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Frequent Itemset Border Approximation by Dualization

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Abstract. The approach *FIBAD* is introduced with the purpose of computing approximate borders of frequent itemsets by leveraging dualization and computation of approximate minimal transversals of hypergraphs. The distinctiveness of the *FIBAD*'s theoretical foundations is the approximate dualization where a new function \tilde{f} is defined to compute the approximate negative border. From a methodological point of view, the function \tilde{f} is implemented by the method *AMTHR* that consists of a reduction of the hypergraph and a computation of its minimal transversals. For evaluation purposes, we study the sensibility of *FIBAD* to *AMTHR* by replacing this latter by two other algorithms that compute approximate minimal transversals. We also compare our approximate dualization-based method with an existing approach that computes directly, without dualization, the approximate borders. The experimental results show that our method outperforms the other methods as it produces borders that have the highest quality.

Keywords: frequent itemsets, borders, hypergraph transversals, dualization, approximation.

1 Introduction

The discovery of frequent itemsets was initiated by Agrawal et al. [3]. This research field has quickly become an important task of data mining. The problem is to find the sets of items (i.e. attribute values) that appear together in at least a certain number of transactions (i.e. objects) recorded in a database. These sets of items are called frequent itemsets. In that regard, other types of itemsets have been proposed, for instance, closed itemsets [33], free itemsets [8] and emerging itemsets [11]. Even more important, such itemsets play a key role in the generation of association rules [3], supervised classification [34], clustering [14] and are useful in a broad of range of application fields [22].

Two issues are important in the discovery of frequent itemsets: (1) the reduction of the search space due to combinatorial explosion and (2) the reduction of the number of generated itemsets to improve efficiency. In this paper, we consider only the second point, with a focus on the set of maximal frequent

itemsets corresponding to a subset of frequent closed itemsets [33]. The maximal frequent itemsets represent a reduced collection of frequent itemsets, but they are not considered as a condensed representation contrary to the frequent closed itemsets. Indeed, the regeneration of all the frequent itemsets is possible from the maximal frequent itemsets but the database must be read to compute the frequencies. The problem of mining maximal frequent itemsets is NP-hard [40]. The algorithms discovering these itemsets are, for instance, *MaxMiner* [5], *Pincer-Search* [29], *MAFIA* [9], *GenMax* [20], *Dualize&Advance* [21], *IBE* [37] and *ABS* [18]. Maximal frequent itemsets are used a lot in recommendation systems where long itemsets describe trends. For example, a list of movies (i.e., a maximal itemset) that many people like can be used as recommendations for other users that share a large overlap of movies with this list.

The maximal frequent itemsets and the minimal infrequent itemsets correspond respectively to the positive border and the negative border of the set of frequent itemsets [30]. These two borders are linked together by the computation of minimal hypergraph transversals (also called "minimal hitting sets") [30, 10]. Thus, it is possible to switch to a border from the other one. This is called *dualization*. Algorithms like *Dualize&Advance* [21], *IBE* [37] and *ABS* [18] use dualization to discover maximal frequent itemsets. Indeed, *Dualize&Advance* computes the maximal frequent itemsets one by one, *IBE* is an improvement of the previous algorithm as it avoids the redundant frequency checks, and *ABS* starts computing the positive border and uses an adaptative strategy to decide when it is interesting to switch to the other border. Let us remark that the number of itemsets of the borders can be huge.

This paper extends our previous work in [15]. We propose the approach *FIBAD*¹ with the purpose of computing approximate borders of frequent itemsets by leveraging dualization and computation of approximate minimal transversals of hypergraphs. The aim is to reduce the size of borders. The originality of the proposed approach comes from a new function we have defined to compute approximate negative borders. From a methodological point of view, the function is implemented by the method *AMTHR*² that consists of a reduction of the hypergraph and a computation of its minimal transversals. Next, the approximate positive borders are computed from the approximate negative borders by an exact function. To the best of our knowledge, this is the first time that such approach is proposed leveraging dualization and border approximation. For evaluation purposes, we conduct experiments realized on different data sets, and evaluated the quality of the computed approximate borders by using the distance between the computed approximate borders and the exact ones. Moreover, we study the sensibility of *FIBAD* to *AMTHR*. For this, we replace *AMTHR* with two other methods that compute approximate minimal transversals. We also compare our dualization-based method with an existing approach which computes directly, without dualization, the approximate positive borders.

¹ Frequent Itemset Border Approximation by Dualization

² Approximate Minimal Transversals by Hypergraph Reduction

The rest of the paper is organized as follows. Section 2 defines notations and basic notions necessary for understanding the paper. Related works are discussed in Sect. 3. The proposed approach is detailed in Sect. 4. Section 5 presents our method that computes approximate minimal transversals. The experiments and the results are presented in Sect. 6. We conclude and present some future work in Sect. 7.

2 Preliminaries

Let $\mathcal{D} = (\mathcal{T}, \mathcal{I}, \mathcal{R})$ be a data mining context, \mathcal{T} a set of transactions, \mathcal{I} a set of items (denoted by capital letters), and $\mathcal{R} \subseteq \mathcal{T} \times \mathcal{I}$ is a binary relation between transactions and items. Each couple $(t, i) \in \mathcal{R}$ denotes the fact that the transaction t is related to the item i . A transactional database is a finite and nonempty multi-set of transactions. Table 1 provides an example of a transactional database consisting of 6 transactions (each one identified by its "Id") and 8 items (denoted $A \dots H$).

Table 1. Example of transactional database.

Id	Items					
t_1	A		C		E	G
t_2		B	C		E	G
t_3	A		C		E	H
t_4	A			D	F	H
t_5		B	C		F	H
t_6		B	C		E	F

An *itemset* is a subset of \mathcal{I} (note that we use a string notation for sets, e.g., AB for $\{A, B\}$). The complement of an itemset X (according to \mathcal{I}) is noted \overline{X} . A transaction t supports an itemset X iff $\forall i \in X, (t, i) \in \mathcal{R}$. An itemset X is *frequent* if the number of transactions which support it, is greater than (or is equal to) a minimum threshold value, noted *minsup*. The set of all-frequent itemsets, noted S , is presented in Definition 1.

Definition 1 (Set of all-frequent itemsets).

Let $\mathcal{D} = (\mathcal{T}, \mathcal{I}, \mathcal{R})$ be a data mining context and *minsup* be the minimum threshold value. The set of all-frequent itemsets, noted S , is:

$$S = \{X \subseteq \mathcal{I}, |\{t \in \mathcal{T}, \forall i \in X (t, i) \in \mathcal{R}\}| \geq \text{minsup}\}.$$

The notion of frequent itemset border was introduced by Mannila & Toivonen in [30] (see Definition 2). The borders can be visualized by using the itemset lattice, i.e., the partial order naturally defined on the powerset of \mathcal{I} (see Fig. 1). The itemset lattice represents the complete search space.

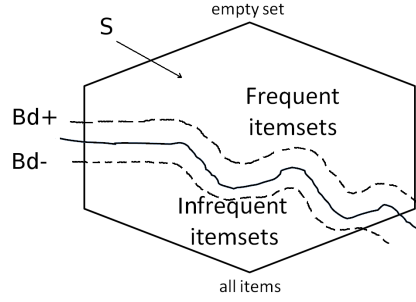


Fig. 1. Lattice of all subsets of items of \mathcal{I}

Definition 2 (Positive border and negative border). *The set of all maximal frequent itemsets (resp. minimal infrequent itemsets), w.r.t. set inclusion, in \mathcal{D} is the positive border (resp. negative border) of S and is noted $Bd^+(S)$ (resp. $Bd^-(S)$).*

$$Bd^+(S) = \{X \in S \mid \forall Y \supset X, Y \notin S\}.$$

$$Bd^-(S) = \{X \in 2^{\mathcal{I}} \setminus S \mid \forall Y \subset X, Y \in S\}.$$

Example 1. Let us take the example of Table 1, if $minsup=3$ then the itemset H is frequent because 4 transactions support it (t_3, t_4, t_5 and t_6). AE is not frequent because only t_1 and t_3 support it. The set of all-frequent itemsets is $S = \{A, B, C, E, F, H, BC, CE, CH, FH\}$. The positive border and the negative border are: $Bd^+(S) = \{A, BC, CE, CH, FH\}$ and $Bd^-(S) = \{D, G, AB, AC, AE, AF, AH, BE, BF, BH, CF, EF, EH\}$.

Before the presentation of the relationship between the positive border and the negative border of frequent itemsets, we need to introduce the notion of hypergraph (see Definition 3), the notion of simple hypergraph (see Definition 4) and the notion of minimal transversals of a hypergraph (see Definition 5) [6].

Definition 3 (Hypergraph and degree of a vertex). *A hypergraph $\mathcal{H} = (V, E)$ is composed of a set V of vertices and a set E of hyperedges [6]. Each hyperedge $e \in E$ is a set of vertices included or equal to V . The degree of a vertex v in \mathcal{H} , denoted $deg_{\mathcal{H}}(v)$, is the number of hyperedges of \mathcal{H} containing v .*

Definition 4 (Simple hypergraph and minimal hyperedges). *Let $\mathcal{H}=(V, E)$ be a hypergraph. \mathcal{H} is simple if for every pair $e_i, e_j \in E, e_j \subseteq e_i \Rightarrow j = i$. $\min(\mathcal{H})$ is the set of minimal hyperedges of \mathcal{H} w.r.t. set inclusion, i.e., $\min(\mathcal{H}) = \{e_i \in E \mid (\forall e_j \in E, i \neq j, e_j \subseteq e_i) : e_j = e_i\}$. The hypergraph $\min(\mathcal{H})$ is simple.*

Definition 5 (Transversal and minimal transversal). *Let \mathcal{H} be a hypergraph and τ be a set of vertices ($\tau \subseteq V$). τ is a transversal of \mathcal{H} if it intersects all the hyperedges of \mathcal{H} . A transversal is also called a "hitting set". The set of all the transversals of \mathcal{H} is $Tr(\mathcal{H}) = \{\tau \subseteq V \mid \forall e_i \in E, \tau \cap e_i \neq \emptyset\}$. A transversal τ of \mathcal{H} is minimal if no proper subset is a transversal of \mathcal{H} . The set of all minimal transversals of \mathcal{H} is noted $MinTr(\mathcal{H})$.*

The following Proposition 1 results from the definition of the minimal transversals of a hypergraph. Considering a hypergraph \mathcal{H} , the minimal transversals of \mathcal{H} are the same as the minimal transversals of the set of minimal hyperedges of \mathcal{H} [28].

Proposition 1. *Let $\mathcal{H}=(V, E)$ be a hypergraph. Then, $MinTr(\mathcal{H})$ is a simple hypergraph and $MinTr(\mathcal{H}) = MinTr(\min(\mathcal{H}))$.*

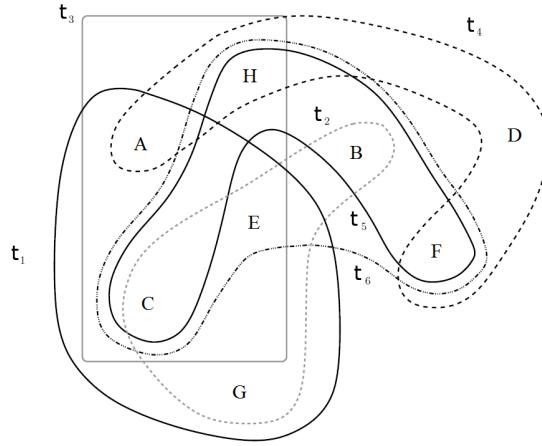


Fig. 2. The hypergraph from the example of Table 1

Example 2. Let us consider the example of Table 1 as a hypergraph \mathcal{H} . The transactions are the hyperedges and the items are the vertices (see Fig. 2). Here, $\min(\mathcal{H}) = \{t_1, t_2, t_3, t_4, t_5\}$. The set of vertices BC is not a transversal. ABC is a transversal but is not minimal (by removing B , it remains a transversal). AC and EF are minimal transversals. The set of all minimal transversals of \mathcal{H} is $\{AB, AC, CD, CF, CH, EF, EH, GH, AFG, BDE\}$.

The computation of the negative border from the positive border and vice versa are presented in Property 1 [30] and Property 2 [10].

Property 1 (Negative border and minimal transversals)

$Bd^-(S) = \overline{MinTr(Bd^+(S))}$ where $Bd^+(S)$ is the hypergraph formed by the items of \mathcal{I} (i.e. the vertices) and the complements of the itemsets of the positive border of S (i.e. the hyperedges).

Property 2 (Positive border and minimal transversals)

$Bd^+(S) = \overline{MinTr(Bd^-(S))}$ where $Bd^-(S)$ is the hypergraph formed by the items of \mathcal{I} (i.e. the vertices) and the itemsets of the negative border of S (i.e. the hyperedges).

Example 3. Let us compute $Bd^-(S)$ from $Bd^+(S)$ with our running example.
 $Bd^-(S) = \overline{MinTr(Bd^+(S))} = \overline{MinTr(\{\overline{A}, \overline{BC}, \overline{CE}, \overline{CH}, \overline{FH}\})}$
 $= \overline{MinTr(\{BCDEFGH, ADEFGH, ABDFGH, ABDEFG, ABCDEG\})}$
 $= \{D, G, AB, AC, AE, AF, AH, BE, BF, BH, CF, EF, EH\}.$
Let us compute $Bd^+(S)$ from $Bd^-(S)$. $Bd^+(S) = \overline{MinTr(Bd^-(S))}$
 $= \overline{MinTr(\{D, G, AB, AC, AE, AF, AH, BE, BF, BH, CF, EF, EH\})}$
 $= \{\overline{BCDEFGH}, \overline{ADEFGH}, \overline{ABDFGH}, \overline{ABDEFG}, \overline{ABCDEG}\}$
 $= \{A, BC, CE, CH, FH\}.$

The term *dualization* refers to the use of the two previous properties to compute the negative border from the positive border, and vice versa.

Due to the exponential size of the search space, the size of the borders can be huge according to *minsup*. In this paper, we propose an approach to approximate the borders and to reduce their size. In this way, the exploitation of the itemsets of the borders will be easier. The next section presents the related works and positions our propositions.

3 Related Works

3.1 Approximation of the Frequent Itemset Border

The approximation of both positive and negative borders of frequent itemsets has been examined on the basis algorithmic dimension [7]. At first, the computation of the approximate borders is performed directly from the data. In fact, it was shown that there is no algorithm of approximation to compute the positive border with a reasonable approximation factor. Furthermore, the computation of the negative border can be approximated by a greedy algorithm in polynomial time.

Numerous methods have been proposed to reduce the number of itemsets of the positive border. In [23], the authors have proposed an algorithm to obtain k representative itemsets by uniformly sampling from the pool of all maximal frequent itemsets (i.e., the positive border). They have used a variant of Markov Chain Monte Carlo algorithm. The algorithm simulates a random walk over the frequent itemset partial order graph with a prescribed transition probability matrix, whose values are computed locally during the simulation. In [31], the authors have studied the problem of randomly sampling maximal itemsets without explicit enumeration of the complete itemset search space. They have employed a simple random walk that only allows additions of singletons to the current set until a maximal itemset is found. An approximation measure, given as input of the algorithm, is used to guide the search for maximal frequent itemsets to different parts of the output space.

In [2], the approximation of a collection of frequent itemsets by the k best covering sets has been studied. The proposed algorithm input is the whole collection of the frequent itemsets or the positive border. The authors have explained the difficulties to use a greedy algorithm to obtain k covering sets belonging to

the initial collection. In [41], the authors have proposed the notion of colossal frequent itemsets. Some small frequent itemsets (defined as core itemsets) are fused into colossal itemsets by taking leaps in the itemset search space. The fusion process is designed to give an approximation to the colossal frequent itemsets. The maximal number of colossal itemsets to discover is given as input. In [25], the authors have proposed a concise representation for a collection of frequent itemsets, called cartesian contour, based on the cartesian product. They have linked the computation of the cartesian contour into a generalized minimum biclique cover problem and they have designed an approximate algorithm with bound. This algorithm takes in input the set of all maximal frequent itemsets and produces a cover. Let us remark that, with this approach, there is no need to set the number of itemsets, contrary to the previous presented works.

Other approaches exist to reduce the number of itemsets but the link with maximal frequent itemsets is not direct. In [39], the authors have used the Minimum Description Length (MDL) principle: the best set of itemsets is that set that compresses the database best. A post-treatment step would be needed to obtain maximal itemsets.

Our approach computes an approximate border from a border given as input. In that respect, other algorithms presented in [2] and [25] have borders as input. Nevertheless, we do not try to find some covering sets, but our primary goal is to approximate the border that may contains itemsets, which are not necessarily belong to the initial collection. Our approach has the advantage to generate both the approximate positive border and the corresponding approximate negative border. We have an understanding mapping between the exact border and the approximate border. Consequently, we have the possibility to use several other strategies to approximate borders (see Sect. 4). Moreover, contrary to the most of the previous presented works, we do not need to fix the size of the searched borders.

3.2 Approximate Minimal Transversals

The computation of minimal transversals is a central point in hypergraph theory [6] and represents a NP-hard problem. The algorithms that address this need have been developed by different communities like graph theory [6, 28], logic [19, 16] and data mining [12, 4, 24]. Some works approximate the minimal transversals in order to obtain several ones or only one [36]. Some works are based on an evolutionary computation [38] where the transversality and the minimality are transcribed both in a fitness function.

In [1], the *Staccato* algorithm computes low-cost approximate minimal transversals with a depth-first search strategy. It has been designed for model-based diagnosis. We have adapted *Staccato* in order to compute approximated minimal transversals in general and to use it in the experiments (see Sect. 6). The adaptation consists of defining the *Staccato*'s cost function using properties of vertices belonging to the hypergraph (i.e., their degrees). This adaptation allows the definition of vertex's cost. The algorithm sorts the vertices according to their cost value in decreasing order. This vertex ranking is exploited to guide

the search. At each selection step, only the first λ (%) vertices of the remaining hypergraph are used. For instance, the algorithm starts by selecting the vertex having the highest cost value. Then, it selects the vertex having the highest cost value in the hypergraph formed by the hyperedges not intersected yet. Etc. The more the λ value is high, the more the result is close to the set of all minimal transversals.

The algorithm presented in [35], that we call δ -*MTminer* in reference to *MTminer* [24], produces minimal transversals which can miss at most δ hyperedges. It uses a breadth-first search strategy and several itemset discovery techniques (candidate generation, anti-monotone constraint, ...). The search space corresponds to the lattice of all subsets of items (i.e., the vertices). The transactions correspond to the hyperedges. The algorithm uses the notion of "anti-support", where the anti-support of an itemset is the number of transactions having no item in common with this itemset. If the anti-support of an itemset is equals to 0 then this itemset is a transversal. The minimality is achieved by using free itemsets [8]. A δ -minimal transversal is a free itemset having an anti-support lower than or equals to δ . Let us note that δ -*MTminer* is especially efficient on dense hypergraphs (i.e., hypergraphs which strongly intersect).

Staccato and δ -*MTminer* present two approaches to compute approximate minimal transversals. Alternatively, we propose a method that performs a hypergraph reduction and then computes the minimal transversals of the reduced hypergraph. These transversals are considered as the approximate minimal transversals of the initial hypergraph. The number of missed hyperedges is free, contrary to δ -*MTminer*. Moreover, using our method, there are no parameters to set in advance.

3.3 Hypergraph Reduction

The classical approaches to reduce a hypergraph are edge and hyperedge coarsening [27]. In the edge coarsening approach, pairs of vertices that are present in the same hyperedges are selected to group the vertices. On the contrary, in the hyperedge coarsening approach, a set of independent hyperedges is selected and the vertices that belong to individual hyperedges are contracted together. The hyperedges are initially sorted, and they are visited according to the resulted order. Variants of these two approaches exist. For more information, we refer the reader to [27]. Recently, in [13], two algorithms of hypergraph reduction have been proposed. The first algorithm, called *HR-IH*, uses hyperedge intersections. Whereas, the second algorithm, called *HR-MST*, uses minimum spanning tree. The hypergraph reduction is a step of the proposed hypergraph clustering method which is applied to image segmentation.

Our hypergraph reduction algorithm uses, as *HR-IH*, the hyperedge intersections. Nevertheless, only the step of computation of hyperedge intersections is in common. In fact, *HR-IH* computes a cover of the set of hyperedge intersections and our algorithm selects the most interesting intersections according to a heuristic for finding minimal transversals.

4 Proposed Approach of Border Approximation

The *FIBAD* approach is introduced with the purpose of computing approximate borders by dualization. Let f and g be the functions that allow to compute respectively the negative border from the positive border and vice versa:

$$f : \begin{array}{l} 2^{\mathcal{I}} \rightarrow 2^{\mathcal{I}} \\ x \mapsto \text{MinTr}(\bar{x}) \end{array} \qquad g : \begin{array}{l} 2^{\mathcal{I}} \rightarrow 2^{\mathcal{I}} \\ x \mapsto \overline{\text{MinTr}(x)} \end{array}$$

The following diagram allow us to visualize the dualizations between the positive and negative borders:

$$\begin{array}{ccc} & f & \\ & \curvearrowright & \\ \text{Bd}^+(S) & & \text{Bd}^-(S) \\ & \curvearrowleft & \\ & g & \end{array}$$

The principle of *FIBAD* is to replace the function f by a function \tilde{f} which performs an approximate computation of the negative border. We define the following new function \tilde{f} that uses an approximate minimal transversals computation, noted $\widetilde{\text{MinTr}}$:

$$\tilde{f} : \begin{array}{l} 2^{\mathcal{I}} \rightarrow 2^{\mathcal{I}} \\ x \mapsto \widetilde{\text{MinTr}}(\bar{x}) \end{array}$$

From the positive border, the function \tilde{f} computes an approximate negative border, noted $\widetilde{\text{Bd}}^-(S)$ (see Definition 6).

Definition 6 (Approximate negative border). Let $\text{Bd}^+(S)$ be a positive border of frequent itemsets. The approximate negative border, noted $\widetilde{\text{Bd}}^-(S)$, is defined as follows:

$$\widetilde{\text{Bd}}^-(S) = \tilde{f}(\text{Bd}^+(S)) = \widetilde{\text{MinTr}}(\overline{\text{Bd}^+(S)}).$$

The return to a positive border (via the function g) allows to obtain an approximate positive border, noted $\widetilde{\text{Bd}}^+(S)$ (see Definition 7).

Definition 7 (Approximate positive border). Let $\widetilde{\text{Bd}}^-(S)$ be an approximate negative border of frequent itemsets. The approximate positive border, noted $\widetilde{\text{Bd}}^+(S)$, is defined as follows:

$$\widetilde{\text{Bd}}^+(S) = g(\widetilde{\text{Bd}}^-(S)) = \overline{\widetilde{\text{MinTr}}(\widetilde{\text{Bd}}^-(S))}.$$

From the positive border, the proposed approach produces the approximate negative border $\widetilde{Bd}^-(S)$ and the corresponding approximate positive border $\widetilde{Bd}^+(S)$. The following diagram presents all the dualization process:

$$\begin{array}{ccc} Bd^+(S) & \xrightarrow{\tilde{f}} & \widetilde{Bd}^-(S) \\ \widetilde{Bd}^+(S) & \xleftarrow{g} & \end{array}$$

Let us remark that we still have an exact dualization between the two approximated borders:

$$\begin{array}{ccc} \widetilde{Bd}^+(S) & \xrightarrow{f} & \widetilde{Bd}^-(S) \\ \widetilde{Bd}^-(S) & \xleftarrow{g} & \end{array}$$

In order to give a general overview of *FIBAD*, we introduce Algorithm 1 by noting that its main Steps 2 and 3 are treatments considered in the next section. Next, we illustrate *FIBAD* with Example 4.

Algorithm 1 General overview of *FIBAD*

Require: a positive border of frequent itemsets, $Bd^+(S)$

Ensure: the approximate negative and positive borders, $\widetilde{Bd}^-(S)$ and $\widetilde{Bd}^+(S)$

- {Steps 1, 2 and 3: Dualization using function \tilde{f} }
 - 1: $\mathcal{H} = \widetilde{Bd}^+(S)$; {Computation of the hypergraph from $Bd^+(S)$ }
 - 2: $\mathcal{H}_R = \text{HR}(\mathcal{H})$; {Reduction of the hypergraph \mathcal{H} }
 - 3: $\widetilde{Bd}^-(S) = \widetilde{\text{MinTr}}(\mathcal{H}) = \text{MinTr}(\mathcal{H}_R)$ {Computation of the approximate negative border ; The approximate minimal transversals of \mathcal{H} are the exact minimal transversals of \mathcal{H}_R }
 - {Step 4 and 5: Dualization using function g }
 - 4: $P = \text{MinTr}(\widetilde{Bd}^-(S))$; {Computation of the exact minimal transversals of $\widetilde{Bd}^-(S)$ }
 - 5: $\widetilde{Bd}^+(S) = \widetilde{P}$; {Computation of the approximate positive border}
 - 6: **return** $\widetilde{Bd}^-(S)$ and $\widetilde{Bd}^+(S)$;
-

Example 4. Let us take the example of Table 1 with $\text{minsup}=3$, $Bd^+(S) = \{A, BC, CE, CH, FH\}$, and let us compute the approximate borders with *FIBAD*.

Step 1: $\mathcal{H} = \widetilde{Bd}^+(S) = \{\overline{A}, \overline{BC}, \overline{CE}, \overline{CH}, \overline{FH}\} = \{BCDEFGH, ADEFGH, ABDFGH, ABDEFG, ABCDEG\}$

Steps 2 and 3: $\widetilde{Bd}^-(S) = \widetilde{\text{MinTr}}(\mathcal{H}) = \text{MinTr}(\mathcal{H}_R)$. Let us assume that this computation provides the following result: $\widetilde{Bd}^-(S) = \{D, E, G, AF, AH, BF, BH\}$.

Step 4: $P = \text{MinTr}(\widetilde{Bd}^-(S)) = \text{MinTr}(\{D, E, G, AF, AH, BF, BH\}) = \{ABDEG, DEFGH\}$

Step 5: $Bd^+(S) = \overline{P} = \{\overline{ABDEG}, \overline{DEFGH}\} = \{CFH, ABC\}$.

We can remark that A , B , C and BC are frequent itemsets and here ABC is considered as a frequent itemset. CFH is not frequent (its support is equal to 2) but it is almost frequent. These two itemsets can be interesting for applications like document recommendation. For instance, without our approach, FH is frequent and CFH is not frequent. The item C is potentially interesting. If the items are documents, with our approach, the item C can be recommended to a user.

In short, the distinctiveness of the *FIBAD*'s theoretical foundations is the approximate dualization where the new approximate function \tilde{f} is defined to compute the approximate negative border from the exact positive one. From a methodological point of view, the approximate function is defined as a sequence of two subfunctions: (1) reduction of the hypergraph formed by the complements of the itemsets of the exact positive border (Step 2 of Algorithm 1) and (2) computation of the exact minimal transversals of the reduced hypergraph (Step 3 of Algorithm 1). Thus, the resulted transversals are the approximate minimal transversals of the initial hypergraph. They correspond to the approximate negative border as the input hypergraph is formed by the complements of the itemsets of the exact positive border. Afterwards, the approximate positive border is computed from the approximate negative border by an exact function g . In the remainder of this paper, the sequence of the two subfunctions is called *AMTHR* and is detailed in the next section.

It should be noted that the function \tilde{f} can be supported by any other functions that compute approximate minimal transversals. The use of each alternative functions, instead of *AMTHR*, leads to a new strategy for computing approximate borders using dualization. In this paper, the function g is unchanged and its experimental section is dedicated to the comparison of our proposal with other alternative strategies.

5 Computation of Approximate Minimal Transversals

We propose the method *AMTHR* to compute the approximate minimal hypergraph transversals (Steps 2 and 3 of Algorithm 1). This method is based on a new algorithm that reduces the initial hypergraph. Our goal is to compute the minimal transversals on the reduced hypergraph which is smaller than the initial one. The proposed algorithm of reduction is specially designed to compute minimal transversals. It exploits the fact that the hyperedges formed by the complements of the itemsets of the positive border, strongly intersect (i.e. the average degree of a vertex is high). For instance, in Example 4, this hypergraph is: $\{BCDEFGH, ADEFGH, ABDFGH, ABDEFG, ABCDEG\}$. The proposed method is composed of two steps: (1) Reduction of the hypergraph and (2) Computation of the (exact) minimal transversals of the reduced hypergraph. At the end, the minimal transversals obtained from the reduced hypergraph are declared as the approximate minimal transversals of the initial hypergraph.

5.1 Reduction of the Hypergraph

The reduction method is based on both the intersections of the hyperedges and the degree of each vertex. The representative graph [6] (also called "line-graph") of the hypergraph is thus generated. Let us recall that the representative graph of the hypergraph \mathcal{H} is a graph whose vertices represent the hyperedges of \mathcal{H} and two vertices are adjacent if and only if the corresponding hyperedges in \mathcal{H} intersect. In our algorithm, we add values to the edges of the representative graph.

Algorithm 2 presents the reduction of a hypergraph \mathcal{H} . This algorithm is composed of three steps: (1) Computation of the degree of each vertex in \mathcal{H} (lines 1-3), (2) Generation of the valued representative graph of \mathcal{H} (lines 4-9), and (3) Generation of the reduced hypergraph from the valued representative graph (lines 10-17).

Algorithm 2 HR (Hypergraph Reduction)

Require: a hypergraph $\mathcal{H}=(V, E)$ where $|V|=n$ and $|E|=m$

Ensure: the reduced hypergraph \mathcal{H}_R

```

1: for all  $v \in V$  do
2:   Compute  $deg_{\mathcal{H}}(v)$ 
3: end for
4:  $V' \leftarrow \{v'_i \mid i = 1, \dots, m\}$ ; {each  $v'_i \in V'$  represents  $e_i \in E$ }
5:  $E' \leftarrow \{\}$ ;
6: for all  $v'_i \cap v'_j \neq \emptyset$  do
7:    $E' \leftarrow E' \cup \{(v'_i, v'_j)\}$ ;
8:    $w_{(v'_i, v'_j)} \leftarrow \sum_{v \in \{\psi^{-1}(v'_i) \cap \psi^{-1}(v'_j)\}} deg_{\mathcal{H}}(v)$ ;
9: end for
10:  $V_R \leftarrow \{\}$ ;
11:  $E_R \leftarrow \{\}$ ;
12: while  $E' \neq \emptyset$  do
13:   Select  $e'_{max} = (v'_{max_i}, v'_{max_j})$  having the maximal weight value
14:    $V_R \leftarrow V_R \cup \{\psi^{-1}(v'_{max_i}) \cap \psi^{-1}(v'_{max_j})\}$ ;
15:    $E_R \leftarrow E_R \cup \{\{\psi^{-1}(v'_{max_i}) \cap \psi^{-1}(v'_{max_j})\}\}$ ;
16:   Delete the edges  $e' \in E'$  where  $v'_{max_i}$  or  $v'_{max_j}$  is present
17: end while
18: return  $\mathcal{H}_R$ ;

```

Valued Representative Graph Generation (lines 1-9). Let be $\mathcal{H} = (V, E)$ a hypergraph ($|V| = n$ and $|E| = m$). The algorithm constructs a valued graph $G=(V', E')$ where $V' = \{v'_i\}$ ($i = 1, \dots, m$) and $E' = \{e'_k\}$ ($k = 1, \dots, l$). A vertex v'_i represents a hyperedge e_i from \mathcal{H} . Let be $\psi : E \rightarrow V'$ the bijective function that associates a hyperedge e_i to a vertex v'_i . A hyperedge between v'_i and v'_j shows that the intersection between the hyperedges $\psi^{-1}(v'_i)$ and $\psi^{-1}(v'_j)$

(e_i and e_j from \mathcal{H}) is not empty. The weight of an edge is based on the degree of each vertex in the corresponding intersection.

To evaluate the weight of a generated edge, we use the degree of each vertex from the initial hypergraph. The idea is that a vertex very present has a good chance to be in a minimal transversal. This expresses a "degree" of transversality. If the degree of a vertex is equal to the number of hyperedges then this vertex is a minimal transversal. Let us note that this heuristic is used by several algorithms that compute transversals [1, 36].

The weight of an edge $e'_k = (v'_i, v'_j)$, noted $w_{e'_k}$, is the sum of the degree of the vertices present in the intersection which has led to create this edge (see (1)).

$$w_{e'_k} = \sum_{v \in \{\psi^{-1}(v'_i) \cap \psi^{-1}(v'_j)\}} \text{deg}_{\mathcal{H}}(v). \quad (1)$$

Generation of the Reduced Hypergraph (lines 10-17). After the creation of the valued representative graph, the algorithm performs a selection of edges with a greedy approach. It selects the edge having the higher weight value while there are edges left in the valued representative graph G . Each selected edge is transformed to a hyperedge of the reduced hypergraph. This hyperedge contains the vertices from \mathcal{H} corresponding to the intersection of the two vertices of the edge.

We obtain, at the end, a set of hyperedges corresponding to the reduced hypergraph $\mathcal{H}_R=(V_R, E_R)$. Let us remark that if several edges have the same weight, the first edge found is selected.

Example 5. Let us consider the example of Table 1 as a hypergraph \mathcal{H} (6 hyperedges, 8 vertices). The transactions are the hyperedges e_k and the items are the vertices v_i . First, the degree of each vertex is evaluated: $\text{occur}[A] = 3$, $\text{occur}[B] = 3$, $\text{occur}[C] = 5$, $\text{occur}[D] = 1$, $\text{occur}[E] = 4$, $\text{occur}[F] = 3$, $\text{occur}[G] = 2$ and $\text{occur}[H] = 4$. Then, the intersections between the hyperedges are computed. For instance, $\psi^{-1}(v'_1) \cap \psi^{-1}(v'_3) = e_1 \cap e_3 = \{A, C, E\}$. The sum of the numbers of occurrences of A , C and E is equals to 12. This is the weight of the edge (v'_1, v'_3) generated in G . The adjacency matrix of the generated valued graph is:

$$\begin{array}{c} v'_1 \ v'_2 \ v'_3 \ v'_4 \ v'_5 \ v'_6 \\ \begin{array}{c} v'_1 \\ v'_2 \\ v'_3 \\ v'_4 \\ v'_5 \\ v'_6 \end{array} \begin{pmatrix} 0 & 11 & 12 & 3 & 5 & 9 \\ 11 & 0 & 9 & 0 & 8 & 12 \\ 12 & 9 & 0 & 7 & 9 & 13 \\ 3 & 0 & 7 & 0 & 7 & 7 \\ 5 & 8 & 9 & 7 & 0 & 15 \\ 9 & 12 & 13 & 7 & 15 & 0 \end{pmatrix} \end{array}$$

The edge (v'_5, v'_6) is selected because its weight is maximal (i.e., 15). The edges where v'_5 or v'_6 are present, are deleted. Thus, we have $V_R = \{B, C, F$,

$H\}$ and $E_R = \{\{B, C, F, H\}\}$). The adjacency matrix of the remaining valued graph is:

$$\begin{array}{c} v'_1 \\ v'_2 \\ v'_3 \\ v'_4 \\ v'_5 \\ v'_6 \end{array} \begin{pmatrix} v'_1 & v'_2 & v'_3 & v'_4 & v'_5 & v'_6 \\ 0 & 11 & 12 & 3 & 0 & 0 \\ 11 & 0 & 9 & 0 & 0 & 0 \\ 12 & 9 & 0 & 7 & 0 & 0 \\ 3 & 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The next selected edge is (v'_1, v'_3) . After the deletion of the edges where v'_1 or v'_3 are present, there are no remaining edges and the algorithm ends. The reduced hypergraph is $\mathcal{H}_R = (V_R, E_R)$ where $V_R = \{A, B, C, E, F, H\}$ and $E_R = \{\{A, C, E\}, \{B, C, F, H\}\}$. There are 6 vertices (instead of 8) and 2 hyperedges (instead of 6).

Remarks. Algorithm 2 can be implemented in time $O(m^3)$ where m is the number of hyperedges of the initial hypergraph. Let us remark that the selection of the edge having the maximal weight value (line 13) can be changed to the selection the first edge of the list of all edges, if the algorithm performs a quicksort of this list according to the weight before line 12. In the worst case, there are $\frac{(m^2-m)}{2}$ hyperedges in the reduced hypergraph. Let us remark that we could also compute $\min(\mathcal{H}_R)$ because there are no consequences on the next step, ie., the computation of minimal transversals (see Proposition 1).

5.2 Minimal Transversal Computation

The last step is the computation of the (exact) minimal transversals of the reduced hypergraph. These transversals correspond to the approximate minimal transversals of the initial hypergraph:

$$\widetilde{MinTr}(\mathcal{H}) = MinTr(\mathcal{H}_R).$$

Example 6. Let us continue Example 5. The minimal transversals of \mathcal{H}_R are: $\{C, AB, AF, AH, BE, EF, EH\}$. We consider them as the approximate minimal transversals of \mathcal{H} . Let us remark that the (exact) minimal transversals of \mathcal{H} are: $\{AB, AC, CD, CF, CH, EF, EH, GH, AFG, BDE\}$.

Let us note that an approximate minimal transversal is not necessarily a subset of an exact minimal transversal, for instance AH .

6 Experiments

As noted at the end of Sect. 4, the function \widetilde{f} proposed in *FIBAD* can be supported by any other functions that compute approximate minimal transversals.

In these experiments, we evaluate our dualization-based method that computes approximate borders by using *AMTHR*. How much this latter is efficient, i.e., the quality of the computed borders? In answering this question, we compute the distance between the computed approximate borders and the exact ones. We also ask the following question: How much *FIBAD* is sensitive to *AMTHR*? To reply to this question, we replace *AMTHR* with other methods that compute approximate minimal transversals. In practice, we consider two alternative strategies based respectively on the δ -*MTminer* [35] and *Staccato* [1] algorithms. Finally, we compare our dualization-based method with the *CartesianContour* algorithm [25] which computes directly, without dualization, the cover of $Bd^+(S)$ that corresponds to an approximate positive border. All the experiments have been performed on a Intel Xeon X5560 2.8 GHz with 16GB of memory. The implementations have been developed in JAVA.

6.1 Data and Protocol

Data. Four data sets have been used: Mushroom, Chess, Connect and Kosarak. They have been downloaded from the FIMI web site³. Mushroom contains data on 23 species of gilled mushrooms. Chess contains some strategies for chess sets. Connect contains strategies for the game of connect-4. Kosarak contains anonymized click-stream data of a hungarian on-line news portal.

Table 2. Data sets used in the experiments.

Data set	$ \mathcal{T} $	$ \mathcal{I} $	Avg. $ t_i $	Gouda & Zaki	Flouvat et al.
Mushroom	8,124	119	23.0	type 4	type II
Chess	3,196	75	37.0	type 1	type I
Connect	67,557	129	43.0	type 2	type II
Kosarak	990,002	41,270	8.1	type 3	type III

These data sets (see Table 2) have been chosen to cover the different types of existing data sets according to two classifications: Gouda & Zaki [20] and Flouvat et al. [17]. The classification proposed by Gouda & Zaki [20] (types 1, 2, 3 et 4) is based on the density and on the distribution of the positive border according to the size of the itemsets and the value of the minimum support threshold. Let us note that a data set is dense when it produces long itemsets for high values of the minimum support threshold. The classification proposed by Flouvat et al. [17] (types I, II et III) studies both the positive border and the negative border. Type I corresponds to data sets having long itemsets in the two borders and most of the itemsets in the two borders have approximately the same size. Type II corresponds to data sets having long itemsets in the positive border and the itemsets of the negative border are much smaller than those of the positive border. Type III is a special case of type I: the itemset size in the two

³ Frequent Itemset Mining Implementations, <http://fimi.ua.ac.be/data/>

borders is very close for very low minimal support values. Type III captures the notion of sparseness. Type I and Type II correspond to dense data sets. Chess and Connect are dense data sets. Kosarak is a sparse data set. Let us remark that Mushroom is special: Flouvat et al. classify it in Type II (as Connect) but Gouda & Zaki classify it in an other type, Type 4. Mushroom is between dense data sets and sparse data sets.

Protocol. For each data set and for some minimum support threshold values, we adopt the following protocol:

- 1) Exact border computation:
The exact positive border $Bd^+(S)$ is the input of the evaluated methods. Both the exact positive and negative borders, $Bd^+(S)$ and $Bd^-(S)$, will be used to evaluate the precision of the resulted approximate borders. The more an approximate border is close to the exact one, the more its precision is high. We compute $Bd^+(S)$ using the *IBE* algorithm [37]. Afterwards, we compute $Bd^-(S)$ from $Bd^+(S)$ according to Property 1.
- 2) Approximate border computation:
Firstly, we compute $Bd^-(S)$ and $Bd^+(S)$ from $Bd^+(S)$ by dualization using *AMTHR*. Next, this computation is re-done twice by replacing *AMTHR* by the δ -*MTminer* algorithm [35] and the *Staccato* algorithm [1], respectively.
- 3) Direct computation of approximate positive border:
We compute the cover of $Bd^+(S)$ using the *CartesianContour* algorithm [25]. This cover corresponds to an approximate positive border.
- 4) Evaluation of the quality of the computed borders.

In this experiments, we use the *Border-Diff* algorithm [12] to compute exact minimal transversals when it is needed. Alternatively, we can replace this algorithm by any other one which computes exact minimal transversals without however any change of the resulted borders.

For δ -*MTminer* (see Sect. 3), the best results have been obtained with δ set to 1, so we have selected this value. Let us recall that for *Staccato* (see Sect. 3), the more λ is high, the more *Staccato* is close to the exact solution. Nevertheless, the more λ is high, the more the execution time is high. Thus, we have chosen the highest values of λ before being impracticable: $\lambda=0.8$ for Mushroom, $\lambda=0.65$ for Chess, $\lambda=0.7$ for Connect, and $\lambda=0.95$ for Kosarak. *CartesianContour* (developed in C++) has been downloaded from the web page of one of the authors⁴. There is no need to set the number of itemsets of the computed borders, and the exact positive border is given as input (see Sect. 3).

Some statistics are computed: the number of itemsets of the computed border, the average size of the itemsets of the computed border, and the distance between the set of the itemsets of the computed border and the set of itemsets of the exact border.

⁴ <http://www.cs.kent.edu/~lliu/sourceCode.html>

To evaluate the distance between two borders, we have used the distance of Karonski & Palka [26] based on the Hausdorff distance. The cosine distance (see (2)) has been chosen to compute the distance between two elements (i.e., two itemsets). The distance D between two set of itemsets \mathcal{X} and \mathcal{Y} is defined in (3).

$$d(X, Y) = 1 - \frac{|X \cap Y|}{\sqrt{|X| \times |Y|}}. \quad (2)$$

$$D(\mathcal{X}, \mathcal{Y}) = \frac{1}{2} \{h(\mathcal{X}, \mathcal{Y}), h(\mathcal{Y}, \mathcal{X})\} \quad (3)$$

$$\text{where } h(\mathcal{X}, \mathcal{Y}) = \max_{X \in \mathcal{X}} \{ \min_{Y \in \mathcal{Y}} d(X, Y) \}.$$

Example 7. Let us consider Example 4 (see Sect. 4). The distance between $Bd^-(S)$ and $Bd^-(S)$ is equal to 0.395 ($\frac{1}{2}(0.29 + 0.5)$). The distance between $Bd^+(S)$ and $Bd^+(S)$ is equal to 0.385 ($\frac{1}{2}(0.18 + 0.59)$).

Let us recall that our main goal is to produce approximate borders smaller than exact borders, while having the lowest values of the distance to the exact borders.

6.2 Results and Discussion

Hypergraph Reduction. Table 3 presents the hypergraphs and the transversals computed by *FIBAD*. Let us recall that for an experiment (a data set and a *minsup* value) the computed hypergraph, noted $\mathcal{H} = (V, E)$, correspond to the complement of the itemsets of the positive border ($V \subseteq \mathcal{I}$ and $E = Bd^+(S)$). The reduced hypergraph of \mathcal{H} , noted \mathcal{H}_R , is computed with Algorithm 2 ($\mathcal{H}_R = HR(\mathcal{H})$). For each data sets and *minsup* values, we have the number of hyperedges of \mathcal{H} ($|E|$), the number of vertices of \mathcal{H} ($|V|$), the average degree of a vertex ($deg_{\mathcal{H}}(v)$), the number of minimal transversals of \mathcal{H} ($|MinTr(\mathcal{H})|$), the number of hyperedges of $\min(\mathcal{H}_R)$ ($|E_R|$), the number of vertices of $\min(\mathcal{H}_R)$ ($|V_R|$) and the number of minimal transversals of \mathcal{H}_R ($|MinTr(\mathcal{H}_R)|$). In order to better show the impact of the reduction, we consider here $\min(\mathcal{H}_R)$ instead of \mathcal{H}_R . The computation of $\min(\mathcal{H}_R)$ has no consequences on the computation of minimal transversals (see Proposition 1).

We can observe that the generated hypergraphs \mathcal{H} strongly intersect, i.e., the average degree of a vertex is close to the number of hyperedges. These results confirm the observation made at the beginning of Sect. 5.

The number of vertices is almost the same in the initial hypergraph and the reduced hypergraph. The number of hyperedges of the reduced hypergraph is much lower than the number of hyperedges of the initial hypergraph. For instance, on Chess with *minsup* equals to 50%, there are 3,351 hyperedges instead of 11,463. The "space savings" (i.e., $1 - \frac{|E_R|}{|E|}$) is equal to 70.7%. In average, over all the data sets, the space savings is equal to 63%.

We can also see that, in general, the number of minimal transversals of the reduced hypergraph is lower than the number of minimal transversals of the initial

Table 3. Computed hypergraphs and transversals.

Data set	Minsup (%)	Initial hypergraph				Reduced hypergraph		
		$ E $	$ V $	$deg_{\mathcal{H}}(v)$ (avg.)	$ MinTr $ (\mathcal{H})	$ E_R $ ($\min(\mathcal{H}_R)$)	$ V_R $ ($\min(\mathcal{H}_R)$)	$ MinTr $ (\mathcal{H}_R)
Mushroom	25	105	115	100.1	652	47	115	581
	20	158	115	150.4	1,012	63	115	879
	15	321	115	303.8	1,816	118	115	1,478
	10	558	118	525.1	3,103	235	118	3,125
	5	1,452	118	1,347.3	9,034	580	118	8,801
	3	2,627	118	2,414.1	16,361	1,075	118	15,674
	2	3,761	118	3,422.6	23,208	1,600	118	21,571
Chess	80	226	66	200.1	398	73	66	284
	75	489	74	437.3	742	155	73	446
	70	898	74	796.1	1,318	273	73	794
	65	1,779	74	1,568.7	2,644	594	73	1,713
	60	3,374	74	2,949.3	4,793	1,017	74	2,809
	55	6,261	74	5,429.4	8,435	1,955	73	4,943
	50	11,463	74	9,875.5	15,224	3,351	74	8,864
Connect	80	676	127	611.8	975	203	127	594
	75	961	127	861.6	1,317	292	127	797
	70	1,220	127	1,083.2	1,672	374	127	989
	65	1,588	127	1,400.1	2,196	467	127	1,289
	60	2,104	127	1,849.3	2,937	657	127	1,758
	55	2,836	127	2,481.1	3,859	827	127	2,145
	50	3,748	127	3,259.2	5,041	1,072	127	2,871
	45	4,720	127	4,079.1	6,364	1,448	127	3,747
40	6,213	127	5,342.9	8,358	1,887	127	4,735	
Kosarak	5	8	218	7.9	240	4	218	234
	4	13	218	12.8	268	5	218	244
	3	17	294	16.8	380	7	294	365
	2	30	490	29.8	780	15	490	759
	1	88	987	87.7	2,298	41	987	2,241
	0.5	305	2,952	304.6	14,791	146	2,952	14,524
	0.4	468	2,952	467.4	25,082	228	2,952	24,938
	0.3	814	3,804	813.2	51,770	394	3,804	51,567

hypergraph. This is not always true. For instance, on Mushroom with *minsup* equals to 10%, there are 3,103 minimal transversals for \mathcal{H} and 3,125 for \mathcal{H}_R . Let us take an example to illustrate this point. The minimal transversals of the hypergraph $\{ABC, ABD, ABE, ABF, ABG\}$ are $\{A, B, CDEFG\}$. There are 3 minimal transversals. Now, let us consider the hypergraph $\{AC, ABD, BG\}$, there are 4 minimal transversals ($\{AB, AG, BC, CDG\}$).

The hypergraph reduction is efficient in view of the space saving. Let us recall, that the hyperedges of the reduced hypergraph are selected using a heuristic which favors the search of approximate minimal transversals (see Sect. 5). Thus, our algorithm reduces the hypergraph while keeping the most important parts to find approximate minimal transversals.

Approximate Negative Borders. Figures 3, 4, 5 and 6 present, for each data sets, (a) the number of itemsets of the computed negative borders, (b) the distance between the computed negative borders and the exact negative borders, (c) the average size of an itemset of the computed negative borders, and (d) the execution time, according to the *minsup* value.

We can observe that the cardinality of $B\widetilde{d}^-(S)$ is lower than the cardinality of $Bd^-(S)$ for each data sets. The itemsets of $B\widetilde{d}^-(S)$ are shorter than the itemsets of $Bd^-(S)$. On Mushroom and Kosarak, the cardinality of $B\widetilde{d}^-(S)$ produced by *AMTHR* is very close to the cardinality of $Bd^-(S)$. The generated itemsets with *AMTHR* are a little shorter than the itemsets of the exact borders. For *1-MTminer*, the cardinality of $B\widetilde{d}^-(S)$ is the smallest on Mushroom and Kosarak. *Staccato* has generated the shortest itemsets on Mushroom but they are numerous. The itemsets produced by *1-MTminer* and *Staccato* are very short on Kosarak. On Chess and Connect, *AMTHR* and *1-MTminer* have produced a similar number of itemsets. These itemsets have a very close average size. We can remark that *Staccato* has produced a very small number of itemsets, and the average size of the itemsets is very small, on Chess and Connect.

Regarding the distance (between $B\widetilde{d}^-(S)$ and $Bd^-(S)$), *AMTHR* is not the best but it is close to the best algorithm for each data sets. *Staccato* has obtained the closest borders on Mushroom and Kosarak. This can be explained by the fact that λ has been set to high values for these data sets. This was not possible for Chess and Connect (dense data sets) and this explains why *Staccato* has the worst results on these data sets. *1-MTminer* has produced the closest borders on Chess and Connect. These data sets are dense and they have many long itemsets in the positive borders (see Sect. 6.1). Let us remember that *δ -MTminer* produces δ -minimal transversals: minimal transversals which can miss at most δ hyperedges (see Sect. 3). *δ -MTminer* does not control where are the missed hyperedges. On dense data sets, this is not a problem because the possibilities to miss a hyperedge are few many. Let us also note that *δ -MTminer* is particularly fast on dense data sets.

Concerning the execution time to compute the approximate negative borders, *Staccato* is the slowest and *1-MTminer* is the fastest. *AMTHR* is relatively fast on Mushroom and Kosarak but not on Chess and Connect.

We can conclude that *Staccato* and *1-MTminer* are the best to compute $B\widetilde{d}^-(S)$ respectively on sparse and dense data sets. Nevertheless, *AMTHR* is close to the best algorithm for each data sets. We can also note that *AMTHR* is not sensitive to the type of data sets, contrary to the other algorithms. Only the execution time varies (it is higher on dense data sets).

Approximate Positive Borders. Figures 7, 8, 9 and 10 present, for each data sets, (a) the number of itemsets of the computed positive borders, (b) the distance between the computed positive borders and the exact positive borders, (c) the average size of an itemset of the computed positive borders, and (d) the execution time, according to the *minsup* value.

The number of itemsets of $Bd^+(S)$ with *AMTHR* is the lowest on Mushroom. On the other data sets, *Staccato* has generated the lowest number of itemsets. *1-MTminer* have produced more itemsets than *AMTHR*, except for Kosarak. The itemsets of $Bd^+(S)$ are longer than the itemsets of $Bd^+(S)$. They are the longest with *Staccato* or *AMTHR* on each data sets.

AMTHR has obtained the closest $Bd^+(S)$ to $Bd^+(S)$ on all the data sets. On Connect, *1-MTminer* has also obtained good results. On Kosarak, *Staccato* and *1-MTminer* have produced bad results. This can be explained by the small number of itemsets of $Bd^-(S)$ and these itemsets are too short. The transition to $Bd^+(S)$ produces too few itemsets and these itemsets are too long. The other bad results can be explained by the same remark.

The itemsets of $Bd^+(S)$ are longer than the itemsets of $Bd^+(S)$ and *AMTHR* has obtained low distance values. We can say that the approximate positive borders generated by *AMTHR* contain some itemsets with new items, while being close to the itemsets of the exact positive border. These new items could be interesting for some applications like document recommendation.

Regarding the execution time, on Mushroom, Chess and Connect, *1-MTminer* is the slowest and *Staccato* is the fastest. On Kosarak, *1-MTminer* is the fastest and *Staccato* is the slowest. The execution time of *AMTHR* is not the best but still correct.

We can conclude that *AMTHR* is the best to compute $Bd^+(S)$. The distances are the lowest, the number is reduced, and the execution time is correct. We explain these results by the characteristics of $Bd^-(S)$ produced by *AMTHR*. We have previously seen that the cardinality of $Bd^-(S)$ is a little lower than the cardinality of $Bd^-(S)$, and the itemsets of $Bd^-(S)$ are a little shorter than the itemsets of $Bd^-(S)$. The transition to $Bd^+(S)$ produces some itemsets longer than the itemsets of $Bd^+(S)$ while being close to $Bd^+(S)$. The other algorithms have not done that.

Table 4. Global results obtained on all the computed approximate borders.

Method	Avg. distance	Avg. number of itemsets	Avg. time (sec.)
<i>AMTHR</i>	0.323	3342.6	15.05
<i>1-MTminer</i>	0.384	2191.1	1.53
<i>Staccato</i>	0.452	1989.4	41.63
<i>Border-Diff</i>	-	4492.1	28.88

Global results. Table 4 presents the average distance, the average number of itemsets, and the average execution time over all the computed borders of all the data sets. The results obtained by the computation of the exact borders by dualization using *Border-Diff*, are presented at the last line of the table.

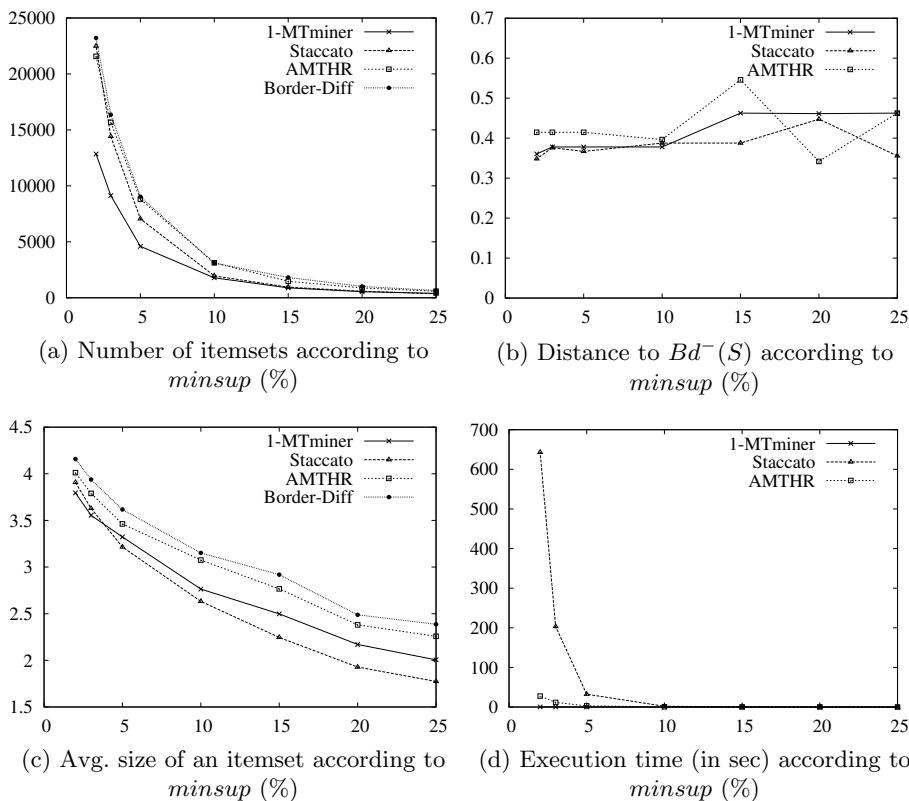


Fig. 3. Negative borders computed on Mushroom

We can observe that *AMTHR* has obtained the lowest average distance, and an average number of itemsets smaller than the average number of itemsets of the exact borders. The average execution time is correct. We can conclude that *AMTHR* is the best of the used methods in overall. Moreover, we have observed that *AMTHR* is robust according to the different types of data sets, contrary to *1-MTminer* which fails on sparse data sets and *Staccato* which does not produce good results on dense data sets. Let us remark that we have used *Border-Diff* to compute the exact minimal transversals of the reduced hypergraph (Step 3 of Algorithm 1). We have also used it for the computation of the approximate positive border (Step 4 of Algorithm 1). This is possible to use another more efficient algorithm, for instance one of the two algorithms presented in [32], in order to decrease the execution time.

FIBAD vs. CartesianContour. Table 5 presents the average distance, the average number of itemsets, and the average execution time over several approximate positive borders computed by *FIBAD* and *CartesianContour*. Indeed,

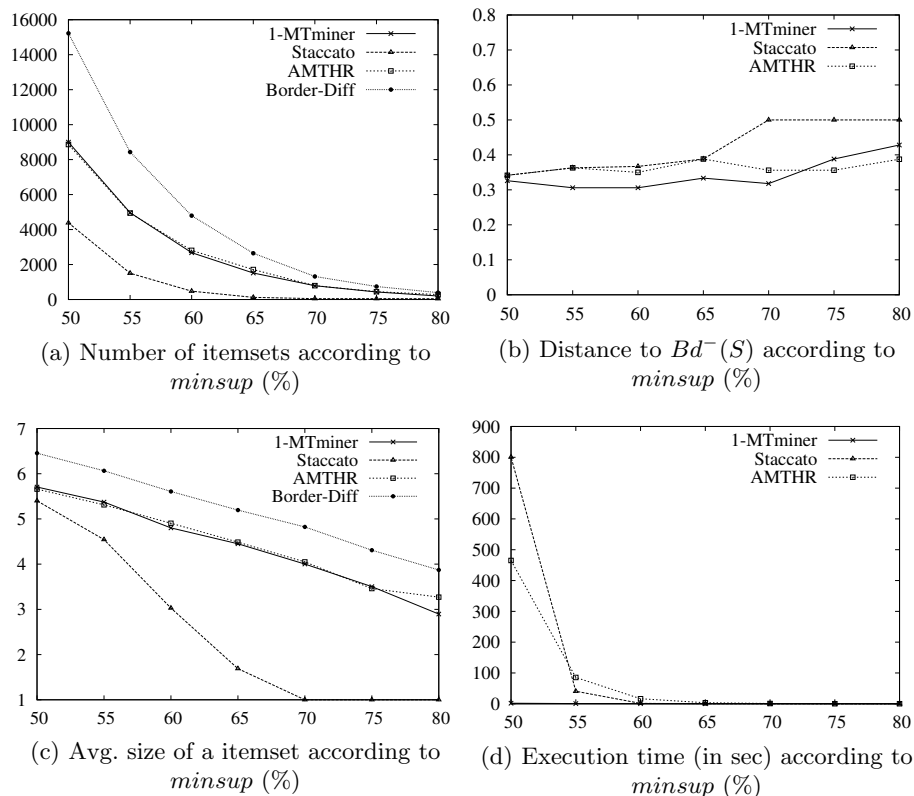


Fig. 4. Negative borders computed on Chess

Table 5. *FIBAD* vs. *CartesianContour*.

Approach	Avg. distance	Avg. number of itemsets	Avg. time (sec.)
<i>FIBAD</i>	0.289	286.6	9.67
<i>CartesianContour</i>	0.283	606.2	49.31

CartesianContour has not been able to compute the approximate positive borders on Chess ($minsup < 70\%$), Connect ($minsup < 75\%$) and Kosarak ($minsup < 1\%$). 17 of the 31 positive borders have been computed. Thus, the values presented in Table 5 have been computed only on these borders. This is why *CartesianContour* is not present in the previous results and discussion.

We observe that the average distances are very close. Nevertheless, the average number of itemsets of the approximate positive borders is the lowest with *FIBAD* (for information, there are 728.7 itemsets in average in the 17 exact positive borders). We also see that *FIBAD* is faster than *CartesianContour*. We can conclude that *FIBAD* is better than *CartesianContour* to generate

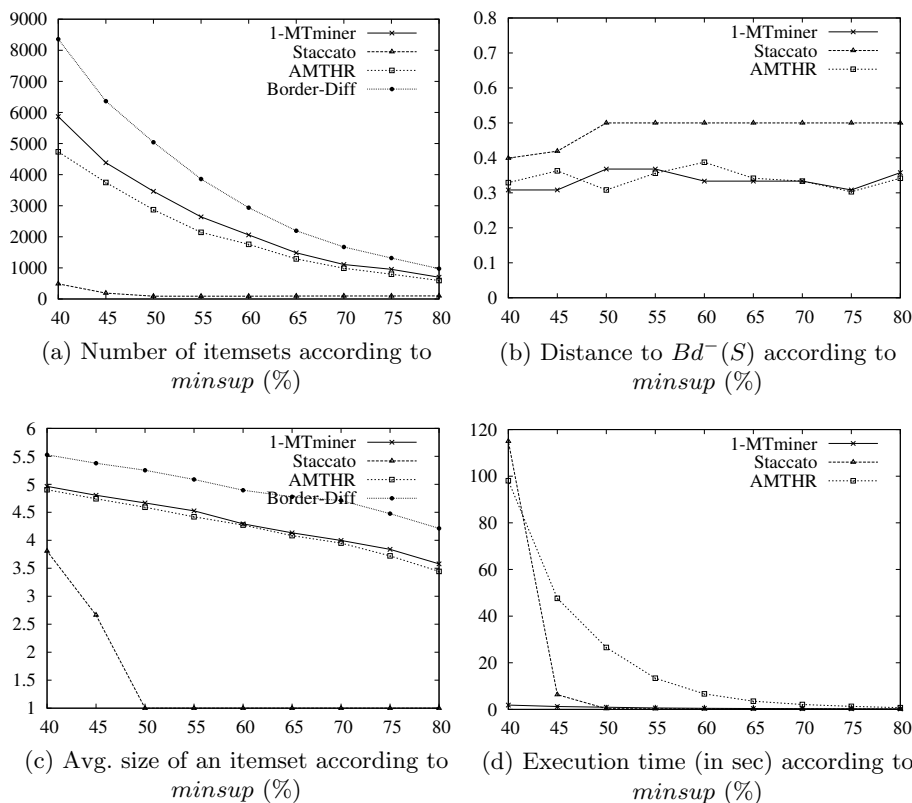


Fig. 5. Negative borders computed on Connect

approximate positive borders. Moreover, *CartesianContour* has some problems with dense data sets, and the $minsup$ value can not be low.

7 Conclusion

This paper deals with the problem of approximate borders computed by dualization. At the same time, this is a challenging theoretical problem which may play a valuable role in a wide range of applications. To achieve this goal, we introduced here the *FIBAD* approach leveraging dualization and computation of approximate minimal transversals of hypergraphs. Its originality comes from a new function we have defined to compute approximate negative borders. For this purpose, we start by reducing the initial hypergraph and, then, we compute its exact minimal transversals. This processing is implemented by the function *AMTHR* and used by *FIBAD* as an approximate dualization. To evaluate our approximate dualization method, we replaced *AMTHR* with other methods that compute approximate minimal transversals. In particular, we considered

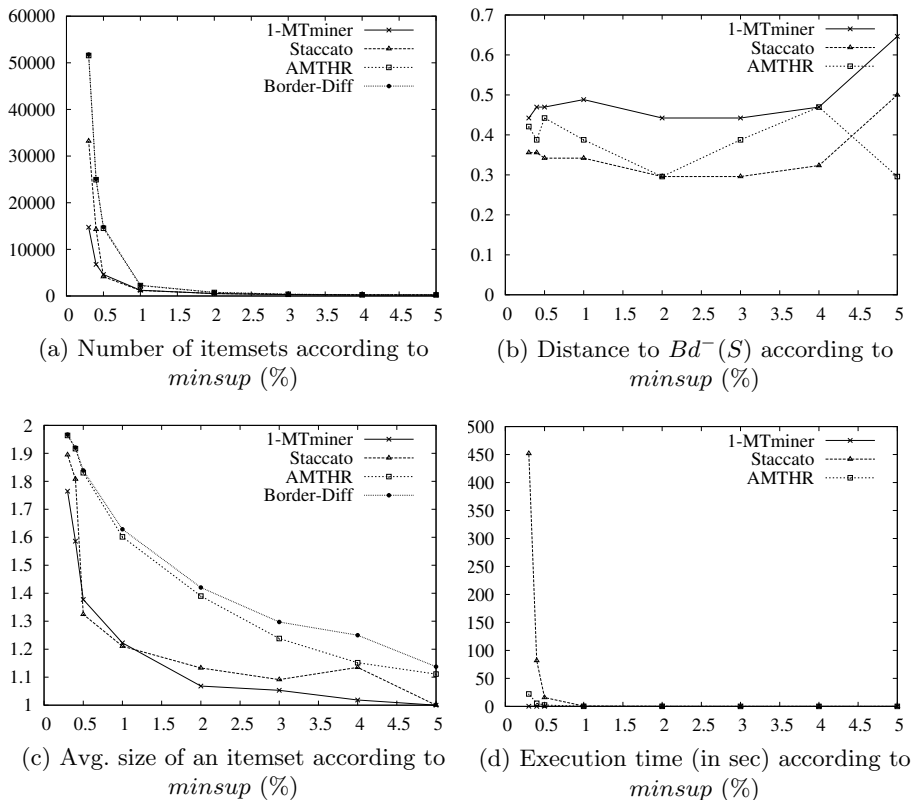


Fig. 6. Negative borders computed on Kosarak

two alternative methods based on the δ -*MTminer* and *Staccato* algorithms, respectively. We also compared our method with *CartesianContour* that computes directly, without dualization, the approximate borders. The experimental results have showed that our method outperforms the other methods as it produces borders which have the highest quality. It produces an approximate positive border smaller than the exact positive border, while keeping a low distance with the exact border. Through these experiments, we have observed that our approach is robust according to the different types of data sets. We can note that for sparse data sets, it is particularly efficient. This point is very interesting for future applications on the Web where most of the constructed data sets are sparse and very large (for instance, data from a web server log file). We have also seen that the proposed approach is able to find potentially interesting items for some applications like document recommendation, for instance. In the future, we will develop a recommendation system using the approximate positive borders generated by *FIBAD*. In that way, we will be able to evaluate the quality of the generated borders in an applicative context.

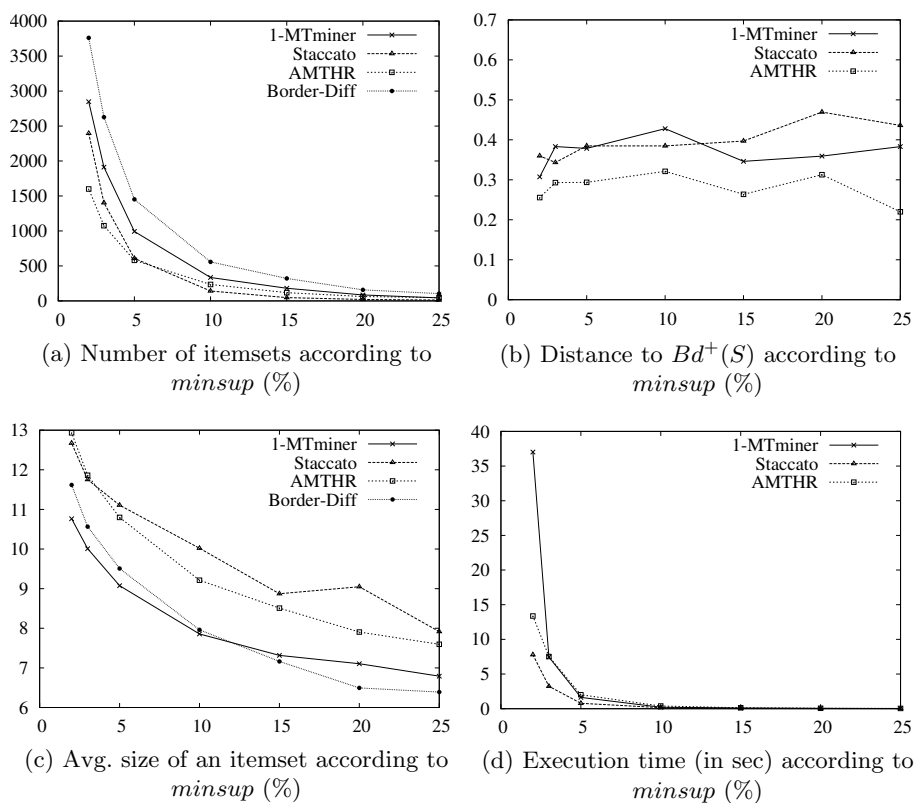


Fig. 7. Positive borders computed on Mushroom

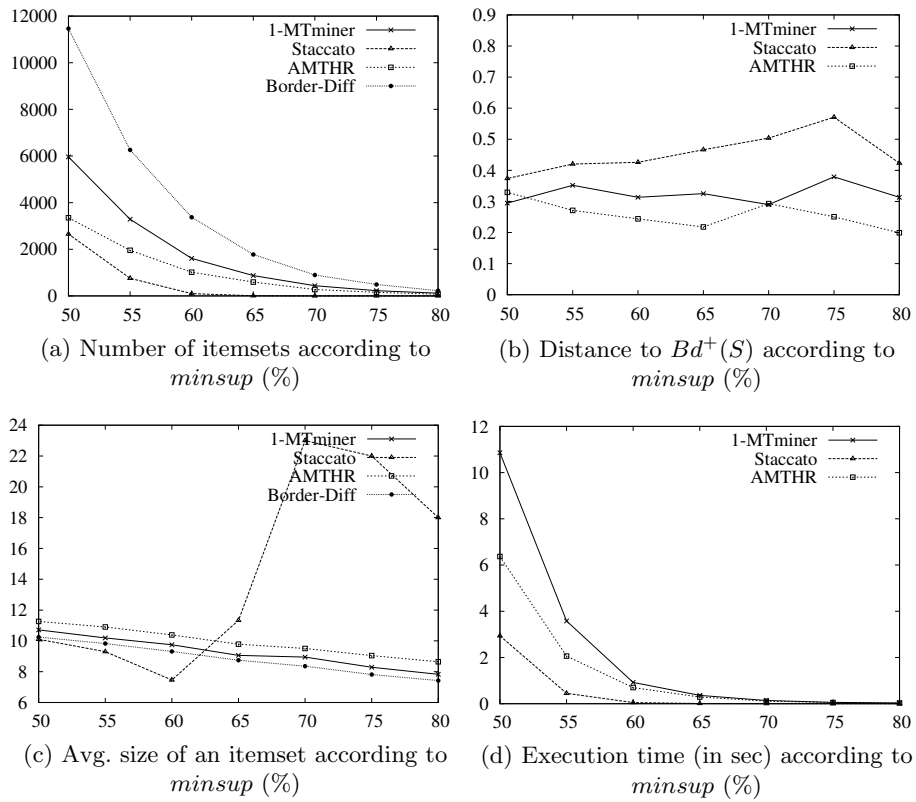


Fig. 8. Positive borders computed on Chess

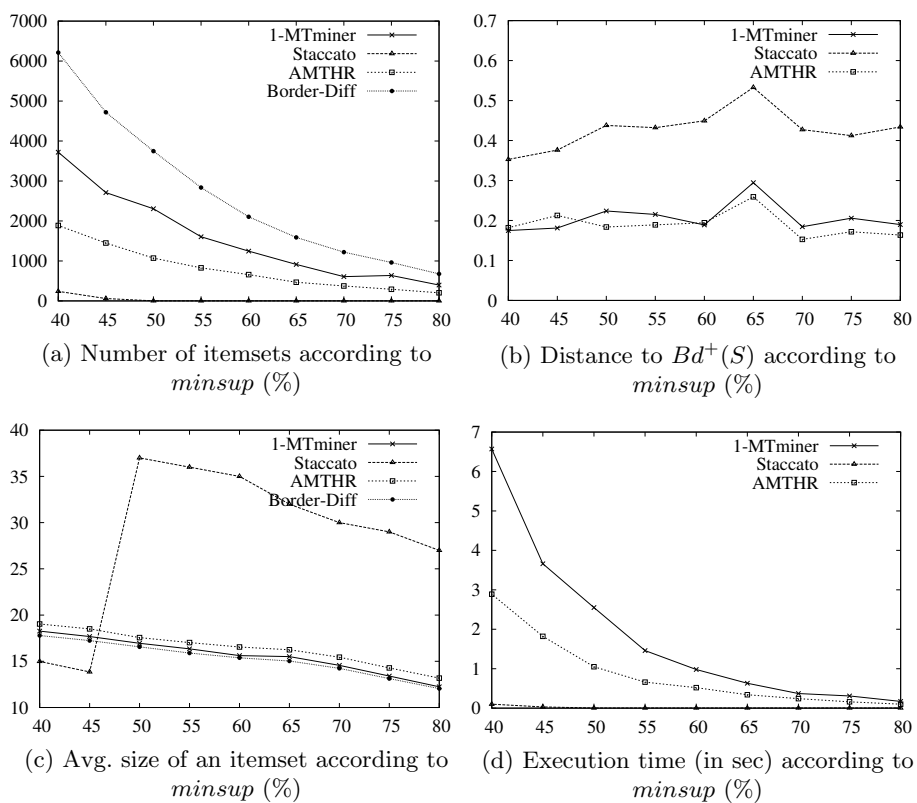


Fig. 9. Positive borders computed on Connect

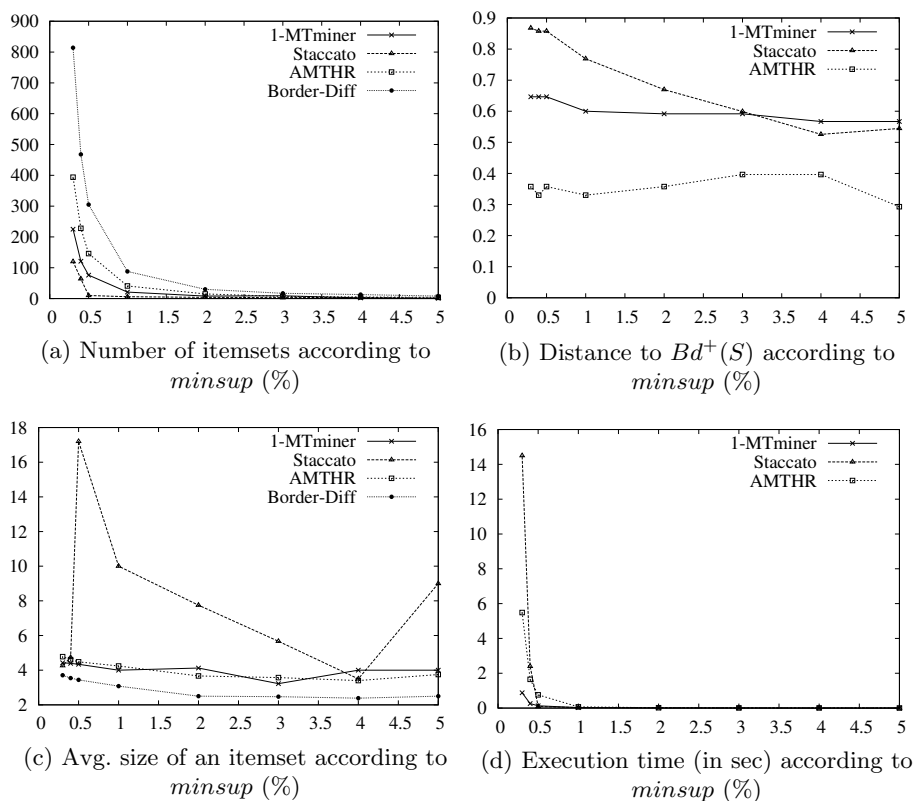


Fig. 10. Positive borders computed on Kosarak

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