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POSITIVE LOOPS OF LOOSE LEGENDRIAN EMBEDDINGS AND APPLICATIONS

GUOGANG LIU

ABSTRACT. In this paper, we prove that there exist contractible positive loops of Legendrian embeddings based at any loose Legendrian submanifold. As an application, we define a partial order on $\widetilde{Cont}_0(M, \xi)$, called strong orderability, and prove that overtwisted contact manifolds are not strongly orderable.

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INTRODUCTION

In this paper, we focus on the study of positive contact and Legendrian isotopies in a co-oriented contact manifold (M, ξ) .

A contact manifold (M^{2n+1}, ξ) is a $2n + 1$ dimensional smooth manifold M with a non-integrable hyperplane field ξ which is called a contact structure. When ξ is co-oriented, it is given by the kernel of a *contact* 1-form α . For example, in \mathbb{R}^4 with the usual coordinates (x_1, x_1, y_1, y_2) , the sphere \mathbb{S}^3 carries a contact form $\alpha_{std} = (y_1 dx_1 - x_1 dy_1 + y_2 dx_2 - x_2 dy_2)|_{\mathbb{S}^3}$. We denote ξ_{std} the contact structure defined by α_{std} . It induces a contact structure on the quotient $\mathbb{R}P^3$ which is also denoted by ξ_{std} .

One class of submanifolds of (M^{2n+1}, ξ) with an interesting behavior is that of Legendrian submanifolds. A n -dimensional submanifold $L \subset M^{2n+1}$ is called a Legendrian submanifold if $\alpha|_L = 0$. A contactomorphism of (M, ξ) is a diffeomorphism which preserves ξ and a contact isotopy $(\varphi_t)_{t \in [0,1]}$ is a path of contactomorphisms with $\varphi_0 = id$. We say a contact isotopy $(\varphi_t)_{t \in [0,1]}$ is positive if $\alpha(\partial_t \varphi_t) > 0$. That is to say, the infinitesimal generator of the isotopy is positively transverse to ξ everywhere. An isotopy $(\varphi_t)_{t \in [0,1]}$ based at a Legendrian submanifold L is said to be a Legendrian isotopy if $\varphi_t(L)$ is a Legendrian submanifold for all t . Similarly, We say φ_t is positive if $\alpha(\partial_t \varphi_t) > 0$. This notion of positivity does only depend on the image

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$L_t = \varphi_t(L)$ of the isotopy. For us, a Legendrian isotopy will be such a family of unparametrized Legendrian submanifolds.

With the concept of positive contact isotopy, Eliashberg and Polterovich defined a partial order on the universal cover $\widetilde{Cont}_0(M, \xi)$ of the identity component of the contactomorphisms group of (M, ξ) . A class of contact isotopy $[(\psi_t)_{t \in [0,1]}]$ is greater than another class $[(\varphi_t)_{t \in [0,1]}]$ if there exists a positive contact isotopy from φ_1 to ψ_1 which is homotopic to the concatenation of the opposite of $(\varphi_t)_{t \in [0,1]}$ and $(\psi_t)_{t \in [0,1]}$.

Proposition 0.1. [EP99] *If (M, ξ) is a contact manifold, the following conditions are equivalent:*

- (i). (M, ξ) is non-orderable;
- (ii). There exists a contractible positive loop of contactomorphisms for (M, ξ) .

This order is closely related to *squeezing* properties in contact geometry [EP99] as well as to the existence of bi-invariant metrics on $\widetilde{Cont}_0(M, \xi)$ or on the space of Legendrian submanifolds [CS12].

From the beginning of the 80's, it is known that the world of contact structures splits in two classes with opposite behaviors. Following Eliashberg, we say that a contact structure ξ on M^3 is overtwisted if there exists an overtwisted disk $D_{OT} \subset M$, i.e. an embedded disk which is tangent to ξ along its boundary. The overtwisted contact structures are flexible and classified by an adequate h-principle [Eli89]. We denote α_{OT} a contact form for an overtwisted contact structure ξ defined on a neighborhood of an overtwisted disk. More recently, the work of Niederküger [Nie06] and Borman-Eliashberg-Murphy [BEM15] have described a similar dichotomy in the higher dimensional case. Following a suggestion of Niederküger, we say a contact structure ξ is overtwisted if (M^{2n+1}, α) contains $D^3 \times D^{2n-2}(r)$ with $\alpha|_{D^3 \times D^{2n-2}(r)} = \alpha_{OT} - (ydx - xdy)$ for some constant $r > 0$ large enough depending on the dimension of M [CMP15]. As in dimension three, Borman, Eliashberg and Murphy [BEM15] have shown that overtwisted contact structures are purely topological objects and are flexible.

On the contrary, we say ξ is a tight contact structure if it is not overtwisted. For example, the contact manifolds $(\mathbb{S}^3, \xi_{std})$ and $(\mathbb{R}P^3, \xi_{std})$ are tight according to the fundamental result of Bennequin [Ben83]. Similar results hold in higher dimension, where holomorphic methods give that a Liouville fillable contact structure is tight, see [Nie06].

The orderability property is not shared by all contact manifolds (see the work of Albers, Frauenfelder, Fuchs and Merry [AF12, AM13, AFM15] for more examples).

Theorem 0.2. (i). $(\mathbb{S}^3, \xi_{std})$ is non-orderable while $(\mathbb{R}P^3, \xi_{std})$ is orderable [EKP06];
(ii). There are some overtwisted contact manifolds which are non-orderable [CPS14].

It is interesting to see that tight contact manifolds can be orderable or not despite their rigid nature. At the same time we guess overtwisted contact manifolds are non-orderable.

Question 0.3. *Are all overtwisted contact manifolds non-orderable?*

In order to answer the above question, we transfer the study of positive contact isotopies to that of positive Legendrian isotopies by the trick of contact product. Indeed, a positive contact isotopy of (M, ξ) can be lifted to a negative Legendrian isotopy of the diagonal $\Delta_{M \times M} \times \{0\}$ in the contact product $(M \times M \times \mathbb{R}, \alpha_1 - e^s \alpha_2)$. Here α_1 and α_2 denote the pull-backs of α by the first and second projection from $M \times M \times \mathbb{R}$ to M . The advantage is that the study of positive Legendrian isotopies should be easier than that of contact isotopies.

In that context, there is a natural question regarding positive Legendrian isotopies:

Question 0.4. *Let (M, ξ) be a contact manifold and let L_0 and L_1 be Legendrian submanifolds in (M, ξ) which are Legendrian isotopic. Does there exist a positive Legendrian isotopy connecting them?*

Example 0.5. *Let (\mathbb{S}^2, g) be the 2-sphere with the round metric g , and let $ST^*\mathbb{S}^2$ be the space of contact elements on \mathbb{S}^2 . Denoting S, N the poles, then the geodesic flow of g induces a positive Legendrian isotopy L_t connecting the Legendrian fibers $ST_N^*\mathbb{S}^2$ and $ST_S^*\mathbb{S}^2$.*

Generally, the answer to Question 0.4 is negative.

Theorem 0.6. *Let M^n , $n > 1$ be a manifold with open universal cover. Then*

- (i). *the fibers of ST^*M are not in a positive loop of Legendrian embeddings [CFP10, CN10];*
- (ii). *the zero-section of $(T^*M \times \mathbb{R}, dz - ydx)$ is not in a positive loop of Legendrian embeddings [CFP10].*

We let F be the front projection $(T^*M \times \mathbb{R}, dz - ydx) \rightarrow M \times \mathbb{R} : (x, y, z) \mapsto (x, z)$. For a Legendrian submanifold $L \subset (T^*M \times \mathbb{R}, dz - ydx)$, the subset $L_F := F(L) \subset M \times \mathbb{R}$ is the front of L . We usually identify L_F with L , since the y coordinates are given by the slopes of the front. In the case where L and M are of dimension 1, we can replace a smooth segment of L_F by a zigzag with two cusps. The zig-zag either has a z -shape, as in Figure 1, or an s -shape (the symmetric of Figure 1 by the vertical axis). The Legendrian submanifold obtained by this operation is denoted by $S(L)$ and is called a stabilization of L . When we want to make it clearer, we will discriminate between the z -shape/positive stabilization denoted $S_+(L)$ and the s -shape/negative stabilization $S_-(L)$.

We have:

Proposition 0.7. [CFP10] *Let L be the zero-section of $T^*\mathbb{S}^1 \times \mathbb{R}$ and $S(L)$ a stabilization of L . Then there exists a loop of positive Legendrian embeddings based at $S(L)$.*

For a contact manifold (M, ξ) of dimension strictly higher than three, Murphy [Mur12] introduced the class of loose Legendrian submanifolds. This is a higher dimensional generalization of the stabilized $S(L)$ in dimension three. Loose Legendrian submanifolds satisfy a h-principle discovered by Murphy which make them flexible. The main result of this article extends this flexible behavior.

Theorem 0.8. *Let (M, ξ) be a contact manifold of dimension ≥ 5 and $L \subset (M, \xi)$ be a Legendrian submanifold. If L is loose then there exists a contractible positive loop of Legendrian embeddings based at L .*

Without the looseness assumption, F.Laudenbach has proven that there always exist positive loops of Legendrian immersions [Lau07].

As an application of Theorem 0.8, we obtain a holomorphic curve free proof of the existence of tight (i.e. non overtwisted in the Borman-Eliashberg-Murphy sense [BEM15]) contact structures in every dimensions. The “hard part” of the argument uses Theorem 0.6 whose proof relies on the existence of a generating function for a specific class of Legendrians (in that case the Legendrian fibers of the Legendrian fibration in $(\mathbb{R}^n \times \mathbb{S}^{n-1}, \xi_{std})$).

Corollary 0.9. [MNPS13] *The contact manifold $(\mathbb{R}^n \times \mathbb{S}^{n-1}, \xi_{std})$ is tight.*

This corollary is proved in Subsection 3.1.

In the last section, we define a new partial order on certain groups $\widetilde{Cont}_0(M, \xi)$, called strong orderability, based on the transfer of an isotopy of contactomorphisms to a Legendrian isotopy of their graphs in the contact product. We then drop the graph condition to stick to Legendrian isotopies and get a (possibly) different notion than that of Eliashberg-Polterovich’s [EP99].

Proposition 0.10. *Let (M, ξ) be a contact manifold. Then (M, ξ) is strongly orderable if and only if there does not exist a contractible positive loop of Legendrian embeddings based at the diagonal of the contact product of (M, ξ) .*

As an example we prove that the contact manifold $(\mathbb{S}^1, d\theta)$ is strongly orderable.

In that context, we explain the following result which was first suggested by Klaus Niederkrüger and also observed by Casals and Presas.

Proposition 0.11. *Let (M^{2n+1}, α) be a compact overtwisted contact manifold. Then the contact product $(M \times M \times \mathbb{R}, \alpha_1 - e^s \alpha_2)$ is also overtwisted and the diagonal $\Delta \subset (M \times M \times \mathbb{R}, \alpha_1 - e^s \alpha_2)$ is loose.*

Therefore, according to Proposition 0.11, we have the following result:

Theorem 0.12. *Overtwisted contact manifolds are not strongly orderable.*

Organisation of the paper: In section 1, we recall some basic definitions including Murphy's loose Legendrian embeddings. In section 2, we give the proof of Theorem 0.8. Finally, we prove all the other results mentioned above in the last section.

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1. BASIC DEFINITIONS IN CONTACT GEOMETRY

Let $L : Y \hookrightarrow (J^1(Y), \alpha)$ be a smooth Legendrian embedding. We denote its front map by $L_F : Y \rightarrow Y \times \mathbb{R}$.

Given a Legendrian submanifold L' , there is a neighborhood $U(L')$ of L' contactomorphic to a neighborhood of the zero section in $J^1(L', \alpha)$, according to the Weinstein neighborhood theorem. If L a Legendrian submanifold close to L' then we can talk about the front L_F of L in this Weinstein neighborhood.

If $\phi_t : Y \rightarrow Y \times \mathbb{R}$ is an homotopy of fronts (with $\phi_t(Y)$ transverse to the \mathbb{R} factor), we denote $\tilde{\phi}_t$ its Legendrian lift and write v_{ϕ_t} and $v_{\tilde{\phi}_t}$ for the corresponding generating time dependent vector fields.

1.1. Positive Legendrian isotopies.

Definition 1.1. [CFP10, CN10](**Positive Legendrian isotopy**) Let $(M, \xi = \ker \alpha)$ be a contact manifold, $L \subset M$ a Legendrian submanifold, $\varphi : L \times [0, 1] \rightarrow M$ a Legendrian isotopy and let $X_t = \frac{d\varphi}{dt}$ where $t \in [0, 1]$. We say φ is **positive** if X_t is transverse to ξ positively, i.e.

$$\alpha(X_t) > 0.$$

Moreover, φ is said to be a **positive loop** if in addition $\varphi_0(L) = \varphi_1(L)$.

The following remark is the starting point of our study.

Remark 1.2. [CFP10] Let $L : \mathbb{S}^1 \hookrightarrow (J^1(\mathbb{S}^1), \xi_{std})$ be a Legendrian embedding whose front have positive slopes everywhere. Then there exists a positive Legendrian loop based at L .

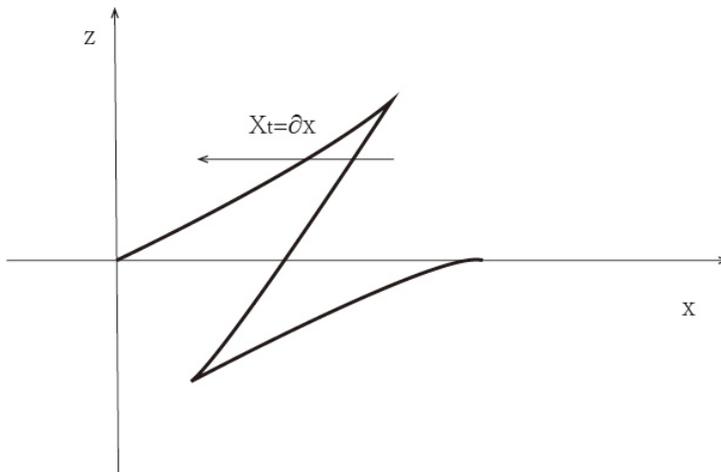


FIGURE 1. A positive stabilized front.

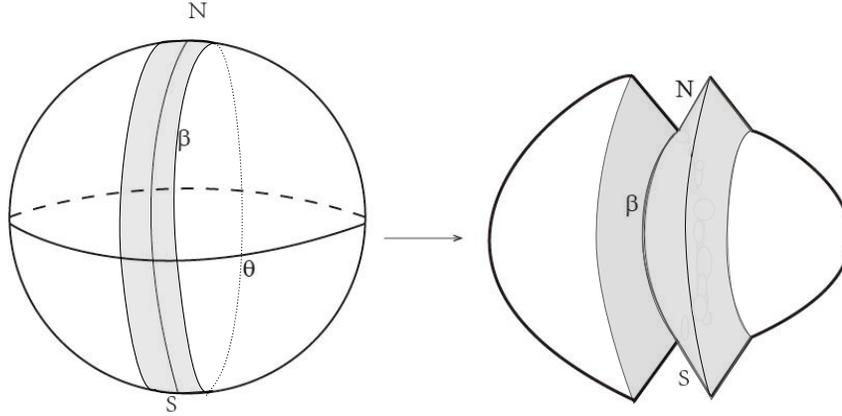
Proof. Regard \mathbb{S}^1 as \mathbb{R}/\mathbb{Z} with coordinate x . Denote $Z = L_F(\mathbb{S}^1)$. On Z , the slopes $\partial z/\partial x > 0$ are positive. See figure 1. Consider the vector field $X_t := -\partial_x$ on $J^1(\mathbb{S}^1)$ and its flow φ_t .

Because $\alpha(X_t) > 0$ on $\varphi_t(Z)$ for every $t \in [0, 1]$, then φ_t is a positive Legendrian isotopy. Since $\varphi_1 = Id$, then we have a positive loop. □

Remark 1.3. If the front of L has negative slopes everywhere, we can choose $v = \partial_x$ so that its flow is a positive loop.

1.2. Loose Legendrian embeddings. In this section, we recall Murphy's notion of loose Legendrian embeddings, wrinkled Legendrian embeddings and the idea for resolving wrinkles [Mur12]. For simplicity, we give the following equivalent definition of a loose Legendrian.

Definition 1.4. Let $L : Y^n \hookrightarrow (J^1(Y^n), \xi_{std})$ be a Legendrian embedding. Let Λ be a one dimensional zigzag and N be a closed $n - 1$ dimensional manifold. We say L is loose if its front contains $\Lambda \times N$. In particular, it is obtained from a Legendrian L' by replacing a neighborhood of $N \subset L'$ by N times a zigzag. We denote $L = S_{\pm}^N(L')$, where \pm stands for the z - or s -shape of the zig-zag.

FIGURE 2. A loose embedding of \mathbb{S}^2 .

Definition 1.5. [EM11](**Wrinkled embeddings**) See figure 3. Let $W : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ be a smooth, proper map, which is a topological embedding. Suppose W is a smooth embedding away from a finite collection of spheres $\{S_j^{n-1}\}$. Suppose, in some coordinates near these spheres, that W can be parametrized by

$$W(u, v) = (v, u^3 - 3u(1 - |v|^2), \frac{1}{5}u^5 - \frac{2}{3}u^3(1 - |v|^2) + u(1 - |v|^2)^2),$$

where our domain coordinates lies in a small neighborhood of the sphere $\{|v|^2 + u^2 = 1\} \subset \mathbb{R}^n$. Then W is called a *wrinkled embedding*, and the spheres S_j^{n-1} are called the *wrinkles*.

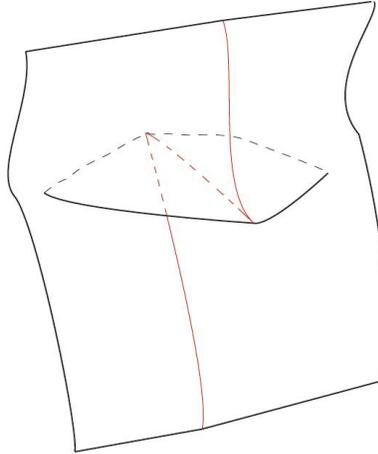


FIGURE 3. Wrinkled embedding.

Definition 1.6. [Mur12](**Wrinkled Legendrians**) Let Y^n be a closed and connected manifold and (M^{2n+1}, ξ) be a contact manifold. A wrinkled Legendrian is a smooth map $L : Y \rightarrow M$, which is a topological embedding, satisfying the following properties: The image of dL is contained in ξ everywhere and dL is full rank outside a subset of codimension 2. This singular set is required to be diffeomorphic to a disjoint union of $(n - 2)$ -spheres $\{S_j^{n-2}\}$, whose images

are called *Legendrian wrinkles*. We assume the image of each S_j^{n-2} is contained in a Darboux chart U_j , so that the front projection of $L(Y) \cap U_j$ is a wrinkled embedding, smooth outside of a compact set.

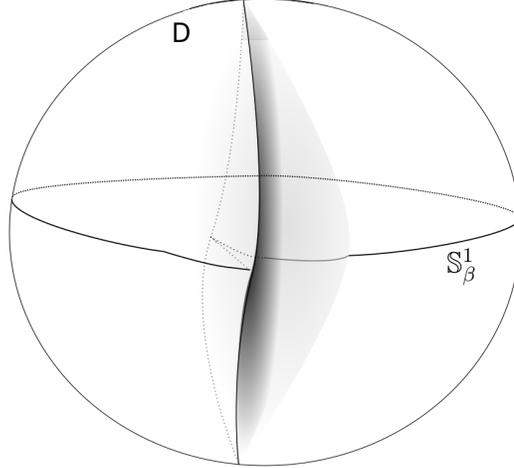


FIGURE 4. A wrinkled sphere.

Definition 1.7. [Mur12](**Twist marking**) Let $L : Y \rightarrow (M, \xi)$ be a wrinkled Legendrian embedding, and $\{S_j^{n-2}\}$ be the set of singular spheres. Let $N \subset Y$ be a submanifold with $\partial N = \cup_j S_j^{n-2}$. Denote $\Phi := L|_N$. Then (Φ, N) is called a *twist marking*.

Remark 1.8. We will put the C^∞ -topology on the space of wrinkled Legendrian embeddings. Thus we can talk about a smooth family of wrinkled embeddings (L_t, Φ_t, N_t) .

Given a Legendrian L , we denote L^w a wrinkled Legendrian obtained by adding some wrinkles to L . Given a twist marking N on L^w and $\eta > 0$, we denote $W_{\eta, N}^{-1}(L^w)$, or $W_\eta^{-1}(L^w)$ when N is understood, be the operation of resolving the wrinkles along N with an η -small operation. To measure proximity, we can first perform an immersed resolution $W_0^{-1}(L^w)$, where along N we incorporate a completely flat zig-zag, covering a segment times N . Then $W_\eta^{-1}(L^w)$ is η - C^∞ -close to $W_0^{-1}(L^w)$. This operation can be done parametrically, as summarized in the following theorem of [Mur12].

Theorem 1.9. [Mur12] *Let L_t^w be a smooth family of wrinkled Legendrian embeddings, let (Φ_t, N_t) be the twist markings. Then there is a smooth family of Legendrian embeddings L_t , such that L_t is identical to L_t^w outside of any small neighborhood of N_t for all t . Also, the resolution L_t can be taken to be as close as we want from L_t^w .*

Theorem 1.10. [Mur12] *Let L^n be a loose Legendrian and $N \subset L$ be a closed codimension 1 submanifold of Euler characteristic 0. If we stabilize L positively and negatively along N , we obtain a Legendrian $S_-^N(S_+^N(L))$ which is isotopic to L .*

Sketch of proof. First of all, the stabilized Legendrian $S_-^N(S_+^N(L))$ is loose. By Murphy's h -principle, have to show that $S_-^N(S_+^N(L))$ is formally Legendrian isotopic to L . This is a consequence of the fact that $S_-^N(S_+^N(L))$ is obtained from a Murphy N -stabilization of L by an

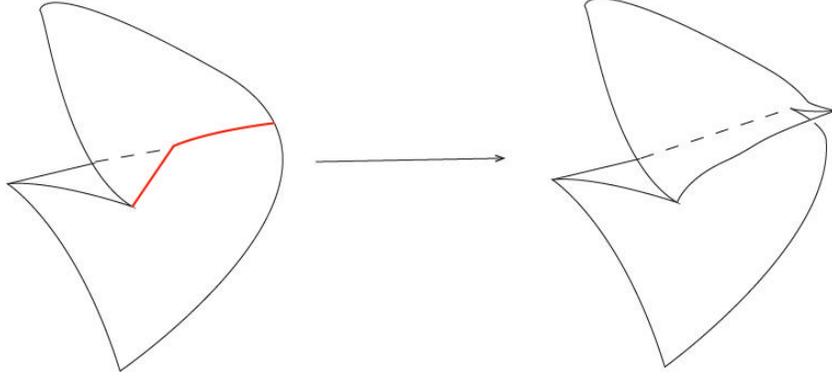


FIGURE 5. Resolve a wrinkle.

isotopy, which doesn't change the formal Legendrian isotopy class when $\chi(N) = 0$ (Proposition 2.6 in [Mur12]). \square

In our case, we note that the stabilization operation passing from L to $S_+^N(L)$ might change the formal isotopy class of the Legendrian L , even if N has Euler characteristic zero. However, we can go back to the original formal class by stabilizing again to $S_-^N(S_+^N(L))$. This fact will be used later on in the proof of our main theorem to correct formal classes.

2. CONTRACTIBLE POSITIVE LEGENDRIAN LOOPS

In this section we prove our main theorem 0.8 in a geometric way.

Proof. We start with a loose Legendrian L and work in a compact region of its standard neighborhood $J^1(L)$.

A. Construction of a positive loop

We first describe an elementary operation that will be applied repeatedly. Recall L_F is the front projection of L . Since L is the zero-section in $J^1(L)$, one can canonically identify L_F with L .

We consider a n -disk $D_0^n \subset L_F$, written as $D_0^{n-2} \times D^2(2)$ together with coordinates (u, ρ, θ) , where $(\rho \leq 3, \theta)$ are polar coordinates on D^2 . We let L^w be the wrinkled Legendrian obtained by adding one wrinkled disk D_0^w along the $(n-1)$ -disk $D_0 = \{1 \leq \rho \leq 2, \theta = 0\}$ to L_F , so that $D_0^w \subset \{1 \leq \rho \leq 2\}$. We moreover slightly modify L^w along $D_0 \times S^1 = \{1 \leq \rho \leq 2\}$, where S^1 corresponds to the θ direction, by propagating the slope of the wrinkle in the θ -direction so that:

- every circle $L^w \cap \{\rho = \rho_0 \in (1, 2), u = u_0\}$, contained in the $\{(u_0, \rho_0, \theta, z)\}$ cylinder, has a positive slope, i.e. is positively transverse to ∂_θ as in Figure 1;
- L^w is equal to L_F away from $\{1 \leq \rho \leq 2\}$.

The situation is pictured in Figure 6. We take a twist marking $N \subset D_0^w$ for D_0^w so that if N' is the closed submanifold $N \cup D_0 \subset L_F$ then the Euler characteristic $\chi(N')$ of N' is zero.

Step 1. We rotate the wrinkle positively in the θ -direction.

Given some constant $K_0 > 0$, we rotate positively the wrinkle in the θ -direction with (large) speed $K_0 \in \mathbb{N}$: We take a z -invariant path of diffeomorphisms ϕ_t^0 in the front such that $\phi_t^0(u, \rho, \theta, z) = (u, \rho, \theta - 2K_0\pi t, z)$ and ϕ_t^0 globally preserves $L^w \setminus \{1 \leq \rho \leq 2\}$. By construction

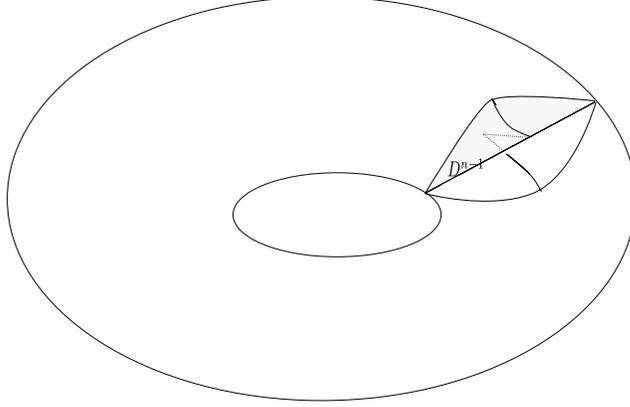


FIGURE 6. Rotation of the wrinkle.

$L_{t,K_0}^w = \phi_t^0(L^w)$ is a loop of wrinkled fronts, even if ϕ_t^0 is only a path. Its lift $\widetilde{L_{t,K_0}^w}$ is a non-negative loop of wrinkled Legendrians based in L^w . Non-negativity comes from the fact that the infinitesimal generator of the isotopy ϕ_t^0 is either tangent (outside $\{1 \leq \rho \leq 2\}$) or positively transverse (inside $\{1 \leq \rho \leq 2\}$) to the front of L_{t,K_0}^w .

Step 2. We resolve the wrinkle. We now parametrically resolve the wrinkle $\phi_t^0(D_0^w) \subset L_{t,K_0}^w$ along the marking $\phi_t^0(N) \subset L_{t,K_0}^w$ to get a loop $W_{\eta_0}^{-1}(L_{t,K_0}^w)$ of Legendrian fronts (notice that $\phi_1^0(N) = N$). Doing so, we might introduce some negative displacement near $\phi_t^0(N)$, but, taking the size of the resolution η_0 small enough in front of K_0 and the slope of the circles $\{\rho = \rho_0 \in [5/4, 7/4], u = u_0\}$, we can make sure that the isotopy is still positive in the region $\{5/4 \leq \rho \leq 7/4\}$.

Here, we notice that the loop of fronts $W_{\eta_0}^{-1}(L_{t,K_0}^w)$ is obtained from L_F by replacing a neighborhood of $\phi_t^0(N')$ by the product of N' with a z -shape. In particular, the loop of fronts $W_{\eta_0}^{-1}(L_{t,K_0}^w)$ admits a parametrization by a loop of homeomorphisms (which are diffeomorphisms except at the cusps of the fronts) $\psi_{t,K_0}^0 : L \rightarrow W_{\eta_0}^{-1}(L_{t,K_0}^w)$ which is constant away from D_0^n . When we lift the loop of fronts $W_{\eta_0}^{-1}(L_{t,K_0}^w)$ to a loop of Legendrians L_{t,K_0}^0 in $J^1(L)$, we can lift the loop of parametrizations ψ_{t,K_0}^0 to a loop of smooth parametrizations Ψ_{t,K_0}^0 , which is constant away from D_0^n . The loop L_{t,K_0}^0 is positive along $\Psi_{t,K_0}^0(\{5/4 \leq \rho \leq 7/4\})$.

Step 3. We adjust the formal class. In steps 1 and 2, we have constructed a loop for a stabilization $S_+^{N'}(L)$ of L , which might not be formally Legendrian isotopic to L . Following Proposition 1.10, we correct this by stabilizing again parametrically along a parallel copy of $\psi_{t,K_0}^0(N')$ to obtain a loop of Legendrians based at $S_-^{N'}(S_+^{N'}(L))$. This second stabilization can be made small at will (in the sense that the s -shape is squashed) so that the positivity property of step 2 is still unchanged and we still have a loop of parametrizations, that we persist to write Ψ_{t,K_0}^0 .

This preparatory work been done, the proof starts from a covering of L by open sets $A_i \subset D_i^n$, $i = 0, \dots, k$, of the form $S^1 \times D^{n-1} = \{5/4 \leq \rho \leq 7/4\} \subset D_i^n$ as before.

We then construct our loop by induction: by step 1,2,3, we construct a loop of Legendrians $L_{t,K_0}^0 = \Psi_{t,K_0}^0(L)$ which is positive along $\Psi_{t,K_0}^0(A_0)$, by rotating a wrinkle with speed K_0 . We now take a loop of standard Weinstein neighborhoods $N(L_{t,K_0}^0)$ of L_{t,K_0}^0 parametrized by $t \in S^1$, in which L_{t,K_0}^0 is the zero section diffeomorphic to L . This is given by a family of embeddings $J^1(L) \rightarrow J^1(L)$ sending the zero section to $L_{t,K_0}^0 = \Psi_{t,K_0}^0(L)$. These embeddings can be chosen to extend Ψ_{t,K_0}^0 , so we still denote them $\Psi_{t,K_0}^0 : J^1(L) \rightarrow N(L_{t,K_0}^0)$.

We then apply steps 1,2,3 to A_1, D_1^n in $J^1(L)$, though of as the source of Ψ_{t,K_0}^0 , by rotating a wrinkle with (relative) speed K_1 and resolving it with size η_1 . This means we are performing this step in the moving neighborhood $\Psi_{t,K_0}^0(J^1(L))$ of L_{t,K_0}^0 . We get a loop of Legendrians $L_{t,K_1}^1 = \Psi_{t,K_1}^1(L)$ in the moving neighborhood $J^1(L)$. Viewed in the original jet-space, we are considering the loop $\Psi_{t,K_0}^0(L_{t,K_1}^1) = \Psi_{t,K_0}^0(\Psi_{t,K_1}^1(L))$.

We see that if we take K_1 large enough, in particular with respect to η_0 and K_0 , then the loop $\Psi_{t,K_0}^0(\Psi_{t,K_1}^1(L))$ becomes positive along $\Psi_{t,K_0}^0(\Psi_{t,K_1}^1(A_1))$ – where it was before possibly negative. We also have to take η_1 small enough so that the isotopy remains positive along $\Psi_{t,K_0}^0(\Psi_{t,K_1}^1(A_0))$ after resolving the wrinkle with size η_1 .

Precise computations are described by the following composition of speeds:

Since

$$v_{\tilde{\Psi}_{t,K_0}^0 \circ \tilde{\Psi}_{t,K_1}^1}(x) = v_{\tilde{\Psi}_{t,K_0}^0}(\tilde{\Psi}_{t,K_1}^1(x)) + D\tilde{\Psi}_{t,K_0}^0(v_{\tilde{\Psi}_{t,K_1}^1}(x)),$$

we have

$$\alpha(v_{\tilde{\Psi}_{t,K_0}^0 \circ \tilde{\Psi}_{t,K_1}^1}(x)) = \alpha(v_{\tilde{\Psi}_{t,K_0}^0}(\tilde{\Psi}_{t,K_1}^1(x))) + (\tilde{\Psi}_{t,K_0}^0)_* \alpha(v_{\tilde{\Psi}_{t,K_1}^1}(x)).$$

Now, we have that $\alpha(v_{\tilde{\Psi}_{t,K_0}^0}) > -k_0$ independent of K_1 . Moreover, since the isotopy of Legendrians is compactly supported, there exists some $c_0 > 0$ independent of K_1 such that $\tilde{\Psi}_{t,K_0}^{0*} \alpha = f\alpha$, where $f > c_0 > 0$ in a neighborhood of the original L which contains all the deformations.

We can thus see that in the neighborhood of $\Psi_{t,K_0}^0(\Psi_{t,K_1}^1(A_1))$ where the slope of the front is larger than some $c_1 > 0$, $\alpha(v_{\tilde{\Psi}_{t,K_1}^1 \circ \tilde{\Psi}_{t,K_0}^0}) > -k_0 + c_0 c_1 K_1$. Thus, for K_1 large enough $\tilde{\Psi}_{t,K_0}^0 \circ \tilde{\Psi}_{t,K_1}^1$ is positive in the neighborhood of $\Psi_{t,K_0}^0(\Psi_{t,K_1}^1(A_1))$. Near A_0 where the loop was already positive, we do not alter positivity if the size η_1 of the resolution is small enough.

Once this is understood, it is clear that we can repeat the process until we get a loop which is positive everywhere. At each step the rotation speed has to be higher and higher with respect to previous operations.

To conclude, we observe that we have been producing a loop based at a loose Legendrian which is formally isotopic to L , and thus by Murphy's theorem [Mur12] Legendrian isotopic to L .

B. Contractibility We show that the positive loop that we have been constructing is contractible amongst Legendrian loops.

The construction was inductive on the set of annuli (A_i) and thus it is enough to check that the first loop $W_{\eta_0}^{-1}(L_{t,K_0}^w)$ is homotopic to a constant loop.

We first treat the case when the dimension of the Legendrian L is greater or equal to 3. Define $\phi_{s,t}^0$ such that $\phi_{s,t}^0(u, \rho, \theta, z) = (u, \rho - s, \theta - 2K_0\pi t, z)$. We can see that $\phi_{s,t}^0(L^w)$ is a homotopy from $\phi_{t,K_0}^0(L^w)$ to $\phi_{1,t}^0(L^w)$ which is a loop of rotation of a wrinkled disk D^w around some point, says x_0 . Up to homotopy, the wrinkled disk $\phi_{1,t}^0(D^w)$ is completely determined by its normal vector in L at x_0 , and thus by a map $S^1 \rightarrow S^{n-1}$. Since $n \geq 3$, this map is homotopic to a

point and thus we can deform our loop of wrinkled Legendrians to a constant loop. Moreover, this homotopy can be extended to a homotopy of twist markings from the original loop of twist markings to a constant loop. Resolving parametrically the markings, we get a homotopy from $W_{\eta_0}^{-1}(L_{t,K_0}^w)$ to a constant loop. The extra stabilization of step 3 to fix the formal isotopy class enters the same scheme and can be also homotoped to a constant operation. This concludes the proof.

The case when the dimension of L is two follows the same scheme, except that we homotope the loop of resolved wrinkles $\Psi_{t,K_0}^0(U)$, where U is a circle times a z -shape segment, to a constant annuli around the circle $\{\rho = 1\}$. \square

3. APPLICATIONS

In this chapter, we give some applications of our main theorem. First, we reprove tightness of $(\mathbb{S}^{n-1} \times \mathbb{R}^n, \xi_{std})$. Second, we define a partial order on the universal cover $\widetilde{Cont}_0(M, \xi)$ of the identity component of the group of contactomorphisms of a contact manifold (M, ξ) and prove that overtwisted contact structures are not orderable.

3.1. Tightness of $(\mathbb{S}^{n-1} \times \mathbb{R}^n, \xi_{std})$. In this section we prove Corollary 0.9. A similar proof for $\mathbb{S}^1 \times \mathbb{R}^2$ was given in [CFP10].

Proof. Assume $(\mathbb{S}^{n-1} \times \mathbb{R}^n, \xi_{std})$ is overtwisted, and $D_{OT} \subset (\mathbb{S}^{n-1} \times \mathbb{R}^n, \xi_{std})$ is an overtwisted disk. Denote $\pi : \mathbb{S}^{n-1} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ the projection. There exists some point $x \in \mathbb{R}^n$ such that the fiber $\pi^{-1}(x) \cap D_{OT} = \emptyset$. According to [CMP15], the fiber $\pi^{-1}(x)$ is loose. Thus, there exists a positive loop based at it by Theorem 0.8. That contradicts Theorem 0.6. Therefore, the manifold $(\mathbb{S}^{n-1} \times \mathbb{R}^n, \xi_{std})$ is tight. \square

3.2. Positive loops and orderings.

Definition 3.1. Given a contact manifold (M, α) , the manifold $(\Gamma_M, \tilde{\alpha}) = (M \times M \times \mathbb{R}, \alpha_1 - e^s \alpha_2)$ is called a contact product. Here $\alpha_i = \pi_i^* \alpha$ where π_i project Γ_M to the i -th factor. The Legendrian submanifold of $(\Gamma_M, \tilde{\alpha})$ $\Delta = \{(x, x, 0)\}$ is called the diagonal.

The contact product Γ_M is a special case of a contact fibration. We recall the definition from [Pre07].

Definition 3.2. Let $(E, \xi = \ker \alpha)$ be a contact manifold, and $E \rightarrow B$ is a fibration with fiber F . Then $(E, \xi = \ker \alpha) \rightarrow B$ is called a contact fibration if $(F, \alpha|_F)$ is a contact manifold. Let $(E, \xi = \ker \alpha) \rightarrow B$ be contact fibration we say that the horizontal distribution $H = (TF \cap \xi)^{\perp d\alpha}$ is the contact connection associated to the fibration.

Remark 3.3. The horizontal distribution depends on the contact form α .

The connection defined above has the following properties:

Proposition 3.4. [Pre07] *For a path $\gamma : [0, 1] \rightarrow B$, the monodromy $m_\gamma : F(\gamma(0)) \rightarrow F(\gamma(1))$ induced by γ is a contactomorphism.*

Corollary 3.5. *Let $\phi \in \text{Diff}_0(B)$. Then it lifts to a contactomorphism $\tilde{\phi}$.*

Note that Γ_M is a contact fibration with $F = (M, \alpha)$ and $B = M \times \mathbb{R}$.

We now explain the following result which was first suggested by Klaus Niederkrüger and also observed by Casals and Presas.

Proposition 3.6. *Let (M^{2n+1}, α) be a compact overtwisted contact manifold and let $(\Gamma_M, \tilde{\alpha})$ be the associated contact product. Then $(\Gamma_M, \tilde{\alpha})$ is also overtwisted and the diagonal $\Delta \subset \Gamma_M$ is loose.*

Proof. We apply the overtwisted criterion from [CMP15]. If $\lambda = ydx - xdy$, it is enough to construct a higher dimensional overtwisted ball $D = (B_{OT}^{2n+1} \times D^{2n+2}(r), \alpha_{OT} - \lambda) \subset (\Gamma_M, \tilde{\alpha})$ for some r large enough, such that D does not intersect Δ .

Let $\mathbb{S}^{2n+1} = \{(x, y) \mid x^2 + y^2 = 1\}$ with its standard contact form α_{std} , and let $\varphi_0 : \mathbb{S}^{2n+1} \times \mathbb{R} \rightarrow \mathbb{R}^{2n+2}, (x, y, s) \mapsto (e^s x, e^s y)$. Note that $\varphi_0^* \lambda = \alpha_{std}$. We take a Darboux ball $B \subset (M, \alpha)$ and we regard it as a subset of $(\mathbb{S}^{2n+1}, \alpha_{std})$. Then we can construct a contact embedding $\varphi : (M \times B \times \mathbb{R}, \tilde{\alpha}) \hookrightarrow (M \times \mathbb{R}^{2n+2}, \alpha_1 - \lambda)$ by the following series of contact embeddings

$$(M \times B \times \mathbb{R}, \tilde{\alpha}) \xrightarrow{i} (M \times \mathbb{S}^{2n+1} \times \mathbb{R}, \alpha_1 - e^s \alpha_{std}) \xrightarrow{id \times \varphi_0} (M \times \mathbb{R}^{2n+2}, \alpha_1 - \lambda).$$

Let $B_{OT}^{2n+1} \subset M$ be a overtwisted ball, then $D_0 = (B_{OT}^{2n+1} \times \mathbb{D}^{2n+2}(r), \alpha_1 - \lambda)$ is the overtwisted ball in $(M \times \mathbb{R}^{2n+2}, \alpha_1 - \lambda)$. We can move D_0 away from $\varphi(\Delta)$ by Corollary 3.5. More precisely, we take the vector field $V = 2r\partial_x + 2r\partial_y$ on \mathbb{R}^{2n+2} , then lift it to a contact vector field $V' = V + 2r(y - x)R_\alpha$ on $M \times \mathbb{R}^{2n+2}$ where R_α is the Reeb vector field of (M, α) . Let ϕ_t be the contact isotopy of V' . Denote $C = \{rx \mid x \in B, r > 0\}$ the cone defined by B . Then $D_1 = \phi_1(D_0) \subset M \times (C \setminus \{0\}) = \varphi(M \times B)$ does not intersect $\varphi(\Delta)$. Therefore $D = \varphi^{-1}(D_1)$ is one of the overtwisted balls we want. \square

Corollary 3.7. *Let (M, α) be a compact overtwisted contact manifold and $(\Gamma_M, \tilde{\alpha})$ the contact product. Then there exists a positive loop of Legendrian embeddings based at Δ .*

Let $Leg(M, \Gamma_M)$ be the set of Legendrian embeddings $M \hookrightarrow (\Gamma_M, \tilde{\alpha})$. Given $\phi \in Cont_0(M, \xi = ker\alpha)$ with $\phi^* \alpha = e^{g(x)} \alpha$, it induces a contactomorphism

$$\bar{\phi}(x, y, s) := (x, \phi(x), s - g(y))$$

on $(\Gamma_M, \tilde{\alpha})$. We denote $gr(\phi) = \bar{\phi}|_\Delta$ which is in $Leg(M, \Gamma_M)$. In fact, given a positive contact isotopy ϕ_t , we can see that $gr(\phi_t)$ is a negative Legendrian isotopy. Therefore, we would like to transfer the study of positive contact isotopies to that of negative Legendrian isotopies.

Definition 3.8. Let $f = [f_t]$ and $g = [g_t]$ be two elements in $\widetilde{Cont}_0(M, \xi)$. We say $f \succeq g$ if there exists a non-positive path $L_t \in Leg(M, \Gamma_M)$ from $gr(g_1)$ to $gr(f_1)$ and $gr(g_t) * L_t$ is homotopic to $gr(f_t)$. The space $\widetilde{Cont}_0(M, \xi)$ and (M, ξ) are said to be **strongly orderable** if \succeq defines a partial order ¹ on it. Otherwise, they are said to be non strongly orderable.

Remark 3.9. Let C be the set generated by all the homotopy classes of non-positive paths in $Leg(M, \Gamma_M)$. Then $f \succeq g$ equals to $gr(g^{-1}f) \in C$. Given $[L_t] \in C$ and $\phi \in Cont_0(M, \xi)$, then we have $[\bar{\phi}L_t] \in C$. Therefore, the order \succeq is left invariant, that is to say, given f and g in $\widetilde{Cont}_0(M, \xi)$, if $f \succeq g$, then $hf \succeq hg$ for all $h \in \widetilde{Cont}_0(M, \xi)$. Because if L_t is a non-positive path from g_1 to f_1 , then $\bar{h}_1 L_t$ is a non-positive path from $h_1 g_1$ to $h_1 f_1$.

Proposition 3.10. *Let (M, ξ) be a contact manifold. Then (M, ξ) is strongly orderable if and only if there does not exist a contractible negative loop of Legendrian embeddings based at Δ .*

Proof. Let $f = [f_t], g = [g_t]$ and $h = [h_t]$ be elements in $\widetilde{Cont}_0(M, \xi)$. The relation \succeq is reflective, since we have $f \succeq f$ by the definition of \succeq . If there are two non-positive paths L_t^1 from $gr(g_1)$ to $gr(f_1)$ and L_t^2 from $gr(h_1)$ to $gr(g_1)$, then $L_t^2 * L_t^1$ is a non-positive path

¹in the sense of a partial order on sets

from $gr(h_1)$ to $gr(f_1)$. Thus, the relation is transitive. Now we check the antisymmetry of \succeq . According to [CN13][Proposition 4.5], the existence of contractible non-positive non-trivial loop of Legendrian embeddings is equivalent to the existence of contractible negative loop of Legendrian embeddings. Thus, for any $f \neq 1$, on one hand, if there does not exist any negative loop based at Δ , we can not find a non-negative path L_t^1 and a non-positive path L_t^1 in the homotopy class of $gr(f_t)$ at the same times. Otherwise, $L_t^1 * L_t^2$ would be a contractible non-negative loop. On the other hand, if there exists a non-positive loop f_t based at Δ , then $f_{1/2} \succeq 1$ and $1 \succeq f_{1/2}$. That means (M, ξ) is not strongly orderable. \square

Our definition is stronger than that of [EP99], since we do not require the path of Legendrian embeddings $\tilde{\phi}_t$ to be graphical for all t .

Corollary 3.11. *Let (M, ξ) be a contact manifold. If (M, ξ) is strongly orderable, then it is orderable.*

A contact manifold which is not strongly orderable is said to be weakly non-orderable. Immediately, according to Proposition 3.6 and Corollary 3.7, we deduce theorem 0.12 saying that overtwisted contact manifolds are weakly non-orderable.

We have the following example of strong orderability.

Theorem 3.12. *$(\mathbb{S}^1, \xi_{std})$ is strongly orderable.*

Proof. Denote $d\theta$ the standard contact form for \mathbb{S}^1 . We have a contactomorphism $\varphi : (\Gamma_{\mathbb{S}^1}, d\theta_1 - e^s d\theta_2) \rightarrow (\mathbb{S}^1 \times T^*\mathbb{S}^1, dz - ydx), (\theta_1, \theta_2, s) \mapsto (z = \theta_1 - \theta_2, x = \theta_2, y = e^s - 1)$ such that $\varphi(\Delta)$ is the zero-section. Assume there exists a contractible positive loop based at the zero-section of $(\mathbb{S}^1 \times T^*\mathbb{S}^1, dz - ydx)$, then it lifts to a positive loop based at the zero-section of $(\mathbb{R}^1 \times T^*\mathbb{S}^1, dz - ydx)$. However, such loops do not exist according to [CFP10] (notice this is not a trivial result). Thus $(\mathbb{S}^1, \xi_{std})$ is strongly orderable. \square

Question 3.13. *Is $(\mathbb{R}P^3, \xi_{std})$ strongly orderable?*

REFERENCES

- [AF12] Peter Albers and Urs Fauenhfelder, *A Variational Approach to Givental's Nonlinear Maslov Index*, Geometric and Functional Analysis **22** (2012), no. 5, 1033–1050 (en).
- [AFM15] Peter Albers, Urs Fuchs, and Will J. Merry, *Orderability and the Weinstein Conjecture*, Compositio Mathematica **151** (2015), no. 12, 2251–2272, arXiv: 1310.0786.
- [AM13] Peter Albers and Will J. Merry, *Orderability, contact non-squeezing, and Rabinowitz Floer homology*.
- [BEM15] Matthew Strom Borman, Yakov Eliashberg, and Emmy Murphy, *Existence and classification of overtwisted contact structures in all dimensions*, Acta Math (2015), 215–281.
- [Ben83] D. Bennequin, *Entrelacements et équations de pfaff*, Astérisque **107-108** (1983), 87–161.
- [CFP10] Vincent Colin, Emmanuel Ferrand, and Petya Pushkar, *Positive isotopies of Legendrian submanifolds and applications*.
- [CMP15] Roger Casals, Emmy Murphy, and Francisco Presas, *Geometric criteria for overtwistedness*, arXiv:1503.06221 [math] (2015), arXiv: 1503.06221.
- [CN10] Vladimir Chernov and Stefan Nemirovski, *Non-negative Legendrian isotopy in $\mathbb{S}T^*M\mathbb{S}$* , Geometry & Topology **14** (2010), no. 1, 611–626 (en).
- [CN13] ———, *Universal orderability of Legendrian isotopy classes*, arXiv:1307.5694 [gr-qc] (2013), arXiv: 1307.5694.
- [CPS14] Roger Casals, Francisco Presas, and Sheila Sandon, *On the non-existence of small positive loops of contactomorphisms on overtwisted contact manifolds*, arXiv:1403.0350 [math] (2014), arXiv: 1403.0350.
- [CS12] Vincent Colin and Sheila Sandon, *The discriminant and oscillation lengths for contact and Legendrian isotopies*.
- [EKP06] Yakov Eliashberg, Sang Seon Kim, and Leonid Polterovich, *Geometry of contact transformations and domains: orderability versus squeezing*, Geometry & Topology **10** (2006), no. 3, 1635–1747 (en).

- [Eli89] Y. Eliashberg, *Classification of overtwisted contact structures on 3-manifolds*, *Inventiones Mathematicae* **98** (1989), no. 3, 623–637 (en).
- [EM11] Yakov M. Eliashberg and Nikolai M. Mishachev, *Wrinkled Embeddings*, *Contemporary Mathematics* **498** (2011), 207–232.
- [EP99] Yakov Eliashberg and Leonid Polterovich, *Partially ordered groups and geometry of contact transformations*, arXiv:math/9910065 (1999), arXiv: math/9910065.
- [Lau07] Francois Laudenbach, *Positive Legendrian regular homotopies*, vol. 49, American Mathematical Society, June 2007, pp. 299–305 (en).
- [MNPS13] Emmy Murphy, Klaus Niederkrüger, Olga Plamenevskaya, and András I Stipsicz, *Loose Legendrians and the plastikstufe*, *Geometry & Topology* **17** (2013), no. 3, 1791–1814 (en).
- [Mur12] Emmy Murphy, *Loose Legendrian Embeddings in High Dimensional Contact Manifolds*, arXiv:1201.2245 [math] (2012), arXiv: 1201.2245.
- [Nie06] Klaus Niederkrüger, *The plastikstufe – a generalization of the overtwisted disk to higher dimensions*, *Algebraic & Geometric Topology* **6** (2006), no. 5, 2473–2508 (en).
- [Pre07] Francisco Presas, *A class of non-fillable contact structures*, *Geometry & Topology* **11** (2007), no. 4, 2203–2225 (en).

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