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Decomposition of small diagonals and Chow rings of hypersurfaces and Calabi-Yau complete intersections

Lie Fu

*Département de Mathématiques et Applications,
École Normale Supérieure,
45 Rue d'Ulm, 75230 Paris Cedex 05, France*

Abstract

On the one hand, for a general Calabi-Yau complete intersection X , we establish a decomposition, up to rational equivalence, of the small diagonal in $X \times X \times X$, from which we deduce that any *decomposable* 0-cycle of degree 0 is in fact rationally equivalent to 0, up to torsion. On the other hand, we find a similar decomposition of the smallest diagonal in a higher power of a hypersurface, which provides us an analogous result on the multiplicative structure of its Chow ring.

Keywords: decomposition of diagonal, Calabi-Yau complete intersection, decomposable 0-cycle, Chow ring, intersection theory, Hodge structure

2010 MSC: 14C25, 14C30, 14C15, 14N99

0. Introduction

For a given smooth projective complex algebraic variety X , we can construct very few subvarieties or algebraic cycles of $X \times X$ in an *a priori* fashion. Besides the divisors and the exterior products of two algebraic cycles of each factor, the diagonal $\Delta_X := \{(x, x) \mid x \in X\} \subset X \times X$ is essentially the only one that we can canonically construct in general. Despite of its simplicity, the diagonal in fact contains a lot of geometric information of the original variety. For instance, its normal bundle is the tangent bundle of X ; its self-intersection number is its topological Euler characteristic and so on. Besides these obvious facts, we would like to remark that the Bloch-Beilinson-Murre conjecture (*cf.* [4] [17] [11]), which is considered as one of the deepest conjectures in the study of algebraic cycles, claims a conjectural decomposition of the diagonal, up to rational equivalence, as the sum of certain orthogonal idempotent correspondences related to the Hodge structures on its Betti cohomology groups.

The idea of using decomposition of diagonal to study algebraic cycles is initiated by Bloch and Srinivas [5]; we state their main theorem in the following form.

Theorem 0.1 (Bloch, Srinivas [5]). *Let X be a smooth projective complex algebraic variety of dimension n . Suppose that $\mathrm{CH}_0(X)$ is supported on a closed algebraic subset Y , i.e. the natural morphism $\mathrm{CH}_0(Y) \rightarrow \mathrm{CH}_0(X)$ is surjective. Then there exist a positive integer $m \in \mathbf{N}^*$ and a proper closed algebraic subset $D \subsetneq X$, such that in $\mathrm{CH}_n(X \times X)$ we have*

$$m \cdot \Delta_X = \mathcal{L}_1 + \mathcal{L}_2 \tag{1}$$

where \mathcal{L}_1 is supported on $Y \times X$, and \mathcal{L}_2 is supported on $X \times D$.

The above decomposition in the case that Y is a point, or equivalently $\mathrm{CH}_0(X) \simeq \mathbf{Z}$ by degree map, is further generalized by Paranjape [18] and Laterveer [12] for varieties with small Chow groups in the following form.

Theorem 0.2 (Paranjape [18], Laterveer [12]). *Let X be a smooth projective n -dimensional variety. If the cycle class map $\mathrm{cl} : \mathrm{CH}_i(X)_{\mathbf{Q}} \rightarrow H^{2n-2i}(X, \mathbf{Q})$ is injective for any $0 \leq i \leq c-1$. Then there exist a positive integer $m \in \mathbf{N}^*$, a closed algebraic subset T of dimension $\leq n-c$, and for each $i \in \{0, 1, \dots, c-1\}$, a pair of closed algebraic subsets V_i, W_i with $\dim V_i = i$ and $\dim W_i = n-i$, such that in $\mathrm{CH}_n(X \times X)$, we have*

$$m \cdot \Delta_X = \mathcal{L}_0 + \mathcal{L}_1 + \dots + \mathcal{L}_{c-1} + \mathcal{L}' \quad (2)$$

where \mathcal{L}_i is supported on $V_i \times W_i$ for any $0 \leq i < c$, and \mathcal{L}' is supported on $X \times T$.

For applications of such decompositions, the point is that we consider (1) and (2) as equalities of correspondences from X to itself, which yield decompositions of the identity correspondence. This point of view allows us to deduce from (1) and (2) many interesting results like generalizations of Mumford's theorem (cf. [16] [5] [19]).

Most of this paper is devoted to the study of the class of the *small diagonal*

$$\delta_X := \{(x, x, x) \in X^3 \mid x \in X\} \quad (3)$$

in $\mathrm{CH}_n(X^3)_{\mathbf{Q}}$, where X is an n -dimensional Calabi-Yau variety. The interest of the study is motivated by the obvious fact that while the diagonal seen as a self-correspondence of X controls $\mathrm{CH}^*(X)_{\mathbf{Q}}$ as an additive object, the small diagonal seen as a correspondence between $X \times X$ and X controls the multiplicative structure of $\mathrm{CH}^*(X)_{\mathbf{Q}}$.

The first result in this direction is due to Beauville and Voisin [3], who find a decomposition of the *small diagonal* $\delta_S := \{(x, x, x) \mid x \in S\}$ in $\mathrm{CH}_2(S \times S \times S)$ for S an algebraic K3 surface.

Theorem 0.3 (Beauville, Voisin [3]). *Let S be a projective K3 surface, and $c_S \in \mathrm{CH}_0(S)$ be the well-defined¹ 0-dimensional cycle of degree 1 represented by any point lying on any rational curve of S . Then in $\mathrm{CH}_2(S \times S \times S)$, we have*

$$\delta_S = \Delta_{12} + \Delta_{13} + \Delta_{23} - S \times c_S \times c_S - c_S \times S \times c_S - c_S \times c_S \times S \quad (4)$$

where Δ_{12} is represented by $\{(x, x, c_S) \mid x \in S\}$, and Δ_{13}, Δ_{23} are defined similarly.

As is mentioned above, we regard (4) as an equality of correspondences from $S \times S$ to S . Applying this to a product of two divisors $D_1 \times D_2$, one can recover the following corollary, which is in fact a fundamental observation in [3] for the proof of the above theorem.

Corollary 0.4 (Beauville, Voisin [3]). *Let S be a projective K3 surface. Then the intersection product of any two divisors is always proportional to the class c_S in $\mathrm{CH}_0(S)$, i.e.*

$$\mathrm{Im}\left(\mathrm{Pic}(S) \times \mathrm{Pic}(S) \xrightarrow{\bullet} \mathrm{CH}_0(S)\right) = \mathbf{Z} \cdot c_S.$$

¹The fact that c_S is well-defined relies on the result of Bogomolov-Mumford (cf. the appendix of [14]) about the existence of rational curves in any ample linear system, cf. [3].

The next result in this direction, which is also the starting point of this paper, is the following partial generalization of Theorem 0.3 due to Voisin:

Theorem 0.5 (Voisin [21]). *Let $X \subset \mathbf{P}^{n+1}$ be a general smooth hypersurface of Calabi-Yau type, that is, the degree of X is $n + 2$. Let $h := c_1(\mathcal{O}_X(1)) \in \text{CH}^1(X)$ be the hyperplane section class, $h_i := \text{pr}_i^*(h) \in \text{CH}^1(X^3)$ for $i = 1, 2, 3$, and $c_X := \frac{h^n}{n+2} \in \text{CH}_0(X)_{\mathbf{Q}}$ be a \mathbf{Q} -0-cycle of degree 1. Then we have a decomposition of the small diagonal in $\text{CH}_n(X^3)_{\mathbf{Q}}$*

$$\delta_X = \frac{1}{(n+2)!} \Gamma + \Delta_{12} + \Delta_{13} + \Delta_{23} + P(h_1, h_2, h_3), \quad (5)$$

where $\Delta_{12} = \Delta_X \times c_X$, and Δ_{13}, Δ_{23} are defined similarly; P is a homogeneous polynomial of degree $2n$; and $\Gamma := \bigcup_{t \in F(X)} \mathbf{P}_t^1 \times \mathbf{P}_t^1 \times \mathbf{P}_t^1 \subset X^3$, where $F(X)$ is the variety of lines of X , and \mathbf{P}_t^1 is the line corresponding to $t \in F(X)$.

Applying (5) as an equality of correspondences, she deduces the following

Corollary 0.6 (Voisin [21]). *In the same situation as the above theorem, the intersection product of any two cycles of complementary and strictly positive codimensions is always proportional to c_X in $\text{CH}_0(X)_{\mathbf{Q}}$, i.e. for any $i, j \in \mathbf{N}^*$ with $i + j = n$, we have*

$$\text{Im} \left(\text{CH}^i(X)_{\mathbf{Q}} \times \text{CH}^j(X)_{\mathbf{Q}} \xrightarrow{\bullet} \text{CH}_0(X)_{\mathbf{Q}} \right) = \mathbf{Q} \cdot c_X.$$

In particular, for any $i = 1, \dots, m$, let Z_i, Z'_i be algebraic cycles of strictly positive codimension with $\dim Z_i + \dim Z'_i = n$, then any equality on the cohomology level $\sum_{i=1}^m \lambda_i [Z_i] \cup [Z'_i] = 0$ in $H_0(X, \mathbf{Q})$ is in fact an equality modulo rational equivalence: $\sum_{i=1}^m \lambda_i Z_i \bullet Z'_i = 0$ in $\text{CH}_0(X)_{\mathbf{Q}}$.

The main results of this paper are further generalizations of Voisin's theorem and its corollary in two different directions. The first direction of generalization is about smooth Calabi-Yau complete intersections in projective spaces:

Theorem 0.7 (=Theorem 1.12+Theorem 1.13). *Let X be a general Calabi-Yau complete intersection of multi-degree $d_1 \geq \dots \geq d_r$, in a projective space. Then*

(i) *We have in $\text{CH}_n(X^3)$ a decomposition of the small diagonal:*

$$\left(\prod_{i=1}^r (d_i!) \right) \cdot \delta_X = \Gamma + j_{12*}(Q(h_1, h_2)) + j_{13*}(Q(h_1, h_2)) + j_{23*}(Q(h_1, h_2)) + P(h_1, h_2, h_3),$$

where Q and P are symmetric homogeneous polynomials with \mathbf{Z} -coefficients; Γ is defined as in Theorem 0.5; $h_i \in \text{CH}^1(X^3)$ or $\text{CH}^1(X^2)$ is the pull-back of $h = c_1(\mathcal{O}_X(1))$ by the i^{th} projection; and j_{12}, j_{13}, j_{23} are the inclusions of big diagonals $X^2 \hookrightarrow X^3$.

(ii) *For any $k, l \in \mathbf{N}^*$ with $k + l = n$,*

$$\text{Im} \left(\bullet : \text{CH}^k(X)_{\mathbf{Q}} \times \text{CH}^l(X)_{\mathbf{Q}} \rightarrow \text{CH}_0(X)_{\mathbf{Q}} \right) = \mathbf{Q} \cdot h^n,$$

where $h = c_1(\mathcal{O}_X(1)) \in \text{CH}^1(X)$.

The above degeneration property of the intersection product is remarkable since CH_0 is *a priori* very huge ('of infinite dimension' in the sense of Mumford, *cf.* [16]). To emphasize the principle that the small diagonal controls the intersection product, we point out that part (ii) of the preceding theorem is obtained by applying the equality of part (i) as correspondences to an exterior product of two algebraic cycles.

We will make a comparison of our result (iii) for Calabi-Yau complete intersections with Beauville's 'weak splitting principle' in [2] for holomorphic symplectic varieties, see Remark 1.14. We also give in §1.4 an example of a surface S in \mathbf{P}^3 of general type, such that

$$\text{Im}(\bullet : \text{CH}^1(S)_{\mathbf{Q}} \times \text{CH}^1(S)_{\mathbf{Q}} \rightarrow \text{CH}_0(S)_{\mathbf{Q}}) \cong \mathbf{Q} \cdot h^2,$$

where $h = c_1(\mathcal{O}_S(1)) \in \text{CH}^1(S)$, in contrast to the result for K3 surfaces proved in [3]. This example supports the feeling that the Calabi-Yau condition gives some strong restrictions on the multiplicative structure of the Chow ring.

The second direction of generalization is about higher powers (≥ 3) of hypersurfaces with ample or trivial canonical bundle. The objective is to decompose the smallest diagonal in a higher self-product, and deduce from it some implication on the multiplicative structure of the Chow ring of the variety. That such a decomposition should exist was suggested by Nori to be the natural generalization of Theorem 0.5. We also refer to [9] for similar results in the case of curves. Now we state our result precisely:

Theorem 0.8 (=Theorem 2.12 + Theorem 2.13). *Let X be a smooth hypersurface in \mathbf{P}^{n+1} of degree d with $d \geq n + 2$. Let $k = d + 1 - n \geq 3$. Then*

(i) *One of the following two cases occurs:*

(a) *There exist rational numbers λ_j for $j = 2, \dots, k-1$, and a symmetric homogeneous polynomial P of degree $n(k-1)$, such that in $\text{CH}_n(X^k)_{\mathbf{Q}}$ we have:*

$$\delta_X = (-1)^{k-1} \frac{1}{d!} \cdot \Gamma + \sum_{i=1}^k D_i + \sum_{j=2}^{k-2} \lambda_j \sum_{|I|=j} D_I + P(h_1, \dots, h_k),$$

where Γ is defined similarly as in Theorem 0.5 (see (39)) and the bigger diagonals D_I are defined in (32) or (33), and $D_i := D_{\{i\}}$.

Or

(b) *There exist a (smallest) integer $3 \leq l < k$, rational numbers λ_j for $j = 2, \dots, l-2$, and a symmetric homogeneous polynomial P of degree $n(l-1)$, such that in $\text{CH}_n(X^l)_{\mathbf{Q}}$ we have:*

$$\delta_X = \sum_{i=1}^l D_i + \sum_{j=2}^{l-2} \lambda_j \sum_{|I|=j} D_I + P(h_1, \dots, h_l).$$

Moreover, $\Gamma = 0$ if $d \geq 2n$.

(ii) *For any strictly positive integers $i_1, i_2, \dots, i_{k-1} \in \mathbf{N}^*$ with $\sum_{j=1}^{k-1} i_j = n$, the image*

$$\text{Im}\left(\text{CH}^{i_1}(X)_{\mathbf{Q}} \times \text{CH}^{i_2}(X)_{\mathbf{Q}} \times \dots \times \text{CH}^{i_{k-1}}(X)_{\mathbf{Q}} \xrightarrow{\bullet} \text{CH}_0(X)_{\mathbf{Q}}\right) = \mathbf{Q} \cdot h^n$$

The main line of the proofs of the above theorems is the same as in Voisin's paper [21]: one proceeds in three steps:

- Firstly, one ‘decomposes’ the class of Γ (see Theorem 0.5 for its definition) restricting to the complement of the small diagonal, by means of a careful study of the geometry of collinear points on the variety.
- Secondly, by the localization exact sequence for Chow groups, we obtain a decomposition of certain multiple of the small diagonal in terms of Γ and other cycles of diagonal type or coming from the ambient space.
- Thirdly, once we have a decomposition of small diagonal, regarded as an equality of correspondences, we will draw consequences on the multiplicative structure of the Chow ring.

We remark that at some point of the proof, one should verify that the ‘multiple’ appeared in the decomposition is non-zero to get a genuine decomposition of the small diagonal, and also some coefficients should be distinct to deduce the desired conclusion on the multiplicative structure of Chow rings. These are too easy to be noticed in [21], but become the major difficulties in our present paper.

The paper consists of two parts. The first part deals with the first direction of generalization explained above (Theorem 0.7), where we start by the geometry of collinear points to deduce a decomposition of a certain multiple of the small diagonal; then we deduce from it our result on the multiplicative structure of Chow rings; after that we treat the complete intersection case to obtain the main results; finally we construct an example of surface in \mathbf{P}^3 such that the image of intersection product of line bundles is ‘non-trivial’, in contrast to the Calabi-Yau case. The second part establishes Theorem 0.8, the second direction of generalization, and we also follow the line of geometry of collinear points, decomposition of the smallest diagonal, and finally consequence on Chow ring’s structure.

We will work over the complex numbers throughout this paper for simplicity, but all the results and proofs go through for any uncountable algebraic closed field of characteristic zero.

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1. Calabi-Yau complete intersections

The main goal of this section is to prove Theorem 0.7 in the introduction. First of all, we will set up the basic situation. Let E be a rank r vector bundle on the complex projective space $\mathbf{P} := \mathbf{P}^{n+r}$. Although we will mostly be interested in the splitting case where $E = \bigoplus_{i=1}^r \mathcal{O}(d_i)$ with $d_1 \geq \dots \geq d_r \geq 2$, however to simplify the notation as well as to make the construction canonical and generalizable, we will firstly work in general with the vector bundle E by imposing only the following

Positivity Assumption (*): The evaluation map $H^0(\mathbf{P}, E) \rightarrow \bigoplus_{i=1}^3 E_{y_i}$ is surjective for any three collinear points y_1, y_2, y_3 in \mathbf{P} , where E_y means the fiber of E over y .

We note that this condition implies in particular:

(*') E is globally generated.

(*'') The restriction of $E(-2) := E \otimes \mathcal{O}_{\mathbf{P}}(-2)$ to each line is globally generated².

²Equivalently speaking, along any line \mathbf{P}^1 , the splitting type $E|_{\mathbf{P}^1} = \bigoplus_{i=1}^r \mathcal{O}_{\mathbf{P}^1}(d_i)$ satisfies $d_1 \geq d_2 \geq \dots \geq d_r \geq 2$.

Let $d \in \mathbf{N}^*$ be such that $\det(E) = \mathcal{O}_{\mathbf{P}}(d)$. Let $f \in H^0(\mathbf{P}^{n+r}, E)$ be a general global section of E , and

$$X := V(f) \subset \mathbf{P}^{n+r}$$

be the subscheme of \mathbf{P} defined by f . Here f is general implies that X is smooth of expected dimension n .

We are interested in the case when X is of *Calabi-Yau type*: $K_X = 0$, or equivalently,

Calabi-Yau Assumption: $d = n + r + 1$.

Throughout this section, we will always work in the above setting. A typical example of such situation is when $E = \bigoplus_{i=1}^r \mathcal{O}_{\mathbf{P}}(d_i)$ with $d_1 \geq d_2 \geq \dots \geq d_r \geq 2$, X is hence an n -dimensional smooth Calabi-Yau complete intersection of multi-degree (d_1, \dots, d_r) and $d = \sum_{i=1}^r d_i = n + r + 1$. Since the case of K3 surfaces is well treated in [3], we assume $n \geq 3$ from now on.

1.1. Decomposition of small diagonals

Like in the paper [21], our strategy is to express the class of the small diagonal by investigating the lines contained in X . The decomposition result is Corollary 1.8 and see Theorem 1.10 for its more precise form.

Let $G := \text{Gr}(\mathbf{P}^1, \mathbf{P}^{n+r})$ be the Grassmannian of projective lines in \mathbf{P} , and for a point $t \in G$, we denote by \mathbf{P}_t^1 the corresponding line. Define the *variety of lines* of X :

$$F(X) := \{t \in G \mid \mathbf{P}_t^1 \subset X\}.$$

Firstly, let us recall the following basic fact, see for example [7].

Lemma 1.1. *If f is general, $F(X)$ is non-empty, smooth and of dimension $n - 3$.*

We define

$$\Gamma := \bigcup_{t \in F(X)} \mathbf{P}_t^1 \times \mathbf{P}_t^1 \times \mathbf{P}_t^1 \subset X^3. \quad (6)$$

It is then an n -dimensional subvariety of X^3 .

To get a decomposition of the small diagonal δ_X of X^3 , we first decompose or calculate the class of Γ_o in $\text{CH}_n(X^3 \setminus \delta_X)$, where Γ_o is the restriction of Γ to $X^3 \setminus \delta_X$.

Before doing so, let us make some preparatory geometric constructions. Let $\delta_{\mathbf{P}}$ be the small diagonal of $\mathbf{P}^{\times 3} := \mathbf{P} \times \mathbf{P} \times \mathbf{P}$. Define the following closed subvariety of $\mathbf{P}^{\times 3} \setminus \delta_{\mathbf{P}}$:

$$W := \{(y_1, y_2, y_3) \in \mathbf{P}^{\times 3} \mid y_1, y_2, y_3 \text{ are collinear}\} \setminus \delta_{\mathbf{P}}.$$

In other words, if we denote by $L \rightarrow G$ the universal line over the Grassmannian G of projective lines, then in fact $W = L \times_G L \times_G L \setminus \delta_L$. In particular W is a smooth variety with

$$\dim W = \dim G + 3 = 2n + 2r + 1.$$

Similarly, let δ_X be the small diagonal of X^3 . We define a closed subvariety V of $X^3 \setminus \delta_X$ by taking the closure of $V_o := \{(x_1, x_2, x_3) \in X^3 \mid x_1, x_2, x_3 \text{ are collinear and distinct}\}$ in $X^3 \setminus \delta_X$:

$$V := \overline{V_o}.$$

We remark that the boundary $\partial V := V \setminus V_o$ consists, up to a permutation of the three coordinates, of points of the form (x, x, x') with $x \neq x'$ such that the line joining x, x' is tangent to X at x .

We will also need the ‘big’ diagonals in $X^3 \setminus \delta_X$:

$$\Delta_{12} := \{(x, x, x') \in X^3 \mid x \neq x'\},$$

and Δ_{13}, Δ_{23} are defined in the same way.

Lemma 1.2. *Consider the intersection of W and $X^3 \setminus \delta_X$ in $\mathbf{P}^{\times 3} \setminus \delta_{\mathbf{P}}$. The intersection scheme has four irreducible components:*

$$W \cap (X^3 \setminus \delta_X) = V \cup \Delta_{12} \cup \Delta_{13} \cup \Delta_{23}. \quad (7)$$

The intersection along V is transversal, in particular $\dim V = 2n - r + 1$. The intersection along Δ_{ij} is not proper, having excess dimension $r - 1$, but the multiplicity of Δ_{ij} in the intersection scheme is 1, where $1 \leq i < j \leq 3$. In particular, the intersection scheme is reduced and the above identity (7) also holds scheme-theoretically:

$$\begin{array}{ccc} V \cup \Delta_{12} \cup \Delta_{13} \cup \Delta_{23} & \longrightarrow & X^3 \setminus \delta_X \\ \downarrow & \square & \downarrow \\ W & \longrightarrow & \mathbf{P}^{\times 3} \setminus \delta_{\mathbf{P}} \end{array}$$

Proof. It is obvious that (7) holds set-theoretically. To verify (7) scheme-theoretically, let

$$W_o := \{(y_1, y_2, y_3) \in \mathbf{P}^{\times 3} \mid y_1, y_2, y_3 \text{ are collinear and distinct}\}.$$

Consider the incidence variety

$$I := \{([f], (y_1, y_2, y_3)) \in \mathbf{P}(H^0(\mathbf{P}^{n+r}, E)) \times W_o \mid f(y_1) = f(y_2) = f(y_3) = 0\}.$$

Let $q : I \rightarrow \mathbf{P}(H^0(\mathbf{P}^{n+r}, E))$ and $p : I \rightarrow W_o$ be the two natural projections. The positivity assumption (*) means precisely that p is a $\mathbf{P}^{h^0(\mathbf{P}, E) - 1 - 3r}$ -bundle, therefore I is smooth of dimension $h^0(\mathbf{P}, E) + 2n - r$. Since for general f , the corresponding variety X contains a line (Lemma 1.1), q is of dominant. By the theorem of generic smoothness, the fiber of q , which is exactly V_o , is smooth of dimension $2n - r + 1$. In particular, V_o is reduced, of locally complete intersection in W_o of codimension $3r$ as expected. In other words, the intersection is transversal along a general point³ of V . The assertions concerning the big diagonals are easier: by passing to a general point of Δ_{ij} , it amounts to prove that the intersection scheme of $\Delta_{\mathbf{P}}$ and X^2 in $\mathbf{P}^{\times 2}$ is Δ_X with multiplicity 1, (and of excess dimension r). \square

Now we construct a vector bundle F on W . Let S be the tautological rank 2 vector bundle on W , with fiber $S_{y_1 y_2 y_3}$ over a point $(y_1, y_2, y_3) \in W$ the 2-dimensional vector space corresponding to the projective line $\mathbf{P}_{y_1 y_2 y_3}^1$ determined by these three collinear points. Therefore $p : \mathbf{P}(S) \rightarrow W$ is the \mathbf{P}^1 -bundle of universal line, and it admits three tautological sections $\sigma_i : W \rightarrow \mathbf{P}(S)$ determined by the points y_i , where $i = 1, 2, 3$. Let $q : \mathbf{P}(S) \rightarrow \mathbf{P}^{n+r}$ be the natural morphism. We summarize the situation by the following diagram:

$$\begin{array}{ccc} \mathbf{P}(S) & \xrightarrow{q} & \mathbf{P} \\ \sigma_i \uparrow \uparrow \uparrow \downarrow p & & \\ W & & \end{array}$$

³This suffices for the scheme-theoretical assertions concerning V , since we work over the complex numbers, there are enough (closed) points such that any algebraic condition satisfied by a general point is also satisfied by the generic point.

Let D_i be the image of section σ_i , which is a divisor of $\mathbf{P}(S)$, for $i = 1, 2, 3$. We define the following sheaf on W :

$$F := p_*(q^*E \otimes \mathcal{O}_{\mathbf{P}(S)}(-D_1 - D_2 - D_3)) \quad (8)$$

Lemma 1.3. *F is a vector bundle on W of rank $n - r + 1$, with fiber*

$$F_{y_1, y_2, y_3} = H^0(\mathbf{P}^1_{y_1, y_2, y_3}, E|_{\mathbf{P}^1_{y_1, y_2, y_3}} \otimes \mathcal{O}(-y_1 - y_2 - y_3)).$$

Proof. For any $(y_1, y_2, y_3) \in W$, it is obvious that the restriction of the vector bundle $q^*E \otimes \mathcal{O}(-D_1 - D_2 - D_3)$ to the fiber $p^{-1}(y_1, y_2, y_3) =: \mathbf{P}^1_{y_1, y_2, y_3}$ is exactly $E|_{\mathbf{P}^1_{y_1, y_2, y_3}} \otimes \mathcal{O}(-y_1 - y_2 - y_3)$. By the positivity assumption (*), the splitting type of E at $\mathbf{P}^1_{y_1, y_2, y_3}$ is $\oplus_{i=1}^r \mathcal{O}(a_i)$ with $a_1 \geq a_2 \geq \dots \geq a_r \geq 2$, we find that

$$h^0\left(\mathbf{P}^1_{y_1, y_2, y_3}, E|_{\mathbf{P}^1_{y_1, y_2, y_3}} \otimes \mathcal{O}(-y_1 - y_2 - y_3)\right) = h^0\left(\mathbf{P}^1, \oplus_{i=1}^r \mathcal{O}(a_i - 3)\right) = \sum_{i=1}^r (a_i - 2) = d - 2r = n - r + 1,$$

which is independent of the point (y_1, y_2, y_3) in W . Now the lemma is a consequence of Grauert's base-change theorem. \square

Here is the motivation to introduce the vector bundle F : the section $f \in H^0(\mathbf{P}, E)$ gives rise to a section $s_f \in H^0(V, F|_V)$ in the way that for any $(x_1, x_2, x_3) \in V$ the value $s_f(x_1, x_2, x_3)$ is simply given by $f|_{\mathbf{P}^1_{x_1, x_2, x_3}} \in F_{x_1, x_2, x_3} = H^0(\mathbf{P}^1_{x_1, x_2, x_3}, E|_{\mathbf{P}^1_{x_1, x_2, x_3}} \otimes \mathcal{O}(-x_1 - x_2 - x_3))$ (see Lemma 1.3), because f vanishes on x_i by definition. As a result,

Lemma 1.4. $c_{n-r+1}(F|_V) = \Gamma_o \in \text{CH}_n(X^3 \setminus \delta_X)$, where Γ_o is the restriction of the variety Γ constructed in (6) to the open subset $X^3 \setminus \delta_X$.

Proof. By construction, Γ_o is exactly the zero locus of the section s_f of $F|_V$. By Lemma 1.1, Γ_o is n -dimensional, thus represents the top Chern class of $F|_V$. \square

Now consider the cartesian diagram (Lemma 1.2):

$$\begin{array}{ccc} V \cup \Delta_{12} \cup \Delta_{13} \cup \Delta_{23} & \xrightarrow{i_2} & X^3 \setminus \delta_X \\ \downarrow i_4 & \square & \downarrow i_1 \\ W & \xrightarrow{i_3} & \mathbf{P}^{\times 3} \setminus \delta_{\mathbf{P}} \end{array} \quad (9)$$

Since i_1 is clearly a regular embedding, we can apply the theory of refined Gysin maps of [8] to the cycle $c_{n-r+1}(F) \in \text{CH}^{n-r+1}(W)$ in the above diagram. Before doing so, recall that in Lemma 1.2 we have observed that the intersections along Δ_{ij} 's are not proper. Let us first calculate the excess normal bundles of them.

Lemma 1.5. *For any $1 \leq i < j \leq 3$, the excess normal sheaf along $\Delta_{ij} \setminus V$ is a rank $r - 1$ vector bundle isomorphic to a quotient $\frac{\text{pr}_1^* E|_X}{\text{pr}_1^* \mathcal{O}_X(1) \otimes \text{pr}_2^* \mathcal{O}_X(-1)}$, where we identify Δ_{ij} with $X \times X \setminus \Delta_X$, and pr_i are the natural projections to two factors.*

Proof. For simplicity, assume $i = 1, j = 2$, and write the inclusion $j : \Delta_{12} = X \times X \setminus \Delta_X \hookrightarrow X^3 \setminus \delta_X$, which sends (x, x') to (x, x, x') , where $x \neq x'$. Now we are in the following situation:

$$\begin{array}{ccc}
X \times X \setminus \Delta_X & \xrightarrow{j} & X^3 \setminus \delta_X \\
\downarrow & & \downarrow i_1 \\
\mathbf{P}^{\times 2} \setminus \Delta_{\mathbf{P}} & \xrightarrow{i''_3} & \mathbf{P}^{\times 3} \setminus \delta_{\mathbf{P}} \\
\downarrow i'_3 & & \downarrow \\
W & \xrightarrow{i_3} & \mathbf{P}^{\times 3} \setminus \delta_{\mathbf{P}}
\end{array}$$

The normal bundle of j is obviously $\text{pr}_1^* TX$. And the normal bundle of i_3 sits in the exact sequence:

$$0 \rightarrow N_{i'_3} \rightarrow N_{i''_3} \rightarrow N_{i_3} \rightarrow 0.$$

The normal bundle of i''_3 is $\text{pr}_1^* T\mathbf{P}$. As for the normal bundle of i'_3 , let us reinterpret i'_3 as:

$$\begin{array}{ccc}
L \times_G L \setminus \Delta_L & \xrightarrow{i'_3} & L \times_G L \times_G L \setminus \delta_L \\
& \searrow & \swarrow \\
& & G
\end{array}$$

where $L \rightarrow G$ is the universal \mathbf{P}^1 -fibration over the Grassmannian of projective lines G . From this we see that the normal bundle of i'_3 is the same as the quotient of the two relative (over G) tangent sheaves, thus the fiber of $N_{i'_3}$ at $(y, y') \in \mathbf{P}^{\times 2} \setminus \Delta_{\mathbf{P}}$ is canonically isomorphic to $T_y \mathbf{P}^1_{yy'}$. Therefore at the point $(x, x') \in X \times X \setminus \Delta_X$, the fiber of the excess normal bundle is canonically isomorphic to

$$\frac{N_{i''_3, (x, x')}}{N_{i'_3, (x, x')} + N_{j, (x, x')}} = \frac{T_x \mathbf{P}}{T_x \mathbf{P}^1_{xx'} + T_x X}.$$

As long as the line $\mathbf{P}^1_{xx'}$ is not tangent to X at x , i.e. $(x, x') \notin V$, the sum in the denominator is a direct sum, and the fiber of the excess bundle at this point is canonically isomorphic to the $(r-1)$ -dimensional vector space

$$\frac{N_{X/\mathbf{P}, x}}{T_x \mathbf{P}^1_{xx'}} = \frac{E_x}{\text{Hom}_{\mathbf{C}}(\mathbf{C}\dot{x}, \mathbf{C}\dot{x}')},$$

where $\mathbf{C}\dot{x}$ is the 1-dimensional sub-vector space corresponding to $x \in \mathbf{P}$. Therefore along $\Delta_{ij} \setminus V$, the excess normal bundle is isomorphic to $\frac{\text{pr}_1^* E|_X}{\text{pr}_1^* \mathcal{O}_X(1) \otimes \text{pr}_2^* \mathcal{O}_X(-1)}$. \square

Now we consider the Gysin map $i_1^!$ in the diagram (9) to get the following.

Proposition 1.6. *There exists a symmetric homogeneous polynomial P of degree $2n$ with integer coefficients, such that in $\text{CH}_n(X^3 \setminus \delta_X)$,*

$$c_{n-r+1}(F|_V) + j_{12*}(\alpha) + j_{13*}(\alpha) + j_{23*}(\alpha) + P(h_1, h_2, h_3) = 0, \quad (10)$$

where $h_i = \text{pr}_i^*(h) \in \text{CH}^1(X^3 \setminus \delta_X)$ with $h = c_1(\mathcal{O}_X(1))$, $i = 1, 2, 3$; the cycle α is defined by

$$\alpha = c_{n-r+1}(F|_{\Delta_{12}}) \cdot c_{r-1} \left(\frac{\text{pr}_1^* E|_X}{\text{pr}_1^* \mathcal{O}_X(1) \otimes \text{pr}_2^* \mathcal{O}_X(-1)} \right) \in \text{CH}_n(X^2 \setminus \Delta_X); \quad (11)$$

and the morphisms $j_{12}, j_{13}, j_{23} : X^2 \setminus \Delta_X \hookrightarrow X^3 \setminus \delta_X$ are defined by

$$\begin{aligned} j_{12} &: (x, x') \mapsto (x, x, x'); \\ j_{13} &: (x, x') \mapsto (x, x', x); \\ j_{23} &: (x, x') \mapsto (x', x, x). \end{aligned}$$

Proof. By the commutativity of Gysin map and push-forwards ([8] Theorem 6.2(a)):

$$i_{2*} \left(i_1^! c_{n-r+1}(F) \right) = i_1^! \left(i_{3*} c_{n-r+1}(F) \right) \text{ in } \text{CH}_n(X^3 \setminus \delta_X). \quad (12)$$

While in the right hand side, $i_{3*} c_{n-r+1}(F) \in \text{CH}_{n+3r}(\mathbf{P}^{\times 3} \setminus \delta_{\mathbf{P}}) \simeq \text{CH}_{n+3r}(\mathbf{P}^{\times 3})$, and the Chow ring of $\mathbf{P}^{\times 3}$ is well-known:

$$\text{CH}^*(\mathbf{P}^{\times 3}) = \mathbf{Z}[H_1, H_2, H_3] / (H_i^{n+r+1}; i = 1, 2, 3),$$

where $H_i = \text{pr}_i^*(H)$ with $H \in \text{CH}^1(\mathbf{P})$ being the hyperplane section class. Hence there exists a symmetric homogeneous polynomial P of degree $2n$ with integer coefficients, such that

$$i_{3*} c_{n-r+1}(F) = -P(H_1, H_2, H_3) \text{ in } \text{CH}_{n+3r}(\mathbf{P}^{\times 3} \setminus \delta_{\mathbf{P}}).$$

Combining this with (12), and denoting $h_i = H_i|_{X^3} \in \text{CH}^1(X^3)$, we obtain the following equality

$$i_{2*} \left(i_1^! c_{n-r+1}(F) \right) + P(h_1, h_2, h_3) = 0 \text{ in } \text{CH}_n(X^3 \setminus \delta_X). \quad (13)$$

In the left hand side, by [8] Proposition 6.3, we have:

$$i_1^! c_{n-r+1}(F) = i_1^! (c_{n-r+1}(F) \cdot [W]) = c_{n-r+1}(F|_{V \cup \Delta_{12} \cup \Delta_{13} \cup \Delta_{23}}) \cdot i_1^!([W]), \quad (14)$$

where $[W]$ is the fundamental class of W . Note here $i_1^!([W])$ is a $(2n - r + 1)$ -dimensional cycle, but $V \cap \Delta_{ij}$ is of dimension strictly less than $2n - r + 1$, thus we can use the excess intersection formula ([8] §6.3) component by component in the open subsets $\Delta_{ij} \setminus V$ to get (the excess normal bundle is given in Lemma 1.5):

$$i_1^!([W]) = [V] + \sum_{1 \leq i < j \leq 3} [\Delta_{ij}] \cdot c_{r-1} \left(\frac{\text{pr}_1^* E|_X}{\text{pr}_1^* \mathcal{O}_X(1) \otimes \text{pr}_2^* \mathcal{O}_X(-1)} \right).$$

Therefore, by omitting all the push-forwards induced by inclusions of subvarieties of $X^3 \setminus \delta_X$,

$$i_{2*} \left(c_{n-r+1}(F|_V) \cdot i_1^!([W]) \right) = c_{n-r+1}(F|_V);$$

$$i_{2*} \left(c_{n-r+1}(F|_{\Delta_{12}}) \cdot i_1^!([W]) \right) = c_{n-r+1}(F|_{\Delta_{12}}) \cdot c_{r-1} \left(\frac{\text{pr}_1^* E|_X}{\text{pr}_1^* \mathcal{O}_X(1) \otimes \text{pr}_2^* \mathcal{O}_X(-1)} \right),$$

putting these in (13) and (14) we get the desired formula. \square

Let us now deal with the equality (10) term by term. Firstly, by Lemma 1.4, $c_{n-r+1}(F|_V) = \Gamma_o$ in $\text{CH}_n(X^3 \setminus \delta_X)$. Secondly, we would like to calculate $F|_{\Delta_{12}}$ of Proposition 1.6. We remark that this bundle is the pull-back by the inclusion $X^2 \setminus \Delta_X \hookrightarrow \mathbf{P}^{\times 2} \setminus \Delta_{\mathbf{P}}$ of the bundle

$$M := F|_{\Delta_{12, \mathbf{P}}}$$

where $\Delta_{12, \mathbf{P}} = \{(y, y, y') \in \mathbf{P}^{\times 3} \mid y \neq y'\} \subset W$, and we identify $\Delta_{12, \mathbf{P}}$ with $\mathbf{P}^{\times 2} \setminus \Delta_{\mathbf{P}}$.

We still use S to denote the tautological rank 2 vector bundle on $\mathbf{P}^{\times 2} \setminus \Delta_{\mathbf{P}}$, hence $p : \mathbf{P}(S) \rightarrow \mathbf{P}^{\times 2} \setminus \Delta_{\mathbf{P}}$ is the universal line, which admits two tautological sections σ, σ' , and we call $q : \mathbf{P}(S) \rightarrow \mathbf{P}$ the natural morphism:

$$\begin{array}{ccc} \mathbf{P}(S) & \xrightarrow{q} & \mathbf{P} \\ \left\{ \begin{array}{c} \uparrow \sigma \\ \uparrow \sigma' \end{array} \right. & \downarrow p & \\ & & \mathbf{P}^{\times 2} \setminus \Delta_{\mathbf{P}} \end{array} \quad (15)$$

Lemma 1.7. *Notations as in the diagram (15) above, then $M \simeq (\mathcal{O}_{\mathbf{P}}(1) \boxtimes \mathcal{O}_{\mathbf{P}}(2)) \otimes p_*(q^*E(-3))$.*

Proof. By construction (or see (8)),

$$M = p_*(q^*E \otimes \mathcal{O}_{\mathbf{P}(S)}(-2D - D')), \quad (16)$$

where D, D' is the images of the sections σ, σ' . Since p is a projective bundle and the intersection number of D with the fiber is 1, we can assume that $\mathcal{O}_{\mathbf{P}(S)}(-D) = p^*(\mathcal{O}_{\mathbf{P}}(a) \boxtimes \mathcal{O}_{\mathbf{P}}(b)) \otimes \mathcal{O}_{\mathbf{P}(S)}(-1)$. Pushing forward by p_* the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbf{P}(S)}(-D) \otimes \mathcal{O}_{\mathbf{P}(S)}(1) \rightarrow \mathcal{O}_{\mathbf{P}(S)}(1) \rightarrow \mathcal{O}_{\mathbf{P}(S)}(1)|_D \rightarrow 0,$$

we find an exact sequence:

$$0 \rightarrow \mathcal{O}_{\mathbf{P}}(a) \boxtimes \mathcal{O}_{\mathbf{P}}(b) \rightarrow S^\vee \rightarrow \mathcal{O}_{\mathbf{P}}(1) \boxtimes \mathcal{O}_{\mathbf{P}} \rightarrow 0,$$

where the last term comes from the fact that $p_*(\mathcal{O}_{\mathbf{P}(S)}(1)|_D) = \sigma^*(\mathcal{O}_{\mathbf{P}(S)}(1))$ whose fiber at (y, y') is $(\mathbf{C}y)^*$. Now noting $S^\vee = \text{pr}_1^* \mathcal{O}_{\mathbf{P}}(1) \oplus \text{pr}_2^* \mathcal{O}_{\mathbf{P}}(1)$, and restricting to $\mathbf{P} \times \{\text{pt}\}$ and $\{\text{pt}\} \times \mathbf{P}$, we get $a = 0, b = 1$, i.e.

$$\mathcal{O}_{\mathbf{P}(S)}(-D) = p^*(\text{pr}_2^* \mathcal{O}_{\mathbf{P}}(1)) \otimes \mathcal{O}_{\mathbf{P}(S)}(-1).$$

Similarly, $\mathcal{O}_{\mathbf{P}(S)}(-D') = p^*(\text{pr}_1^* \mathcal{O}_{\mathbf{P}}(1)) \otimes \mathcal{O}_{\mathbf{P}(S)}(-1)$. Putting these into (16), the projection formula finishes the proof of Lemma. \square

Combining Proposition 1.6, Lemma 1.4, Lemma 1.7 and the localization exact sequence

$$\text{CH}_n(X) \xrightarrow{\delta_*} \text{CH}_n(X^3) \rightarrow \text{CH}_n(X^3 \setminus \delta_X) \rightarrow 0,$$

we deduce the following decomposition of the small diagonal:

Corollary 1.8. *There exists an integer N , such that in $\text{CH}_n(X^3)$:*

$$N \cdot \delta_X = \Gamma + j_{12*}(Q(h_1, h_2)) + j_{13*}(Q(h_1, h_2)) + j_{23*}(Q(h_1, h_2)) + P(h_1, h_2, h_3), \quad (17)$$

where Γ is as in (6), P and Q are homogeneous polynomials with integer coefficients of degree $2n$ and n respectively; $h_i \in \text{CH}^1(X^3)$ or $\text{CH}^1(X^2)$ is the pull-back of $h = c_1(\mathcal{O}_X(1))$ by the i^{th} projection; the inclusions $j_{12}, j_{13}, j_{23} : X^2 \hookrightarrow X^3$ are defined by the same formula in Proposition 1.6.

For later use, we note that more precisely, P is symmetric and Q is determined by⁴:

$$Q(H_1, H_2) = c_{n-r+1}(M) \cdot c_{r-1} \left(\frac{\text{pr}_1^* E}{\text{pr}_1^* \mathcal{O}_{\mathbf{P}}(1) \otimes \text{pr}_2^* \mathcal{O}_{\mathbf{P}}(-1)} \right) \in \text{CH}^n(\mathbf{P}^{\times 2} \setminus \Delta_{\mathbf{P}}) \simeq \text{CH}^n(\mathbf{P}^{\times 2}); \quad (18)$$

where $M \simeq (\mathcal{O}_{\mathbf{P}}(1) \boxtimes \mathcal{O}_{\mathbf{P}}(2)) \otimes p_*(q^*E(-3))$ as in Lemma 1.7.

In the equation of Corollary 1.8 above, we make the following observation of relations between N and the coefficients of P and Q by using the *non-existence* of decomposition of diagonal $\Delta_X \subset X \times X$ in the sense of Bloch-Srinivas (see Theorem 0.1 of the introduction) for smooth projective varieties with $H^{n,0} \neq 0$, for example varieties of Calabi-Yau type.

Lemma 1.9. *Write the \mathbf{Z} -coefficient polynomials*

$$P(H_1, H_2, H_3) = \sum_{\substack{i+j+k=2n \\ i,j,k \geq 0}} b_{ijk} H_1^i H_2^j H_3^k \quad \text{with } b_{ijk} \text{ symmetric on the indexes;}$$

$$Q(H_1, H_2) = a_n H_1^n + a_{n-1} H_1^{n-1} H_2 + \cdots + a_0 H_2^n.$$

Then we have

$$N = a_0 \cdot \deg(X); \quad (19)$$

$$a_i + a_j = -b_{ijn} \cdot \deg(X) \quad \text{for any } i + j = n. \quad (20)$$

Here the degree of X is given by $\deg(X) = \left(\underbrace{h \cdot h \cdots h}_n \right)_X$, which is in fact the top Chern number of E .

Proof. Applying to the equation in Corollary 1.8 the push-forward induced by the projection to the first two factors $\text{pr}_{12} : X^3 \rightarrow X^2$, we obtain that in $\text{CH}_n(X^2)$,

$$(N - a_0 \cdot \deg(X)) \cdot \Delta_X = \sum_{i+j=n} (b_{ijn} \deg(X) + a_i + a_j) \cdot h_1^i h_2^j. \quad (21)$$

Here the push-forward of Γ vanishes since Γ has relative dimension 1 for pr_{12} . If $N - a_0 \cdot \deg(X) \neq 0$, then (21) gives a nontrivial decomposition of diagonal $\Delta_X \in \text{CH}_n(X^2)$ of Bloch-Srinivas type, but this is impossible: we regard (21) as an equality of cohomological correspondences from $H^n(X)$ to itself, then the left hand side acts by multiplying a non-zero constant $(N - a_0 \cdot \deg(X))$, while the action of the right hand side has image a sub-Hodge structure of coniveau at least 1, which contradicts to the non-vanishing of $H^{n,0}(X)$. As a result, we have (19), and hence (20) since $\{h_1^i h_2^j\}_{i+j=n}$ are linearly independent. \square

Using Lemma 1.9, we obtain the following improved version of Corollary 1.8.

Theorem 1.10. *Let \mathbf{P}, E, X be as in the basic setting. Then in $\text{CH}_n(X^3)$ we have:*

$$a_0 \deg(X) \cdot \delta_X = \Gamma + j_{12*}(Q(h_1, h_2)) + j_{13*}(Q(h_1, h_2)) + j_{23*}(Q(h_1, h_2)) + P(h_1, h_2, h_3), \quad (22)$$

⁴Here $\frac{\text{pr}_1^* E}{\text{pr}_1^* \mathcal{O}_{\mathbf{P}}(1) \otimes \text{pr}_2^* \mathcal{O}_{\mathbf{P}}(-1)}$ is no more a quotient vector bundle, but only an element in the Grothendieck group of vector bundles on $\mathbf{P}^{\times 2}$ on which the Chern classes are however still well-defined.

where Γ is defined in (6) and $Q(H_1, H_2) = a_n H_1^n + a_{n-1} H_1^{n-1} H_2 + \cdots + a_0 H_2^n$ is the homogeneous polynomial with \mathbf{Z} -coefficients determined by (18), and P is the symmetric polynomial with \mathbf{Z} -coefficients

$$P(H_1, H_2, H_3) = \sum_{i+j+k=2n} b_{ijk} H_1^i H_2^j H_3^k,$$

with (20): $a_i + a_j = -b_{ijn} \cdot \deg(X)$ for any $i + j = n$; and $h_i \in \text{CH}^1(X^3)$ or $\text{CH}^1(X^2)$ is the pull-back of $h = c_1(\mathcal{O}_X(1))$ by the i^{th} projection, and $j_{12}, j_{13}, j_{23} : X^2 \hookrightarrow X^3$ are the inclusions of big diagonals as before.

This theorem generalizes the result of Voisin [21] for Calabi-Yau hypersurfaces (see Theorem 0.5 of the introduction) except for a small point: to get a non-trivial decomposition of the small diagonal and thus applications like Corollary 0.6, we need to verify that $a_0 \neq 0$. It is the case when E is splitting, *i.e.* when X is Calabi-Yau complete intersection:

1.2. Splitting case: Calabi-Yau complete intersections

In this subsection, we deal with the case that E is of splitting type:

$$E = \bigoplus_{i=1}^r \mathcal{O}_{\mathbf{P}}(d_i),$$

where $d_1 \geq d_2 \geq \cdots \geq d_r \geq 2$, with the Calabi-Yau condition $d = \sum_{i=1}^r d_i = n + r + 1$. Hence X is a smooth Calabi-Yau complete intersection of multi-degree (d_1, \dots, d_r) . In particular, $\deg(X) = \prod_{i=1}^r d_i$.

Lemma 1.11. *If E is of splitting type as above, then*

1. *The vector bundle M in Lemma 1.7 is isomorphic to the restriction to $\mathbf{P}^{\times 2} \setminus \Delta_{\mathbf{P}}$ of*

$$\bigoplus_{i=1}^r \bigoplus_{j=1}^{d_i-2} \mathcal{O}_{\mathbf{P}}(j) \boxtimes \mathcal{O}_{\mathbf{P}}(d_i - j).$$

In particular in (18),

$$c_{n-r+1}(M) = \prod_{i=1}^r \prod_{j=1}^{d_i-2} (jH_1 + (d_i - j)H_2). \quad (23)$$

2. *In (18), $c_{r-1}\left(\frac{\text{pr}_1^* E}{\text{pr}_1^* \mathcal{O}_{\mathbf{P}(1)} \otimes \text{pr}_2^* \mathcal{O}_{\mathbf{P}(-1)}}\right)$ is the degree $(r-1)$ part of the formal series*

$$\frac{\prod_{i=1}^r (1 + d_i H_1)}{1 - (H_2 - H_1)}.$$

3. *The coefficient of H_2^n in the polynomial Q is given by $a_0 = \prod_{i=1}^r ((d_i - 1)!)$.*
4. *The coefficient of $H_1 H_2^{n-1}$ in the polynomial Q is given by*

$$a_1 = \left(\prod_{i=1}^r (d_i - 1)! \right) \cdot \left(\left(\sum_{i=1}^r \sum_{j=1}^{d_i-2} \frac{j}{d_i - j} \right) + n + 2 \right),$$

in particular $a_1 \neq a_0$.

Proof. In the situation as in diagram (15),

$$\begin{aligned}
p_*(q^*E(-3)) &= \bigoplus_{i=1}^r p_* q^* \mathcal{O}_{\mathbf{P}}(d_i - 3) \\
&= \bigoplus_{i=1}^r p_* \mathcal{O}_{\mathbf{P}(S)}(d_i - 3) && \text{(since } q^* \mathcal{O}_{\mathbf{P}}(1) = \mathcal{O}_{\mathbf{P}(S)}(1)\text{)} \\
&= \bigoplus_{i=1}^r \text{Sym}^{d_i-3} S^\vee && (d_i - 3 \geq -1, \text{ define } \text{Sym}^0 = \mathcal{O}, \text{Sym}^{-1} = 0) \\
&= \bigoplus_{i=1}^r \bigoplus_{j=0}^{d_i-3} (\mathcal{O}_{\mathbf{P}}(j) \boxtimes \mathcal{O}_{\mathbf{P}}(d_i - 3 - j)) && \text{(recall } S \simeq \text{pr}_1^* \mathcal{O}_{\mathbf{P}}(-1) \oplus \text{pr}_2^* \mathcal{O}_{\mathbf{P}}(-1)\text{)}
\end{aligned}$$

Thus $M \simeq (\mathcal{O}_{\mathbf{P}}(1) \boxtimes \mathcal{O}_{\mathbf{P}}(2)) \otimes p_*(q^*E(-3)) = \bigoplus_{i=1}^r \bigoplus_{j=1}^{d_i-2} (\mathcal{O}_{\mathbf{P}}(j) \boxtimes \mathcal{O}_{\mathbf{P}}(d_i - j))$, and the top Chern class follows immediately.

The second point is obvious. As for the coefficient a_0 , by (18) it is the product of the coefficient of H_2^{n-r+1} in $c_{n-r+1}(M)$ and the coefficient of H_2^{r-1} in $c_{r-1}\left(\frac{\text{pr}_1^* E}{\text{pr}_1^* \mathcal{O}_{\mathbf{P}}(1) \otimes \text{pr}_2^* \mathcal{O}_{\mathbf{P}}(-1)}\right)$, which are $\prod_{i=1}^r (d_i - 1)!$ and 1 respectively, by the first two parts of this lemma. Remembering $(\sum_{i=1}^r d_i) - r + 1 = n + 2$, the calculation for a_1 is also straightforward. \square

As a result, in this complete intersection case the decomposition Theorem 1.10 reads as following:

Theorem 1.12. *Let X be a general Calabi-Yau complete intersection in a projective space, then in $\text{CH}_n(X^3)$ we have a decomposition of the small diagonal:*

$$\left(\prod_{i=1}^r (d_i!) \right) \cdot \delta_X = \Gamma + j_{12*}(Q(h_1, h_2)) + j_{13*}(Q(h_1, h_2)) + j_{23*}(Q(h_1, h_2)) + P(h_1, h_2, h_3), \quad (24)$$

where the notations are as in Theorem 1.10.

1.3. Applications to the multiplicative structure of Chow rings

We continue dealing with the Calabi-Yau complete intersection case of the previous subsection. Now we can regard (24) as an equality of correspondences from $X \times X$ to X . Combining with (20), we get the following result in the same spirit of Corollary 0.6 of the introduction.

Theorem 1.13. *Let X be a general Calabi-Yau complete intersection in a projective space, then for any strictly positive integers $k, l \in \mathbf{N}^*$, with $k + l = n$.*

$$\text{Im}(\bullet : \text{CH}^k(X)_{\mathbf{Q}} \times \text{CH}^l(X)_{\mathbf{Q}} \rightarrow \text{CH}_0(X)_{\mathbf{Q}}) = \mathbf{Q} \cdot h^n,$$

where $h = c_1(\mathcal{O}_X(1)) \in \text{CH}^1(X)$.

Proof. Let $Z \in \text{CH}^k(X), Z' \in \text{CH}^l(X)$ be two algebraic cycles of X of codimension $k, l \in \mathbf{N}$ with $k + l = n$. We apply the equality of correspondences in (24) to the cycle $Z \times Z' \in \text{CH}^n(X \times X)$. Remembering (20), we have the following equality in $\text{CH}_0(X)$, here \bullet means the intersection product in $\text{CH}^*(X)$:

$$\left(\prod_{i=1}^r (d_i!) \right) \cdot Z \bullet Z' = \left(\prod_{i=1}^r (d_i!) \right) \frac{\deg(Z \bullet Z')}{\deg(X)} \cdot h^n + a_l \deg(Z') \cdot Z \bullet h^l + a_k \deg(Z) \cdot Z' \bullet h^k - (a_k + a_l) \frac{\deg(Z) \deg(Z')}{\deg(X)} \cdot h^n, \quad (25)$$

where the *degree* of an algebraic cycle \mathcal{Z} is defined to be the intersection number $(\mathcal{Z} \cdot h^{\dim \mathcal{Z}})_X$. Here $\Gamma_*(Z \times Z') = 0$ by dimension reason.

To simplify (25) further, we make the following observation: given any $\mathcal{Z} \in \mathrm{CH}^k(X)_{\mathbf{Q}}$, replacing Z by $\mathcal{Z} \bullet h^{n-k-1}$ and Z' by h in (25), we get:

$$(a_0 - a_1) \deg(X) \cdot \left(Z \bullet h^{n-k} - \frac{\deg(\mathcal{Z})}{\deg(X)} h^n \right) = 0.$$

By the last point of Lemma 1.11, $a_0 \neq a_1$, so we can divide out $(a_0 - a_1) \deg(X)$, obtaining that for any $\mathcal{Z} \in \mathrm{CH}^k(X)_{\mathbf{Q}}$, $\mathcal{Z} \bullet h^{n-k} = \frac{\deg(\mathcal{Z})}{\deg(X)} h^n \in \mathrm{CH}_0(X)_{\mathbf{Q}}$.

Therefore, the last three terms of (25) simplify with each other, and the proof is complete. \square

Remark 1.14. Theorem 1.13 can be reformulated as: for a general Calabi-Yau complete intersection in a projective space, any *decomposable* 0-cycle with \mathbf{Q} -coefficient is \mathbf{Q} -rational equivalent to zero if and only if it has degree 0. Here a 0-cycle is called *decomposable* if it is in the sum of the images:

$$\sum_{\substack{k+l=n \\ k,l>0}} \mathrm{Im} \left(\mathrm{CH}^k(X)_{\mathbf{Q}} \times \mathrm{CH}^l(X)_{\mathbf{Q}} \xrightarrow{\bullet} \mathrm{CH}_0(X)_{\mathbf{Q}} \right).$$

It is interesting to compare this result for Calabi-Yau varieties with Beauville's '*weak splitting principle*' conjecture in the holomorphic symplectic case. In [3], Beauville and Voisin reinterpreted their result (see Corollary 0.4 in §0 Introduction) as some sort of compatibility with splitting of the conjectural Bloch-Beilinson filtration. In [2], Beauville proposed (and checked some examples of) a weak form of such compatibility for higher dimensional irreducible holomorphic symplectic varieties to check, namely his '*weak splitting property*' conjecture. Later in [20], Voisin formulates a stronger version of this conjecture: *for an irreducible holomorphic symplectic projective variety, any polynomial relation between the cohomological Chern classes of lines bundles and the tangent bundle holds already for their Chow-theoretical Chern classes*. She also proved this conjecture for the variety of lines in a cubic four-fold and for Hilbert schemes of points on K3 surfaces in certain range (cf. [20]).

Note that the Chern classes of the tangent bundle of a complete intersection is given by

$$c(T_X) = \left(\frac{c(T_{\mathbf{P}})}{c(E)} \right) \Big|_X,$$

which is clearly a cycle coming from the ambient projective space. Therefore comparing to the '*weak splitting principle*' for holomorphic symplectic varieties, our result for Calabi-Yau varieties is on the one hand stronger, in the sense that besides the divisors and Chern classes of the tangent bundle, cycles of all strictly positive codimension are allowed in the polynomial; and on the other hand weaker since only the polynomials of (weighted) degree n are taken into account.

Remark 1.15 (non-splitting case). In the author's upcoming thesis, the general case where E is not necessarily splitting is discussed. In the previous proof, everything goes through in general except the verification of $a_0 \neq 0$ and $a_0 \neq a_1$. However these two conditions are numerical conditions on the Chern numbers of E , which are presumably implied by our positivity assumption.

1.4. A contrasting example

The results of Corollary 0.4 ([3]), Corollary 0.6 ([21]), and the generalization Theorem 1.13 in this paper suggest the following

Question.: To what extent such degeneracy of intersection products in Chow ring can be generalized to other smooth projective varieties?

In this subsection, we construct a contrasting example, showing that in the results above the Calabi-Yau condition is essential, while the complete intersection assumption is not sufficient. More precisely, we will construct a *smooth surface* S in \mathbf{P}^3 which is of general type, such that the image of the intersection product map

$$\bullet : \text{Pic}(S) \times \text{Pic}(S) \rightarrow \text{CH}_0(S)_{\mathbf{Q}}$$

has some elements not proportional to h^2 , where $h = c_1(\mathcal{O}_S(1))$.

Let Σ be a general smooth quartic surface in $\mathbf{P}^3 =: \mathbf{P}$. Thus Σ is a K3 surface. Let us denote by $V_{k,g}$ its Severi variety:

$$V_{k,g} := \overline{\{C \in |\mathcal{O}_{\Sigma}(k)| : C \text{ is irreducible and nodal with } g(\tilde{C}) = g\}},$$

where \tilde{C} means the normalization of C , and the closure is taken in $|\mathcal{O}_{\Sigma}(k)|$. One knows that $V_{k,g}$ is smooth non-empty of dimension g if $0 \leq g \leq 2k^2 + 1$, whose general member has $\delta := 2k^2 + 1 - g$ nodes (cf. [14] [6]).

In particular, $V_{3,2}$ is of dimension 2, and its general member has 17 nodes. We first show that the nodes of the irreducible curves parameterized by $V_{3,2}$ sweep out a 2-dimensional part of Σ . Indeed, consider general members $C_1 \in V_{1,1}$ and $C_2 \in V_{2,1}$, then C_1, C_2 are both irreducible, of normalization genus 1 and they intersect transversally at 8 points. Note that any morphism from an elliptic curve to K3 surface $f : E \rightarrow \Sigma$ with nodal image, the normal bundle of f is $\frac{f^*T_{\Sigma}}{T_E}$, which is clearly trivial. Therefore C_1 and C_2 both vary in a 1-dimensional family with intersection points running over a 2-dimensional part of Σ , and thus the unions $C_1 \cup C_2$ give a 2-dimensional family of (reducible) curves in $|\mathcal{O}_{\Sigma}(3)|$ with 18 nodes, where at least one node (in $C_1 \cap C_2$) sweeps out a 2-dimensional part of Σ . Keeping this node, we smoothify another node we get a 2-dimensional family of irreducible curves (with 17 nodes) in $V_{3,2}$ with one node sweeping out a 2-dimensional part as desired.

As $\text{CH}_0(\Sigma)_{\mathbf{Q}}$ is different from \mathbf{Q} and generated by the points of Σ , we can find $C' \in V_{3,2}$ irreducible with 17 nodes N_1, \dots, N_{17} , such that at least one of its nodes has class in $\text{CH}_0(\Sigma)_{\mathbf{Q}}$ different from $c_{\Sigma} := \frac{1}{4}c_1(\mathcal{O}_{\Sigma}(1))^2$. Quite obviously, there exist thus 16 of them, say the first 16, with sum not rationally equivalent to $16c_{\Sigma}$. Since $V_{3,3}$ is also smooth, containing $V_{3,2}$ as a smooth divisor, we can deform C' in $|\mathcal{O}_{\Sigma}(3)|$ by keeping the first 16 nodes, and smoothifying the last one, to get $C \in V_{3,3}$ with the sum $Z := N_1 + \dots + N_{16}$ of its 16 nodes not rationally equivalent to $16c_{\Sigma}$.

Now we take another copy of $\mathbf{P}^3 =: \mathbf{P}'$, and construct a finite cover $\pi : \mathbf{P}' \rightarrow \mathbf{P}$ by taking $[X_0 : X_1 : X_2 : X_3]$ to $[q_0(X) : q_1(X) : q_2(X) : q_3(X)]$ with q_i quadratic polynomials in X_0, \dots, X_3 without base points, the q_i will be given later. We want to have an embedding $\tilde{C} \hookrightarrow \mathbf{P}'$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{C} & \dashrightarrow & \mathbf{P}' \\ n \downarrow & & \downarrow \pi \\ C & \longrightarrow & \mathbf{P} \end{array} \quad (26)$$

We consider the square roots of $n^* \mathcal{O}_C(1)$ in $\text{Jac}(\tilde{C})$, i.e. $L \in \text{Jac}(\tilde{C})$ such that $L^{\otimes 2} \simeq n^* \mathcal{O}_C(1)$. Note that such L is a degree 6 divisor on the genus 3 curve \tilde{C} . We can choose one of these square roots L which is very ample on \tilde{C} . Indeed, if none of them is very ample, then each square root is of the form $\mathcal{O}_{\tilde{C}}(K_{\tilde{C}} + x + y)$

for some $x, y \in \widetilde{C}$. Therefore all the 2-torsion points of $\text{Jac}(\widetilde{C})$ are contained in a translation of the image of $u : \text{Sym}^2 \widetilde{C} \rightarrow \text{Jac}(\widetilde{C})$, which is again a translation of the theta divisor by Poincaré's formula. However, it is known that for a principally polarized abelian variety, any translation of a theta divisor cannot contain all the 2-torsion points, see for example [13] [15] [10].

L being chosen as above, the corresponding embedding $i : \widetilde{C} \hookrightarrow \mathbf{P}' := \mathbf{P}[L]^* \simeq \mathbf{P}^3$, (with $\mathcal{O}_{\widetilde{C}}(1) = L$), induces a morphism $i^* : H^0(\mathbf{P}', \mathcal{O}_{\mathbf{P}'}(2)) \rightarrow H^0(\widetilde{C}, 2L)$. We make the following

Claim(★): for the smoothing $C \in V_{3,3}$ chosen generically, i^* above is an isomorphism.

Since both vector spaces have the same dimension 10, we only need to verify the injectivity of i^* , that is, \widetilde{C} is not contained in a quadric of \mathbf{P}' . Suppose on the contrary that there exists a quadric $Q \subset \mathbf{P}'$ containing \widetilde{C} , then we have:

Lemma 1.16. \widetilde{C} is hyperelliptic.

Proof. If $Q \simeq \mathbf{P}^1 \times \mathbf{P}^1$ is smooth, we denote the class of its two fibers by l_1, l_2 , and assume $\widetilde{C} = a \cdot l_1 + b \cdot l_2$ for $a, b \in \mathbf{Z}$. Since \widetilde{C} is of degree 6 and $\mathcal{O}_Q(1) = \mathcal{O}_{\mathbf{P}^1}(1) \boxtimes \mathcal{O}_{\mathbf{P}^1}(1)$, we have $a + b = 6$. On the other hand, the fact that \widetilde{C} is of genus 3 implies that $(\widetilde{C}^2) + ((-2l_1 - 2l_2) \cdot \widetilde{C}) = 4$, which is equivalent to $2ab - 2(a + b) = 4$. Therefore $a = 2, b = 4$ (or $a = 4, b = 2$), i.e. $\widetilde{C} \in |\mathcal{O}_{\mathbf{P}^1}(2) \boxtimes \mathcal{O}_{\mathbf{P}^1}(4)|$. In particular, the projection to the first (or second) ruling of Q shows that \widetilde{C} is hyperelliptic.

If Q is the projective cone over a conic with the singular point O , then \widetilde{C} must pass through O . Indeed, if $O \notin \widetilde{C}$, then \widetilde{C} is a Cartier divisor of Q . Since $\text{Pic}(Q) = \mathbf{Z} \cdot \mathcal{O}_Q(1)$ and $\deg(\widetilde{C}) = 6$, we find that \widetilde{C} is the (smooth) intersection of Q with a cubic. However, the smooth intersection of a cubic and a quadric in \mathbf{P}^3 should have genus 4 by the adjunction formula. This contradiction shows $O \in \widetilde{C}$. Then (after the blow-up of \widetilde{C} at O), the projection from O to the conic provides a degree 2 morphism from \widetilde{C} to the conic, showing that \widetilde{C} is hyperelliptic.

If Q is the union of two projective planes, then \widetilde{C} is a plane curve of degree 6, but then its genus should be 10 instead of 3, so this case can not happen at all. \square

Since the smoothing C can be chosen in a 3-dimensional family B , and the normalization \widetilde{C} as well as the choice of square root L can also be carried over this base variety B (by shrinking B if necessary), we obtain a 3-dimensional family of hyperelliptic curves mapping to the K3 surface Σ . Their hyperelliptic involutions then yield a family of rational curves on $\Sigma^{[2]}$ parameterized by B , where $\Sigma^{[2]}$ is the Hilbert scheme of 0-dimensional subschemes of length 2 on Σ , which is an irreducible holomorphic symplectic variety (cf. [1]). Namely, we have the following diagram:

$$\begin{array}{ccc} P & \xrightarrow{f} & \Sigma^{[2]} \\ \downarrow & & \\ B & & \end{array}$$

where P is a \mathbf{P}^1 -bundle over B and f is the natural morphism. We now exclude this situation case by case:

- If f is generically finite, or equivalently dominant, then $\Sigma^{[2]}$ would be dominated by a ruled variety, thus uniruled, contradicts to the fact that its canonical bundle is trivial.

- If the image of f is of dimension 3, *i.e.* f dominates a prime divisor D of $\Sigma^{[2]}$. Let \widetilde{D} be a suitable resolution of singularities of D , such that the rationally connected quotient of \widetilde{D} is a morphism $q : \widetilde{D} \rightarrow T$. After some blow-ups of P if necessary, we have the following commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{f} & \widetilde{D} \\ \downarrow & & \downarrow q \\ B & \xrightarrow{\bar{f}} & T \end{array}$$

Firstly, we note that $\dim(T) \leq 2$ since \widetilde{D} is covered by rational curves. Secondly, since the fibers of q are rationally connected, the morphism $q^* : H^{2,0}(T) \rightarrow H^{2,0}(\widetilde{D})$ is surjective. However, let $\sigma \in H^{2,0}(\Sigma^{[2]})$ be the holomorphic symplectic form (*cf.* [1]), then its restriction (more precisely, its pull-back) to \widetilde{D} is non-zero (since $\dim(D) = 3$). Therefore $H^{2,0}(T) \neq 0$, which implies in particular $\dim(T) \geq 2$.

The above argument shows that $\dim(T) = 2$. As a result, the fibers of q are unions of rational curves, and the dimension of the fibers of \bar{f} is at least 1. Therefore f maps some 1-dimensional family of \mathbf{P}^1 into a union of (finitely many) rational curves on $\Sigma^{[2]}$, which implies that all the rational curves in this 1-dimensional family are mapped into a same rational curve, contradicting to the fact that all the rational curves parameterized by B are distinct from each other.

- If the image of f is contained in a surface of $\Sigma^{[2]}$, then a dimension counting shows that through a general point of the surface passes a 1-dimensional family of rational curves, so it is actually a rational surface. As a result, the corresponding points of $\Sigma^{[2]}$ are all equal up to rational equivalence; in other words, the push-forwards of the g_2^1 's of the family of hyperelliptic curves, viewed as elements in $\text{CH}_0(\Sigma)_{\mathbf{Q}}$, are all equal. However, for any hyperelliptic curve \widetilde{C} in this family, let $\iota : \widetilde{C} \rightarrow \Sigma$ be the composition of the normalization n with the natural inclusion, then

$$\iota_* (2g_2^1) = \iota_* (K_{\widetilde{C}}) = \iota_* (\iota^* \mathcal{O}_{\Sigma}(C)|_{\widetilde{C}} - n^*(Z)) = \mathcal{O}_{\Sigma}(3)|_C - 2Z,$$

which is non-constant by our construction, where Z is the sum of the nodes of C . This is a contradiction.

In conclusion, we have proved the claim (★).

Thanks to the isomorphism $i^* : H^0(\mathbf{P}', \mathcal{O}_{\mathbf{P}'}(2)) \rightarrow H^0(\widetilde{C}, 2L)$, we can define the morphism $\pi : \mathbf{P}' \rightarrow \mathbf{P}$, hence also the q_0, \dots, q_3 as promised, by the following commutative diagram:

$$\begin{array}{ccc} H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(1)) & \longrightarrow & H^0(C, \mathcal{O}_C(1)) \\ \downarrow \pi^* & & \downarrow n^* \\ H^0(\mathbf{P}', \mathcal{O}_{\mathbf{P}'}(2)) & \xrightarrow{i^*} & H^0(\widetilde{C}, 2L) \end{array}$$

achieving the commutative diagram (26). Define $S := \pi^{-1}(\Sigma)$, then \widetilde{C} is a curve in S :

$$\begin{array}{ccccc} \widetilde{C} & \longrightarrow & S & \longrightarrow & \mathbf{P}' \\ \downarrow n & & \downarrow p & \square & \downarrow \pi \\ C & \longrightarrow & \Sigma & \longrightarrow & \mathbf{P} \end{array}$$

By the adjunction formula, we have

$$\mathcal{O}_S(\widetilde{C})|_{\widetilde{C}} + K_S|_{\widetilde{C}} = K_{\widetilde{C}} = n^* \left(\mathcal{O}_\Sigma(C)|_C \otimes \mathcal{O}_C(-Z) \right).$$

Note that $\deg(S) = \deg(p) = \deg(\pi) = 8$, hence $K_S = \mathcal{O}_S(4) = p^* \mathcal{O}_\Sigma(2)$. Therefore the above equality implies

$$\mathcal{O}_S(\widetilde{C})|_{\widetilde{C}} = n^* (\mathcal{O}_C(1) \otimes \mathcal{O}_C(-Z)).$$

Pushing this forward to Σ , we deduce that

$$p_*(\widetilde{C}^2) = 12c_\Sigma - 2Z.$$

Since $Z \neq 16c_\Sigma$, \widetilde{C}^2 is not proportional to h^2 as desired, where $h = c_1(\mathcal{O}_S(1))$.

In the above construction, we have not yet verified the smoothness of S . However, as above the smoothing C can be chosen in a 3-dimensional family B , and the construction of \widetilde{C} , L , π and finally S can also be carried over this base B (by shrinking B if necessary). Generically, the family \mathcal{S} is the transverse pull-back

$$\begin{array}{ccc} \mathcal{S} & \longrightarrow & \mathbf{P}' \times B \\ \downarrow p_B & & \downarrow \bar{\pi} \\ \Sigma \times B & \longrightarrow & \mathbf{P} \times B \end{array}$$

which is generically smooth over B , thus a general fiber S is smooth.

2. Higher powers of hypersurfaces of general type

We now turn to the study of the smallest diagonal in higher self-products of a non-Fano hypersurface in the projective space. Our goal is to establish Theorem 0.8 in the introduction.

First of all, let us fix the basic set-up for this section. Let $\mathbf{P} := \mathbf{P}^{n+1}$ be the projective space, and X be a general hypersurface in \mathbf{P} of degree d , with $d \geq n + 2$. Thus X is an n -dimensional smooth variety. Since $K_X = \mathcal{O}_X(d - n - 2)$, when $d > n + 2$, X has ample canonical bundle thus is of general type; when $d = n + 2$, X is of Calabi-Yau type and we will recover the results of [21]. Let $k := d + 1 - n \geq 3$. Our first objective is to express in the Chow group of X^k the class of the *smallest* diagonal

$$\delta_X := \{(x, x, \dots, x) \mid x \in X\}$$

in terms of bigger diagonals. Since there will be various types of diagonals involved, we need some systematic notation to treat them.

Definition 2.1. For any positive integers $s \leq r$,

1. we define the set N_s^r to be the set of all possible *partitions*⁵ of r elements into s non-empty *non-ordered* parts. In other words:

$$N_s^r := \{\{1, 2, \dots, r\} \rightarrow \{1, 2, \dots, s\} \text{ surjective maps}\} / \mathfrak{S}_s.$$

⁵Here we use the terminology ‘partition’ in an unusual way.

Here the action of the symmetric groups is induced from the action on the target. The action is clearly free, and we take the quotient in the naive (set-theoretical) way. We will view such a surjective map as some sort of *degeneration* of r points into s points. In every equivalent class, we have a canonical representative $\dot{\alpha} : \{1, 2, \dots, r\} \rightarrow \{1, 2, \dots, s\}$ such that $\dot{\alpha}(1), \dot{\alpha}(2), \dots, \dot{\alpha}(r)$ is alphabetically minimal.

2. For positive integers $t \leq s \leq r$, let $\alpha \in N_s^r$ and $\alpha' \in N_t^s$ be two partitions, then the *composition* $\alpha\alpha' \in N_t^r$ is defined in the natural way, and obviously $\dot{\alpha}' \circ \dot{\alpha}$ is the minimal representative of $\alpha\alpha'$.
3. For every $\alpha \in N_s^r$, and any set (or algebraic variety) Y , we have a natural morphism denoted still by α ,

$$\begin{aligned} \alpha : Y^s &\rightarrow Y^r \\ (y_1, y_2, \dots, y_s) &\mapsto (y_{\dot{\alpha}(1)}, y_{\dot{\alpha}(2)}, \dots, y_{\dot{\alpha}(r)}) \end{aligned}$$

where $\dot{\alpha}$ is the minimal representative of α defined above. Note that the morphism induced by their composition $\alpha\alpha'$ is exactly the composition of the morphisms induced by α and α' , so there is no ambiguity in the notation $\alpha\alpha' : Y^t \rightarrow Y^r$.

4. The *pull-back* of an r -tuple $\underline{a} = (a_1, a_2, \dots, a_r)$ by an element $\alpha \in N_s^r$ is defined by

$$\alpha^*(\underline{a}) := (b_1, \dots, b_s),$$

where $b_j := \sum_{\dot{\alpha}(i)=j} a_i$ for any $1 \leq j \leq s$. We note that pull-backs are functorial: $(\alpha\alpha')^*(\underline{a}) = \alpha'^*\alpha^*(\underline{a})$.

Let us explain this definition in a concrete example: if $r = 5, s = 3$, and the partition of $\{1, 2, 3, 4, 5\}$ is $\alpha = (\{1, 3\}; \{4\}; \{2, 5\}) \in N_3^5$, then the representative $\dot{\alpha} : 1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 1, 4 \mapsto 3, 5 \mapsto 2$, and thus the corresponding morphism for any Y is

$$\begin{aligned} \alpha : Y^3 &\rightarrow Y^5 \\ (y_1, y_2, y_3) &\mapsto (y_1, y_2, y_1, y_3, y_2). \end{aligned}$$

And also the pull-back of a 5-tuple is $\alpha^*(a_1, \dots, a_5) = (a_1 + a_3, a_2 + a_5, a_4)$. If we have another $t = 2, \alpha' \in N_2^3$ defined by $(\{1, 3\}; \{2\})$, then $\alpha\alpha'$ is $(\{1, 2, 3, 5\}; \{4\})$, and $\alpha\alpha' : Y^2 \rightarrow Y^5$ maps (y_1, y_2) to $(y_1, y_1, y_2, y_1, y_1)$.

This definition is nothing else but all the diagonal inclusions we need in the sequel: for instance, the unique partition in N_1^2 is the diagonal $\Delta_Y \subset Y \times Y$; the three elements in N_2^3 is the so-called *big* diagonals of $Y \times Y \times Y$ in the preceding section, and the unique partition in N_1^r is the smallest diagonal $\delta_Y \subset Y^r$. Note that the morphisms α also induce morphisms from $Y^s \setminus \delta_Y$ to $Y^r \setminus \delta_Y$.

Now return to our geometric setting. As in the preceding section, for any integer $r \geq 2$, we define

$$W_r := \{(y_1, y_2, \dots, y_r) \in \mathbf{P}^{\times r} \mid y_i \text{ are collinear}\} \setminus \delta_{\mathbf{P}}.$$

In other words, if we denote by $L \rightarrow G$ the universal line over the Grassmannian $G := \text{Gr}(\mathbf{P}^1, \mathbf{P})$, then in fact $W_r = \underbrace{L \times_G \dots \times_G L}_r \setminus \delta_L$. In particular W_r is a smooth variety with $\dim W_r = \dim G + r = 2n + r$.

Similarly, for any $r \geq 2$, we define a closed subvariety V_r of $X^r \setminus \delta_X$ by taking the closure in $X^r \setminus \delta_X$ of

$$V_r^o := \{(x_1, x_2, \dots, x_r) \in X^r \mid x_i \text{ are collinear and distinct}\}$$

$$V_r := \overline{V_r^o}.$$

Then we can prove as in Lemma 1.2 (in fact more easily) or as in [21] the following lemma. See also [9].

Lemma 2.2. *Consider the intersection of W_r and $X^r \setminus \delta_X$ in $\mathbf{P}^{\times r} \setminus \delta_{\mathbf{P}}$. The intersection scheme has the following irreducible component decomposition:*

$$W_r \cap (X^r \setminus \delta_X) = V_r \cup \bigcup_{2 \leq s < r} \bigcup_{\alpha \in N_s^r} \alpha(V_s) \quad (27)$$

Moreover, the intersection along each component is transversal, in particular the intersection scheme is of pure dimension $2n$, that is $\dim V_s = 2n$ for any s . In particular, (27) also holds scheme-theoretically:

$$\begin{array}{ccc} V_r \cup \bigcup_{2 \leq s < r} \bigcup_{\alpha \in N_s^r} \alpha(V_s) & \longrightarrow & X^r \setminus \delta_X \\ \downarrow & \square & \downarrow \\ W_r & \longrightarrow & \mathbf{P}^{\times r} \setminus \delta_{\mathbf{P}} \end{array}$$

We now define some vector bundles on V_r for any $r \in \{2, 3, \dots, k\}$, which are analogues of the vector bundle F defined in the preceding section. Let S be the tautological rank 2 vector bundle on W_r , such that $p : \mathbf{P}(S) \rightarrow W_r$ is the tautological \mathbf{P}^1 -bundle, which admits r tautological sections $\sigma_i : W_r \rightarrow \mathbf{P}(S)$, where $i = 1, \dots, r$. Let $q : \mathbf{P}(S) \rightarrow \mathbf{P}^{n+1}$ be the natural morphism. We summarize the situation by the following diagram:

$$\begin{array}{ccc} & \mathbf{P}(S) & \xrightarrow{q} & \mathbf{P} \\ \sigma_i \uparrow \cdots \uparrow & \downarrow p & & \\ & W_r & & \end{array}$$

Let B_i be the image of the section σ_i , which is a divisor of $\mathbf{P}(S)$, for $i = 1, \dots, r$. For any r -tuple $\underline{a} := (a_1, a_2, \dots, a_r)$ such that $\sum_{i=1}^r a_i = k$, we make the following constructions:

1. A sheaf on W_r by

$$\widetilde{F}(\underline{a}) := p_* (q^* \mathcal{O}_{\mathbf{P}}(d) \otimes \mathcal{O}_{\mathbf{P}(S)}(-a_1 B_1 - \dots - a_r B_r)).$$

As in Lemma 1.3, we can prove that $\widetilde{F}(\underline{a})$ is a vector bundle on W_r of rank $d + 1 - k = n$, with fiber

$$\widetilde{F}(\underline{a})_{y_1 \cdots y_r} = H^0(\mathbf{P}^1_{y_1 \cdots y_r}, \mathcal{O}(d) \otimes \mathcal{O}(-a_1 y_1 - a_2 y_2 - \dots - a_r y_r)).$$

2. A rank n vector bundle on V_r by restriction:

$$F(\underline{a}) := \widetilde{F}(\underline{a})|_{V_r}.$$

3. An n -dimensional algebraic cycle on

$$\gamma_{\underline{a}} := \gamma_{a_1, \dots, a_r} := i_{r*} c_n(F(a_1, \dots, a_r)) \in \text{CH}_n(X^r \setminus \delta_X),$$

where for any integer $r \geq 2$, we denote the natural inclusion $i_r : V_r \rightarrow X^r \setminus \delta_X$.

Recall that for any $2 \leq s \leq r$ and any $\alpha \in N_s^r$, the morphisms $\alpha : X^s \setminus \delta_X \rightarrow X^r \setminus \delta_X$ induces a diagonal inclusion $\alpha : V_s \hookrightarrow W_r$, whose formula is given by repeating some coordinates. We observe the following relation:

Lemma 2.3. *For any r -tuple $\underline{a} := (a_1, a_2, \dots, a_r)$ such that $\sum_{i=1}^r a_i = k$, the restriction of the vector bundle $\widetilde{F}(\underline{a})$ on W_r to the image $\alpha(V_s)$ gives the vector bundle $F(\alpha^*(\underline{a}))$ on V_s , i.e.*

$$\widetilde{F}(\underline{a})|_{\alpha(V_s)} = F(\alpha^*(\underline{a})), \quad (28)$$

where $\alpha^*(\underline{a})$ is defined in Definition 2.1.

Proof. To avoid heavy notation, let us explain in the simplest case that $s = r - 1$ and the partition α is given by $(\{1, 2\}; \{3\}; \{4\}; \dots; \{r\})$, then $\alpha : V_s \rightarrow W_r$ maps (x_1, x_2, \dots, x_s) to $(x_1, x_1, x_2, \dots, x_s)$. Therefore the fiber of $\widetilde{F}(\underline{a})|_{\alpha(V_s)}$ over the point (x_1, x_2, \dots, x_s) is exactly the fiber of $\widetilde{F}(\underline{a})$ over the point $(x_1, x_1, x_2, \dots, x_s)$, which is

$$H^0(\mathbf{P}_{x_1 \dots x_s}^1, \mathcal{O}(d) \otimes \mathcal{O}(-(a_1 + a_2)x_1 - a_3x_2 - \dots - a_sx_r)).$$

It is nothing else but the fiber of $F(\alpha^*(\underline{a}))$ over the point $(x_1, x_2, \dots, x_s) \in V_s$. \square

Using the same trick as in Proposition 1.6, we obtain the following recursive relations.

Proposition 2.4. *The algebraic cycles γ satisfy some recursive equalities: for any integer $r \geq 3$ and any r -tuple $\underline{a} = (a_1, \dots, a_r)$ with $\sum_{i=1}^r a_i = k$, we have in $\text{CH}_n(X^r \setminus \delta_X)$,*

$$\gamma_{\underline{a}} + \sum_{2 \leq s < r} \sum_{\alpha \in N_s^r} \alpha_* \gamma_{\alpha^*(\underline{a})} + P_{\underline{a}}(h_1, \dots, h_r) = 0, \quad (29)$$

where $P_{\underline{a}}$ is a homogeneous polynomial of degree $n(r - 1)$ depending on \underline{a} . Moreover the starting data are given by

$$\gamma_{a,b} = \prod_{i=0}^{n-1} \left((b + i)h_1 + (n - 1 - i + a)h_2 \right) \quad (30)$$

for any $a + b = k$. Here h_i is the pull-back of $h = c_1(\mathcal{O}_X(1))$ by the i -th projection.

Proof. Consider the cartesian square in Lemma 2.2:

$$\begin{array}{ccc} V_r \cup \bigcup_{2 \leq s < r} \bigcup_{\alpha \in N_s^r} \alpha(V_s) & \xrightarrow{j_3} & X^r \setminus \delta_X \\ j_4 \downarrow & \square & \downarrow j_2 \\ W_r & \xrightarrow{j_1} & \mathbf{P}^{\times r} \setminus \delta_{\mathbf{P}} \end{array}$$

Thanks to Lemma 2.2, there is no excess intersection here, we thus obtain:

$$j_2^* j_1^* c_n(\widetilde{F}(\underline{a})) = j_3^* j_4^* c_n(\widetilde{F}(\underline{a})).$$

In the left hand side, $j_1^* c_n(\widetilde{F}(\underline{a})) \in \text{CH}_{n+r}(\mathbf{P}^{\times r} \setminus \delta_{\mathbf{P}}) = \text{CH}_{n+r}(\mathbf{P}^{\times r})$, which can be written as a homogeneous polynomial $-P(H_1, \dots, H_r)$ of degree $n(r - 1)$. Applying j_2^* , H_i restricts to h_i . While in the right hand side,

$$\begin{aligned} j_4^* c_n(\widetilde{F}(\underline{a})) &= \sum_{2 \leq s \leq r} \sum_{\alpha \in N_s^r} \alpha_* c_n(\widetilde{F}(\underline{a})|_{\alpha(V_s)}) \\ &= \sum_{2 \leq s \leq r} \sum_{\alpha \in N_s^r} \alpha_* \gamma_{\alpha^*(\underline{a})} \quad (\text{by Lemma 2.3}) \end{aligned}$$

Putting them together, and noting that N_r^r has only one element inducing the identity map, we have (29). As for the starting data, we only need to calculate $c_n(F(a, b))$. By the same computations in the proof of Lemma 1.7, we find that on $V_2 = X \times X \setminus \Delta_X$,

$$F(a, b) = \bigoplus_{i=0}^{n-1} \mathcal{O}_X(i + b) \boxtimes \mathcal{O}_X(n - 1 - i + a),$$

and the formula (30) follows. \square

Before we exploit the recursive formula (29) further, we need the following easy lemma which allows us to simplify the push-forwards by some diagonal maps. This lemma essentially appeared in [21] (Lemma 3.3), but for the convenience of the readers we give a proof.

Lemma 2.5. *Let X be a hypersurface in $\mathbf{P} := \mathbf{P}^{n+1}$ of degree d . Then in $\mathrm{CH}_{n-1}(X \times X)$,*

$$d\Delta_*(h) = (i \times i)^!(\Delta_{\mathbf{P}}) = \sum_{j=1}^n h_1^j h_2^{n+1-j},$$

where $i : X \rightarrow \mathbf{P}$ is the natural inclusion, $\Delta : X \rightarrow X \times X$ is the diagonal inclusion, and h_i is the pull-back of $h := c_1(\mathcal{O}_X(1))$ by the i -th projection.

Proof. Consider the cartesian diagram:

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times X \\ i \downarrow & \square & \downarrow i \times i \\ \mathbf{P} & \xrightarrow{\Delta_{\mathbf{P}}} & \mathbf{P} \times \mathbf{P} \end{array}$$

Its excess normal bundle is exactly $N_{X/\mathbf{P}} \simeq \mathcal{O}_X(d)$. Therefore

$$\begin{aligned} d\Delta_*(h) &= \Delta_*(c_1(\mathcal{O}_X(d))) = \Delta_*(i^![\mathbf{P}] \cdot c_1(\text{excess normal bundle})) \\ &= \Delta_*((i \times i)^![\mathbf{P}]) = (i \times i)^!(\Delta_{\mathbf{P}}). \end{aligned}$$

Finally, we know that in $\mathrm{CH}_{n+1}(\mathbf{P} \times \mathbf{P})$, we have a decomposition $\Delta_{\mathbf{P}} = \sum_{j=0}^{n+1} H_1^j H_2^{n+1-j}$. Restricting this to $X \times X$, H_i becomes h_i , and $h_i^{n+1} = 0$, proving the lemma. \square

For the rest of this section, we will consider only \mathbf{Q} -coefficient cycles. Let $c := c_X := \frac{1}{d}h^n \in \mathrm{CH}_0(X)_{\mathbf{Q}}$ be the 0-cycle of degree 1, where $h = c_1(\mathcal{O}_X(1))$. Then note that Lemma 2.5 implies the following simple equation in $\mathrm{CH}_0(X \times X)_{\mathbf{Q}}$:

$$\Delta_*(c) = c \times c, \tag{31}$$

where $\Delta : X \rightarrow X \times X$ is the diagonal inclusion.

To make use of the recursive formula (29), we need to introduce some terminology and notation. Let $r \geq 2$ be an integer. For any non-empty subset J of $\{1, 2, \dots, r\}$, define the diagonal

$$\Delta_J := \{(x_1, \dots, x_r) \in X^r \mid x_j = x_{j'}, \forall j, j' \in J\},$$

which is a cycle of dimension $n(r+1-|J|)$. For example, $\Delta_{\{1,2,\dots,r\}} = \delta_X$ is the smallest diagonal, and $\Delta_{\{i\}} = X^r$ for any i . Then for any proper subset I of $\{1, 2, \dots, r\}$, we can define the n -dimensional cycle

$$D_I := \Delta_{I^c} \cdot \prod_{i \in I} \mathrm{pr}_i^* c, \tag{32}$$

where $I^c = \{1, 2, \dots, r\} \setminus I$ is the complementary set. Informally, we could write

$$D_I = \{(x_1, \dots, x_r) \in X^r \mid x_i = c, \forall i \in I; x_j = x_{j'}, \forall j, j' \notin I\}. \tag{33}$$

For example, $D_{\emptyset} = \delta_X$; $D_{\{1,2,\dots,r-1\}} = \underbrace{c \times \dots \times c}_{r-1} \times X$. And for any i , $D_i := D_{\{i\}}$ is called the i -th *secondary diagonal*. The crucial idea of the calculation of this section is to focus on the coefficients of these secondary diagonals in the expression of γ 's.

Definition 2.6. An algebraic cycle in $\text{CH}_n(X^r)_{\mathbf{Q}}$ is called of

- *Type A*: if it is a \mathbf{Q} -coefficient homogeneous polynomial in h_1, \dots, h_r of degree $n(r-1)$, such that each h_i appears in every monomial, *i.e.* if it is a \mathbf{Q} -linear combination of $\left\{ \prod_{j=1}^r h_j^{m_j}; m_j > 0, \sum_j m_j = n(r-1) \right\}_{i=1}^r$;

And for any $0 \leq j \leq r-1$:

- *Type B_j* : if it is a \mathbf{Q} -linear combination of the cycles D_I for I proper subsets of $\{1, 2, \dots, r\}$ with $|I| = j$.

For example:

- *Type B_0* : if it is a multiple of the cycle $D_\emptyset = \delta_X$. We will not need this type.
- *Type B_1* : if it is a \mathbf{Q} -linear combination of the secondary diagonals, *i.e.* the cycles D_i for $1 \leq i \leq r$;
- ...
- *Type B_{r-1}* : if it is a \mathbf{Q} -linear combination of $\left\{ \prod_{j \neq i} h_j^n \right\}_{i=1}^r$;

We remark that these notions of types also make sense (except B_0 becomes zero) when viewed as cycles in $\text{CH}_n(X^r \setminus \delta_X)$ by restricting to this open subset. We sometimes write *Type $B_{\geq 2}$* for a sum of the form *Type $B_2 + \text{Type } B_3 + \dots + \text{Type } B_{r-1}$* .

Now we study the behaviors of the various types under push-forwards by diagonal maps. In the simplest case, we have:

Lemma 2.7. *Let $r \geq 2$ be an integer and $\alpha \in N_r^{r+1}$ be a partition, inducing a diagonal map $\alpha : X^r \setminus \delta_X \rightarrow X^{r+1} \setminus \delta_X$. Then*

1. $\alpha_*(\text{Type } A) = \text{Type } A$;
2. $\alpha_*(\text{Type } B_j) = \text{Type } B_j + \text{Type } B_{j+1}$, for any $1 \leq j \leq r-1$. In particular,

$$\alpha_*(\text{Type } B_1) = \text{Type } B_1 + \text{Type } B_2;$$

$$\alpha_*(\text{Type } B_2) = \text{Type } B_{\geq 2};$$

Proof. For simplicity, one can suppose $\alpha : (x_1, x_2, \dots, x_r) \mapsto (x_1, x_1, x_2, \dots, x_r)$. Then the proofs are just some straightforward computations making use of Lemma 2.5 and (31). \square

Since any $\alpha \in N_s^r$ is a composition of several one-step-degenerations treated in the above lemma, as a first corollary of Proposition 2.4, any (a_1, \dots, a_r) with $\sum_{i=1}^r a_i = k$, $\gamma_{\underline{a}}$ is of the form: *Type A + Type $B_1 + \dots + \text{Type } B_{r-1}$* .

We need something more precise: our first objective is to determine the coefficients of secondary diagonals, *i.e.* cycles of type B_1 , in the expression of $\gamma_{\underline{a}}$ determined by the recursive relations in Proposition 2.4. However, Lemma 2.7 tells us that the coefficients of B_1 -cycles in the γ 's for a certain r are determined only by the coefficients of B_1 -cycles in γ 's for strictly smaller r 's. Now let us work them out.

Proposition 2.8. Let $r \geq 2$ be an integer, (a_1, \dots, a_r) be an r -tuple with $\sum_{i=1}^r a_i = k$, then

$$\gamma_{a_1, \dots, a_r} = \mu_r \sum_{i=1}^r \psi(a_i) D_i + \text{Type } B_{\geq 2} + \text{Type } A, \quad (34)$$

where the constants $\mu_r = (-1)^r (r-2)!$, and $\psi(a_i) := d \cdot \frac{(n-1+\sum_{j \neq i} a_j)!}{(\sum_{j \neq i} a_j - 1)!} = d \cdot \frac{(d-a_i)!}{(k-1-a_i)!}$.

In particular,

$$\gamma_{1^k} = (-1)^k d! \sum_{i=1}^k D_i + \text{Type } B_{\geq 2} + \text{Type } A, \quad (35)$$

where $1^k = \underbrace{(1, 1, \dots, 1)}_k$.

Proof. Rewrite the recursive formula (29): for $r \geq 3$,

$$\gamma_{\underline{a}} + \sum_{2 \leq s < r} \sum_{\alpha \in N_s^r} \alpha_* \gamma_{\alpha^*(\underline{a})} = \text{Type } A + \text{Type } B_{r-1} = \text{Type } A + \text{Type } B_{\geq 2};$$

and the starting data (30): for any $a + b = k$,

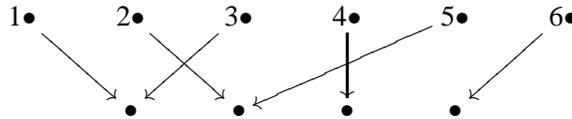
$$\gamma_{a,b} = \frac{(b+n-1)!}{(b-1)!} h_1^n + \frac{(a+n-1)!}{(a-1)!} h_2^n + \text{Type } A = \psi(a) D_1 + \psi(b) D_2 + \text{Type } A.$$

Hence (34) holds for $r = 2$. Now we will prove the result by induction on r . Thanks to Lemma 2.7, Type A and Type $B_{\geq 2}$ are preserved by α_* , hence we can work throughout this proof ‘modulo’ these two types, and we will use ‘ \equiv ’ in the place of ‘=’ to indicate such simplification.

In the recursive formula

$$\gamma_{\underline{a}} + \sum_{2 \leq s < r} \sum_{\alpha \in N_s^r} \alpha_* \gamma_{\alpha^*(\underline{a})} \equiv 0, \quad (36)$$

to calculate a typical term $\alpha_* \gamma_{\alpha^*(\underline{a})}$, we first make an elementary remark that we can choose *any* representative of α to define the pull-back and the induced push-forward morphism instead of sticking to the minimal representative as we did before: α_* , α^* compensate each other for the effect of renumbering the parts. An example would be helpful: let $\alpha \in N_4^6$ be as following:



then no matter how one renumbers the four points on the second row, we always have (assuming the induction hypothesis (34) for $r = 4$):

$$\begin{aligned} \alpha_* \gamma_{\alpha^*(a_1, \dots, a_6)} &= \mu_4 \cdot \left(\psi(a_1 + a_3) D_{\{1,3\}} + \psi(a_2 + a_5) D_{\{2,5\}} + \psi(a_4) D_4 + \psi(a_6) D_6 \right) \\ &\equiv \mu_4 \cdot \left(\psi(a_4) D_4 + \psi(a_6) D_6 \right) \quad (\text{mod Type } A + \text{Type } B_{\geq 2}). \end{aligned}$$

The second observation is that the contribution to Type B_1 of a typical term $\alpha_* \gamma_{\alpha^*(\underline{a})}$ with $\alpha \in N_s^r$, is exactly the sum of $\mu_s \cdot \psi(a_i) D_i$ for those i stays ‘isolated’ in the partition defined by α . One can check this in the

above example too, the isolated points are 4 and 6, and the contribution is $\mu_4 \cdot (\psi(a_4)D_4 + \psi(a_6)D_6)$. Therefore, the recursive formula (36) reads as (by the induction hypothesis for all $2 \leq s < r$):

$$\gamma_{\underline{a}} + \sum_{i=1}^r \sum_{2 \leq s < r} m_{r,s} \mu_s \cdot \psi(a_i) D_i \equiv 0, \quad (37)$$

where $m_{r,s}$ is the cardinality of the set $\{\alpha \in N_s^r \mid i \text{ stays isolated in the partition defined by } \alpha\}$, here $i \in \{1, 2, \dots, r\}$, and obviously $m_{r,s}$ is independent of i . However, by ignoring the isolated part, it is easy to see that $m_{r,s}$ is exactly the cardinality of N_{s-1}^{r-1} . Therefore to complete the proof, it suffices to show the following identity:

$$\mu_r + \sum_{2 \leq s < r} \#N_{s-1}^{r-1} \cdot \mu_s = 0,$$

which is an immediate consequence of the following elementary lemma. \square

Lemma 2.9. *For any positive integer $m \geq 2$, we have*

$$\sum_{1 \leq j \leq m} (-1)^j (j-1)! \cdot \#N_j^m = 0.$$

Proof. For any positive integer $1 \leq j \leq m$, let S_j^m be the set of *surjective* maps from $\{1, 2, \dots, m\}$ to $\{1, 2, \dots, j\}$. By definition 2.1, N_j^m is the quotient of the action of \mathfrak{S}_j on S_j^m induced by the action on the target. This action is clearly free, thus $\#N_j^m = \frac{1}{j!} \#S_j^m$. Denoting $s_j^m := \#S_j^m$, we have to show the following identity:

$$\sum_{1 \leq j \leq m} (-1)^j \frac{1}{j} \cdot s_j^m = 0. \quad (38)$$

Now for any integer $1 \leq l \leq m$, we consider the number of all maps from $\{1, 2, \dots, m\}$ to $\{1, 2, \dots, l\}$, which is obviously l^m . However on the other hand, we could count this number by classifying these maps by the cardinality of their images: the number of maps with $\#\text{image} = j$ is exactly $\binom{l}{j} \cdot s_j^m$. Hence,

$$l^m = \sum_{j=1}^m \binom{l}{j} \cdot s_j^m.$$

Since this identity holds for $l = 0, 1, \dots, m$, it is in fact an identity of polynomials of degree m :

$$T^m = \sum_{j=1}^m \binom{T}{j} \cdot s_j^m,$$

where T is the variable. Simplifying T from both sides:

$$T^{m-1} = 1 + \sum_{j=2}^m s_j^m \cdot \frac{(T-1)(T-2) \cdots (T-j+1)}{j!}.$$

Let $T = 0$, we obtain (38), and the lemma follows. \square

Now we relate the cycle γ_{1^k} in (35) to a geometric constructed cycle. Like before, let $F(X)$ be the variety of lines of X . Since X is general, $F(X)$ is a smooth variety of dimension $2n - d - 1 = n - k$ if $k \leq n$, and empty if $k > n$. Define also the subvariety of X^k

$$\Gamma := \bigcup_{t \in F(X)} \underbrace{\mathbf{P}_t^1 \times \cdots \times \mathbf{P}_t^1}_k, \quad (39)$$

which is of dimension n if $k \leq n$ and empty if $k > n$.

Write $\Gamma_o := \Gamma|_{X^k \setminus \delta_X} \in \text{CH}_n(X^k \setminus \delta_X)$. As in Lemma 1.4,

Lemma 2.10. *Let 1^k be the k -tuple $(1, 1, \dots, 1)$, then $\Gamma_o = \gamma_{1^k}$ in $\text{CH}_n(X^k \setminus \delta_X)$.*

Proof. The defining function of X gives rise to a section of the rank n vector bundle $F(1, 1, \dots, 1)$ on V_k by restricting to lines. Its zero locus defines exactly the cycle Γ_o in $X^k \setminus \delta_X$. Then the geometrical meaning of top Chern classes proves the desired equality. \square

Combining the results of Proposition 2.8 and Lemma 2.10, we get a decomposition of the class of the smallest diagonal of X^k , except that the multiple λ_0 appearing below could be zero.

Proposition 2.11. *There exist rational numbers λ_j for $j = 0, \dots, k-2$, and a symmetric homogeneous polynomial P of degree $n(k-1)$, such that in $\text{CH}_n(X^k)_{\mathbf{Q}}$ we have:*

$$\Gamma = \sum_{j=0}^{k-2} \lambda_j \sum_{|I|=j} D_I + P(h_1, \dots, h_k), \quad (40)$$

where $\lambda_1 = (-1)^k d!$ is non-zero. More concretely,

$$\Gamma = \lambda_0 \delta_X + (-1)^k d! \sum_{i=1}^k D_i + \lambda_2 \sum_{|I|=2} D_I + \cdots + \lambda_{k-2} \sum_{|I|=k-2} D_I + P(h_1, \dots, h_k). \quad (41)$$

Proof. Putting Lemma 2.10 into (35), we obtain $\Gamma_o = (-1)^k d! \sum_{i=1}^k D_i + \text{Type } B_{\geq 2} + \text{Type } A$ in $\text{CH}_n(X^k \setminus \delta_X)_{\mathbf{Q}}$. Thanks to the localization exact sequence

$$\text{CH}_n(X)_{\mathbf{Q}} \xrightarrow{\delta_*} \text{CH}_n(X^k)_{\mathbf{Q}} \rightarrow \text{CH}_n(X^k \setminus \delta_X)_{\mathbf{Q}} \rightarrow 0,$$

and the symmetry of Γ , we can write it in the way as stated (remember that Type B_{k-1} is in fact a polynomial of the h_i 's). \square

A second reflection on (40) or (41) gives the main result of this section, which is a generalization of Theorem 0.5 in the introduction:

Theorem 2.12. *Let X be a general smooth hypersurface in \mathbf{P}^{n+1} of degree d with $d \geq n+2$. Let $k = d+1-n \geq 3$. Then one of the following two cases occurs:*

1. *There exist rational numbers λ_j for $j = 2, \dots, k-1$, and a symmetric homogeneous polynomial P of degree $n(k-1)$, such that in $\text{CH}_n(X^k)_{\mathbf{Q}}$ we have:*

$$\delta_X = (-1)^{k-1} \frac{1}{d!} \cdot \Gamma + \sum_{i=1}^k D_i + \sum_{j=2}^{k-2} \lambda_j \sum_{|I|=j} D_I + P(h_1, \dots, h_k), \quad (42)$$

where D_l is defined in (32) or (33).

Or

2. There exist a (smallest) integer $3 \leq l < k$, rational numbers λ_j for $j = 2, \dots, l-2$, and a symmetric homogeneous polynomial P of degree $n(l-1)$, such that in $\text{CH}_n(X^l)_{\mathbf{Q}}$ we have:

$$\delta_X = \sum_{i=1}^l D_i + \sum_{j=2}^{l-2} \lambda_j \sum_{|I|=j} D_I + P(h_1, \dots, h_l). \quad (43)$$

Moreover, $\Gamma = 0$ if $d \geq 2n$.

Proof. In (40) or (41), if $\lambda_0 = -\lambda_1 (= (-1)^{k-1} d!)$ which is non-zero in particular, then we can divide on both sides by λ_1 to get (42) in Case 1, up to a rescaling of the numbers λ_i and the polynomial P .

If $\lambda_0 + \lambda_1 \neq 0$, then we project both sides onto the first $k-1$ factors. Since Γ has relative dimension 1 for this projection, it vanishes after the projection. Therefore we get an equality in $\text{CH}_n(X^{k-1})_{\mathbf{Q}}$ of the form:

$$0 = (\lambda_0 + \lambda_1)\delta_X + \lambda'_1 \sum_{i=1}^k D_i + \lambda'_2 \sum_{|I|=2} D_I + \dots + \lambda'_{k-2} \sum_{|I|=k-2} D_I + P'(h_1, \dots, h_{k-1}).$$

Dividing both sides by $\lambda_0 + \lambda_1$, which is non-zero, we get a decomposition of the smallest diagonal in $\text{CH}_n(X^l)_{\mathbf{Q}}$ for $l = k-1$:

$$\delta_X = \lambda_1 \sum_{i=1}^l D_i + \sum_{j=2}^{l-2} \lambda_j \sum_{|I|=j} D_I + P(h_1, \dots, h_l). \quad (44)$$

by reverting to the old notation λ_i and P .

If such a decomposition does not exist for $l = k-2$, then by a further projection to the first $k-2$ factors of (44), we find $\lambda_1 = 1$ in (44) for $l = k-1$. Hence we obtain a decomposition (43) in Case 2 for $l = k-1$.

If there does exist such decomposition for $l = k-2$, *i.e.* we have an identity of the form (44) for $l = k-2$. By projecting to the first $k-3$ factors, if such decomposition as (44) for $l = k-3$ does not exist, then we find $\lambda_1 = 1$ in (44) for $l = k-2$. Hence we obtain a decomposition (43) in Case 2 for $l = k-2$. If there does exist such decomposition for $l = k-3$, we continue doing the same argument.

Since $H^{n,0}(X) \neq 0$, as in the proof of Lemma 1.9, the non-existence of a decomposition of the diagonal $\Delta_X \subset X \times X$ implies that the minimal l for the existence of a decomposition of the form (44) is at least 3. Therefore the above argument must stop at some $l \geq 3$, and gives the decomposition (43) in Case 2 for this minimal l .

As for the vanishing of Γ , we note that $d \geq 2n$ if and only if $k > n$, in which case we know that Γ is empty. \square

Now we draw the following consequence on the ring structure of $\text{CH}^*(X)_{\mathbf{Q}}$ from the above decomposition theorem, generalizing Corollary 0.6:

Theorem 2.13. *Let X be a general smooth hypersurface in \mathbf{P}^{n+1} of degree d with $d \geq n+2$. Let $m = d-n \geq 2$. Then for any strictly positive integers $i_1, i_2, \dots, i_m \in \mathbf{N}^*$ with $\sum_{j=1}^m i_j = n$, the image*

$$\text{Im} \left(\text{CH}^{i_1}(X)_{\mathbf{Q}} \times \text{CH}^{i_2}(X)_{\mathbf{Q}} \times \dots \times \text{CH}^{i_m}(X)_{\mathbf{Q}} \xrightarrow{\bullet} \text{CH}_0(X)_{\mathbf{Q}} \right) = \mathbf{Q} \cdot h^n$$

Proof. In our notation before, $m = k - 1$. Let $z_j \in \text{CH}^{i_j}(X)_{\mathbf{Q}}$ for $1 \leq j \leq m$. By Theorem 2.12, (42) or (43) holds. Suppose first that we are in Case 1, *i.e.* (42). We view its both sides as correspondences from X^m to X . Apply these correspondences to the algebraic cycle $z := z_1 \times \cdots \times z_m \in \text{CH}^n(X^m)_{\mathbf{Q}}$:

- $\delta_{X^*}(z) = z_1 \bullet \cdots \bullet z_m$;
- $\Gamma_*(z) = 0$ since $\Gamma_*(z)$ is represented by a linear combination of fundamental classes of certain subvarieties of dimension at least 1, but $\Gamma_*(z)$ is a zero-dimensional cycle, thus vanishes;
- $D_{I^*}(z) = 0$ for any $I \neq \{k\}$;
- $D_{k^*}(z) = \deg(z_1 \bullet \cdots \bullet z_m) \cdot c_X$;
- $P(h_1, \dots, h_{m+1})_*(z)$ is always proportional to h^n .

Therefore $z_1 \bullet \cdots \bullet z_m \in \mathbf{Q} \cdot h^{i_1 + \cdots + i_m}$.

If we are in Case 2, the same proof goes through. □

Remark 2.14. When $d = n + 2$, this recovers the result of [21] as in the first part of the paper. When $d > n + 2$, the preceding theorem is actually predicted by the Bloch-Beilinson conjecture (*cf.* [4] [17] [11]), which claims the existence of a functorial filtration of length i on $\text{CH}^i(X)_{\mathbf{Q}}$, whose j -th graded piece is controlled by the Hodge structures $H^{2i-j}(X, \mathbf{Q})$ for any $j \leq i$, and vanishes if $H^{0,2i-j} = \cdots = H^{i-j,i} = 0$. However by the Lefschetz hyperplane section theorem, the only non-Tate-type Hodge structure of $H^*(X, \mathbf{Q})$ is the middle cohomology $H^n(X, \mathbf{Q})$. Therefore, according to the Bloch-Beilinson conjecture, the smallest i such that $\text{CH}^i(X)_{\mathbf{Q}} \supseteq \mathbf{Q} \cdot h^i$ is $\lceil \frac{n}{2} \rceil$. Since in the corollary, $\sum_{j=1}^m i_j = n$, thus the conjecture implies that there is at most one j such that z_j is not proportional to h^{i_j} . Now the above corollary follows from the easy fact that the intersection of any algebraic cycle z with the hyperplane section class h is always \mathbf{Q} -proportional to a power of h : write ι for the inclusion of the hypersurface X into the projective space, then $z \bullet dh = \iota^* \iota_*(z)$, which is the pull-back of a cycle of the projective space, thus must be proportional to a power of h .

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