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Linear pantographic sheets: existence and uniqueness of weak solutions

Victor A. Eremeyev · Francesco dell'Isola · Claude Boutin · David Steigmann

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Abstract The well-posedness of the boundary value problem for second gradient elasticity has been studied under the assumption of strong ellipticity of the dependence on the second placement gradients (see e.g. [9,32]). In [18] the equilibrium of pantographic lattices is studied via a homogenised second gradient deformation energy and the predictions obtained with such a model are successfully compared with experiments. This energy is not strongly elliptic in its dependence on second gradients. This circumstance motivates the present paper, where we address the well-posedness for the equilibrium problem for

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homogenised pantographic lattices. To do so: i) we introduce a class of subsets of anisotropic Sobolevs space as energy space E relative to assigned boundary conditions; ii) we prove that the considered strain energy density is coercive and positive definite in E; iii) we prove that the set of placements for which is the strain energy is vanishing (so-called floppy modes) strictly include rigid motions; iv) we determine the restrictions on displacement boundary conditions which assure existence and uniqueness of linear static problems. The presented results represent one of the first mechanical applications of the concept of Anisotropic Sobolev space, initially introduced only on the basis of purely abstract mathematical considerations.

Keywords strain gradient elasticity · linear pantographic sheets · existence · uniqueness · anisotropic Sobolev's space

1 Introduction

Mechanical scientists have been recently attracted to the formulation of design and construction criteria of new materials whose behaviour is established *a priori*. One can say that the aim of this stream of researches is to produce **Materials on Demand**. More precisely: once fixed the peculiar behaviour of a material which is desirable for optimising its use in a given application, the aim of aforementioned researches is to find the way for constructing such a material. Materials designed in order to get a specific behaviour are often called metamaterials.

The role of mathematical sciences in the design and constructions of metamaterials recently increased for two reasons: i) the development of the technology of 3D printing allowed for the transformation of mathematically conceived structures, geometries and material properties into the reality of precisely built specimens; ii) the way in which one specifies the set of properties to be realised is specifically mathematical, as it consists in choosing the equations one assumes are governing the mechanical response of the conceived metamaterial.

Once more we can say that mathematics is shaping our world, as it is allowing us to design new technological solutions and tools. The present paper deals with a mathematical problem arising in a specific context involving the design of second gradient metamaterials. More precisely: in order to find a class of materials whose deformation energy depends on both first and second gradient of placement field in [18] a microstructured (pantographic) fabric is introduced and its homogenised continuum model (which we call pantographic sheet) is determined. Various aspects of modelling of pantographic lattices are considered in [3, 37, 39, 41, 45–49] where discrete and homogenized models are considered. Let us note that one of the sources of generalized continua and models of metamaterials is the homogenization of heterogeneous materials, see e.g. [14, 38, 40, 44] and reference therein. Homogenization may lead to strain gradient models [8, 11]. While the ideas underlying the definition of pantographic microstructures have been exploited up to now only in the context

of purely mechanical phenomena, it is expected that when introducing multiphysics effects (as the piezoelectric coupling phenomena exploited as explained in [12,13,19,27,36] including surface-related phenomena [21,34]) the designed meta materials could have even more interesting features. We expect a fortiori that the mathematical tools used in the present context will be of use also in the envisioned more general context.

The linearised equilibrium equations valid in the neighbourhood of a stress free configuration for such pantographic sheets cannot be immediately studied by using the results available in the literature. However the standard strategy involving the use of Poincaré inequality, Lax-Milgram Theorem and coercivity of bilinear deformation energy form do apply also in the present more generalised context.

What has to be modified is the Energy space where the solutions, relative to suitable well-posed boundary conditions, are looked for and the Sobolev space which include this Energy Space.

Indeed the concept of Anisotropic Sobolev Space, whose definition was conceived on purely logical grounds by Sergei M. Nikol'skii, see [35], has to be used in order to apply the abstract Hilbertian setting of solution strategy.

We expect that further developments will lead us to study the complete nonlinear problem of deformation of pantographic sheets.

2 Postulated deformation energy for "long-fibers" pantographic sheets

Pantographic sheets are bidimensional continua whose microstructure is constituted by a lattice of extensible and continuous fibers having bending stiffness and interconnected by pivots (i.e. pin joints). It has to be explicitly remarked that, in general, we are not considering trusses. A truss, by definition, is assumed to comprise a set of independent beams that are connected by means of pin joints connecting only ending points of the beams. This means that if the truss is loaded only with concentrated forces applied to pin joints then each beam (or fiber) can only be either in compression or in extension. We call *lattices of beams* the most general beams structure involving pin joints (but also possibly clamping devices, or rollers or glyphs).

Roughly speaking, pantographic sheets can be characterized as those lattices of fibers whose microstructure, once pivots are assumed to be ideal and no external constraints are applied, allows for the existence of some homogeneous deformations which do not store deformation energy. These deformations are sometimes called "floppy-modes". In Fig. 3 such a structure is schematically described, while in Fig. 2 a picture of a 3D printed specimen in polyamide is shown.

The main feature of the considered pantographic structure consists in the presence of "long" continuous fibers constituting two arrays: at each intersection point of one fiber with all fibers of the other array it is present a pin joint

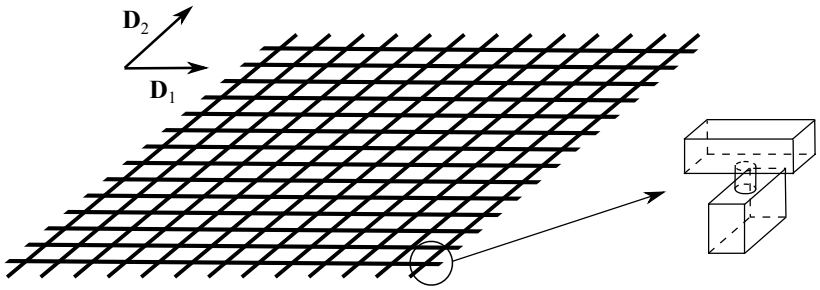


Fig. 1 Scheme of a pantographic sheet and beams connection trough a pivot.

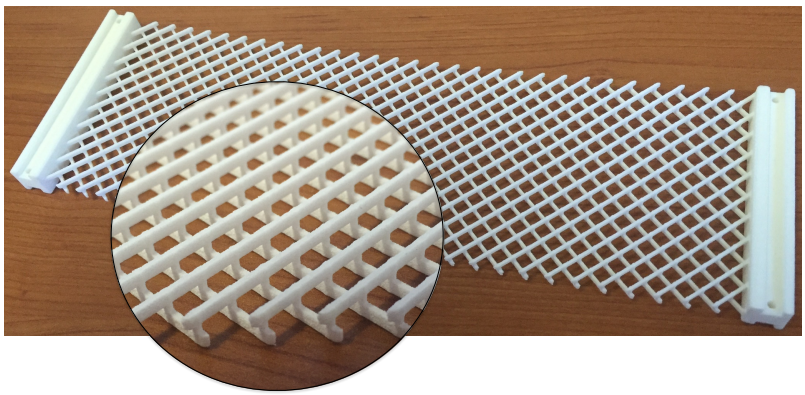


Fig. 2 3D printed specimen of a pantographic sheet.

which is not interrupting the mechanical and geometrical continuity of both interconnecting fibers.

We assume that in the reference configuration the two arrays of fibers are initially orthogonal and we denote $\mathbf{D}_\alpha$, $\alpha = 1, 2$, the unit vectors of their current directions.

As a macro model of the system described before we consider a continuum whose reference configuration is given by a (suitably regular) domain $\omega \subset \mathbb{R}^2$. By assuming planar motion, the actual configuration of ω is described by the planar macro-placement

$$\boldsymbol{\chi} : \omega \to \mathbb{R}^2 \tag{1}$$

whose gradient $\nabla\boldsymbol{\chi}$ will be denoted by $\mathbf{F}$.

For considered "long fibers" pantographic sheets a possible expression for deformation energy is given by (the two possible methods for getting this

expression are described in [18] or in [7]):

$$
\begin{aligned}
U(\boldsymbol{\chi}(\cdot)) = &\iint_{\omega} \sum_{\alpha} \frac{\mathbb{K}_e^{\alpha}}{2} \left(\|\boldsymbol{F}\mathbf{D}_{\alpha}\| - 1\right)^2 \, d\omega \\
&+ \iint_{\omega} \sum_{\alpha} \frac{\mathbb{K}_b^{\alpha}}{2} \left[\frac{\nabla \boldsymbol{F} : \mathbf{D}_{\alpha} \otimes \mathbf{D}_{\alpha} \cdot \nabla \boldsymbol{F} : \mathbf{D}_{\alpha} \otimes \mathbf{D}_{\alpha}}{\|\boldsymbol{F}\mathbf{D}_{\alpha}\|^2} \right. \\
&\qquad \left. - \left(\frac{\boldsymbol{F}\mathbf{D}_{\alpha}}{\|\boldsymbol{F}\mathbf{D}_{\alpha}\|} \cdot \frac{\nabla \boldsymbol{F} : \mathbf{D}_{\alpha} \otimes \mathbf{D}_{\alpha}}{\|\boldsymbol{F}\mathbf{D}_{\alpha}\|} \right)^2 \right] d\omega \\
&+ \iint_{\omega} \frac{\mathbb{K}_p}{2} \left| \arccos\left(\frac{\boldsymbol{F}\,\boldsymbol{D}_1}{\|\boldsymbol{F}\,\boldsymbol{D}_1\|} \cdot \frac{\boldsymbol{F}\,\boldsymbol{D}_2}{\|\boldsymbol{F}\,\boldsymbol{D}_2\|} \right) - \frac{\pi}{2} \right|^2 d\omega
\end{aligned} \tag{2}
$$

which accounts for stretching (first integral) and bending deformations of fibers (second integral) as well as for the resistance to shear distortion (third integral) related to the variation of the angle between the fibers. The twisting deformations in fibers are not considering here. coefficients $\mathbb{K}_e^{\alpha} > 0$ and $\mathbb{K}_b^{\alpha} > 0$ are related respectively to the extensional and bending stiffnesses of the fibers at the interpivot scale, while the coefficient $\mathbb{K}_p \geq 0$ models, at macro-level, the shear stiffness of the pantographic sheet and is related to the interaction between the two arrays of fibers via their interconnecting pivots: when these pivots are perfect this interaction is vanishing and $\mathbb{K}_p$ vanishes. $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^2$ and : is the double dot product.

In this paper we will start considering small deformations of the sheet in the neighborhood of the reference configuration. Therefore we calculate the second order Taylor expansion for the energy $U(\boldsymbol{\chi}(\cdot))$ in terms of the small parameter η controlling the amplitude of the displacement $\mathbf{u}$ starting from the reference configuration. In formulas

$$
\boldsymbol{\chi}(X) = \mathbf{X} + \eta \mathbf{u}(\mathbf{X}), \qquad \mathbf{X} \in \omega. \tag{3}
$$

By introducing the notations $\mathbf{H} := \nabla \mathbf{u}$, $\mathbf{H} =: \mathbf{E} + \mathbf{W}$, where $\mathbf{E}$ is symmetric and $\mathbf{W}$ is skew-symmetric we get formally (where $\mathbf{I}$ denotes the identity tensor)

$$
\mathbf{F} = \mathbf{I} + \eta \mathbf{H}, \qquad \nabla \mathbf{F} = \eta \nabla \mathbf{H}.
$$

As a consequence

$$
\begin{gathered}
\mathbf{F}\mathbf{D}_{\alpha} = \mathbf{D}_{\alpha} + \eta \mathbf{H}\mathbf{D}_{\alpha}, \\
\mathbf{F}\mathbf{D}_{\alpha} \cdot \mathbf{F}\mathbf{D}_{\alpha} = 1 + 2\eta \mathbf{H}\mathbf{D}_{\alpha} \cdot \mathbf{D}_{\alpha} + \eta^2 \mathbf{H}\mathbf{D}_{\alpha} \cdot \mathbf{H}\mathbf{D}_{\alpha}, \\
\|\mathbf{F}\mathbf{D}_{\alpha}\| = \sqrt{1 + 2\eta \mathbf{E}\mathbf{D}_{\alpha} \cdot \mathbf{D}_{\alpha} + \eta^2 \mathbf{H}\mathbf{D}_{\alpha} \cdot \mathbf{H}\mathbf{D}_{\alpha}} \simeq 1 + \eta \mathbf{H}\mathbf{D}_{\alpha} \cdot \mathbf{D}_{\alpha}, \\
\frac{1}{\|\mathbf{F}\mathbf{D}_{\alpha}\|} \simeq 1 - \eta \mathbf{H}\mathbf{D}_{\alpha} \cdot \mathbf{D}_{\alpha}, \\
\frac{1}{\|\mathbf{F}\mathbf{D}_{\alpha}\|^2} \simeq 1 - 2\eta \mathbf{H}\mathbf{D}_{\alpha} \cdot \mathbf{D}_{\alpha},
\end{gathered}
$$

$$\frac{\mathbb{K}_e^\alpha}{2}\left(\|\mathbf{F}\mathbf{D}_\alpha\|-1\right)^2 \simeq \frac{\mathbb{K}_e^\alpha}{2}\left(\eta\mathbf{H}\mathbf{D}_\alpha\cdot\mathbf{D}_\alpha\right)^2,$$

$$\frac{\boldsymbol{F}\,\boldsymbol{D}_1}{\|\boldsymbol{F}\,\boldsymbol{D}_1\|}\cdot\frac{\boldsymbol{F}\,\boldsymbol{D}_2}{\|\boldsymbol{F}\,\boldsymbol{D}_2\|} \simeq \left(\eta\mathbf{D}_1\cdot\mathbf{H}\mathbf{D}_2+\eta\mathbf{H}\mathbf{D}_1\cdot\mathbf{D}_2\right),$$

$$\frac{\mathbb{K}_p}{2}\left|\arccos\left(\frac{\boldsymbol{F}\,\boldsymbol{D}_1}{\|\boldsymbol{F}\,\boldsymbol{D}_1\|}\cdot\frac{\boldsymbol{F}\,\boldsymbol{D}_2}{\|\boldsymbol{F}\,\boldsymbol{D}_2\|}\right)-\frac{\pi}{2}\right|^2 \simeq \frac{\mathbb{K}_p}{2}\left(\eta\mathbf{D}_1\cdot\mathbf{H}\mathbf{D}_2+\eta\mathbf{H}\mathbf{D}_1\cdot\mathbf{D}_2\right)^2,$$

$$\frac{\nabla\mathbf{F}:\mathbf{D}_\alpha\otimes\mathbf{D}_\alpha\cdot\nabla\mathbf{F}:\mathbf{D}_\alpha\otimes\mathbf{D}_\alpha}{\|\mathbf{F}\mathbf{D}_\alpha\|^2} \simeq \eta^2\left(\nabla\mathbf{H}:\mathbf{D}_\alpha\otimes\mathbf{D}_\alpha\cdot\nabla\mathbf{H}\mathbf{D}_\alpha\otimes\mathbf{D}_\alpha\right),$$

$$\frac{\mathbf{F}\mathbf{D}_\alpha}{\|\mathbf{F}\mathbf{D}_\alpha\|}\cdot\frac{\nabla\mathbf{F}\mathbf{D}_\alpha\otimes\mathbf{D}_\alpha}{\|\mathbf{F}\mathbf{D}_\alpha\|} \simeq \left(1-\eta\mathbf{H}\mathbf{D}_\alpha\cdot\mathbf{D}_\alpha\right)^2\left(\mathbf{D}_\alpha+\eta\mathbf{H}\mathbf{D}_\alpha\right)\cdot\eta\nabla\mathbf{H}\mathbf{D}_\alpha\otimes\mathbf{D}_\alpha,$$

$$\frac{\mathbf{F}\mathbf{D}_\alpha}{\|\mathbf{F}\mathbf{D}_\alpha\|}\cdot\frac{\nabla\mathbf{F}\mathbf{D}_\alpha\otimes\mathbf{D}_\alpha}{\|\mathbf{F}\mathbf{D}_\alpha\|} \simeq \mathbf{D}_\alpha\cdot\eta\nabla\mathbf{H}\mathbf{D}_\alpha\otimes\mathbf{D}_\alpha.$$

Since $\mathbf{D}_\alpha$ are unit orthogonal vector we may replace them by standard basis vectors $\mathbf{i}_\alpha$, $\mathbf{D}_\alpha=\mathbf{i}_\alpha$. As a consequence if the displacement $\mathbf{u}$ is represented in the basis $\mathbf{D}_\alpha$ by means of its components

$$\mathbf{u}=u_1\mathbf{i}_1+u_2\mathbf{i}_2,\quad u_\alpha=u_\alpha(x_1,x_2),\quad \mathbf{X}=x_1\mathbf{i}_1+x_2\mathbf{i}_2,$$

then the second order Taylor expansion of the energy (2) is given by:

$$U\left(\mathbf{u}(\cdot)\right)=\iint\limits_\omega\left[\frac{\mathbb{K}_e^1}{2}u_{1,1}^2+\frac{\mathbb{K}_e^2}{2}u_{2,2}^2+\frac{\mathbb{K}_p}{2}\left(u_{1,2}+u_{2,1}\right)^2+\frac{\mathbb{K}_b^1}{2}u_{1,22}^2+\frac{\mathbb{K}_b^2}{2}u_{2,11}^2\right]d\omega, \tag{4}$$

see also [7] for a direct derivation of this energy by a rigorous homogenization procedure. Here indices after comma denote derivatives, so $f_{,\alpha}$ is the partial derivative of f with respect to x_α, $f_{,\alpha}=\partial_\alpha f\equiv\frac{\partial f}{\partial x_\alpha}$.

A mathematically interesting case is represented by pantographic structures whose shear stiffness is vanishing: this singular limit case will be addressed in the following sections.

3 Energy for pantographic sheets and equilibrium conditions

Let us consider the deformation energy relative to pantographic structures having vanishing shear stiffness. The deformation energy becomes

$$U\left(\mathbf{u}(\cdot)\right)=\iint\limits_\omega W\,d\omega, \tag{5}$$

where the strain energy density W is given by

$$W=\frac{\mathbb{K}_e^1}{2}u_{1,1}^2+\frac{\mathbb{K}_e^2}{2}u_{2,2}^2+\frac{\mathbb{K}_b^1}{2}u_{1,22}^2+\frac{\mathbb{K}_b^2}{2}u_{2,11}^2.$$

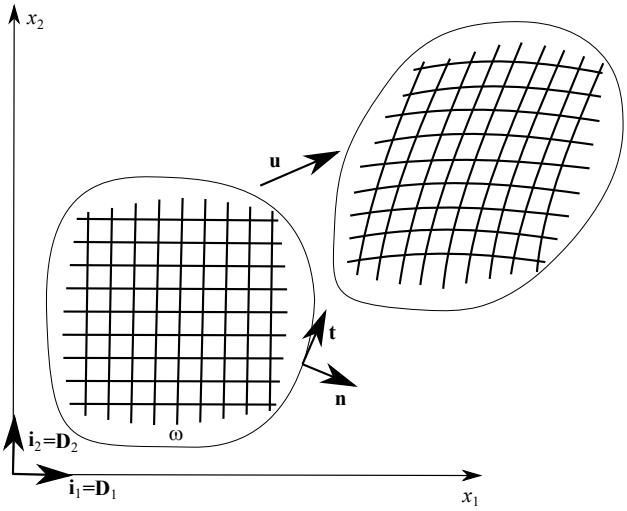


Fig. 3 Deformation of a pantographic sheet.

For derivation of the equilibrium conditions we consider the first variation of U. First we obtain

$$\delta U = \iint\limits_{\omega} \left(\mathbb{K}^1_e u_{1,1}\delta u_{1,1} + \mathbb{K}^2_e u_{2,2}\delta u_{2,2} + \mathbb{K}^1_b u_{1,22}\delta u_{1,22} + \mathbb{K}^2_b u_{2,11}\delta u_{2,11}\right)\, d\omega.$$

Then, integrating by parts we transform δU into

$$\begin{aligned}
\delta U = &\iint\limits_{\omega} \left(-\mathbb{K}^1_e u_{1,11} + \mathbb{K}^1_b u_{1,2222}\right)\delta u_1\, d\omega \\
&+ \iint\limits_{\omega} \left(-\mathbb{K}^2_e u_{2,22} + \mathbb{K}^2_b u_{2,1111}\right)\delta u_2\, d\omega \\
&+ \int\limits_{\partial\omega} \left[\left(n_1\mathbb{K}^1_e u_{1,1} - n_2\mathbb{K}^1_b u_{1,222}\right)\delta u_1 + n_2 u_{1,22}\delta u_{1,2}\right]\, ds \\
&+ \int\limits_{\partial\omega} \left[\left(n_2\mathbb{K}^2_e u_{2,2} - n_1\mathbb{K}^2_b u_{2,111}\right)\delta u_2 + n_1 u_{2,11}\delta u_{2,1}\right]\, ds. \qquad (6)
\end{aligned}$$

Here $n_\alpha = \mathbf{i}_\alpha \cdot \mathbf{n}$, $t_\alpha = \mathbf{i}_\alpha \cdot \mathbf{t}$, $\mathbf{n}$ and $\mathbf{t}$ are the unit external normal and unit tangent to the boundary $\partial\omega$ vectors, respectively, see Fig. 3. Representing $\delta u_{1,2}$ and $\delta u_{2,1}$ by formulas

$$\delta u_{1,2} = \mathbf{i}_2 \cdot \nabla\delta u_1, \quad \delta u_{2,1} = \mathbf{i}_1 \cdot \nabla\delta u_2$$

with ∇ defined at the boundary through normal and tangent derivatives

$$\nabla = \mathbf{n}\frac{\partial}{\partial n} + \mathbf{t}\frac{\partial}{\partial s},$$

where $\partial/\partial s$ and $\partial/\partial n$ are derivatives with respect to arc length s and normal coordinate, respectively, we obtain that

$$\delta u_{1,2} = n_2 \frac{\partial \delta u_1}{\partial n} + t_2 \frac{\partial \delta u_1}{\partial s}, \quad \delta u_{2,1} = n_1 \frac{\partial \delta u_2}{\partial n} + t_1 \frac{\partial \delta u_2}{\partial s}.$$

Substituting the latter formulae into (6) and again integration by parts with respect to s we obtain that

$$\begin{aligned}
\delta U = &\iint\limits_{\omega} \left[\left(-\mathbb{K}_e^1 u_{1,11} + \mathbb{K}_b^1 u_{1,2222} \right) \delta u_1 + \left(-\mathbb{K}_e^2 u_{2,22} + \mathbb{K}_b^2 u_{2,1111} \right) \delta u_2 \right] \, d\omega \\
&+ \int\limits_{\partial\omega} \left(n_1 \mathbb{K}_e^1 u_{1,1} - n_2 \mathbb{K}_b^1 u_{1,222} - \frac{\partial}{\partial s} (n_2 t_2 \mathbb{K}_b^1 u_{1,22}) \right) \delta u_1 \, ds \\
&+ \int\limits_{\partial\omega} \left(n_2 \mathbb{K}_e^2 u_{2,2} - n_1 \mathbb{K}_b^2 u_{2,111} - \frac{\partial}{\partial s} (n_1 t_1 \mathbb{K}_b^2 u_{2,11}) \right) \delta u_2 \, ds \\
&+ \int\limits_{\partial\omega} \left(\mathbb{K}_b^1 u_{1,22} n_2^2 \frac{\partial}{\partial n} \delta u_1 + \mathbb{K}_b^2 u_{2,11} n_1^2 \frac{\partial}{\partial n} \delta u_2 \right) \, ds. \qquad (7)
\end{aligned}$$

Here for simplicity we assumed that boundary contour $\partial\omega$ is a plane curve which is smooth enough, i.e. differentiable and without corner points. The form of δU requires that only a class of external loads can be applied: indeed the virtual work of external loads must be consistent with it. So, we must assume that the virtual work of external loads δA is given in the following form:

$$\begin{aligned}
\delta A = &\iint\limits_{\omega} \left(f_1 \delta u_1 + f_2 \delta u_2 \right) \, d\omega + \int\limits_{\partial\omega} \left(\varphi_1 \delta u_1 + \varphi_2 \delta u_2 \right) \, ds \\
&+ \int\limits_{\partial\omega} \left(n_2 \mu_1 \frac{\partial}{\partial n} \delta u_1 + n_1 \mu_2 \frac{\partial}{\partial n} \delta u_2 \right) \, ds. \qquad (8)
\end{aligned}$$

Here f_α are surface loads. Moreover φ_α and μ_α are forces and couples respectively assigned on the part of the boundary $\partial\omega$ where u_α and/or $\frac{\partial u_\alpha}{\partial n}$ are not assigned. Therefore we introduce a suitably regular partition of $\partial\omega$ into two disjoint subsets $\partial_e\omega_\alpha$ and $\partial_n\omega_\alpha$ (or $\partial_e\omega_\alpha^\perp$ and $\partial_n\omega_\alpha^\perp$) on which either displacements (or normal derivatives of displacements) are assigned or their dual quantities are assigned respectively (the index $\alpha = 1, 2$ refers to the displacement component u_α).

Finally, from the principle of virtual action $\delta U - \delta A = 0$, and by assuming the following essential boundary conditions,

$$u_1 = u_1^0, \quad (x_1, x_2) \in \partial_e\omega_1, \qquad (9)$$

$$u_2 = u_2^0, \quad (x_1, x_2) \in \partial_e\omega_2, \qquad (10)$$

$$n_2 \partial_n u_1 = \vartheta_1 n_2, \quad (x_1, x_2) \in \partial_e\omega_1^\perp, \qquad (11)$$

$$n_1 \partial_n u_2 = \vartheta_2 n_1, \quad (x_1, x_2) \in \partial_e\omega_2^\perp, \qquad (12)$$

where u_1^0, u_2^0, ϑ_1, and ϑ_2 are given functions at $\partial\omega$ and $\partial_n = \partial/\partial n$, we obtain the equilibrium equations and natural (static) boundary conditions

$$-\mathbb{K}_e^1 u_{1,11} + \mathbb{K}_b^1 u_{1,2222} - f_1 = 0, \quad (x_1, x_2) \in \omega \tag{13}$$

$$-\mathbb{K}_e^2 u_{2,22} + \mathbb{K}_b^2 u_{2,1111} - f_2 = 0, \quad (x_1, x_2) \in \omega; \tag{14}$$

$$n_1 \mathbb{K}_e^1 u_{1,1} - n_2 \mathbb{K}_b^1 u_{1,222} - \frac{\partial}{\partial s}(n_2 t_2 \mathbb{K}_b^1 u_{1,22}) = \varphi_1, \quad (x_1, x_2) \in \partial_n \omega_1 \tag{15}$$

$$n_2 \mathbb{K}_e^2 u_{2,2} - n_1 \mathbb{K}_b^2 u_{2,111} - \frac{\partial}{\partial s}(n_1 t_1 \mathbb{K}_b^2 u_{2,11}) = \varphi_2, \quad (x_1, x_2) \in \partial_n \omega_2 \tag{16}$$

$$\mathbb{K}_b^1 u_{1,22} n_2^2 = n_2 \mu_1, \quad (x_1, x_2) \in \partial_n \omega_1^{\perp} \tag{17}$$

$$\mathbb{K}_b^2 u_{2,11} n_1^2 = n_1 \mu_2, \quad (x_1, x_2) \in \partial_n \omega_2^{\perp} \tag{18}$$

It is interesting that (13) and (14) contain partial derivatives of different orders. For example, (13) contains second derivative with respect to x_1 and fourth derivative with respect to x_2.

Since the energy has a reduced form (that is it has not all derivatives) we have also reduced boundary conditions. For example, for a fixed boundary unlike classic case instead of $\partial_n u_1 = 0$ we have $n_2 \partial_n u_1 = 0$ which has a sense if $n_2 \neq 0$. Let us consider a rectangle $ABCD$ shown in Fig. 4. Note that here the fibers are oriented exactly the rectangle sides directions. Here two sides are free and at two sides the displacements are zero. The corresponding boundary conditions are

$$\begin{aligned}
&\text{along } AB: \quad \mathbb{K}_b^1 u_{1,22} = 0, \quad \mathbb{K}_b^1 u_{1,222} = 0, \quad \mathbb{K}_e^2 u_{2,2} = 0,\\
&\text{along } BC: \quad \mathbb{K}_e^1 u_{1,1} = 0, \quad \mathbb{K}_b^2 u_{2,111} = 0, \quad \mathbb{K}_b^2 u_{2,11} = 0,\\
&\text{along } CD: \quad u_1 = 0, \quad u_2 = 0, \quad \partial_2 u_1 = 0,\\
&\text{along } DA: \quad u_1 = 0, \quad u_2 = 0, \quad \partial_1 u_2 = 0.
\end{aligned}$$

Clearly, this rectangle gives an example of degenerated boundary conditions, since instead of four conditions in general case we have only three. It happens when the boundary or its part is parallel to one of the coordinate axes that is parallel to fiber direction.

4 Heuristics

It is evident that the immediate application of the classic methods used for proving existence and uniqueness of the solution of elastic problem is not possible [10, 22, 24, 31], as coercivity could seem, at first sight, a condition which is not verified as mentioned in [7]. Moreover also the results for second gradient continua proven by Healey et al. [28, 32] are not applicable here as this energy is not coercive with respect to highest order derivatives.

Before framing the problem in the appropriate energy space we present here some heuristic preliminary considerations.

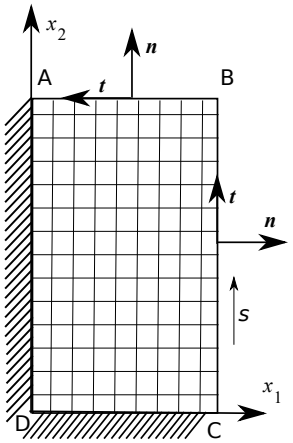


Fig. 4 Pantographic rectangle.

First of all: assume that for a displacement field $\mathbf{u}^*$ the energy (5) vanishes. It is trivial to check that as $u^*_{1,1} = 0$

$$u^*_1 = f(x_2)$$

while, as $u^*_{1,22} = f_{,22} = 0$

$$f = a_1 x_2 + b_1$$

and, finally,

$$u^*_1 = a_1 x_2 + b_1,$$

where a_1 and b_1 are constants. we have that

$$u^*_2 = a_2 x_1 + b_2$$

with constant a_2 and b_2 independent of a_1 and b_1.

Note that in the case of plane infinitesimal deformations the rigid body motion is

$$\mathbf{u}_r = \boldsymbol{\phi} \times \mathbf{X} + \mathbf{b},$$

where $\boldsymbol{\phi} = \phi \mathbf{i}_3$ is a constant rotation vector, $\mathbf{i}_3 = \mathbf{i}_1 \times \mathbf{i}_2$, $\times$ is the cross product, and $\mathbf{b} = b_1 \mathbf{i}_1 + b_2 \mathbf{i}_2$ is a constant vector. So in components the rigid body motion has the form

$$u_1 = \phi x_2 + b_1, \quad u_2 = -\phi x_1 + b_2.$$

It is therefore evident that the "kernel" or "null-space" of strain energy not only include rigid (infinitesimal) motions (corresponding to $a_1 = -a_2$) but also pure shear that corresponds elongation/contraction in the directions at an angle $\pm\pi/4$ with respect to the coordinate axes (when $a_1 = a_2$). The null-space of strain energy density consists of four linear independent modes and their linear combinations

$$\mathbf{u}^*_1 = \mathbf{i}_1, \quad \mathbf{u}^*_2 = \mathbf{i}_2, \quad \mathbf{u}^*_3 = \mathbf{i}_3 \times \mathbf{X}, \quad \mathbf{u}^*_4 = x_2 \mathbf{i}_1.$$

Instead of $\mathbf{u}_4^*$ one can use equivalent mode $x_1\mathbf{i}_2$ or symmetric mode $x_2\mathbf{i}_1+x_1\mathbf{i}_2$. Unlike classic elasticity fourth mode relates with shear in certain directions.

Clearly well-posedness results must take into account such a property.

Second. On the other hand it has to be recalled that boundary conditions producing well-posed problems in the case of second gradient continua are more general than when dealing with first gradient continua (see e.g. [25,26,33,43]). The procedure which is used in the aforementioned papers can be summarized as follows (see [2,15]): i) one postulates the principle of virtual work, i.e. the equality between internal and external work expended on virtual displacements; ii) one determines a class of internal work functional involving second gradients of virtual displacement; iii) one determines, by means of an integration by parts, the class of external work functionals which are compatible with the determined class of internal work functionals.

A consequence (see e.g. [20, 25, 33, 43]) of the just described procedure is that Neumann problems for considered (second gradient deformation energies) must include, to be complete, double symmetric and skew-symmetric boundary forces together with forces concentrated on points. To be more precise: the class of so-called natural boundary conditions must include the dual (with respect to work functionals) quantities of normal gradients of virtual displacements: following Germain the dual of tangential part of normal gradient of virtual displacement is a "couple" (i.e. skew-symmetric contact double forces) while the duals of normal part of normal gradient of virtual displacement is a "double force" (i.e. symmetric contact double forces). For some reasons (initially investigated in [16, 17]dellIsola Hellinger, which surely need further investigations) this kind of boundary conditions has been considered, sometimes and by some schools of mechanicians, unphysical: the reader is referred to the beautiful paper by Sedov, Leonid Ivanovich, [42] for a discussion of this point.

After having identified the displacements which are in the null space for deformation energy a Conjecture about mixed boundary conditions which are likely to produce well-posed problems can consequently be formulated. Indeed let us partition the boundary $\partial\omega$ of the body ω into two disjoint subsets, i.e. $\partial\omega_e$ and $\partial\omega_n$. We assume that the displacements on $\partial\omega_e$ are assigned and that the displacements on $\partial\omega_n$ are free. We call AC the set of C^2 displacements verifying the assigned conditions on $\partial\omega_e$. We say that AC is singular if there exist an element $\mathbf{u}$ in AC and a displacement field $\mathbf{u}_0$ belonging to the null space for deformation energy (i.e. a displacement having vanishing deformation energy) such that $\mathbf{u}+\mathbf{u}_0$ also belongs to AC. We conjecture here (and rigorously prove in the next section) that the considered mixed boundary problem is well-posed if and only if AC is NOT singular. Remark that the aforementioned statement reduces to the standard requirement that in well-posed problems the constraint body cannot undergo rigid displacements when the considered energy is a first gradient one and it is positive defined when regarded as a function of infinitesimal strain tensor. The concept of underconstrained system (see [29]) has to be modified in order to include the treated case of planar second gradient continua: see for instance Figs. 1 and

2 for examples of underconstrained pantographic sheets. This point will need further investigations to include the case of pantographic sheets moving in three-dimensional space and three-dimensional pantographic bodies.

In the present paper we limit ourselves to consider Dirichlet's and mixed boundary problems in which on a part of the body boundary only the displacement is assigned, while the remaining part of the body boundary is left free.

5 Existence and uniqueness of weak solutions

Let us now come back to the first variation of the energy functional. For solution we have the principle of virtual work in the form

$$\delta U - \delta A = 0. \tag{19}$$

For simplicity let us replace $\delta\mathbf{u}$ by new function $\mathbf{v}$. With $\mathbf{v}$ Eq. (19) transforms to

$$\begin{aligned} B(\mathbf{u},\mathbf{v}) \equiv & \iint\limits_{\omega} \left(\mathbb{K}_e^1 u_{1,1} v_{1,1} + \mathbb{K}_e^2 u_{2,2} v_{2,2} + \mathbb{K}_b^1 u_{1,22} v_{1,22} + \mathbb{K}_b^2 u_{2,11} v_{2,11}\right) d\omega \\ & = \iint\limits_{\omega} \left(f_1 v_1 + f_2 v_2\right) d\omega \\ & + \int\limits_{\partial\omega} \left(\varphi_1 v_1 + \varphi_2 v_2 + n_2\mu_1 \frac{\partial v_1}{\partial n} + n_1\mu_2 \frac{\partial v_2}{\partial n}\right) ds. \end{aligned} \tag{20}$$

Here we introduce bilinear form $B(\mathbf{u},\mathbf{v})$ for designation of quadratic terms in (19).

Now we introduce the weak solution of the boundary-value problem (13)–(17) as the vector-function $\mathbf{u}$ such that the variational equation (20) is fulfilled for any test function $\mathbf{v} = v_1\mathbf{i}_1 + v_2\mathbf{i}_2$. The properties of $\mathbf{u}$ and $\mathbf{v}$ we specialize below.

Without loss of generality in what follows we use the following dimensionless form of W

$$2W = u_{1,1}^2 + u_{2,2}^2 + u_{1,22}^2 + u_{2,11}^2. \tag{21}$$

Now the bilinear form is

$$B(\mathbf{u},\mathbf{v}) = \iint\limits_{\omega} \left(u_{1,1} v_{1,1} + u_{2,2} v_{2,2} + u_{1,22} v_{1,22} + u_{2,11} v_{2,11}\right) d\omega$$

Here we keep the same notations u_α and x_α for dimensionless displacements and dimensionless coordinates, respectively. So, W has the form of a seminorm in anisotropic Sobolev spaces, see definition in [4–6], and Appendix. More

precisely: let $u_1 \in W_2^{(1,2)}(\omega)$ and $u_2 \in W_2^{(2,1)}(\omega)$, then we have that $\mathbf{u} \in W_2^{(1,2)}(\omega) \oplus W_2^{(2,1)}(\omega)$ and

$$2U(\mathbf{u}) = \iint_\omega \left[u_{1,1}^2 + u_{2,2}^2 + u_{1,22}^2 + u_{2,11}^2\right] d\omega = |u_1|_{W_2^{(1,2)}} + |u_2|_{W_2^{(2,1)}}. \quad (22)$$

is a seminorm in $W_2^{(1,2)}(\omega) \oplus W_2^{(2,1)}(\omega)$. Here we introduced the following notations for some auxiliary seminorms

$$|f|_{W_2^{(1,2)}} := \|f_{,1}\|_{L_2} + \|f_{,22}\|_{L_2}, \quad |f|_{W_2^{(2,1)}} := \|f_{,11}\|_{L_2} + \|f_{,2}\|_{L_2},$$

whereas it is possible to transform them into norms for instance following the standard choice:

$$\|f\|_{W_2^{(1,2)}} = \|f\|_{L_2} + |f|_{W_2^{(1,2)}}, \quad \|f\|_{W_2^{(2,1)}} = \|f\|_{L_2} + |f|_{W_2^{(2,1)}}. \quad (23)$$

It clear that the functional space $W_2^{(1,2)}(\omega) \oplus W_2^{(2,1)}(\omega)$ is constituted exactly by the set of all functions for which (22) is finite. We will call energy space E for the considered energy functional any subspace of $W_2^{(1,2)}(\omega) \oplus W_2^{(2,1)}(\omega)$ which is the completion of one of the previously introduced space AC relative to NONSINGULAR boundary conditions using the norms (23). Remark that when restricted to an energy space the seminorm given by (22) becomes a norm.

Now the definition of a weak solution for linear pantographic structures can be given as follows.

Definition 1 We call $\mathbf{u} \in E$ a weak solution of the equilibrium problem (19) if (20) is fulfilled for any test function $\mathbf{v}$ from a dense set in E.

The bilinear form $B(\mathbf{u}, \mathbf{v})$ is continuous and the following inequalities are valid

$$B(\mathbf{u}, \mathbf{v}) \leq |u_1|_{W_2^{(1,2)}} |v_1|_{W_2^{(1,2)}} + |u_2|_{W_2^{(2,1)}} |v_2|_{W_2^{(2,1)}} \leq \|\mathbf{u}\|_E \|\mathbf{v}\|_E. \quad (24)$$

For the analysis of existence and uniqueness of weak solutions we start by considering two cases. The simplest case is given by Dirichlet boundary conditions.

5.1 Dirichlet's boundary conditions

We start by proving the existence and uniqueness for the simplest case when the whole boundary is fixed. So we consider the set of equations for the strong formulation of equilibrium problem:

$$-u_{1,11} + u_{1,2222} = f_1, \quad -u_{2,22} + u_{2,1111} = f_2, \quad (x_1, x_2) \in \omega; \quad (25)$$

$$u_1 = 0, \quad u_2 = 0, \quad n_2 \partial_n u_1 = 0, \quad n_1 \partial_n u_2 = 0, \quad (x_1, x_2) \in \partial\omega. \quad (26)$$

Here the weak solution is defined trough the integral equation

$$B(\mathbf{u},\mathbf{v}) = \iint\limits_{\omega} (f_1 v_1 + f_2 v_2)\, d\omega; \quad \forall \quad v_1, v_2 \in C_0^2(\omega), \tag{27}$$

which, when assuming that $\mathbf{f}$ belongs to $L_2(\omega)$, can be written as

$$B(\mathbf{u},\mathbf{v}) - (\mathbf{f},\mathbf{v})_{L_2} = 0; \quad \forall \quad \mathbf{v} \in C_0^2(\omega)$$

Using the Poincaré inequalities (Friedrich's inequality, see e.g. [1]) we get

$$\|u_1\|_{L_2} \le C_1 \|u_{1,1}\|_{L_2}, \quad \|u_2\|_{L_2} \le C_1 \|u_{2,2}\|_{L_2} \tag{28}$$

with some constants C_1 and C_2 and, as a consequence, we can establish that

$$|u_1|_{W_2^{(1,2)}} + |u_2|_{W_2^{(2,1)}} \ge C \left(\|u_1\|_{W_2^{(1,2)}} + \|u_2\|_{W_2^{(2,1)}} \right)$$

with another constant C. In other words we proved that $|\cdot|_{W_2^{(1,2)}}$ and $|\cdot|_{W_2^{(2,1)}}$ play the role of norms in $\overset{\circ}{W}{}_2^{(1,2)}$ and $\overset{\circ}{W}{}_2^{(2,1)}$, respectively, where with the upper ball we denoted the completion of $C_0^2(\omega)$ (or $C_0^\infty(\omega)$) with respect to the corresponding norms. So here the energy space E is the anisotropic Sobolev's space $\overset{\circ}{W}{}_2^{(1,2)}(\omega) \oplus \overset{\circ}{W}{}_2^{(2,1)}(\omega)$. This means that we have proven that $B(\mathbf{u},\mathbf{v})$ is coercive

$$B(\mathbf{u},\mathbf{u}) \ge C\|\mathbf{u}\|_E^2.$$

One can easily prove that $(\mathbf{f},\mathbf{v})_{L_2}$ is a linear bounded functional in E. Thus, by using Lax-Milgram theorem [23], the following theorem can be easily proven

Theorem 1 *Let the Cartesian components f_1 and f_2 of $\mathbf{f}$ belong to the space $L_2(\omega)$. There exists a weak solution $\mathbf{u}^* \in E \equiv \overset{\circ}{W}{}_2^{(1,2)}(\omega) \oplus \overset{\circ}{W}{}_2^{(2,1)}(\omega)$ to the equilibrium problem (25) and (26), which for any $\mathbf{v} \in E$ satisfies the equation*

$$(\mathbf{u}^*,\mathbf{v})_E - \iint\limits_{\omega} \mathbf{f}\cdot\mathbf{v}\, d\omega = 0.$$

Furthermore, $\mathbf{u}^*$ is unique and it is a minimizer of the energy:

$$F(\mathbf{u}^*) = \inf_{\mathbf{u}\in E} F(\mathbf{u}), \quad F(\mathbf{u}) \equiv U(\mathbf{u}) - \iint\limits_{\omega} \mathbf{f}\cdot\mathbf{u}\, d\omega.$$

Remark 1. Since for the coercivity we need inequalities (28) which require that only the functions are zero at the boundary,(i.e. $u_1 = u_2 = 0$ at $\partial\omega$), for uniqueness it is enough to fulfillment of only the boundary conditions concerning displacements, without considering the condition on the normal derivatives(26).

Remark 2. We used here $L_2(\omega)$ as a functional space for $\mathbf{f}$. This condition can be weakened using imbedding theorems of E into anisotropic Lebesgue spaces [4–6] and we omit this for simplicity.

For non homogeneous boundary conditions (12) we seek solution in the form $\mathbf{u} = \mathbf{u}^* + \mathbf{u}_0$, where $\mathbf{u}_0$ is a vector function which satisfies (12) whereas for $\mathbf{u}^*$ boundary conditions (12) to be assumed. Substituting this representation into (25) and (12) we reduce the non homogeneous boundary-value problem to the previous one, for which we already proved the theorem on existence and uniqueness of weak solutions.

5.2 Mixed boundary conditions

Somehow more difficult is the case of mixed boundary conditions. In the linear elasticity it is known that for existence and uniqueness it is enough to request that a part of the boundary is fixed [10, 22, 24, 31]. In our problem this is not the case. Indeed as an example we consider two rectangles with one fixed side whereas others are free, see Fig. 5. More precisely, on side AB the displacements are zero: $\mathbf{u} = \mathbf{0}$. The difference in rectangles consists only on their orientation with respect to coordinate axes, that is to fiber orientations. It is clear that for the first rectangle the solution is not unique since vector

$$\mathbf{u} = ax_2\mathbf{i}_1$$

satisfies equilibrium and boundary conditions for any value of a: the set AC in this circumstance is indeed SINGULAR. For the second rectangle, instead, the set AC is actually NON SINGULAR and the aforementioned displacement is not a solution.

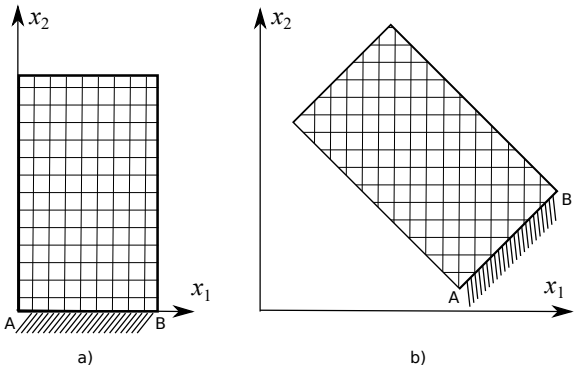


Fig. 5 Two rectangles with clamped edge with different fiber orientation.

In other words, for rectangle a) we have at least two solutions $\mathbf{u} = \mathbf{0}$ and $\mathbf{u} = ax_2\mathbf{i}_1$. Obviously we should avoid such situation since even without

loading we have an infinity non-trivial (deformative) solutions. Thus, in what follows we always assume that the boundary conditions are nonsingular.

Let us consider the mixed boundary-value problem formulated by (12)–(18). Here energy space E is a subspace of $W_2^{(1,2)}(\omega) \oplus W_2^{(2,1)}(\omega)$ obtained as the completion of functions $C^2(\omega)$ which verify (12).

The weak solution is defined as an element $\mathbf{u}$ belonging to E satisfying the equation

$$B(\mathbf{u},\mathbf{v}) = \iint\limits_{\omega} (f_1 v_1 + f_2 v_2)\, d\omega + \int\limits_{\partial\omega} \left(\varphi_1 v_1 + \varphi_2 v_2 + n_2\mu_1 \frac{\partial v_1}{\partial n} + n_1\mu_2 \frac{\partial v_2}{\partial n} \right) ds \tag{29}$$

for any admissible function $\mathbf{v}$ (i.e. a function belonging to a dense subset of E).

Using the same technique we formulate the theorem on existence and uniqueness of the weak solution in E

Theorem 2 *Let the Cartesian components f_1 and f_2 of $\mathbf{f}$ belong to the space $L_2(\omega)$, $\varphi_\alpha \in L_2(\partial_n\omega_\alpha)$, $\mu_\alpha \in L_2(\partial_n\omega_\alpha^\perp)$ and assume that the boundary conditions are nonsingular. There exists a weak solution $\mathbf{u}^* \in E$ to the equilibrium problem (12)–(18) which for any $\mathbf{v} \in E$ satisfies the equation (29).*

Furthermore, $\mathbf{u}^*$ is unique and it is a minimizer of the functional $F(\mathbf{u})$:

$$F(\mathbf{u}^*) = \inf_{\mathbf{u}\in E} F(\mathbf{u}),$$

$$F(\mathbf{u}) \equiv U(\mathbf{u}) - \iint\limits_{\omega} \mathbf{f}\cdot\mathbf{u}\, d\omega + \int\limits_{\partial\omega_1} \left(\varphi_1 u_1 + \varphi_2 u_2 + n_2\mu_1 \frac{\partial u_1}{\partial n} + n_1\mu_2 \frac{\partial u_2}{\partial n} \right) ds.$$

6 Conclusions

The results presented in this paper allow us to prove existence and uniqueness theorems for the elastic problem in the case of planar pantographic sheets and for a variety of boundary conditions. The main difficulties which we had to confront were: i) the existence of floppy modes, i.e. deformations corresponding to zero deformation energy and ii) the absence in the deformation energies of many higher order derivatives.

Therefore the results by Healey and Chambon [9, 28, 32] could not be applied directly and there was the appearance of a lack of coercivity of considered energy. Indeed the second gradient deformation energy for pantographic sheets is not coercive if one considers the standard Sobolev Space, whose norm involves all second order derivatives.

However we prove that the standard Hilbertian abstract setting used for solving the elastic problem does not need to be changed. Instead one has

to change the definition of the Energy spaces which correspond to the various imposed boundary conditions: they must be regarded as subsets of the Anisotropic Sobolev space whose norm is defined by involving only the derivatives appearing in the considered deformation energy. The abstraction effort due to Nikol'skii (and the to Besov and others) which lead him to introduce a wider class of Sobolev spaces was initially motivated only by the need of developing a mathematical theory based on the minimum possible necessary assumptions: Anisotropic Sobolev Spaces include functions which do not posses all higher order weak derivatives.

The abstract tool which he developed allowed us to frame rather naturally the numerical and mathematical problems concerning the equilibrium of linear pantographic sheets.

We are also confident that the same tools will allow us to study non-linear deformations problems.

Appendix. Anisotropic Lebesgue and Sobolev functional spaces

In the paper we used the classic and anisotropic Lebesgue and Sobolev functional spaces. Here we present necessary information on this topic. In plane elasticity and other problems of mechanics the functional spaces such as Lebesgue space $L_2(\omega)$ and Sobolev's spaces $W_2^1(\omega)$ and $W_2^2(\omega)$ are widely used [10, 22, 30, 31]. The norms in these spaces are defined as follows

$$\|f\|_{L_2} = \left(\iint\limits_{\omega} f^2 \, d\omega \right)^{\frac{1}{2}},$$
$$\|f\|_{W_2^1} = \left(\|f\|_{L_2}^2 + \|f_{,1}\|_{L_2}^2 + \|f_{,2}\|_{L_2}^2 \right)^{\frac{1}{2}},$$
$$\|f\|_{W_2^2} = \left(\|f\|_{L_2}^2 + \|f_{,1}\|_{L_2}^2 + \|f_{,2}\|_{L_2}^2 + \|f_{,11}\|_{L_2}^2 + 2\|f_{,12}\|_{L_2}^2 + \|f_{,22}\|_{L_2}^2 \right)^{\frac{1}{2}},$$

where $f = f(x_1, x_2)$ is a function defined in an open set in the two-dimensional Euclidean space, $\omega \subset \mathbb{R}^2$, the boundary of ω is to abe assumed smooth enough. Various useful requirements to the boundary of ω are discussed in [1, 6]. The Greek indices take values $1, 2$. $L_2(\omega)$, $W_2^1(\omega)$ and $W_2^2(\omega)$ are examples of a separable Hilbert space [1].

In what follows we use the various imbedding theorems for Sobolev's spaces. Let us recall the general definition of imbedding. We say the normed space E is imbedded in the normed space H, and we write $E \to H$ to denote this imbedding, if (i) E is a vector subspace of H, and (ii) there exists constant C such that $\|u\|_H \leq C\|u\|_E \ \forall u \in E$. For imbedding theorems in Sobolev's spaces we refer to [1, 30, 31].

In addition to classic Lebesgue and Sobolev's space we introduce the anisotropic Lebesgue and Sobolev spaces. Here we are restricted ourselves by functions defined on a set of $\mathbb{R}^2$. Let $\mathbf{p} = (p_1, p_2)$ be a multiindex, where $1 < p_\alpha <$

∞. Then the norm in the anisotropic Lebesgue space $L_{\mathbf{p}}$ is defined as

$$\|f\|_{L_{\mathbf{p}}} = \left(\left(\int f(x_1, x_2)^{p_1} \, dx_1 \right)^{p_2/p_1} dx_2 \right)^{1/p_2}.$$

If $p_1 = p_2 = p$ we use standard notation $L_{\mathbf{p}} = L_p$. $\|\cdot\|_{L_{\mathbf{p}}}$ is called the mixed norm [1].

An anisotropic Sobolev space consist's of functions having different differential properties in different coordinate directions, so such functions have generalized derivatives of different order and, in general, different L_p in coordinate directions x_1 and x_2. The theory of the anisotropic Sobolev spaces including imbedding theorems, relations with other Sobolev's spaces and analysis of the coercivity of differential operators is presented in [4, 5, 35], see also [6]. We introduce the multiindex $\boldsymbol{\ell} = (l_1, l_2)$ where l_α are natural numbers and the norm

$$\|f\|_{W^{\boldsymbol{\ell}}_{\mathbf{p}}} = \|f\|_{L_{\mathbf{p}}} + \sum_{\alpha=1}^{2} \|\partial_\alpha^{l_\alpha} f\|_{L_{\mathbf{p}}}. \tag{30}$$

So, the set of functions defining on ω and having generalized derivatives such that the introduced norm is finite, is called the anisotropic Sobolev space $W^{\boldsymbol{\ell}}_{\mathbf{p}}(\omega)$. Obviously, when $l_1 = l_2 = l$ and $p_1 = p_2 = p$ we have the classical Sobolev space W^l_p. The anisotropic Sobolev's space $W^{\boldsymbol{\ell}}_{\mathbf{p}}$ is a separable Banach space whereas $W^{\boldsymbol{\ell}}_2$ is a Hilbert space.

We also introduce the anisotropic Sobolev space $\overset{\circ}{W}{}^{\boldsymbol{\ell}}_{\mathbf{p}}$ as the closure of $C_0^2(\omega)$ (or $C_0^\infty(\omega)$) functions in norm (30).

For our purposes we consider two specific anisotropic Sobolev spaces $W_2^{(1,2)}$ and $W_2^{(2,1)}$ with norms

$$\|f\|_{W_2^{(1,2)}} = \|f\|_{L_2} + \|f_{,1}\|_{L_2} + \|f_{,22}\|_{L_2}, \tag{31}$$

$$\|f\|_{W_2^{(2,1)}} = \|f\|_{L_2} + \|f_{,11}\|_{L_2} + \|f_{,2}\|_{L_2}. \tag{32}$$

With certain assumptions on the regularity of ω for these spaces there are following imbedding theorems [6, 35]

$$W^{\boldsymbol{\ell}}_2(\omega) \to W^1_2(\omega), \quad W^{\boldsymbol{\ell}}_2(\omega) \to C(\omega), \quad \boldsymbol{\ell} = \{(1,2),(2,1)\}.$$

Evidently, any function $f \in W^2_2(\omega)$ belongs to $W^{\boldsymbol{\ell}}_2(\omega)$ with $\boldsymbol{\ell} = \{(1,2),(2,1)\}$, but not all elements of $W^{\boldsymbol{\ell}}_2(\omega)$ belong to $W^2_2(\omega)$. For more details on imbeddings in anisotropic Sobolev's spaces and their further generalizations including results on traces of functions we refer to [4–6, 35].

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