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Abstract. We consider left and right Fredholm-decomposably regular operators introduced in [23], and the corresponding holomorphic versions. Using their results established by Zeng in [23], we give new properties of these classes of operators. We introduce the concept of Weyl-decomposably regular operator and the corresponding holomorphic version in the setting of $\mathcal{L}(X)$, where $\mathcal{L}(X)$ is the set of all bounded operators from Banach space X to X, and we give various characterizations of this class of operators.

1. Introduction

Let X denote an infinite dimensional Banach space. We use $\mathcal{L}(X)$ to denote the set of all linear bounded operators on X. Also, $\mathcal{L}_0(X)$ denote the set of all compact operators on X. For $A \in \mathcal{L}(X)$ we use $\mathcal{N}(A)$ and $A(X)$, respectively, to denote the null-space and the range of A. The nullity, $\alpha(A)$, of A is defined as the dimension of $\mathcal{N}(A)$ and the deficiency, $\beta(A)$, of A is defined as the codimension of $A(X)$ in X. The hyper-range $A^\infty(X)$ of A is defined by $A^\infty(X) = \cap_{k\geq 1} A^k(X)$. Let $\sigma(A)$ denote the spectrum of A. Sets of upper and lower Fredholm operators, respectively, are defined as

$$\Phi_+(X) = \{A \in \mathcal{L}(X) \text{ such that } A(X) \text{ is closed in } X \text{ and } \alpha(A) < \infty\},$$

and

$$\Phi_-(X) = \{A \in \mathcal{L}(X) \text{ such that } A(X) \text{ is closed in } X \text{ and } \beta(A) < \infty\}.$$

Opertors in $\Phi_\pm(X) = \Phi_+(X) \cup \Phi_-(X)$ are called semi-Fredholm operators. For such operators the index is defined by $i(A) = \alpha(A) - \beta(A)$. If $A \in \Phi_+(X)\backslash\Phi(X)$ then $i(A) = -\infty$ and if $A \in \Phi_-(X)\backslash\Phi(X)$ then $i(A) = +\infty$. The set of Fredholm operators is defined as $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$.

An operator $A \in \mathcal{L}(X)$ is called Kato non-singular (see, [13, 16, 18]) if $A(X)$ is closed and $\mathcal{N}(A) \subseteq A^\infty(X)$. The Kato non-singular spectrum of A is defined by

$$\sigma_K(A) := \{\lambda \in \mathbb{C} \text{ such that } A - \lambda \text{ is not a Kato non-singular}\}.$$

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An operator $A \in \mathcal{L}(X)$ is relatively regular if there exists $B \in \mathcal{L}(X)$ such that $ABA = A$. It is well-known that A is relatively regular if and only if $A(X)$ and $\mathcal{N}(A)$ are closed and complemented subspaces of X. In particular, if X is a Hilbert space then A is relatively regular operator if and only if $A(X)$ is closed (see [6, 9, 12, 14]). Let $R(X)$ denote the class of all relatively regular operators. A is called Fredholm-decomposably regular operator if there is $B \in \Phi(X)$ such that $ABA = A$ (see [10, 22]), if B is an invertible operator then A is said decomposably regular operator. Let $G(X)$ denote the set of all bounded invertible operators. The class of all Fredholm-decomposably regular (resp. decomposably regular) operators is denoted by $\Phi R(X)$ (resp. $GR(X)$). This class of operators was introduced and investigated in [8, 19]. In particular it is proved that

$$GR(X) = R(X) \cap \overline{G(X)} \text{ and } \Phi R(X) = R(X) \cap \overline{\Phi(X)}.$$

In [5], El Bakkali introduced and studied the holomorphically Fredholm-decomposably regular resolvent $\rho_{hf}(A)$, where

$\rho_{hf}(A) := \{\mu \in \mathbb{C}$: there are V a neighborhood of μ and $F : V \to \mathcal{L}(X)$ analytic such that $(A-\lambda)F(\lambda)(A-\lambda) = A - \lambda$ and $F(\lambda) \in \Phi(X)$ for each $\lambda \in V\}$.

An operator $A \in \mathcal{L}(X)$ is called holomorphically Fredholm-decomposably regular operator if $0 \in \rho_{hf}(A)$. We denote $H\Phi R(X)$ the class of all holomorphically Fredholm-decomposably regular operators.

We use $G_l(X)$ and $G_r(X)$, respectively, to denote the set of all left and right invertible operators on X. Sets of left and right Fredholm operators, respectively, are defined as

$$\Phi_l(X) = \{A \in \mathcal{L}(X) : \ A(X) \text{ is closed and complemented of } X \text{ and } \alpha(A) < \infty\}$$

and

$$\Phi_r(X) = \{A \in \mathcal{L}(X) : \ \mathcal{N}(A) \text{ is complemented of } X \text{ and } \beta(A) < \infty\}.$$

In this paper, we introduce the concept of Weyl-decomposably regular operator and the corresponding holomorphic version in the setting of $\mathcal{L}(X)$, and we give various characterizations of this class of operator.

We organise our paper in the following way: in Section 2 we gather some results and notions from Fredholm theory connected with the sequel of the paper. The main results of section 3 is to prove that for $A \in \mathcal{L}(X)$ where X be a direct sum of closed subspaces X_1 and X_2 which are A-invariant, $A_1 = A_{|X_1} : X_1 \to X_1$ and $A_2 = A_{|X_2} : X_2 \to X_2$, if A_1 and A_2 are left (right) Fredholm-decomposably regular operators then A is left (right) Fredholm-decomposably regular operator (see, Definition 2.6). A similar, if A_1 and A_2 are left (right) holomorphically Fredholm-decomposably regular operators then A is left (right) holomorphically Fredholm-decomposably regular operator. In Section 4, we present the class of holomorphically Weyl-decomposably regular operators and we give various characterizations of this class of operators. More precisely, let $HWR(X)$ be the class of all holomorphically Weyl-decomposably regular operators (see, Definition 4.3), then $HWR(X) = S(X) \cap WR(X)$, where $S(X)$ is the set of all Saphar operator (see, Definition 2.1) and $WR(X)$ is the class of Weyl-decomposably regular operators. Let $A \in \mathcal{L}(X)$. Then $A \in HWR(X)$ if and only if there exist $R \in W(X)$ and sequence $(B_n)_n \subset G(X)$ and $(A_n)_n \subset WR(X)$ such that $ARA = A$, $AB_n = B_nA$, $(A - B_n)A_n(A - B_n) = A - B_n$ for all $n \in \mathbb{N}$ and $\lim_n \left(\|B_n\| + \|A_n - R\|\right) = 0$. Finally, in section 5, we define a new spectrum $\sigma_{hw}(A)$ said holomorphically Weyl-decomposably regular spectrum of $A \in \mathcal{L}(X)$. Let $\rho_{hw}(A) = \mathbb{C} \setminus \sigma_{hw}(A)$. We prove that $0 \in \rho_{hw}(A)$ if and only if $0 \in \mathbb{C}\setminus\sigma_K(A)$ and $A \in WR(X)$.

2. Preliminary Results

In this section we recall some definitions and we give some lemmas that we will need in the sequel.

Definition 2.1. *Let X be a Banach space and let $A \in \mathcal{L}(X)$. The operator A is called Saphar operator if A is Kato non-singular and $A \in R(X)$.*

We denote $S(X)$ the set of all Saphar operators. This class of operators was studied by Saphar in [20].

Definition 2.2. *Let X be a Banach space.*

1) *An operator A is said to be left (resp. right) decomposably regular operator if there exists $B \in G_r(X)$ (resp. $G_l(X)$) such that $ABA = A$.*
2) *An operator A is said to be left (resp. right) Fredholm-decomposably regular operator if there exists $B \in \Phi_r(X)$ (resp. $\Phi_l(X)$) such that $ABA = A$.*

We also denote classes of left decomposably regular operators, right decomposably regular operators, left Fredholm-decomposably regular operators and right Fredholm-decomposably regular operators from X to X by $G_lR(X)$, $G_rR(X)$, $\Phi_lR(X)$ and $\Phi_rR(X)$, respectively.

Let $A \in \mathcal{L}(X)$. Let

$$Com(A) := \{B \in \mathcal{L}(X) \text{ such that } AB = BA\}$$

the commutant of A and

$$Com^{-1}(A) = Com(A) \cap G(X)$$

the invertible commutant of A.

Definition 2.3. [10, Definition 8] *Let $A \in \mathcal{L}(X)$. We say that A is consortedly regular if there are sequences $(B_n)_n$ in $Com^{-1}(A)$ and $(\widehat{A_n})_n$, $\widehat{A}$ in $\mathcal{L}(X)$ for which*

$$A\widehat{A}A = A, \quad \| B_n \| + \| \widehat{A_n} - \widehat{A} \| \longrightarrow 0 \ \text{ and } \ A - B_n = (A - B_n)\widehat{A_n}(A - B_n).$$

Lemma 2.4. [17, Lemma 4, p. 131] *Let $A \in \mathcal{L}(X)$ be a Saphar operator and let $B \in \mathcal{L}(X)$ satisfy $ABA = A$ and let $n \in \mathbb{N}$. Then $A^nB^nA^n = A^n$. In particular, A^n is Saphar operator.*

It is know in [15, Theorem 2.6] and [21, Theorem 1.4] that $A \in S(X)$ if and only if there exist a neighborhood $U \subseteq \mathbb{C}$ of 0 and a holomorphic function $F : U \to \mathcal{L}(X)$ such that

$$(A - \lambda)F(\lambda)(A - \lambda) = A - \lambda, \text{ for all } \lambda \in U. \tag{2.1}$$

In [22], Schmoeger introduced and studied the holomorphically decomposably regular operator $HGR(X)$ given by $A \in HGR(X)$ if and only if there exist a neighborhood $U \subset \mathbb{C}$ of 0 and a holomorphic function $F : U \longrightarrow \mathcal{L}(X)$ such that

$$F(x) \in G(X), \ (A - x)F(x)(A - x) = A - x$$

for all $x \in U$. It is clear that $HGR(X) \subset H\Phi R(X)$.

Theorem 2.5. [10, Theorem 2.6] *Let $A \in \mathcal{L}(X)$. If A is consortedly regular then A is holomorphically decomposably regular.*

To generalized these classes of operators Zeng (see, [23]), introduced the following concept:

Definition 2.6. *Let $A \in \mathcal{L}(X)$. Suppose that there exist a neighborhood $U \subseteq \mathbb{C}$ of 0 and a holomorphic function $F: U \to \mathcal{L}(X)$ such that $(A-\lambda)F(\lambda)(A-\lambda) = A-\lambda$, for all $\lambda \in U$.*
1) *A is said to be left holomorphically decomposably regular (resp. Fredholm-decomposably regular), if $F(\lambda) \in G_r(X)$ (resp. $\Phi_r(X)$) for all $\lambda \in U$.*
2) *A is said to be right holomorphically decomposably regular (resp. Fredholm-decomposably regular), if $F(\lambda) \in G_l(X)$ (resp. $\Phi_l(X)$) for all $\forall \lambda \in U$.*

We denote $HG_lR(X)$, $HG_rR(X)$, $H\Phi_lR(X)$, and $H\Phi_rR(X)$ these class of operators.

Denote by $\mathcal{L}_{00}(X)$ the ideal of (bounded) finite rank operators. Note that

$$\mathcal{L}_{00}(X) \subseteq \mathcal{L}_0(X) \subseteq \mathcal{L}(X).$$

In the sequel we will recall the following important theorem (see, [2, Theorem 1.5]).

Theorem 2.7. *Let X be a Banach space and $A \in \mathcal{L}(X)$. Then the following assertions are equivalent:*
i) $A \in \Phi_l(X)$;
ii) there exist $B \in \mathcal{L}(X)$ such that $I - BA \in \mathcal{L}_{00}(X)$;
iii) there exist $B \in \mathcal{L}(X)$ such that $I - BA \in \mathcal{L}_0(X)$.
Analogously; the following assertions are equivalent:
iv) $A \in \Phi_r(X)$;
v) there exist $B \in \mathcal{L}(X)$ such that $I - AB \in \mathcal{L}_{00}(X)$;
vi) there exist $B \in \mathcal{L}(X)$ such that $I - AB \in \mathcal{L}_0(X)$.

Lemma 2.8. [17, Lemma 6, p. 132] *Let $A \in \mathcal{L}(X)$. If $A \in S(X)$, then there exists $\varepsilon > 0$ such that $A - U$ has a generalized inverse for every operator $U \in \mathcal{L}(X)$ commuting with A such that $\| U \| < \varepsilon$. More precisely, if A is Kato non-singular, $ABA = A$, $UA = AU$ and $\| U \| \leq \| B \|^{-1}$, then $(A-U)B(I-UB)^{-1}(A-U) = A-U$.*

In the next theorem we will recall some well-known properties of the decomposably regular and Fredholm operators (see, [5, 23]).

Theorem 2.9. *Let X be a Banach space.*

1) $G_iR(X) = R(X) \cap \overline{G_i(X)}$, $i = l, r$.

2) $\Phi_iR(X) = R(X) \cap \overline{\Phi_i(X)}$, $i = l, r$.

3) $GR(X) = R(X) \cap \overline{G(X)}$.

4) $\Phi R(X) = R(X) \cap \overline{\Phi(X)}$.

5) $HG_iR(X) = \mathcal{S}(X) \cap \overline{G_i(X)}$, $i = l, r$.

6) $H\Phi_iR(X) = \mathcal{S}(X) \cap \overline{\Phi_i(X)}$, $i = l, r$.

Theorem 2.10. [4, Theorem 2.4] *Let $A, B \in \mathcal{L}(X)$ such that $AB = BA$. If $A, B \in H\Phi R(X)$ and if there exists $C, D \in \mathcal{L}(X)$ such that $AC + DB = I$, then $AB \in H\Phi R(X)$.*

3. On left and right decomposably regular operators

We first prove the following theorem.

Theorem 3.1. *Let $A \in \mathcal{L}(X)$ and let X be a direct sum of closed subspaces X_1 and X_2 which are A-invariant. If $A_1 = A_{|X_1} : X_1 \to X_1$ and $A_2 = A_{|X_2} : X_2 \to X_2$,*

1) *If $A_1 \in \Phi_iR(X_1)$ and $A_2 \in \Phi_iR(X_2)$ then $A \in \Phi_iR(X)$, $i = l, r$.*

2) *If $A_1 \in G_iR(X_1)$ and $A_2 \in G_iR(X_2)$ then $A \in G_iR(X)$, $i = l, r$.*

3) *If* $A_1 \in \Phi R(X_1)$ *and* $A_2 \in \Phi R(X_2)$ *then* $A \in \Phi R(X)$.

4) *If* $A_1 \in GR(X_1)$ *and* $A_2 \in GR(X_2)$ *then* $A \in GR(X)$.

5) *If* $A_1 \in H\Phi_i R(X_1)$ *and* $A_2 \in H\Phi_i R(X_2)$ *then* $A \in H\Phi_i R(X)$, $i = l,\ r$.

6) *If* $A_1 \in HG_i R(X_1)$ *and* $A_2 \in HG_i R(X_2)$ *then* $A \in HG_i R(X)$, $i = l,\ r$.

7) *If* $A_1 \in H\Phi R(X_1)$ *and* $A_2 \in H\Phi R(X_2)$ *then* $A \in H\Phi R(X)$.

8) *If* $A_1 \in HGR(X_1)$ *and* $A_2 \in HGR(X_2)$ *then* $A \in HGR(X)$.

Proof. The operator A has the following matrix form with respect to the decomposition $X = X_1 \oplus X_2$:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \longrightarrow \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

1) Suppose that $A_1 \in \Phi_l R(X_1)$ and $A_2 \in \Phi_l R(X_2)$. Using Theorem 2.9, there exists $B_1 \in \mathcal{L}(X_1)$ and $B_2 \in \mathcal{L}(X_2)$ such that

$$A_1 B_1 A_1 = A_1 \text{ and } A_2 B_2 A_2 = A_2.$$

Let

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

Therefore

$$\begin{aligned} ABA &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \\ &= \begin{bmatrix} A_1 B_1 A_1 & 0 \\ 0 & A_2 B_2 A_2 \end{bmatrix} \\ &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \\ &= A \end{aligned}$$

This prove that $A \in R(X)$. So, to find this proof it suffice to shaw that $A \in \overline{\Phi_l(X)}$. Since $A_1 \in \overline{\Phi_l(X_1)}$ and $A_2 \in \overline{\Phi_l(X_2)}$. Then there are $(A_{1,n})_n \subset \Phi_l(X_1)$ and $(A_{2,n})_n \subset \Phi_l(X_2)$ such that $A_{1,n}$ belong to A_1 and $A_{2,n}$ belong to A_2. Let

$$A_n = \begin{bmatrix} A_{1,n} & 0 \\ 0 & A_{2,n} \end{bmatrix}.$$

Then, we have

$$\mathcal{N}(A_n) = \mathcal{N}(A_{1,n}) \oplus \mathcal{N}(A_{2,n}) \text{ and } A_n(X) = A_{1,n}(X_1) \oplus A_{2,n}(X_2).$$

This shaw that $(A_n)_n \subset \Phi_l(X)$. Since it is clear that A_n belong to A. So, $A \in \overline{\Phi_l(X)}$. Finally, using Theorem 2.9 1) we infer that $A \in \Phi_l R(X)$.
A similar reasoning as before, we prove that if $A_1 \in \Phi_r R(X_1)$ and $A_2 \in \Phi_r R(X_2)$ then $A \in \Phi_r R(X)$.
Their assertions 2), 3) and 4) can be proved similarly.

5) Suppose that $A_1 \in H\Phi R(X_1)$ and $A_2 \in H\Phi R(X_2)$. Then A_1 and A_2 are Kato non-singular operators, $A_1 \in \Phi_l R(X_1)$ and $A_2 \in \Phi_l R(X_2)$. Using 1), we can deduce that $A \in \Phi_l R(X)$. So, it suffice to shaw that A is Kato non-singular. By the decomposing $X = X_1 \oplus X_2$, we have

$$\mathcal{N}(A) = \mathcal{N}(A_1) \oplus \mathcal{N}(A_2) \text{ and } A^n(X) = A_1^n(X_1) \oplus A_2^n(X_2) \text{ for all } n \in \mathbb{N}.$$

Since $A_1(X_1)$, $A_2(X_2)$ are closed, $\mathcal{N}(A_1) \subset A_1^n(X_1)$ and $\mathcal{N}(A_2) \subset A_2^n(X_2)$ for all $n \in \mathbb{N}$. So, $A(X)$ is closed and $\mathcal{N}(A) \subset A^n(X)$ for all $n \in \mathbb{N}$.
A similar as before, we prove that if $A_1 \in H\Phi_r R(X_1)$ and $A_2 \in H\Phi_r R(X_2)$ then $A \in H\Phi_r R(X)$.

Their assertions 6), 7) and 8) can be proved similarly to 5). □

Theorem 3.2. *Let $A \in \mathcal{L}(X)$. Then for $\lambda_0 \in \mathbb{C}\backslash\sigma(A)$ and $\lambda \neq \lambda_0$, we have*

$$A - \lambda \in H\Phi R(X) \textit{ if and only if } (A - \lambda_0)^{-1} - \tfrac{1}{\lambda - \lambda_0} \in H\Phi R(X).$$

Proof.
Since $A \in \mathcal{L}(X)$, then $\mathbb{C}\backslash\sigma(A)$ is not empty. Let $\lambda_0 \in \mathbb{C}\backslash\sigma(A)$ and $\lambda \neq \lambda_0$. We can write

$$(A - \lambda) = (\lambda_0 - \lambda)\left((A - \lambda_0)^{-1} - \frac{1}{\lambda - \lambda_0}\right)(A - \lambda_0) \tag{3.1}$$

Suppose that $A - \lambda \in H\Phi R(X)$. Then, by Eq. (3.1),

$$\left((A - \lambda_0)^{-1} - \tfrac{1}{\lambda - \lambda_0}\right)(A - \lambda_0) \in H\Phi R(X).$$

Moreover

$$\left((A - \lambda_0)^{-1} - \tfrac{1}{\lambda - \lambda_0}\right)(A - \lambda_0) = (A - \lambda_0)\left((A - \lambda_0)^{-1} - \tfrac{1}{\lambda - \lambda_0}\right)$$

and $A - \lambda_0$ is invertible operator. So, by Lemma 2.10, we infer that $(A - \lambda_0)^{-1} - \frac{1}{\lambda - \lambda_0} \in H\Phi R(X)$.

Conversely, Suppose that $(A - \lambda_0) - \frac{1}{\lambda - \lambda_0} \in H\Phi R(X)$. Since $A - \lambda_0$ is invertible operator, then $A - \lambda_0 \in H\Phi R(X)$. Now, let

$$T' = A - \lambda_0 \text{ and } S' = (1 + \lambda - A)(A - \lambda_0)^{-1}.$$

It is easy to see that

$$(\lambda_0 - \lambda)\left((A - \lambda_0)^{-1} - \tfrac{1}{\lambda - \lambda_0}\right)T' + S'(A - \lambda_0) = I.$$

Applying Lemma 2.10, we get $(A - \lambda) \in H\Phi R(X)$. □

4. Holomorphically Weyl-decomposably regular

Let X be a Banach space and $A \in \mathcal{L}(X)$. A is said to be a Weyl operator if A is Fredholm operator having index 0 (see, [1, 3, 11]). Let $W(X)$ be the class of all Weyl operators.

Definition 4.1. *An operator $A \in \mathcal{L}(X)$ is said to be Weyl-decomposably regular operator, in symbol $A \in WR(X)$, provided that there exist $B \in W(X)$ such that $ABA = A$.*

Remark 4.2. *Let $A \in GR(X)$. Then, there exist $B \in G(X)$ such that $ABA = A$. Cleary $B \in \Phi(X)$ and $i(B) = 0$. In this way we see that*

$$GR(X) \subset WR(X) \subset \Phi R(X).$$

Definition 4.3. *Let $A \in \mathcal{L}(X)$. A is said to be holomorphically Weyl-decomposably regular if there exist a neighborhood $U \subset \mathbb{C}$ of 0 and holomorphic function $F : U \longrightarrow \mathcal{L}(X)$ such that*

$$F(x) \in W(X) \text{ and } (A - x)F(x)(A - x) = A - x$$

for all $x \in U$. Let $HWR(X)$ be the class of all holomorphically Weyl-decomposably regular operators.

Theorem 4.4. *Let X be a Banach space. Then $HWR(X) = S(X) \cap WR(X)$.*

Proof. Let $A \in HWR(X)$. Then, there exist a neighborhood $U \subset \mathbb{C}$ of 0 and a holomorphic function $F : U \longrightarrow \mathcal{L}(X)$ such that

$$F(x) \in W(X) \text{ and } (A - x)F(x)(A - x) = A - x \tag{4.1}$$

for all $x \in U$. In particular, we have

$$F(0) \in W(X) \text{ and } AF(0)A = A.$$

This prove that $A \in WR(X)$. By Eq.(4.1), we have $A \in S(X)$. Now, it remains to prove that $S(X) \cap WR(X) \subset HWR(X)$. Let $A \in S(X) \cap WR(X)$, then there exist $B \in W(X)$ such that $ABA = A$. Let

$$F : D(0, \| B \|^{-1}) \longrightarrow \mathcal{L}(X)$$

defined by $F(x) = (I - xB)^{-1}B$ for all $x \in D(0, \| B \|^{-1})$. It is clear that

$$F(x) \in W(x) \text{ and } i(F(x)) = i((I - xB)^{-1}) + i(B) = 0.$$

So, we infer that $F(x) \in W(X)$ for all $x \in D(0, \| S \|^{-1})$. Using Lemma 2.8 we infer that

$$(A - x)F(x)(A - x) = A - x$$

for all $x \in D(0, \| S \|^{-1})$. In this way we see that $A \in HWR(X)$. □

Theorem 4.5. *Let $A \in \mathcal{L}(X)$. Then $A \in HWR(X)$ if and only if there exist $R \in W(X)$, $(B_n)_n \subset G(X)$ and $(A_n)_n \subset W(X)$ such that $ARA = A$, $AB_n = B_nA$, $(A - B_n)A_n(A - B_n) = A - B_n$ for all $n \in \mathbb{N}$ and $\lim_n \left(\|B_n\| + \|A_n - R\|\right) = 0$.*

Proof. Let $A \in HWR(X)$. Then there exist $\varepsilon > 0$ and a holomorphic function $F : D(0, \varepsilon) \longrightarrow \mathcal{L}(X)$ such that

$$F(x) \in W(X) \text{ and } (A - x)F(x)(A - x) = A - x$$

for all $x \in D(0, \varepsilon)$. Let

$$B_n = \frac{\varepsilon}{2^n} I \text{ and } A_n := F(\frac{\varepsilon}{2^n})$$

for all $n \in \mathbb{N}^*$. Let $R = F(0)$. In this way we see that

$$R \in W(X), \ (B_n)_{n \in \mathbb{N}^*} \subset W(X), \ AB_n = B_nA, \ (A - B_n)A_n(A - B_n) = A - B_n$$

and

$$\lim_n \Big(\| B_n \| + \| A_n - R \| \Big) = 0.$$

Conversely, suppose there exist $R \in W(X)$, $(B_n)_n \subset G(X)$ and $(A_n)_n \subset W(X)$ such that $ARA = A$, $AB_n = B_nA$, $(A - B_n)A_n(A - B_n) = A - B_n$ for all $n \in \mathbb{N}^*$. It remain remain to prove that $A \in S(X)$. Using Theorem 2.5, the result follow. $\square$

Theorem 4.6. *Let X be a Banach space. Then $WR(X) \subset R(X) \cap \overline{W(X)}$.*

Proof.
Let $A \in WR(X)$. Then there exist $B_0 \in W(X)$ such that $AB_0A = A$. Since $\Phi(X) = \Phi_l(X) \cap \Phi_r(X)$ and $B_0 \in \Phi(X)$, we can be apply Theorem 2.7, we infer that there exist $B_1 \in \mathcal{L}(X)$, $K_1 \in \mathcal{L}_0(X)$ and $K_2 \in \mathcal{L}_0(X)$ such that

$$B_1B_0 = I + K_1 \ \text{ and } \ B_0B_1 = I + K_2. \tag{4.2}$$

In this way we see that

$$AB_0(B_1 - AK_2) = A$$

So, we can write $A = PC$, where $C = B_1 - AK_2 \in \mathcal{L}(X)$ and $P = AB_0 \in \mathcal{L}(X)$ is an idempotent operator. Again, using Eq.(4.2), we have

$$B_1 - AK_2 \in \Phi(X) \ \text{ and } \ i(B_1B_0) = 0 = i(B_0) + i(B_1) = i(B_1).$$

Hence $B_1 - AK_1 \in W(X)$. Now, let

$$A_n := \Big(P + \frac{I-P}{n}\Big)C \in \mathcal{L}(X), \ \ n \in \mathbb{N}^*.$$

It is clear that

$$\Big(P + \frac{I-P}{n}\Big)\Big(P + n(I-P)\Big) = \Big(P + n(I-P)\Big)\Big(P + \frac{I-P}{n}\Big) = I,$$

for all $n \in \mathbb{N}^*$. This prove that $P + \dfrac{I-P}{n} \in G(X) \subset W(X)$ for all $n \in \mathbb{N}^*$. In this way we say that

$$A_n \in W(X) \ \text{ and } \ i(A_n) = i(P + \frac{I-P}{n}) + i(C) = 0.$$

Then, $A_n \in W(X)$ for all $n \in \mathbb{N}^*$. Since $\lim_n A_n = A$, then $A \in \overline{W(X)}$. $\square$

5. Holomorphically Weyl-decomposably regular spectrum

Let $A \in \mathcal{L}(X)$. Let $\sigma_{rr}(A) = \{x \in \mathbb{C} : \ A - x \notin S(X)\}$ the Saphar spectrum of A and $\rho_{rr}(A) = \mathbb{C}\backslash\sigma_{rr}(A)$ its Saphar resolvent set. It is know (see,[6, 10]) that $x \in \rho_{rr}(A)$ if and only if there is $U(x)$ a neighbourhood of x and there is $F : U(x) \longrightarrow \mathcal{L}(X)$ such that F is analytic and $(A-y)F(y)(A-y) = A-y$ for all $y \in U(x)$. The holomorphically decomposably regular resolvent set $\rho_{gr}(A)$ is given by: $x \in \rho_{gr}(A)$ if and only if there is $V(x)$ a neighbourhood of x and there is $F : V(x) \longrightarrow \mathcal{L}(X)$ such that F is analytic and $(A - y)F(y)(A - y) = A - y$, $F(y) \in G(X)$ for all $y \in V(x)$. The holomorphically decomposably regular spectrum of A is defined by $\sigma_{gr}(A) = \mathbb{C}\backslash\rho_{gr}(A)$.

The holomorphically Weyl-decomposably regular resolvent set of A is defined by

$$\rho_{hw}(A) = \Big\{x \in \mathbb{C} : \ \exists U(x) \ \text{ a neighborhood of } \ x \ \text{ and } \ F : U(x) \longrightarrow \mathcal{L}(X)$$

analytic such that $F(y) \in W(X)$ and $(A - y)F(y)(A - y) = A - y$ for all $y \in U(x)\Big\}$

The holomorphically Weyl-decomposably regular spectrum of A is defined by

$$\sigma_{hw}(A) := \mathbb{C} \setminus \rho_{hw}(A).$$

In this way we see that

$$\sigma_{hf}(A) \subset \sigma_{hw}(A) \subset \sigma_{gr}(A) \subset \sigma(A).$$

Remark 5.1. *Let $A \in \mathcal{L}(X)$. Then $A \in HWR(X)$ if and only if $0 \in \rho_{hw}(A)$.*

Proposition 5.2. *Let $A \in \mathcal{L}(X)$. Then $0 \in \sigma_{hw}(A)$ if and only if $0 \in \sigma_K(A)$ or $A \notin WR(X)$*

Proof. It remain to prove that $A \in HWR(X)$ if and only if A is Kato non-singular and $A \in WR(X)$. Suppose that $A \in HWR(X)$. Then there exist an open disc $D(0, r) \subset \mathbb{C}$ and a holomorphic function $F : D(0, r) \longrightarrow \mathcal{L}(X)$ such that $F(x) \in W(X)$ and $(A - x)F(x)(A - x) = A - x$, for all $x \in D(0, r)$. In this way we see that $0 \in \rho_{rr}(A) \subset \mathbb{C}\backslash\sigma_K(A)$. Since $AF(0)A = A$ and $F(0) \in WR(X)$. So, A is Kato non-singular and $A \in WR(X)$. Conversely, suppose that A is Kato non-singular and there exist $B \in W(X)$ such that $ABA = A$. Let $0 < r <\| B \|^{-1}$ and $F : D(0, r) \longrightarrow \mathcal{L}(X)$ such that $F(x) = (I - xB)^{-1}B$ for all $x \in D(0, r)$. It is clear that

$$(I - xB)^{-1}B = B(I - xB)^{-1}$$

for all $x \in D(0, r)$ and F is analytic. By Lemma 2.8, we have

$$(A - x)F(x)(A - x) = A - x$$

for all $x \in D(0, r)$. On the other hand, $(I - xB)^{-1} \in G(X)$ for all $x \in D(0, r)$. So, $F(x) \in \Phi(X)$ and $i(F(x)) = i(B) + i((I - xB)^{-1}) = 0$ for all $x \in D(0, r)$. This proof is complete. □

Corollary 5.3. *Let $A \in \mathcal{L}(X)$ and $n \in \mathbb{N}^*$. Then If $A \in HWR(X)$ then $A^n \in HWR(X)$.*

Proof. Let $A \in HWR(X)$. Using Proposition 5.2, T is Kato non-singular and $A \in WR(X)$. So, $0 \in \rho_K(A)$ and there exist $B \in W(X)$ such that $ABA = A$. Again we can apply Lemma 2.4, we have $A^nB^nA^n = A^n$. In particular, A^n is Kato non-singular. Since $B \in W(X)$, then

$$B^n \in \Phi(X) \text{ and } i(B^n) = \sum_{i=1}^{n} i(B) = ni(B) = 0.$$

In this way we see that A^n is Kato non-singular and $A^n \in WR(X)$. This proof is complete. □

Corollary 5.4. *Let $A \in \mathcal{L}(X)$ and $B \in G(X)$ such that $AB = BA$. If $AB \in HWR(X)$ then $A \in HWR(X)$.*

Proof. Suppose that $AB \in HWR(X)$. By Proposition 5.2, AB is Kato non-singular and there exist $C \in W(X)$ such that $ABCAB = AB$. It is clear that $AB(X) = A(X)$ and

$$\mathcal{N}(A) = \mathcal{N}(BA) = \mathcal{N}(AB) \subset (AB)^\infty(X) \subset A^\infty(X).$$

In this way we see that, $0 \in \mathbb{C}\backslash\sigma_K(A)$. On the other hand, $B \in G(X)$. Then $ABCA = A$, where $BC \in \Phi(X)$ and $i(BC) = i(B) + i(C) = 0$. So, we give A is Kato non-singular and $A \in WR(X)$. Again, we can apply Proposition 5.2, we have $A \in HWR(X)$. □

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