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On the fourth derivative test for exponential sums

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Abstract

We give an upper bound for the exponential sum $\sum_{m=1}^M \exp(2i\pi f(m))$ where f is a real valued function whose fourth derivative has the order of magnitude $\lambda > 0$ small. Van der Corput's classical bound, in terms of M and λ only, involves the exponent $1/14$. We show how this exponent may be replaced by any $\theta < 1/12$ without further hypotheses. The proof uses a recent result by Wooley on the cubic Vinogradov system.

Let $C \geq 1$ be a real constant fixed once and for all. Let $a \in \mathbb{R}$, $M \in \mathbb{N}$ with $M \geq 10$. Let $0 < \lambda_4 < 1/10$ be a real number and let $f : [a, a + M] \rightarrow \mathbb{R}$ be a C^4 function such that

$$\lambda_4 \leq |f^{(4)}(x)| \leq C\lambda_4 \text{ for } x \in [a, a + M]. \quad (1)$$

We are interested in bounding the exponential sum

$$S_{a,M}(f) := \sum_{a < m \leq a+M} e(f(m)), \quad (2)$$

with the usual notation $e(t) := e^{2i\pi t}$ for $t \in \mathbb{R}$. Under the previous assumptions, van der Corput's result asserts that setting $\theta_4 = 1/14$ and $\beta_4 = 4/7$, one has

$$S_{a,M}(f) \ll_C M\lambda_4^{\theta_4} \quad (3)$$

uniformly for $a \in \mathbb{R}$, $M \geq \lambda_4^{-\beta_4}$ and f satisfying (1).

The purpose of this paper is to improve on the exponent θ_4 in (3). Our result is the following.

Theorem 1 *Let $a \in \mathbb{R}$, $M \in \mathbb{N}$ with $M \geq 10$. Let $0 < \lambda_4 < 1/10$ be a real number and let $f : [a, a + M] \rightarrow \mathbb{R}$ be a C^4 function satisfying (1). Then for each $\varepsilon > 0$, and with the notation (2), one has*

$$S_{a,M}(f) \ll_{C,\varepsilon} M^{1+\varepsilon} \lambda_4^{1/12} + M^{\frac{11}{12}+\varepsilon}$$

uniformly for $a \in \mathbb{R}$, $M \geq \lambda_4^{-3/5}$ and f satisfying (1).

Remark. Theorem 1 is not the first improvement of Van der Corput's result. Here is a brief history of these results under the hypothesis (1) alone :

- Sargos [4] proved that for $\theta_4 = 1/14$, β_4 may be replaced by $3/7$. Moreover, in the same paper, he also proved that θ_4 could be replaced by any $\theta < \frac{3}{40}$ provided that β_4 is replaced by $3/5$.
- Robert & Sargos [1] proved θ_4 may be replaced by any $\theta < \frac{1}{13}$ provided that β_4 is replaced by $8/13$.

Note that our Theorem 1 implies that θ_4 may be replaced by any $\theta < 1/12$ provided that β_4 is replaced by 1.

It should also be mentioned that more generally when $k \geq 2$ is fixed, van der Corput proved that if $f : [a, a + M] \rightarrow \mathbb{R}$ is C^k such that $\lambda_k \leq |f^{(k)}(x)| \leq C\lambda_k$, then $S_{a,M}(f) \ll_C M\lambda_k^{\theta_k}$ for $M \geq \lambda_k^{-\beta_k}$ with $\theta_k := 1/(2^k - 2)$, $\beta_k := 2^{k-1}/(2^k - 2)$. For $k = 2$, the example of the Gauss sums shows at once that $\theta_2 = 1/2$ is optimal. For $k = 3$, the exponent $\theta_3 = 1/6$ is conjectured to be optimal even for $M \geq \lambda_3^{-1}$ (see [4]). For $k \geq 5$, explicit better exponents can be produced (see for example [3] for an exposition of the recent results for $4 \leq k \leq 10$).

Proof of Theorem 1. By a classical series of transforms, the proof reduces to estimating the number $J_s(N)$ of solutions of

$$\sum_{j=1}^s (x_j^r - y_j^r) = 0 \quad (r = 1, 2, 3) \quad 1 \leq x_j, y_j \leq N. \quad (4)$$

The only novelty in the proof of Theorem 1 is the use of Wooley's recent breakthrough [5] where he proved Vinogradov's cubic conjecture, namely that for each integer $s \geq 1$ and each $\varepsilon > 0$ one has

$$J_s(N) \ll_{\varepsilon} N^{s+\varepsilon} + N^{2s-6+\varepsilon} \quad (N \geq 1). \quad (5)$$

The preliminary transform of $S_{a,M}(f)$ is taken care of by the following result, which is a particular case of Lemme 1 of [3] with $k = 4$.

Lemma 1 *Let $a \in \mathbb{R}$, $M \in \mathbb{N}$ with $M \geq 10$. Let $0 < \lambda_4 < 1/10$ be a real number and let $f : [a, a + M] \rightarrow \mathbb{R}$ be a C^4 function satisfying (1). Then setting $N := \lfloor \lambda_4^{-1/5} \rfloor$, and with the notation (2), one has for each $s \geq 1$*

$$|S_{a,M}(f)|^{2s} \ll_{s,C} M^{2s} \lambda_4 \left(1 + \frac{1}{M\lambda_4}\right) \left(\frac{J_s(N)}{N^{2s-6}}\right) \log M$$

uniformly for $a \in \mathbb{R}$, $M \geq \lambda_4^{-3/5}$ and f satisfying (1), where $J_s(N)$ has been defined in (4).

The proof of Theorem 1 is now straightforward : we use Lemma 1 with $s = 6$, and we bound $J_6(N)$ using (5). This completes the proof of Theorem 1. $\square$

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