PDE Models in Macroeconomics

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Abstract

The purpose of this article is to get mathematicians interested in studying a number of PDEs that naturally arise in macroeconomics. These PDEs come from models designed to study some of the most important questions in economics. At the same time they are highly interesting for mathematicians because their structure is often quite difficult. We present a number of examples of such PDEs, discuss what is known about their properties, and list some open questions for future research.

1 Introduction

Macroeconomics is the study of large economic systems. Most commonly this system is the economy of a country. But it may also be a more complex system such as the world as a whole, comprised of a large number of interacting smaller geographic regions. Macroeconomics is concerned with some of the most important questions in economics, for example: What causes recessions? Why are some countries so much poorer than others?

Traditionally macroeconomic theory has focused on studying systems of difference equations or ordinary differential equations describing the evolution of a relatively small number of macroeconomic aggregates. These systems are typically derived from the optimal control problem of a "representative agent". In the last thirty years, however, macroeconomics has seen the development of theories that explicitly model the equilibrium interaction of heterogeneous agents, e.g., heterogeneous consumers, workers and firms (see in particular the early contributions of Bewley, 1986; Aiyagari, 1994; Huggett, 1993; Hopenhayn, 1992). The development of these theories opens up the study of a number of important questions: Why are income and wealth so unequally distributed? How is inequality affected by aggregate economic conditions? Is there a trade-off between inequality and economic growth? What are the forces that lead to the concentration of economic activity in a few very large firms? And why do instabilities in the financial sector seem to matter so much for the macroeconomy?

Heterogeneous agent models are usually set in discrete time. While they are workhorses of modern macroeconomics, relatively little is known about their theoretical properties and they often prove difficult to compute. To make progress, some recent papers have therefore studied continuous time versions of such models. Our paper reviews this literature. Macroeconomic models with heterogeneous agents share a common mathematical

structure which, in continuous time, can be summarized by a system of coupled non-linear PDEs: (1) a Hamilton-Jacobi-Bellman equation describing the optimal control problem of a single atomistic individual; (2) an equation describing the evolution of the distribution of a vector of individual state variables in the population (such as a Fokker-Planck equation, Fisher-KPP equation, or Boltzmann equation). While plenty is known about the properties of each type of equation individually, our understanding of the coupled system is much more limited. Lasry and Lions (2007) have termed such a system a mean field game and obtained some theoretical characterizations for special cases, but many open questions remain. The purpose of this article is to present important examples of these systems of PDEs that arise naturally in macroeconomics, to discuss what is known about their properties, and to highlight some directions for future research.

In Section 2 we present a model describing an economy consisting of a continuum of heterogeneous individuals that face income shocks and can trade a risk-free bond that is in zero net supply. This is the simplest model to illustrate the basic structure of heterogeneous agents framework used in macroeconomics, and it is the building block of many models studying the interaction between macroeconomic aggregates and the distribution of income and wealth. In Section 3 we review PDEs that have been used to describe the distribution of the many economic variables that obey power laws, e.g., city and firm size, wealth and executive compensation. One building block of all of these models is the Fokker-Planck equation for a geometric Brownian motion. This equation is then combined with a model of exit and entry, for instance taking the form of a variational inequality of the obstacle type derived from an optimal stopping time problem. In Section 4 we present a class of models describing processes of economic growth due to experimentation and knowledge diffusion, or alternatively the percolation of information in financial markets. These models generate richer, more non-local dynamics, that give rise to Fisher-KPP or Boltzmann type equations.

In Section 5 we introduce a class of models that is substantially more complicated than those in the preceding sections: models with "aggregate shocks" designed to study business cycle fluctuations. These theories have the property that macroeconomic aggregates, including the distribution of individual states, are stochastic variables rather than just varying deterministically as in the models studied thus far. This creates the difficulty that the distribution – an infinite-dimensional object – has to be included in the state space of the individual optimal control problem. The resulting optimal control problem is no longer a standard HJB equation but instead a "HJB equation in the space of density functions," a very challenging object mathematically. We present the most canonical version of such a theory: the model in Section 2 but now with aggregate income shocks. But in principle, any of the theories in the preceding sections could be enriched by introducing such aggregate shocks. Finally, in section 6 we note that also models with a finite number of agents, rather than a continuum as in the preceding sections, are of interest in certain macroeconomic applications. We present a model of firm dynamics in an oligopolistic industry which takes the form of a differential game.

Space limitations have forced us to leave out other important areas of macroeconomics and economics more broadly where PDEs, and continuous time methods in general, have played an important role in recent years. A good example is the large literature studying the design of optimal dynamic contracts and policies. See for example the recent work by

Sannikov (2008), Williams (2011), and Farhi and Werning (2013). Another area is given by models of the labor market. See for example the recent contribution by Alvarez and Shimer (2011) and the review by Lentz and Mortensen (2010). Finally, throughout this paper we focus on equilibrium allocations in which individuals take as given the actions of others rather than coordinating with them. As a result these equilibrium allocations are in general suboptimal. Optimal allocations in heterogeneous agent models can be analyzed along the lines of Nuño (2013) and Lucas and Moll (2014).

2 Income and Wealth Distribution

The discrete time model of Aiyagari (1994) Bewley (1986) and Huggett (1993) is one of the workhorses of modern macroeconomics. This model captures in a relatively parsimonious way the evolution of the income and wealth distribution and its effect on macroeconomic aggregates. It is a natural framework to study the effect of various policies and institutions on inequality. A huge number of problems in macroeconomics have a similar structure and so this is a particularly useful starting point. The simplest formulation of the model is due to Huggett (1993) and we here present a continuous time formulation of Huggett's model presented in Achdou, Lasry, Lions, and Moll (2014a).

There is a continuum of infinitely lived households that are heterogeneous in their wealth a and their income z. Their income evolves exogenously according to a diffusion process $dz_t = \mu(z_t)dt + \sigma(z_t)dW_t$ in a closed interval $[\underline{z}, \bar{z}]$ (it is reflected at the boundaries if it ever reaches them). Households can borrow and save at an interest rate $r(t)$ which is determined in equilibrium. The evolution of their wealth is chosen optimally as described below. Importantly, there is a *state constraint* $a \geq \underline{a}$ for some scalar $-\infty < \underline{a} \leq 0$. This state constraint has the economic interpretation of a *borrowing constraint*, e.g. if $\underline{a} = 0$ households can only save and cannot borrow at all. Households have utility functions $u(c)$ over consumption c that are strictly increasing and strictly concave (e.g. $u(c) = c^{1-\gamma}/(1-\gamma), \gamma > 0$) and they maximize the present discounted value of utility from consumption. The equilibrium can be characterized in terms of a HJB equation for the value function v and a Fokker-Planck equation for the density of households g. In a *stationary* equilibrium, the unknown functions v and g and the unknown scalar r satisfy the following system of coupled partial differential equations (stationary mean field game) on $(\underline{a}, \infty) \times (\underline{z}, \bar{z})$:

$$\frac{1}{2}\sigma^2(z)\partial_{zz}v + \mu(z)\partial_z v + (z + ra)\partial_a v + H(\partial_a v) - \rho v = 0 \tag{1}$$

$$-\frac{1}{2}\partial_{zz}(\sigma^2(z)g) + \partial_z(\mu(z)g) + \partial_a((z+ra)g) + \partial_a(\partial_p H(\partial_a v)g) = 0 \tag{2}$$

$$\int g(a,z)dadz = 1, \quad g \geq 0 \tag{3}$$

$$\int ag(a,z)dadz = 0 \tag{4}$$

where the Hamiltonian H is given by

$$H(p) = \max_{c \geq 0}\left(-pc + u(c)\right), \tag{5}$$

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The function v satisfies a state constraint boundary condition at $a = \underline{a}$ and Neumann boundary conditions at $z = \underline{z}$ and $z = \bar{z}$. Solutions to the HJB equation (1) will in general be smooth. Given this smoothness, it is possible to show that the state constraint boundary condition is equivalent to

$$z + ra + \partial_p H(\partial_a v) \geq 0, \quad a = \underline{a} \tag{6}$$

so that the trajectory of a points towards the interior of the state space. The interpretation of the equilibrium condition (4) is as follows: wealth a here takes the form of bonds and the equilibrium interest rate r is such that bonds are in *zero net supply*. That is, for every dollar borrowed there is someone else who saves a dollar. Note that the interest rate r is the only variable through which the distribution g enters the HJB equation (7).

The time-dependent analogue of (1) to (4) is also of interest. In the time-dependent equilibrium, the unknown functions v and g satisfy the following system of coupled partial differential equations (time-dependent mean field game) on $(\underline{a}, \infty) \times (\underline{z}, \bar{z}) \times (0, T)$:

$$\partial_t v + \frac{1}{2}\sigma^2(z)\partial_{zz}v + \mu(z)\partial_z v + (z + r(t)a)\partial_a v + H(\partial_a v) - \rho v = 0 \tag{7}$$

$$\partial_t g - \frac{1}{2}\partial_{zz}(\sigma^2(z)g) + \partial_z(\mu(z)g) + \partial_a((z + r(t)a)g) + \partial_a(\partial_p H(\partial_a v)g) = 0 \tag{8}$$

$$\int g(a, z, t)dadz = 1, \quad g \geq 0 \tag{9}$$

$$\int ag(a, z, t)dadz = 0 \tag{10}$$

where the Hamiltonian H is given by (5). The density g satisfies the initial condition $g(a, z, 0) = g_0(a, z)$. For the terminal condition for the value function v we generally take T large and impose $v(a, z, T) = v_\infty(a, z)$ where v_∞ is the stationary value function, i.e. the solution to the stationary problem (1) to (4). The function v also still satisfies the state constraint boundary condition (6) and Neumann boundary conditions at $z = \underline{z}$ and $z = \bar{z}$.

Theoretical Results. Achdou, Lasry, Lions, and Moll (2014a) have analyzed some theoretical properties of both the time-varying and stationary problems. We here briefly review the (rather incomplete) theoretical knowledge of these problems, followed by a list of open question regarding in particular the well-posedness of the problems. Achdou, Lasry, Lions, and Moll (2014a) first analyze the stationary problem (1) to (4) under the additional assumption that the state constraint satisfies $\underline{a} > -\underline{z}/r$. Since (6) can be written as $\partial_a v(\underline{a}, z) \geq u'(z + r\underline{a})$, one can see that this assumption implies that the state constraint will bind for z low enough. That is, the borrowing constraint is "tight." Of particular interest is the stationary *saving policy function*

$$s(a, z) = z + ra + \partial_p H(\partial_a v(a, z)),$$

that is the optimally chosen drift of wealth a, and the behavior of the implied stationary distribution g. Importantly, one can show that the expansion of the function s around $\underline{a}$ satisfies the following property: there exists z^* with $\underline{z} < z^* < \bar{z}$ such that

$$s(a, z) \sim -\bar{s}_z\sqrt{a - \underline{a}}, \quad \bar{s}_z > 0, \qquad \underline{z} \leq z \leq z^*, \tag{11}$$

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meaning that in particular the derivative $\partial_a s$ becomes unbounded when we let a go to $\underline{a}$. It then follows from this property that the stationary distribution g is unbounded and has a Dirac mass at $a = \underline{a}$ for $z \leq z^*$. The existence of a Dirac mass in the stationary version of (7) to (10) of course complicates the mathematics substantially. At the same time, it is also one of the economically most interesting predictions of the model. What fraction of individuals in an economy such as the United States are borrowing constrained and how we would expect this to change when various features of the environment (say the stochastic process for z) change is an important question with wide-reaching policy implications. That interesting economics and challenging mathematics go hand in hand is one of the main themes of this paper.

Achdou, Lasry, Lions, and Moll (2014a) prove the existence of a solution to (1) to (4), i.e. of a stationary equilibrium. The key step in the proof is to analyze solutions v and g to (1) to (3) *for given r* and to show that the corresponding first moment of g, $m(r) = \int ag(a,z)dadz$, goes to $\underline{a}$ as $r \to -\infty$ and that it becomes unbounded as we take r to ρ_-. It follows from this that there exists an r such that (4) holds. Currently open theoretical questions are:

1. uniqueness of a solution to (1) to (4), i.e. of a stationary equilibrium

2. existence of a solution to (7) to (10), i.e. of a time-dependent equilibrium

3. uniqueness of a solution to (7) to (10), i.e. of a time-dependent equilibrium

The main difficulty in the first question, uniqueness of a stationary solution, lies in showing that (or finding conditions under which) the first moment of g, $m(r)$, is monotone as a function of r.

Numerical Methods. Achdou, Lasry, Lions, and Moll (2014a) have also developed numerical methods for solving both the stationary and time-dependent problems, based on Achdou, Camilli, and Capuzzo Dolcetta (2012). Figure 1 plots the optimal stationary saving policy function s and the implied distribution g. These are computed under the assumption that u in (5) is given by $u(c) = c^{1-\gamma}/(1-\gamma)$ with $\gamma = 2$. In the figures, one can see that s satisfies (11) and g has a Dirac mass for low z (the numerical method computes discretized versions of the equations so the Dirac mass corresponds to a finite density). Time-dependent solutions can be computed in a similar fashion and the evolution of the distribution over time can be visualized as "movies," see e.g. `http://www.princeton.edu/~moll/aiyagari.mov`.

3 Models of Power Laws

One of the most ubiquitous regularities in empirical work in economics and finance is that the empirical distribution of many variables can well be approximated by a power law. Examples are the distributions of income and wealth, of the size of cities and firms, stock market returns, trading volume, and executive pay. See Gabaix (2009) who reviews the theoretical and empirical literature on power laws.

(a) Saving as Function of Income and Wealth   (b) Distribution of Income and Wealth
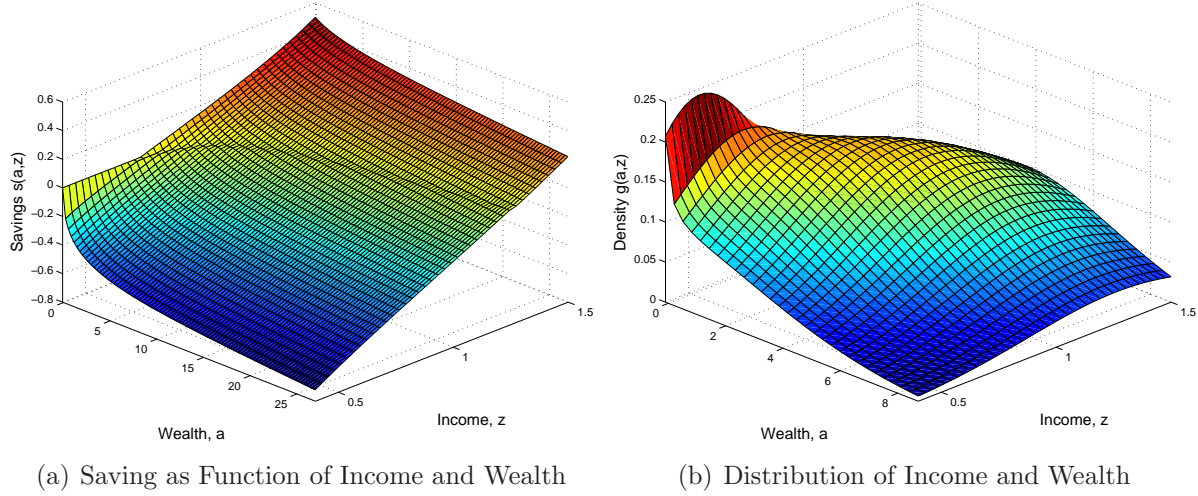
Figure 1: Numerical Solution to Stationary Equilibrium, (7) to (10)

Gabaix (1999) has proposed a simple explanation of power law phenomena that naturally leads to PDEs: many variables follow geometric Brownian motions, combined with a "small friction" such as a minimum size in the form of a reflecting barrier or small "death shocks." The following material is based on Gabaix (2009). Consider a stochastic process

$$\frac{dz_t}{z_t} = \bar{\mu}dt + \bar{\sigma}dW_t, \tag{12}$$

where $\bar{\mu} < 0$ and $\bar{\sigma} > 0$ are scalars. For sake of concreteness, consider the case where z represents the size or productivity of a firm and we are interested in the firm size distribution. But of course z could be city size or any other variable as well. Further assume that there is a minimum firm size $z_{\min}$ in the form of a reflecting barrier (other mechanisms are possible as well and we explore some below). The stationary firm size distribution f satisfies the Fokker-Planck equation

$$\frac{1}{2}\partial_{zz}\left(\bar{\sigma}^2 z^2 f\right) - \partial_z(\bar{\mu}zf) = 0 \tag{13}$$

$$\int f(z)dz = 1, \quad f \geq 0, \tag{14}$$

on $(z_{\min}, \infty)$. It is easy to see that the solution to (13) is

$$f(z) = \zeta z_{\min}^{\zeta} z^{-\zeta-1}, \quad \zeta = 1 - \frac{2\bar{\mu}}{\bar{\sigma}^2}, \tag{15}$$

that is, a power law with exponent $\zeta > 1$. This basic idea can be generalized in a number of ways and applied in a number of different contexts and we here review some of these other applications.

3.1   Entry, Exit and Firm Size Distribution

An important paper by Luttmer (2007) has applied the same logic to the question why the size distribution of firms follows a power law. We here review a simplified version

of Luttmer's model. The problem also corresponds to a continuous time formulation of that originally studied by Hopenhayn (1992). Each firm has a profit function $\pi(z, m[f])$ which is strictly increasing in its own productivity z, and strictly decreasing in a geometric average of all other firms' productivities[1]

$$m[f] = \left(\int z^\theta f(z) dz \right)^{\frac{1}{\theta}}, \quad \theta > 0.$$

The value of a firm is the present discounted value of profits. Firms' productivity and hence profits evolve according to the stochastic process $dz_t = \mu(z_t)dt + \sigma(z_t)dW_t$ which we later specialize to (12), following Luttmer (2007). Firms have only one choice: whether to remain active or whether to exit the industry. If a firm exits the industry it obtains a scrap value ψ, but it can never reenter the industry. When firms exit, they mechanically get replaced by a group of entrants of equal mass whose initial productivity is a random draw from the distribution of productivities of currently active firms. Firms therefore solve a *stopping time* problem and the value function $v(z)$ can be characterized by a *variational inequality of the obstacle type* (see e.g. Bardi and Capuzzo-Dolcetta, 1997). As before the density of firms f satisfies a Fokker-Planck equation. In the stationary version of this problem, the unknown functions v and f satisfy

$$\min \left\{ \rho v - \frac{1}{2}\sigma^2(z)\partial_{zz}v - \mu(z)z\partial_z v - \pi(z, m[f]), v - \psi \right\} = 0 \tag{16}$$

$$\frac{1}{2}\partial_{zz}\left(\sigma^2(z)f\right) - \partial_z(\mu(z)f) = 0 \tag{17}$$

$$\int f(z)dz = 1, \quad f \geq 0, \tag{18}$$

$$m[f] = \left(\int z^\theta f(z) dz \right)^{\frac{1}{\theta}} \tag{19}$$

on $\mathbb{R}^+$.[2] Luttmer (2007) shows that under the assumption that $\mu(z) = \bar{\mu}z, \sigma(z) = \sigma z$ with $\bar{\mu} < 0$ and $\bar{\sigma} > 0$ (i.e. z_t follows (12)) and some other appropriately chosen assumption (e.g. that π is a power function with appropriately chosen exponents), the system can be solved explicitly. Importantly, the stationary distribution still satisfies (15) (where now

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[1]This dependence is motivated as follows. Firms face demand functions $(p(z)/P)^{-\gamma}, \gamma > 1$ where $p(z)$ is the price of firm z and P is a "price index" $P = \left(\int p(z)^{1-\gamma} f(z)dz \right)^{\frac{1}{1-\gamma}}$. Each firm's profit function is given by

$$\pi = \max_p \ p\left(\frac{p}{P}\right)^{-\gamma} - \frac{1}{z}\left(\frac{p}{P}\right)^{-\gamma} = (\bar{p} - 1)z^{\gamma-1}\bar{p}^{-\gamma}P^\gamma, \quad \bar{p} = \frac{\gamma}{\gamma - 1}$$

and the optimal price is $p(z) = \bar{p}/z$ so that $P = \bar{p}\left(\int z^{\gamma-1}f(z)dz \right)^{\frac{1}{1-\gamma}}$ and hence $\pi(z, m[f]) = (\bar{p} - 1)z^{\gamma-1}(m[f])^{-\gamma}$ with $\theta = \gamma - 1$.

[2](16) can also be written somewhat more intuitively as

$$0 = \begin{cases} \rho v - \frac{1}{2}\sigma^2(z)\partial_{zz}v - \mu(z)\partial_z v - \pi(z, m[f]), & v - \psi \geq 0 \\ v - \psi, & \rho v - \frac{1}{2}\sigma^2(z)\partial_{zz}v - \mu(z)\partial_z v - \pi(z, m[f]) \geq 0 \end{cases}$$

$z_{\min}$ is determined from (16)) thereby explaining the empirical regularity that the firm size distribution follows a power law.

While the case in which z_t follows a geometric Brownian motion (12) is very well understood, a natural question is what the exit decision and the firm size distribution look like for more general stochastic processes and perhaps also more general interdepencies between firms $m[f]$. For this more general setup, open questions are:

1. Existence and uniqueness of a stationary equilibrium, i.e. solutions to (16) to (19).

2. Existence and uniqueness of the time-dependent counterpart.

3. Development of numerical methods for solving both stationary and time-dependent equilibria.

Stokey (2009) discusses other examples of stopping time problems in economics, many of them describing richer version of the model of firm dynamics introduced in this section. This includes the problem of firms that set their price subject to an adjustment cost. These models are important in macroeconomics because the existence of frictions to the adjustment of prices is the main motivation for the use of monetary policy to stabilize business cycle fluctuations. Recent examples are given by Golosov and Lucas (2007) and Alvarez and Lippi (2013).

3.2   Other Applications of Theories of Power Laws

The ideas presented in the preceding two sections have been used to understand the emergence of power laws in a number of different contexts. For example, Benhabib, Bisin, and Zhu (2011) and in particular Benhabib, Bisin, and Zhu (2013) develop models of the wealth distribution whose mathematical structure is quite similar to the one presented here. Similarly, Jones (2014) applies the same insights to the question why the top of the income distribution (the infamous "one percent") can be well described by a power law.

4   Knowledge Diffusion and Growth

We now present some models of knowledge diffusion that have recently been used in macroeconomics, international trade and finance. These differ from the mean field games presented in sections 2 and 3 that the law of motion of the distribution does not take the form of a Fokker-Planck equation but instead that of equations that describe richer more "non-local" dynamics, for example Fisher-KPP or Bolztmann type equations. They also differ in that the long-run behavior of the systems they describe are not stationary. Instead these models are designed to feature sustained growth. As such they can be used to try to answer some of the most important questions in economics, for example: what generates long-run growth? What is the relation between growth and inequality? In section 4.1 we first present some problems that are purely "mechanical" in the sense that they do not feature an optimal control problem. We then add such a control problem in section 4.2.

4.1 Diffusion and Experimentation as an Engine of Growth

The following is based on Alvarez, Buera, and Lucas (2008), Lucas (2009) and in particular Luttmer (2012). Consider an economy populated by a continuum of individuals indexed by their productivity or knowledge $z \in \mathbb{R}^+$. The economy is described by its distribution of knowledge with cdf $G(z, t)$. The evolution of G is modeled as as a process of individuals meeting others from the same economy, comparing ideas, improving their own productivity. Meetings happen at Poisson intensity α and from the point of view of an individual a meeting is simply a random draw from the distribution G. When a meeting occurs, a person z compares his productivity with the person he meets and leaves the meeting with the best of the two productivities $\max\{z, z'\}$. Individual productivities also fluctuate in the absence of a meeting. In particular individuals "experiment" and their productivity may either increase or decrease according to the process $d \log z_t = \sigma dW_t, \sigma > 0$. Given this structure it is convenient to work with $x = \log z$ and the corresponding distribution F, and one can show that this distribution satisfies

$$\partial_t F - \frac{\sigma^2}{2} \partial_{xx} F = -\alpha F (1 - F), \tag{20}$$

on $\mathbb{R} \times \mathbb{R}^+$, and with boundary conditions

$$\lim_{x \to -\infty} F(x, t) = 0, \quad \lim_{x \to \infty} F(x, t) = 1, \quad F(x, 0) = F_0(x), \tag{21}$$

where $F_0(x)$ is the initial productivity distribution. As Luttmer (2012) points out this is a Fisher-KPP type equation (Fisher, 1937; Kolmogorov, Petrovskii, and Piskunov, 1937) whose theoretical properties are well understood (see e.g. McKean, 1975). In particular one can show that (20) admits "traveling wave" solutions, i.e. solution of the form

$$F(x, t) = \Phi(x - \gamma t). \tag{22}$$

One can further show that if the initial distribution is a Dirac point mass, the limiting distribution is a traveling wave with $\gamma = \sigma\sqrt{2\alpha}$. If the distribution F is a traveling wave (22), productivity $z = e^x$ is on average growing at the constant rate γ and hence one can say that the economy is on a "balanced growth path" with growth rate γ. The interpretation of the formula for the growth rate $\gamma = \sigma\sqrt{2\alpha}$ is also very natural: it says that it is the combination of "experimentation" parameterized by σ and "diffusion" parameterized by α that is the engine of growth in this economy. Either force in isolation would lead to stagnation, but the two together create sustained growth.

Economists have studied various versions of the Fisher-KPP equation (20). Lucas (2009) and Alvarez, Buera, and Lucas (2008) study the version of (20) with $\sigma = 0$:

$$\partial_t F = -\alpha F (1 - F), \tag{23}$$

on $\mathbb{R} \times \mathbb{R}^+$. To generate sustained growth they assume that the initial distribution satisfies $(1 - F_0(x))/e^{-\zeta x} \to c$ as $x \to \infty$ for some constants $c, \zeta > 0$, meaning that the initial distribution for $z = e^x$ is asymptotically a power law as in (15).[3] Luttmer (2014) studies

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[3] As shown by Luttmer (2012), the traveling wave solution obtained in the case (20) with $\sigma > 0$ satisfies this property and hence this is a relatively innocuous assumption.

the equation

$$\partial_t F - \frac{\sigma^2}{2}\partial_{xx}F = -\alpha\min\{F, 1-F\}$$

on $\mathbb{R}\times\mathbb{R}^+$ which can be solved explicitly.

4.2   Knowledge Diffusion and Search

While the models in the previous section are interesting in that they describe environments in which there is sustained growth, they are somewhat less interesting than those in sections 2 and 3 in that individuals in the economy did not make any choices, i.e. solve optimal control problems. Lucas and Moll (2014) extend the setup in the previous section to feature such an optimal choice. In this extension, one can then ask questions such as: is the equilibrium growth rate of the economy optimal or should policy makers intervene to boost (or perhaps depress) economic growth?

In Lucas and Moll (2014) individuals have one unit of time and they can split it between producing with the knowledge they already have, or they can search for productivity enhancing ideas. Search increases the likelihood of other individuals. In particular the Poisson meeting rate of an individual who searches a fraction s of his time is $\alpha(s)$ which is strictly increasing and concave. Conditional on a meeting the knowledge diffusion process is exactly as described in the previous section. The cost of search is that it interferes with production. In particular, the output of an individual with productivity $z = e^x$ who searches a fraction s of his time is $(1-s)e^x$. Individuals maximize the present discounted value of future output. The equilibrium of this economy can be described in terms of a system of two integro-PDEs for the value function v and the density of the productivity distribution f:

$$\partial_t v + \frac{\sigma^2}{2}\partial_{xx}v + \max_{s\in[0,1]}\left\{(1-s)e^x + \alpha(s)\int_x^\infty (v(y,t)-v(x,t))f(y,t)dy\right\} - \rho v = 0, \quad (24)$$

$$\partial_t f - \frac{\sigma^2}{2}\partial_{xx}f + \alpha(s^*(x,t))f(x,t)\int_x^\infty f(y,t)dy - f(x,t)\int_{-\infty}^x \alpha(s^*(y,t))f(y,t)dy = 0$$
$$(25)$$

$$\int f(z,t)dz = 1, \quad f \geq 0. \tag{26}$$

on $\mathbb{R}\times\mathbb{R}^+$ and where s^* is the maximand of (24). There is also an initial condition $f(x,0) = f_0(x)$. It can be seen that (20) is the special case of (25) in which the optimal control s^* and hence also α are constant across x-types, and written in terms of the cdf $F(x,t) = \int_0^x f(x,t)dx$. However, in general, it will not be true that s^* is constant for all x. Instead, s^* is usually decreasing in x. Lucas and Moll (2014) study the special case of (24) to (26) with $\sigma = 0$. They show that the system admits solutions of the traveling wave type, that is

$$v(x,t) = w(x-\gamma t), \quad f(x,t) = \phi(x-\gamma t)$$

and they develop numerical methods for computing such solutions numerically, and in particular to find the growth rate γ of the system. However, there remain many open theoretical questions, among these:

1. Existence and uniqueness of a solution to (24) to (26).

2. Asymptotic behavior of f for different initial conditions f_0, in particular the one where f_0 is a Dirac point mass. Does the solution converge to a traveling wave $f(x,t) = \phi(x - \gamma t)$? If so, what does this limiting distribution look like? And what is the growth rate γ?

3. Development of numerical methods for solving the time-dependent problem (24) to (26).

Regarding the second question, a natural conjecture would be that the limiting distribution is a traveling wave with growth rate

$$\gamma = \sigma \sqrt{2 \int_{-\infty}^{\infty} \alpha(s^*(x)) \phi(x) dx}.$$

This is the natural generalization of the formula $\gamma = \sigma \sqrt{2\alpha}$ in section 4.1 to the case where s^* varies across productivity types.

Ideas similar to those presented in this section in the context of search and knowledge diffusion have been applied to different contexts. For example, Duffie, Garleanu, and Pedersen (2005) and Lagos and Rocheteau (2009) and others use search theory to model the trading frictions that are characteristic of over-the-counter (OTC) markets, and to examine the effects of these frictions on asset prices and trading volumes.

4.3 Diffusion and International Trade

An alternative route to enrich the model of knowledge diffusion is to consider explicit mechanisms mediating the interactions among individuals. One possible avenue is explored by Alvarez, Buera, and Lucas (2013), who consider a multi country model in which knowledge is transmitted through the interaction with the sellers of goods to a country. In their theory barriers to trade affect the composition of sellers to a country, and therefore, they impact the diffusion of knowledge. The higher trade cost are the more likely it is that sellers in a country are given by relatively inefficient local producers.

The central object in their theory is the distribution of productivities across potential producers of different goods $G(z,t)$, denoting the fraction of goods that can be produced with productivity less than z. Similarly to the previous models, an individual producer meets other producers at the constant Poisson rate α. The main difference is that now draws come from the distribution of sellers, which dependents on the distribution of productivities of producers from all countries in the worlds, and trade costs $1/\kappa$, $\kappa \in [0,1]$. As before, it is convenient to work with $x = \log(z)$ and the corresponding distribution F, and define $\delta = \log(\kappa)$. For the case of a world with n symmetric countries the evolution of the distribution $F(x,t)$ solves the following delayed Fisher-KPP type equation[4]

$$\partial_t F = -\alpha (1 - M) F \tag{27}$$

[4]Related equations have been study by Berestycki, Nadin, Perthame, and Ryzhik (2009).

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on $\mathbb{R} \times \mathbb{R}^+$ where

$$M(x,t) = \int_{-\infty}^{x} \left(F(y - \delta, t)^{n-1} + (n-1)F(y + \delta, t)F(y, t)^{n-2} \right) \partial_x F(y, t) dy \qquad (28)$$

is the distribution of productivity of sellers to a country. The boundary conditions are given by (21). For $\kappa = 1$ ($\delta = 0$) and $n = 1$ this equation simplifies to the one analyzed in Section (a), but more generally only the behavior of the solution for large x is fully understood. One can show that this equation admits solutions of the traveling wave type

$$F(x,t) = \Phi(x - \gamma t), \qquad (29)$$

provided that $(1 - F_0(x))/e^{-\zeta x} \to c$ as $x \to \infty$ for some constants $c, \zeta > 0$, that is the initial distribution of productivity $z = e^x$ follows an asymptotic power law. It can also be shown that the growth rate $\gamma = n\alpha/\zeta$. Natural open questions are:

1. Existence and uniqueness of a solution to (27) and (28).

2. Development of numerical methods for computing both stationary and time dependent solutions.

Another interesting extension could be the addition of noise in the form of a geometric Brownian motion to (27) along the lines of equation (20).

4.4 Information Percolation in Finance

A related class of models arises when studying the distribution of information across individuals in an economy, e.g., beliefs about the value of a particular financial asset. These models are useful to understand the dynamics of asset prices and how these are affected when market participants do not share common beliefs about the "intrinsic" value of a financial asset. A simple example is provided by Duffie and Manso (2007) who consider the beliefs about the realization of a binary random variable. Individuals are initially endowed with a prior about this realization. Over time, individuals randomly meet at a constant Poisson rate α. Upon a meeting individuals exchange their information and update their beliefs. In their example they show that beliefs are characterize by a distribution over a sufficient statistic x, and the updating of beliefs after a meeting with an individual of belief x' is simply given by the sum of x and x'. The evolution of the distribution of the sufficient statistic $F(x, t)$ is given by the PDE

$$\partial_t f(x, t) = -\alpha f(x, t) + \alpha \int_{-\infty}^{+\infty} f(y, t) f(x - y, t) dy$$

This equation can be solved explicitly using Fourier transforms. A natural extension is to endogenize the search effort α, similarly to section 4.2. This is pursued in Duffie, Malamud, and Manso (2009). Other recent contributions in this area includes Amador and Weill (2012) and Golosov, Lorenzoni, and Tsyvinski (2009).

5 Business Cycles: Models with Aggregate Shocks

Some of the most important questions in macroeconomics are concerned with business cycle fluctuations, that is the fluctuations of macroeconomic aggregates like GDP, aggregate investment and asset prices like the interest rate. The models presented so far are not well suited to address these questions because all of them featured macroeconomic aggregates that are deterministic. Instead we want theories in which these aggregates are stochastic. In an important paper Krusell and Smith (1998) have extended theories with heterogeneity at the individual level to feature aggregate risk.[5] We here present a continuous time formulation from Achdou, Lasry, Lions, and Moll (2014b).

To introduce these ideas in the simplest possible way, consider the model of section 2 but assume that income is $z_t A_t$ where z_t is an idiosyncratic income process as before but now income also has an aggregate component A_t. That is, if A_t falls by ten percent, it means that the income of *everyone* in the economy falls by ten percent. For simplicity, assume that $A_t \in \{A_1, A_2\}$ is a two state Poisson process. The process jumps from state 1 to state 2 with intensity ϕ_1 and vice versa with intensity ϕ_2. The introduction of aggregate shocks creates a major difficulty: in contrast to the case without aggregate uncertainty studied in section 2, it becomes necessary to include the entire distribution of income and wealth g as a state variable in the optimal control problem of individuals. This distribution is now itself a random variable and hence calendar time t is no longer a sufficient statistic to describe the behavior of the system.

The aggregate state is $(A_i, g), i = 1, 2$ and the individual state is (a, z) so that the value function of an individual is $v_i(a, z, g)$. This value function satisfies the equation

$$
\begin{aligned}
0 =& \frac{1}{2}\sigma^2(z)\partial_{zz}v_i + \mu(z)\partial_z v_i + (A_i z + r_i(g)a)\partial_a v_i \\
& + \phi_i(v_j(a, z, g) - v_i(a, z, g)) + \int T[g, \partial_a v_i](a, z)\frac{\delta v_i}{\delta g(a, z)}dadz \\
& + H(\partial_a v_i) - \rho v_i
\end{aligned}
\tag{30}
$$

$$
T[g, \partial_a v_i] = \frac{1}{2}\partial_{zz}(\sigma^2(z)g) - \partial_z(\mu(z)g) - \partial_a\left(\partial_p H(\partial_a v_i)g\right)
\tag{31}
$$

for $i = 1, 2, j \neq i$. The domain of this equation is $(\underline{a}, \infty) \times (\underline{z}, \bar{z}) \times \mathcal{S}$ where $\mathcal{S}$ is the space of density functions. The Hamiltonian H is defined in (5) and $\delta v_i/\delta g(a, z)$ denotes the functional derivative of V_i with respect to g at point (a, z). T defined in (31) is the "Fokker-Planck" operator that maps functions g and $\partial_a v_i$ to the time derivative of g. Note that (30) is not an ordinary HJB equation because of the presence of g in the state space. The difficulty, of course, is that g is an infinite-dimensional object.

Achdou, Lasry, Lions, and Moll (2014b) develop methods for approximating (30) numerically. In particular, they assume that aggregate shocks occur only finitely many times and at finite time intervals of length Δ, that is at times $\tau_n = \Delta n, n = 1, ..., N, N = \frac{1}{\Delta}$. It is then possible to represent (30) as a system of mean-field games, and crucially this is a finite-dimensional problem. The hope is that the behavior of this system approximates that of (30) as $\Delta \to 0$.

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[5]Also see Den Haan (1996).

Models with aggregate shocks such as (30) are by far the most challenging in terms of the mathematics, and many open questions remain. Among these are:

1. Existence and uniqueness of solutions to (30).

2. A theoretical understanding of the behavior of g. For example, given a stationary process for A_t (such as the two-state Poisson process), does there exist a "stationary equilibrium" for g? Similarly, are there certain regions of the space of density functions $\mathcal{S}$ in which g lives "most of the time"?

3. Development of efficient and robust approximation schemes to (30) and results regarding their convergence.

One approach to obtaining more tractable formulations of models with aggregate shocks has been to simplify the heterogeneity at the individual level. For example, Brunnermeier and Sannikov (2014), He and Krishnamurthy (2012, 2013), Adrian and Boyarchenko (2012) and Di Tella (2013) all study business cycles in models of financial intermediation with frictions and argue that these frictions give rise to interesting non-linear behavior of macroeconomic aggregates. For example, GDP may have a bi-modal stationary distribution even if the driving stochastic process is uni-modal. These papers all make the assumption that there are only two (or three) types of agents so that the wealth distribution can be summarized by the share of wealth of one of the two types. The big advantage of these two approaches is that this is a one-dimensional rather than an infinite-dimensional object. Related, business cycle fluctuations can also be generated from theories without aggregate shocks. An important early paper by Scheinkman and Weiss (1986) demonstrates that in a model with only a finite number of agents (two in their framework) idiosyncratic shocks (in combination with missing insurance markets) can lead to aggregate fluctuations. See Conze, Lasry, and Scheinkman (1993) and Lippi, Ragni, and Trachter (2013) for other applications of their framework. These authors again make assumptions that avoid dealing with an infinite-dimensional problem. However, for many interesting economic questions it may be necessary to consider richer form of heterogeneity. Our hope is therefore that some progress can be made on infinite dimensional problems such as (30).

6 Models with Finite Number of Agents

In this paper we have mostly focused on models with a continuum of individuals (mean field games). While these frameworks are useful to study a very large class of macroeconomic phenomena, their applicability to other important macro questions is limited. In some industries production is concentrated in a very small numbers of producers, who act strategically when making their production, innovation, and pricing decisions. The strategic nature of their decision could have important aggregate implications. For example, Atkeson and Burstein (2008) consider a model with a continuum of sectors and a finite number of firms in each sector to explain why there are large and systematic deviation of the law of one price across countries. Aghion, Bloom, Blundell, Griffith, and Howitt

(2005) study a model of innovation in duopolist industries to analyze the relationship between competition and innovation.

In this section we introduce a continuous time version of the canonical model of firm dynamics in an oligopolistic industry introduced by Ericson and Pakes (1995), and recently studied by Weintraub, Benkard, and Roy (2008) and Doraszelski and Judd (2012), among many others. We show that this model takes the form of a differential game.

There are two firms $i = 1, 2$ that compete with each other. Firm i has profits $\pi(z_i, q_i, q_j)$ where $j \neq i$. Profits π are increasing in productivity z_i and own quantity q_j, but decreasing in the quantity of the other firm q_j. The quantity choice also affects the evolution of the firm's productivity which evolves as $dz_{it} = \mu(z_{it}, q_{it})dt + \sigma(z_{it})dW_{it}$. We assume that there is "learning-by-doing" so that μ is increasing in q_{it} (of course, other assumptions are possible as well). We assume that the two firms play a *Nash equilibrium*, that is their choices of q_{it} are best responses to each other. Given the symmetry of the problem we look for a symmetric Nash equilibrium. To this end denote by z a firm's own productivity and by x the productivity of the other firm. In a symmetric Nash equilibrium the value function $v(z, x)$ of a firm satisfies

$$\frac{\sigma^2(z)}{2}\partial_{zz}v + \frac{\sigma^2(x)}{2}\partial_{xx}v + \mu(x, q^*(x, z, \partial_x v, \partial_z v))\partial_x v + H(z, x, \partial_z v, \partial_x v) - \rho v = 0 \quad (32)$$

on $\mathbb{R}^+ \times \mathbb{R}^+$, and where the Hamiltonians H and optimal choice q^* jointly satisfy:

$$H(z, x, p_z, p_x) = \max_q \left(\pi(z, q, q^*(x, z, p_x, p_z)) + \mu(z, q)p_z \right)$$

$$q^*(z, x, p_z, p_x) = \arg\max_q \left(\pi(z, q, q^*(x, z, p_x, p_z)) + \mu(z, q)p_z \right)$$

There are many possible extension of this simple framework. Naturally the model can be generalized to $n > 2$. One can also consider version of the model with entry and exit of firms, along the lines of the analysis in Section 3 (a). One way to model this process is to consider a maximum number of potential firms $\bar{n}$. In this case, the relevant "aggregate" state is given by the vector of characteristic of all the active and potential firms, e.g., their respective z. An alternative route, which is the one that is typically followed in the literature, is to assume that the state describing an individual firm takes only a finite set of values. In this case, one can describe the aggregate state with the distribution of firms over these (finite) characteristics. The first route leads naturally to PDE methods. We are not aware of a general characterization of these problems. As in the previous examples, the open questions are:

1. Existence and uniqueness of a solution to (32).

2. Development of numerical methods for computing both stationary and time dependent solutions when the state variable is continuous.

7   Conclusion

We have surveyed a large literature in macroeconomics that studies theories that explicitly model the equilibrium interaction of heterogeneous agents. These theories share a common

mathematical structure which can be summarize by a system of coupled non-linear PDEs or mean field game. Some of our examples are well understood problems in the theory of PDEs, while others present new and challenging mathematical problems. We view this to be a very promising area for future research, or, as economist like to say, wee see large "gains from trade" between macroeconomists and mathematicians working on PDEs.

References

ACHDOU, Y., F. CAMILLI, AND I. CAPUZZO DOLCETTA (2012): "Mean field games: convergence of a finite difference method," *ArXiv e-prints*.

ACHDOU, Y., J.-M. LASRY, P.-L. LIONS, AND B. MOLL (2014a): "Heterogeneous Agent Models in Continuous Time," Working papers, Princeton University.

——— (2014b): "Wealth Distribution and the Business Cycle," Working papers, Princeton University.

ADRIAN, T., AND N. BOYARCHENKO (2012): "Intermediary leverage cycles and financial stability," Discussion paper.

AGHION, P., N. BLOOM, R. BLUNDELL, R. GRIFFITH, AND P. HOWITT (2005): "Competition and Innovation: An Inverted-U Relationship," *The Quarterly Journal of Economics*, 120(2), 701–728.

AIYAGARI, S. R. (1994): "Uninsured Idiosyncratic Risk and Aggregate Saving," *The Quarterly Journal of Economics*, 109(3), 659–84.

ALVAREZ, F., AND F. LIPPI (2013): "Price setting with menu cost for multi-product firms," EIEF Working Papers Series 1302, Einaudi Institute for Economics and Finance (EIEF).

ALVAREZ, F., AND R. SHIMER (2011): "Search and Rest Unemployment," *Econometrica*, 79(1), 75–122.

ALVAREZ, F. E., F. J. BUERA, AND R. E. J. LUCAS (2008): "Models of Idea Flows," NBER Working Papers 14135, National Bureau of Economic Research, Inc.

——— (2013): "Idea Flows, Economic Growth, and Trade," NBER Working Papers 19667, National Bureau of Economic Research, Inc.

AMADOR, M., AND P.-O. WEILL (2012): "Learning from private and public observations of others? actions," *Journal of Economic Theory*, 147(3), 910–940.

ATKESON, A., AND A. BURSTEIN (2008): "Pricing-to-Market, Trade Costs, and International Relative Prices," *American Economic Review*, 98(5), 1998–2031.

BARDI, M., AND I. CAPUZZO-DOLCETTA (1997): *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Springer.

BENHABIB, J., A. BISIN, AND S. ZHU (2011): "The Distribution of Wealth and Fiscal Policy in Economies With Finitely Lived Agents," *Econometrica*, 79(1), 123–157.

——— (2013): "The distribution of wealth in the Blanchard-Yaari model," Working papers, NYU.

BERESTYCKI, H., G. NADIN, B. PERTHAME, AND L. RYZHIK (2009): "The non-local Fisher-KPP equation: travelling waves and steady states," *Nonlinearity*, 22(12), 2813.

BEWLEY, T. (1986): "Stationary Monetary Equilibrium with a Continuum of Independently Fluctuating Consumers," in *Contributions to Mathematical Economics in Honor of Gerard Debreu*, ed. by W. Hildenbrand, and A. Mas-Collel. North-Holland, Amsterdam.

BRUNNERMEIER, M., AND Y. SANNIKOV (2014): "A Macroeconomic Model with a Financial Sector," *American Economic Review*.

CONZE, A., J. M. LASRY, AND J. SCHEINKMAN (1993): "Borrowing Constraints and International Comovements," *Hitotsubashi Journal of Economics*, 34(Special I), 23–47.

DEN HAAN, W. J. (1996): "Heterogeneity, Aggregate Uncertainty, and the Short-Term Interest Rate," *Journal of Business & Economic Statistics*, 14(4), 399–411.

DI TELLA, S. (2013): "Uncertainty Shocks and Balance Sheet Recessions," Working paper, Stanford University.

DORASZELSKI, U., AND K. L. JUDD (2012): "Avoiding the curse of dimensionality in dynamic stochastic games," *Quantitative Economics*, 3(1), 53–93.

DUFFIE, D., N. GARLEANU, AND L. H. PEDERSEN (2005): "Over-the-Counter Markets," *Econometrica*, 73(6), 1815–1847.

DUFFIE, D., S. MALAMUD, AND G. MANSO (2009): "Information Percolation With Equilibrium Search Dynamics," *Econometrica*, 77(5), 1513–1574.

DUFFIE, D., AND G. MANSO (2007): "Information Percolation in Large Markets," *American Economic Review*, 97(2), 203–209.

ERICSON, R., AND A. PAKES (1995): "Markov-Perfect Industry Dynamics: A Framework for Empirical Work," *Review of Economic Studies*, 62(1), 53–82.

FARHI, E., AND I. WERNING (2013): "Insurance and Taxation over the Life Cycle," *Review of Economic Studies*, 80(2), 596–635.

FISHER, R. A. (1937): "The Wave of Advance of Advantageous Genes," *Annals of Eugenics*, 7(4), 355–369.

GABAIX, X. (1999): "Zipf'S Law For Cities: An Explanation," *The Quarterly Journal of Economics*, 114(3), 739–767.

——— (2009): "Power Laws in Economics and Finance," *Annual Review of Economics*, 1(1), 255–293.

Golosov, M., G. Lorenzoni, and A. Tsyvinski (2009): "Decentralized Trading with Private Information," NBER Working Papers 15513, National Bureau of Economic Research, Inc.

Golosov, M., and R. E. Lucas (2007): "Menu Costs and Phillips Curves," *Journal of Political Economy*, 115, 171–199.

He, Z., and A. Krishnamurthy (2012): "A Model of Capital and Crises," *Review of Economic Studies*, 79(2), 735–777.

He, Z., and A. Krishnamurthy (2013): "Intermediary Asset Pricing," *American Economic Review*, 103(2), 732–70.

Hopenhayn, H. A. (1992): "Entry, Exit, and Firm Dynamics in Long Run Equilibrium," *Econometrica*, 60(5), 1127–50.

Huggett, M. (1993): "The risk-free rate in heterogeneous-agent incomplete-insurance economies," *Journal of Economic Dynamics and Control*, 17(5-6), 953–969.

Jones (2014): "A Schumpeterian Model of Top Income Inequality," Working paper, Stanford University.

Kolmogorov, A., I. Petrovskii, and N. Piskunov (1937): "A Study of the Diffusion Equation with Increase in the Amount of Substance, and its Application to a Biological Problem," .

Krusell, P., and A. A. Smith (1998): "Income and Wealth Heterogeneity in the Macroeconomy," *Journal of Political Economy*, 106(5), 867–896.

Lagos, R., and G. Rocheteau (2009): "Liquidity in Asset Markets With Search Frictions," *Econometrica*, 77(2), 403–426.

Lasry, J.-M., and P.-L. Lions (2007): "Mean field games," *Japanese Journal of Mathematics*, 2, 229–260.

Lentz, R., and D. T. Mortensen (2010): "Labor Market Models of Worker and Firm Heterogeneity," *Annual Review of Economics*, 2(1), 577–602.

Lippi, F., S. Ragni, and N. Trachter (2013): "State dependent monetary policy," Discussion paper.

Lucas, R. E. (2009): "Ideas and Growth," *Economica*, 76(301), 1–19.

Lucas, R. E., and B. Moll (2014): "Knowledge Growth and the Allocation of Time," *Journal of Political Economy*.

Luttmer, E. G. (2012): "Eventually, noise and imitation implies balanced growth," Discussion paper.

——— (2014): "Competitive Knowledge Diffusion and Inequality," Working papers, Federal Reserve Bank of Minneapolis.

LUTTMER, E. G. J. (2007): "Selection, Growth, and the Size Distribution of Firms," *The Quarterly Journal of Economics*, 122(3), 1103–1144.

McKEAN, H. P. (1975): "Application of brownian motion to the equation of kolmogorov-petrovskii-piskunov," *Communications on Pure and Applied Mathematics*, 28(3), 323–331.

NUÑO, G. (2013): "Optimal control with heterogeneous agents in continuous time," Working Paper Series 1608, European Central Bank.

SANNIKOV, Y. (2008): "A Continuous-Time Version of the Principal-Agent Problem," *Review of Economic Studies*, 75(3), 957–984.

SCHEINKMAN, J. A., AND L. WEISS (1986): "Borrowing Constraints and Aggregate Economic Activity," *Econometrica*, 54(1), 23–45.

STOKEY, N. L. (2009): *The Economics of Inaction.* Princeton University Press, Princeton, NJ.

WEINTRAUB, G. Y., C. L. BENKARD, AND B. V. ROY (2008): "Markov Perfect Industry Dynamics With Many Firms," *Econometrica*, 76(6), 1375–1411.

WILLIAMS, N. (2011): "Persistent Private Information," *Econometrica*, 79(4), 1233–1275.