Geometric Derivation of the Irradiance of Polygonal Lights

Eric Heitz

Geometric Derivation of the Irradiance of Polygonal Lights

Eric Heitz

Unity Technologies

Abstract

This document shows how to compute the irradiance of polygonal lights. In contrast to other derivations, this one is based solely on geometric concepts and emphasizes the intuition behind the result.

1 Introduction

The goal of this document is to provide a comprehensive derivation of the formula for the irradiance of polygonal light sources:

$$I(v_1, .., v_n) = \frac{1}{2\,\pi} \sum_{i=1}^{n} \text{acos}\left(p_i \cdot p_j\right) \left(\frac{p_i \times p_j}{\|p_i \times p_j\|} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right). \tag{1}$$

This formula dates back to Lambert [Lam60] and has been widely used in the rendering literature, especially at the time of radiosity [BRW89].

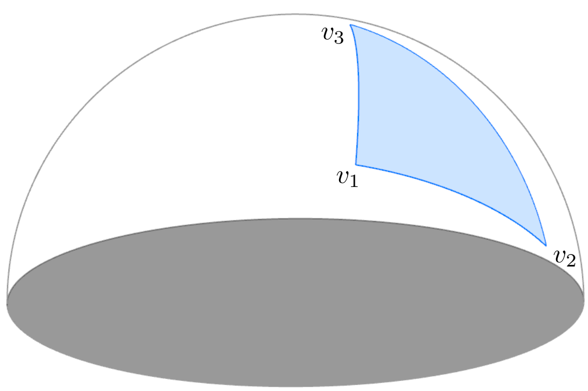

Figure 1: A spherical polygon in the hemisphere.

In the following, we assume that $v_1, .., v_n$ is a spherical polygon (clipped) in the hemisphere of direction $z = [0\,0\,1]^T$, i.e. that for each i

- $|v_i| = 1$, and
- $v_i \cdot z \geq 0$.

2 Irradiance of Hemispherical Domains

Figure 2 shows that the irradiance of a hemispherical domain S_Ω is the fraction of the unit disk covered by its projection $S_\mathscr{C}$:

$$I = \frac{S_\mathscr{C}}{\pi}. \tag{2}$$

where π is the area of the unit disk below the hemisphere. For instance, if a shape covers the unit disk completely, its projected area is $S_\mathscr{C} = \pi$ and its irradiance is $I = 1$.

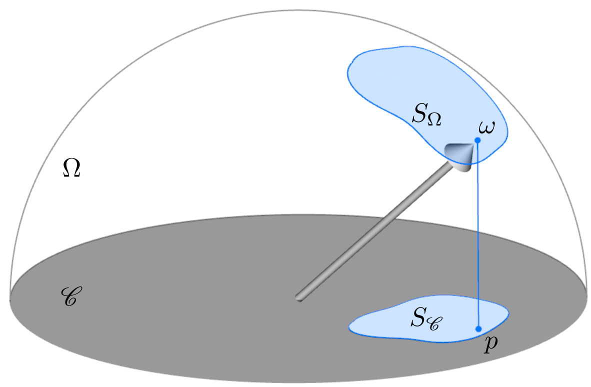

Figure 2: The irradiance of a hemispherical domain is the fraction of the unit disk covered by its projection.

2.1 Proof

The irradiance of a hemispherical domain S_Ω is generally defined as the integral of the cosine of these directions divided by π:

$$I = \frac{1}{\pi} \int_{S_\Omega} \cos\theta \, \mathrm{d}\omega \tag{3}$$

The cosine factor in the expression can also be seen as the Jacobian of the projection onto the unit disk:

$$\frac{\mathrm{d}p}{\mathrm{d}\omega} = \cos\theta, \tag{4}$$

which allows for changing the integration domain:

$$I = \frac{1}{\pi} \int_{S_\Omega} \cos\theta \, \mathrm{d}\omega \tag{5}$$

$$= \frac{1}{\pi} \int_{S_\mathscr{C}} 1 \, \mathrm{d}p \tag{6}$$

$$= \frac{S_\mathscr{C}}{\pi}. \tag{7}$$

3 Irradiance of a Polygon

In Figure 3, the irradiance of the polygon is given by its projected area on the unit disk (in blue). The area of this blue region cannot be computed directly. However, it can be decomposed as the sum of several (signed) areas associated with the edges.

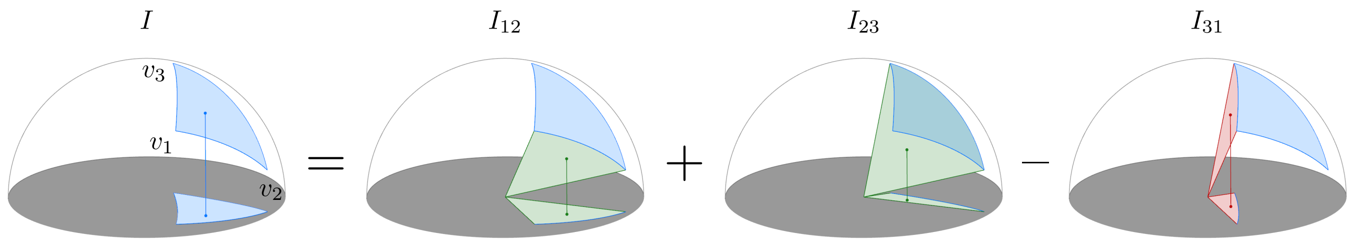

Figure 3: The blue area is the sum of the (positive) green areas and the (negative) red areas.

3.1 Disk Sectors

Each edge $(v_i\, v_j)$ of the polygon defines a disk sector (a “pizza slice”) defined by an angle θ_{ij} and a normal n_{ij}:

$$\theta_{ij} = \text{acos}(v_1 \cdot v_2), \tag{8}$$

$$n_{ij} = \frac{v_i \times v_j}{\|v_i \times v_j\|}. \tag{9}$$

The area of a disk sector is proportional to its angle:

$$A_{ij} = \frac{\theta_{ij}}{2}. \tag{10}$$

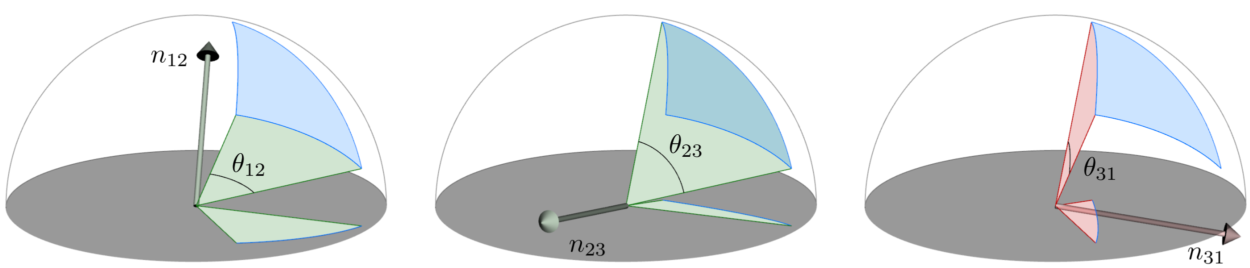

Figure 4: Disk sectors associated to the edges of the polygon.

Signed Irradiance of a Disk Sector The signed projected area of the disk sector is its area A_{ij} multiplied by the signed cosine of its normal $(n_{ij} \cdot z)$. By dividing the signed projected area by π, we obtain the signed irradiance of a disk sector:

$$I_{ij} = \frac{1}{\pi}\,(n_{ij} \cdot z)\, A_{ij} \tag{11}$$

$$= \frac{1}{\pi}\left(\frac{v_i \times v_j}{\|v_i \times v_j\|} \cdot z\right) \frac{\text{acos}(v_i \cdot v_j)}{2}, \tag{12}$$

In Figure 4, the signed projected area of a disk sector is positive (green) if its normal points towards the upper hemisphere and negative (red) otherwise.

3.2 Irradiance of the Polygon

Finally, as shown in Figure 3, the irradiance of the polygon is the sum of the signed irradiances of the disk sectors:

$$I(v_1, v_2, v_3) = I_{12} + I_{23} + I_{13}. \tag{13}$$

By expanding the terms and generalizing the expression we obtain the result of Equation (1).

Comment People often wonder why Equation (1) has a $\frac{1}{2\pi}$ factor. We have seen that π is the classic normalization factor of the irradiance (the area of the unit disk below the hemisphere) and the factor 2 comes from the area of a disk sector in Equation (10) (the angle of the disk sector divided by 2).

4 Bonus: Irradiance of a Sphere Located in the Hemisphere

The same approach provides intuition for the irradiance of spheres. Let us consider a sphere of radius r, located at distance d in direction n. The sphere covers a spherical cap, i.e. a disk, of area

$$A = \pi \frac{r^2}{d^2} \tag{14}$$

The irradiance I of the sphere is the projected area of this disk divided by π:

$$I = \frac{1}{\pi} A \, (n \cdot z) \tag{15}$$

$$= \frac{r^2}{d^2} \, (n \cdot z). \tag{16}$$

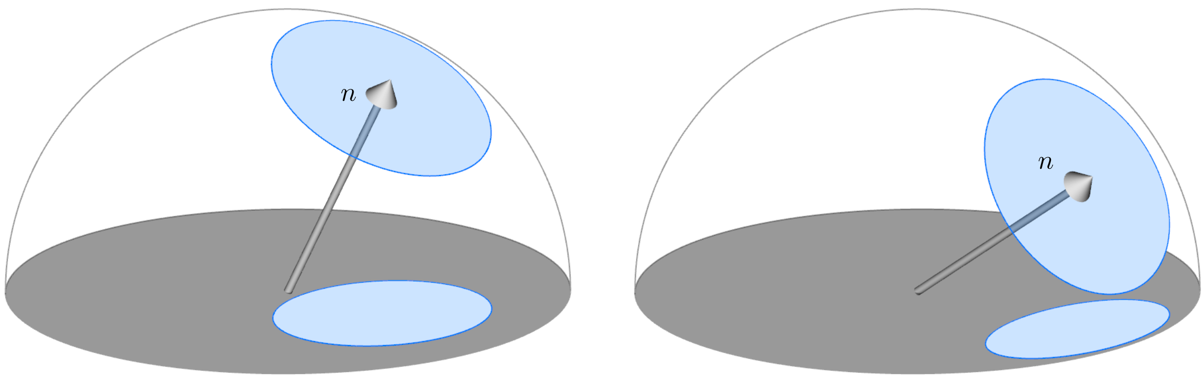

Figure 5: The irradiance of a sphere is given by the projected area of the spherical cap covered by the sphere, which is the area of an ellipse.

Comment 1 The projected spherical cap is an ellipse.

Comment 2 It might be surprising that the irradiance of a sphere depends linearly on the cosine of its direction. Figure 5 makes it obvious that the area A of the disk is the same whatever its direction. Hence, its projected area and its irradiance depend only on its direction n.

Comment 3 This derivation is valid only for spheres located entirely in the upper hemisphere. The general result is more complicated.

References

[BRW89] D. R. Baum, H. E. Rushmeier, and J. M. Winget. Improving radiosity solutions through the use of analytically determined form-factors. *Computer Graphics (Proc. SIGGRAPH)*, 23(3):325–334, 1989. 1

[Lam60] Johann Heinrich Lambert. Photometria, sive de mensura et gradibus luminus, colorum et umbrae. 1760. 1