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Stochastic bilevel network design problems

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1 Introduction

In this paper we develop models for network design in the presence of two types of actors: network owners and network users. The problem is to maximize the profit of a network owner by selecting the tolls or tariffs charged for the network usage by the network owner. The novelty of the paper consists in considering uncertainty in the demand and network costs for the users (for example, travel times). Besides of substantial theoretical and computational interest, this problem has applications in telecommunication and road networks. The modelling framework is based on bilevel programming and Stackelberg games [8]. The bilevel programming (BP for short) considers only two players called leader and follower who seek to optimize their respective objective function within a hierarchical scheme [2]. BP problems are generally difficult to solve as the feasibility region is usually nonconvex, disconnected or even empty in some cases. BP problems are also strongly NP hard even if the functions are linear [9]. Deterministic BP has many applications [3, 4]. Competitive markets require robust solutions for different decision making problems. However, deterministic solutions cannot handle efficiently any change in the input data. To deal with different aspects of uncertainty or risk, stochastic optimization is an efficient tool as it models easily uncertain events, and provides robust solutions. Stochastic bilevel problems have also been recently studied in the literature with successful applications in different areas [1, 6, 7, 10]. The novelty of this paper lies in developing modeling and computational tools for treatment of uncertainty in network design problems. The paper is organized as follows: Section 2 presents the problem formulation. Section 3 presents the MILP formulation and the stochastic gradient method.

2 Problem formulation

We consider a network with the set of nodes N and the set of links $A \subseteq N \times N$. We denote x_a the total flow through link $a \in A$. It is assumed that the network (or part of it) is governed by an actor we call the Network Operator (NO). This actor has the authority to impose tolls on subset of links $\bar{A} \subseteq A$. This toll is proportional: the originator of flow x_a pays to Network Operator toll $T_a x_a$. The Network Operator pursues the aim of maximizing his revenue, i.e., he chooses the toll structure by solving the following optimization problem

$$\max_{T} \quad \Phi(T), \quad \Phi(T) = \sum_{a \in \bar{A}} T_a \mathbb{E} x_a \tag{1}$$

$$s.t. \quad T \in \Upsilon, \Upsilon = \left\{ T | l_a \leq T_a \leq u_a, \quad \forall a \in \bar{A} \right\}. \tag{2}$$

where T is the vector of all tolls T_a. The flows on the network are originated by one or more Flow Originators (FO). In this paper, we consider only one FO. The flows he generates satisfy the flow demand d_{rs} between pairs of nodes $r, s \in N$. Demand d_{rs}, $(r,s) \in \Theta \subseteq N \times N$ is a random variable with known distribution. The Flow Originator decides how to route the demand and which part of it to satisfy in order to minimize the total costs that are composed of tolls T_a, flow routing costs $c_a, a \in A$ and penalty costs q_{rs} for not meeting a unit of demand between nodes r and s. He knows the toll structure that is communicated to him by Network Operator. While routing the demand, he can select between paths $p \in P_{rs}$, where P_{rs} is the set of admissible paths between nodes r and s. His decision how to route the demand is denoted by f_p^{rs}, this is the amount of demand between nodes r, s that is routed on path $p \in P_{rs}$.

In this paper, we consider real time routing, i.e., the demand is known at the time of actual routing decision and no advanced reservation is required. In this case the FO pays routing and toll costs for actually routed demand and the problem becomes the following

$$Q = \min_{f,x} \quad \left\{ \sum_{a \in A} c_a x_a + \sum_{a \in \bar{A}} T_a x_a + \sum_{(r,s) \in \Theta} q_{rs} \left(d_{rs} - \sum_{p \in P_{rs}} f_p^{rs} \right) \right\} \tag{3}$$

$$s.t. \quad \sum_{p \in P_{rs}} f_p^{rs} \leq d_{rs}, forall (r,s) \in \Theta \tag{4}$$

$$x_a = \sum_{(r,s) \in \Theta} \sum_{p \in P_{rs}} \delta_{ap}^{rs} f_p^{rs}, \qquad \forall a \in A \tag{5}$$

$$x_a \leq C_a, \tag{6}$$

$$f_p^{rs} \geq 0, \qquad \forall p \in P_{rs}, \forall (r,s) \in \Theta \tag{7}$$

Taking $T_a = l_a = u_a = 0$ for $a \in A \backslash \bar{A}$ we can take $\bar{A} = A$ and drop the set $\bar{A}$ from our subsequent formal considerations.

Here we consider the case when the routing decision is taken after demand d_{rs} is observed. Then the FO solves the problem (3),(4),(5). As there is no coupling constraints, the FO problem can be decomposed into individual problems for demand between r and s nodes taking the form:

$$\min_{f_p^{rs}} \quad \left[\sum_{p \in P_{rs}} e_p f_p^{rs} + q_{rs} \left(d_{rs} - \sum_{p \in P_{rs}} f_p^{rs} \right) \right] \tag{8}$$

$$s.t. \quad \sum_{p \in P_{rs}} f_p^{rs} \leq d_{rs}, \tag{9}$$

The problem of Network Operator (1),(2) in this case takes the form described in the next Proposition.

Proposition 1. *Suppose that the Flow Originator when faced with alternative decisions that bear the same cost, but different level demand satisfaction always chooses decision with the highest demand satisfaction. Then the problem (1)-(5),(7) is equivalent to the following optimization problem:*

$$\max_{T} \quad \sum_{a \in A} T_a x_a = \sum_{(r,s) \in \Theta rs} \bar{d}_{rs} \mathbb{I}_{q_{rs} \geq e_{p_{rs}(T)}} \sum_{a \in p_{rs}(T)} T_a, \tag{10}$$

where

$$\bar{d}_{rs} = \mathbb{E}d_{rs}, \mathbb{I}_{q_{rs} \geq e_{p_{rs}(T)}} = \begin{cases} 1 & if \quad q_{rs} \geq e_{p_{rs}(T)} \\ 0 & otherwise \end{cases}$$

and $p_{rs}(T)$ *is any path from* P_{rs} *for which* $e_{p_{rs}(T)} \leq e_p$ *for any* $p \in P_{rs}$

Due to the limited size of the conference papers, the proof of 1 will be given in the full version of the paper.

3 MILP formulation and Stochastic gradient method

The problem (10) can be reformulated under the form of linear mixed integer programming problem. Firstly, we introduce the binary variables y_{rs}^p, which equals one if $p = p_{rs}(T)$ and $q_{rs} \geq e_{p_{rs}(T)}$. The objective function of the equivalent problem can be written as $\max_{y_{rs}^p, T_a} \sum_{(r,s)\in\Theta} \bar{d}_{rs} \sum_{p\in P_{rs}} y_{rs}^p \sum_{a\in p} T_a$. After linearizing the nonlinear terms, we obtain the following MILP:

$$\max_{z_{rs}^p, y_{rs}^p, T_a} \sum_{(r,s)\in\Theta} \sum_{p\in P_{rs}} \bar{d}_{rs} z_{rs}^p \tag{11}$$

$$z_{rs}^p - M y_{rs}^p \leq 0, (r,s) \in \Theta, p \in P_{rs}$$

$$z_{rs}^p - \sum_{a\in p} T_a \leq 0, (r,s) \in \Theta, p \in P_{rs}$$

$$M y_{rs}^p + \sum_{a\in p} T_a \leq M + q_{rs} - \sum_{a\in p} c_a, (r,s) \in \Theta, p \in P_{rs}$$

$$M y_{rs}^p + \sum_{a\in p} T_a - \sum_{a\in p'} T_a \leq M + \sum_{a\in p'} c_a - \sum_{a\in p} c_a, (r,s) \in \Theta, p, p' \in P_{rs}, p' \neq p$$

$$\sum_{p\in P_{rs}} y_{rs}^p \leq 1, (r,s) \in \Theta$$

$$l_a \leq T_a \leq u_a, y_{rs}^p \in \{0,1\}$$

where M is a large number.

In order to solve the problem with random travel costs outlined in the previous section, we developed a variant of stochastic gradient method. The general computational scheme of stochastic gradient methods can be stated as follows [5]. The problem to be solved is $\max_{T\in\Upsilon} \Phi(T) = \sum_{a\in A} T_a \mathbb{E}x_a$

iterates T^s according to the following rule: $T^{s+1} = \pi_\Upsilon \left(T^s + \rho_s \xi^s\right), s = 0, 1, ...$ where $\pi_X(\cdot)$ is the projection operator on set Υ, ρ_s is nonnegative step size usually tending to zero and ξ^s is stochastic gradient, that is a vector that satisfies the following stochastic gradient qualification

$$\mathbb{E}\left[\xi^s | T^0, ..., T^s\right] = \nabla F(T) + b_s \tag{12}$$

where b_s diminishes in some sense in the course of iterations. Under quite general conditions the sequence T^s converges to the optimal solution of (1) in a certain probabilistic sense, see [5] for more details.

Briefly, the vector ξ^s is composed from components ξ^s_a, $a \in A$. At iteration s of the stochastic gradient method, ξ^s is computed as follows: $\xi^s_a = \sum_{(r,s)\in\Theta} \bar{d}_{rs}\phi_{rs}(T^s, c^s_{-a})$ where $\phi_{rs}\left(T^s, c^s_{-a}\right)$ is a function studied in the full paper version, and c^s_a is the vector cost generated according to $G(c)$ but c_a. Numerical results comparing our different approaches will be detailed in the full version of the paper.

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