarrow \xi(X \times [0, \tau_2])$ such that $\hat{\zeta}$ is identity on $X \times [0, \tau_1]$ and is $\hat{\xi}$ on $X \times \{\tau_2\}$.*
2. *if for a set $B \subset X$ the equality $\hat{\xi}|_{B \times [0, \tau_2]} = id|_{B \times [0, \tau_2]}$ is true then $\hat{\zeta}|_{B \times [0, \tau_2]} = id|_{B \times [0, \tau_2]}$.*

Proof: Let us choose $l \in (\tau_1, \tau_2)$ such that $X \times [0, l] \subset \hat{\xi}(X \times [0, \tau_2])$. Define a homeomorphism $\kappa : [\tau_1, 1] \rightarrow [0, 1]$ by the formula

$$\kappa(t) = \begin{cases} (x, \frac{l(t-\tau_1)}{l-\tau_1}), & t \in [\tau_1, l]; \\ (x, t), & t \in [l, 1]. \end{cases}$$

Let $\mathcal{K}(x, t) = (x, \kappa(t))$ on $X \times [\tau_1, 1]$. Then the required homeomorphism can be defined by the formula

$$\hat{\zeta}(x, t) = \begin{cases} (x, t), & t \in [0, \tau_1]; \\ \mathcal{K}^{-1}\xi(\mathcal{K}((x, s))), & s \in [\tau_1, \tau_2]. \end{cases}$$

Property 2 automatically follows from this formula. ◇

We now collect some facts of geometric topology in dimension 2 and 3 on which the proof of Proposition 5.1 is based. We begin with the so-called Schönflies Theorem (see Theorem 10.4 in [17]).

Proposition 5.3 *Every topological embedding of S^1 into \mathbb{R}^2 is the restriction of a global homeomorphism of \mathbb{R}^2 which is the identity map outside some compact set of the plane.*

One can derive the Annulus Theorem in dimension 2; we state and prove it in the only case which we use. The coordinates of \mathbb{R}^2 are (x_1, x_2) . The unit closed disc in \mathbb{R}^2 is denoted by \mathbb{D}^2 ; its boundary is S^1 . The annulus $2\mathbb{D}^2 \setminus \text{int}(\mathbb{D}^2)$ is denoted by \mathbb{A} . Finally, \mathbb{I} denotes the arc $\{1 \leq x_1 \leq 2, x_2 = 0\}$.

Lemma 5.4 *Let $g : 2S^1 \cup \mathbb{I} \rightarrow \mathbb{R}^2 \setminus (0, 0)$ be a topological embedding which surrounds the origin in the direct sense and has the next properties: $g(\mathbb{I}) \subset \{x_1 > 0\}$, the image $g(2S^1)$ avoids the circle C of radius $\frac{3}{2}$ and $g(1, 0)$ lies inside $\frac{3}{2}\mathbb{D}^2$. Then $g|_{2\mathbb{D}^2}$ extends to an embedding $G : \mathbb{A} \rightarrow \mathbb{R}^2 \setminus \text{int}(\mathbb{D}^2)$ which coincides with the identity on S^1 and maps \mathbb{I} into $\{x_1 > 0\}$.*

Proof: Let p be the last point on $g(\mathbb{I})$ starting from $g(1, 0)$ which belongs to C . Let q be its inverse image in \mathbb{I} . Define G on the segment $[(1, 0), q]$ as the affine map whose image is $[(1, 0), p]$ and take G coinciding with g on $[q, (2, 0)]$. The image $G(\mathbb{I})$ is a simple arc in $\{x_1 > 0\}$. Because, any simple arc is tame in the plane, this definition of G on \mathbb{I} and the values which are imposed on the two circles S and $2S^1$ extends to a neighborhood N of $S^1 \cup \mathbb{I} \cup 2S^1$ in \mathbb{R}^2 . By taking one boundary component of N one derives a parametrized simple curve C' in $\mathbb{R}^2 \setminus \mathbb{D}^2$ which does

not surround the origin. Therefore, by the Schönflies Theorem, C' bounds a disc D in $\mathbb{R}^2 \setminus \mathbb{D}^2$ and the parametrization of C' extends to a parametrization of D . This yields the complete definition of G . \diamond

We are now going to apply deep theorems of geometric topology in dimension 3 to a concrete situation emanating from the problem we are facing in Proposition 5.1. The setting is the following. We look at the 3-space

$$Y = \mathbb{A} \times [0, 1] = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 1 \leq x_1^2 + x_2^2 \leq 4, 0 \leq x_3 \leq 1\}.$$

Denote Q_0 the solid torus in Y limited by the next two annuli:

- the vertical annulus $A_v := \{x_1^2 + x_2^2 = 1, 0 \leq x_3 \leq 1\}$;
- the *standard curved* annulus $A_0 := \{x_1^2 + x_2^2 + (x_3 - \frac{1}{2})^2 = \frac{5}{4}\} \cap Y$.

Observe that A_0 is contained in Y and contains the two horizontal circles forming ∂A_v .

Proposition 5.5 *Let $g : A_0 \rightarrow Y$ a bi-collared⁹ topological embedding. It is assumed that g coincides with the identity on ∂A_0 and maps the arc $\Gamma_0 := A_0 \cap \{x_1 > 0, x_2 = 0\}$ to an arc Γ in $Y \cap \{x_1 > 0\}$. Then, g extends to an embedding $G : Q_0 \rightarrow Y$ which coincides with the identity on A_v .*

Proof: The image $A := g(A_0)$ separates Y in two components X and X^* . Since A is bi-collared, these two domains of Y are topological 3-manifolds. Therefore, we are allowed to invoke one of the deepest theorems in 3-dimensional topological geometry, namely E. Moise's theorem (See [17, Chap.23 & Theorem 35.3] for existence and [17, 36.2] for uniqueness):

Every 3-manifold has a unique PL-structure, up to an arbitrarily small topological isotopy. Moreover, its topological boundary is a PL-submanifold.

As a consequence, X and X^* have PL -structures which agree on their intersection A . By uniqueness applied for $X \cup X^*$, the PL -structure on the union is the standard one after some C^0 -small ambient isotopy. Denote by P the planar surface $\{x_2 = 0, x_1 > 0\} \cap Y$. After a new C^0 -small ambient isotopy in Y , we may assume that A and P are in general position. In what follows, we borrow the idea of proof from [14, Theorem 3.1]¹⁰.

Since P intersects each connected component of ∂A in one point only, we are sure that in general position $P \cap A$ is made of finitely many simple closed curves c_1, \dots, c_k in $\text{int}A$ and one arc γ which links the two components of ∂A . One of the above curves is *innermost* in P , meaning that it bounds a disc in P whose interior avoids A ; let say that c_1 is so. More precisely, c_1 bounds a disc d in P and a disc δ in A . By innermost position, $d \cup \delta$ is an embedded PL 2-sphere σ . As Y lies in \mathbb{R}^3 , this sphere bounds a 3-ball $\Delta \subset Y$. We are going to use these data in two ways.

First, we use δ for finding an isotopy h_t of A in itself from $Id|_A$ to $h_1 : A \rightarrow A$ such that $h_1(\Gamma) \cap \delta = \emptyset$. This is easily done as $\Gamma \cap \delta$ avoids one point z_δ in δ : one pushes $\Gamma \cap \delta$ along

⁹This means that g extends to an embedding $A_0 \times (-\varepsilon, +\varepsilon)$ for some $\varepsilon > 0$.

¹⁰In this article, it should be meant that the PL -category (or smooth category) is used. Indeed, there is no *general position* statement in topological geometry without more specific assumption.

the rays of δ issued z_δ . Notice that $h_1(\Gamma)$ still lies in $\{x_1 > 0\}$, but this could be not true for $h_t(\Gamma)$, $t \neq 0, 1$, when δ is not contained in $\{x_1 > 0\}$.

Once, this is done, the ball Δ is used for finding an ambient isotopy of Y which is supported in a neighborhood of B , small enough so that $h_1(\Gamma)$ is kept fixed, and which moves $A \cap \Delta$ to the complement of P . Hence, this isotopy cancels c_1 from $A \cap P$; all intersection curves contained in $\text{int } \delta$ are cancelled at the same time. By repeating isotopies similar to the two previous ones, as many times as necessary, we get an embedding $g' : A_0 \rightarrow Y$ which coincides with the identity on ∂A_0 , still maps Γ_0 into $\{x_1 > 0\}$ and fulfils the next property: $A' := g'(A_0) \cap P$ is made of one arc γ' only which links the two components of $g'(\partial A_0)$. The annulus A' divides Y into two (closed) domains X' and X'^* which come from the splitting $X \cup_A X^*$ by an ambient isotopy fixing $\{x_3 = 0, 1\}$ pointwise.

As γ' is the only intersection component of $P \cap A'$, one knows that γ' divides P into a disc $\mu' \subset X'$ (meaning a *meridian* disc in a solid torus) and its complement in P . Removing from X' a regular neighborhood of μ' yields a PL embedded 2-sphere S . According to the Alexander theorem [1], this sphere bounds a ball $B_{X'}$ in \mathbb{R}^3 , as $Y \subset \mathbb{R}^3$. It is not possible that $B_{X'}$ contains μ' in its interior; in the contrary, $B_{X'}$ would get out of X'^* and have a non-bounded interior. As a consequence, X' is a *solid torus* since it is made of a ball and a 1-handle attached. The same holds for X as it is ambient isotopic to X' in Y .

This is not sufficient for concluding. It would be necessary to prove the same for the curve $g'(\Gamma_0)$, after making it a closed curve by adding the vertical arc $\gamma_0^* \subset A_v$ which links the two points of $g'(\Gamma_0) \cap A_v = \Gamma_0 \cap A_v = g(\Gamma_0) \cap A_v = \Gamma_0 \cap A_v = \gamma' \cap A_v$. That is the place where the assumption about $g(\Gamma_0)$ is used.

CLAIM. There exists an ambient isotopy from $\text{Id}|_Y$ to k_1 which is stationary on the vertical annulus A_v , which maps A' into itself and moves $g'(\Gamma_0)$ to γ' .

PROOF OF THE CLAIM. Assume first that the arcs $g'(\Gamma_0)$ and γ' meet in their end points only. Consider the closed curve α which is made of $\gamma \cup g'(\Gamma_0)$; it is contained in $\{x_1 > 0\} \cap A'$. By construction, the homological intersection of α with γ' is zero. Therefore, α bounds a disc $\delta' \subset A'$. Notice that it could be not contained in $\{x_1 > 0\}$. The disc δ' allows one to move $g'(\Gamma_0)$ to γ' by an isotopy of A' into itself with the required properties.

In case where $g'(\Gamma_0) \cap \text{int } \gamma'$ is non-empty, in general position this intersection is made of finitely many points. Among them, choose the point x which is the closest to $\gamma' \cap \{x_3 = 1\}$ when traversing γ' starting from bottom. Denote by x_0 the point $\gamma' \cap \{x_3 = 1\}$. One forms a closed curve $\tau \subset A'$ made of two arcs in A' , from x to x_0 respectively in γ' and $g'(\Gamma_0)$. For the same reason as for α above, the curve τ bounds a disc in A' which allows one to cancel x from $g'(\Gamma_0) \cap \gamma'$ by an isotopy of A' into itself with the required properties. Iterating this process reduces us to the first case. The claim is proved. \diamond

Let $g'' := k_1 g' : A_0 \rightarrow A'$. As a consequence of the claim, the closed curve $g''(\Gamma_0) \cup \gamma_0^*$ is the boundary of the *meridian* disc μ' . We are going to show that g'' extends to an embedding $G'' : Q_0 \rightarrow X'$ which coincides with the identity on A_v . In this aim, we denote by μ_0 the meridian of Q_0 defined by $\mu_0 = Q_0 \cap P$. For beginning with, we consider regular neighborhoods $N(\mu_0)$ and

$N(\mu')$ of both meridians and we extend $g''|_{N(\mu_0) \cap A_0}$ to a homeomorphism $G''_0 : N(\mu_0) \rightarrow N(\mu')$ which is the identity on $N(\mu_0) \cap A_v$.

Let B_{Q_0} be the ball in Q_0 which is the closure of $Q_0 \setminus N(\mu_0)$. The restricted map $G''_0|_{\partial B_{Q_0}}$ glued with the restriction of g'' to the closure of $A_0 \setminus N(\mu_0)$ yields a homeomorphism

$$G''_1 : \partial B_{Q_0} \rightarrow \partial B_{X'}.$$

The desired G'' is obtained by extending G''_1 to B_{Q_0} by the cone construction (seeing a ball as the cone on its boundary). Since g and g'' are related one to the other by an ambient isotopy fixing A_v pointwise, an extension of g follows from an extension of g'' . \diamond

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