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Uniform ball condition and existence of optimal shapes for geometric functionals involving boundary-value problems

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Abstract

In this article, we are interested in shape optimization problems where the functionals are defined on the boundary of the domain, involving the geometry of the associated surface and the boundary values of the solution to a state equation posed on the inner domain enclosed by the shape. Hence, we pursue here the study initiated in a previous work by considering a specific class admissible shapes. Given \( \varepsilon > 0 \) and a fixed non-empty large bounded open hold-all \( B \subset \mathbb{R}^n \), \( n \geq 2 \), we define \( O_\varepsilon(B) \) as the class of open sets \( \Omega \subset B \) satisfying a \( \varepsilon \)-ball condition, which has an equivalent characterization in terms of uniform \( C^{1,1} \)-regularity of the boundary \( \partial \Omega \). The main contribution of this paper is to prove the existence of a minimizer in the class \( O_\varepsilon(B) \) for problems of the form:

\[
\inf_{\Omega \in O_\varepsilon(B)} \int_{\partial \Omega} j \left[ u_\Omega(x), \nabla u_\Omega(x), x, n(x), H(x) \right] \, dA(x),
\]

where \( u_\Omega \) denotes the solution of the Dirichlet Laplacian posed on the domain \( \Omega \) or to the one associated with a Neumann or Robin boundary condition, where \( n \) is the unit outward normal vector, and where \( H \) can refer either to the the scalar mean curvature, to the Gaussian curvature, or more generally to any of the symmetric polynomials in the principal curvatures.

We only assume here the continuity of \( j \) with respect to the set of variables, convexity with respect to the last variable, and quadratic growth regarding the first two variables. We give various applications in the field of partial differential equations such as existence for:

\[
\inf_{\Omega \in O_\varepsilon(B)} \int_{\Omega} j \left[ x, u_\Omega(x), \nabla u_\Omega(x), \text{Hess } u_\Omega(x) \right] \, dV(x),
\]

and boundary shape identifications in the area of inverse and control problems:

\[
\inf_{\Omega \in O_\varepsilon(B)} \int_{\Gamma_0} \left[ (\partial_n u_\Omega - f_0)^2 + (u_\Omega - g_0)^2 \right] \, dA.
\]

Keywords: shape optimization, uniform ball condition, elliptic partial differential equations, geometric functionals, existence theory, boundary shape identification problems.


1 Introduction

In mathematical engineering, many practical applications are modelled by minimization processes. It often happens that the quantity of interest is given by the shape of some optimal domains. Hence, a first natural question arises from this setting: is our problem well posed i.e. does there exists such a design? To answer this question, it is therefore necessary to study the existence of minimizers to the following kind of shape optimization problems:

\[
\inf_{\Omega \in A} J(\Omega),
\]

where \( J : \Omega \mapsto J(\Omega) \) is a real-valued functional defined over a set \( A \) of admissible shapes \( \Omega \subset \mathbb{R}^n \) that may also include additional constraints. In the theory of partial differential equations (PDE), a typical range of functionals is given by:

\[
J(\Omega) = \int_{\Omega} j \left[ x, u_\Omega(x), \nabla u_\Omega(x) \right] \, dV(x),
\]
where the integration on \( \Omega \) is done with respect to the \( n \)-dimensional Lebesgue measure \( V(\bullet) \), and where \( u_\Omega : x \mapsto u_\Omega(x) \) is the solution of a state equation posed on the domain \( \Omega \). For example, the map \( u_\Omega \) can refer to the solution of the Dirichlet Laplacian posed on a bounded domain \( \Omega \):

\[
- \Delta u_\Omega = f \quad \text{in } \Omega, \quad u_\Omega = 0 \quad \text{on } \partial \Omega; \quad (3)
\]

and similarly, to the one associated with a Neumann boundary condition

\[
- \Delta u_\Omega + \lambda u_\Omega = f \quad \text{in } \Omega, \quad \partial_n (u_\Omega) = 0 \quad \text{on } \partial \Omega; \quad (4)
\]

or a Robin boundary condition

\[
- \Delta u_\Omega = f \quad \text{in } \Omega, \quad \partial_n (u_\Omega) + \lambda u_\Omega = 0 \quad \text{on } \partial \Omega, \quad (5)
\]

where \( \lambda \geq 0 \) and \( f : \mathbb{R} \to \mathbb{R} \) are given. In general, one cannot expect to obtain the existence of minimizers to (1) for general functionals (2). Indeed, many counterexamples are available in the vast literature on the subject [1, 2, 9, 14, 17, 18]. However, it is possible to recover the existence in some cases by relaxing the formulation (1) using e.g. the homogenization method [1], by assuming that \( J : \Omega \to J(\Omega) \) is non-increasing for inclusion [4], by adding some topological requirements like convexity or as in [19] for two-dimensional shapes, and by imposing a uniform regularity condition such as capacity constraints [3] or an \( \alpha \)-cone property [5].

In this article, we are interested in the existence of solutions to such shape optimization problems when the functional is now defined on the boundary of the domain and also depends on the first- and second-order geometric properties of the associated (hyper-)surface:

\[
J(\Omega) = \int_{\partial \Omega} j(x, u_\Omega(x), \nabla u_\Omega(x), n(x), H(x), K(x)) \, dA(x), \quad (6)
\]

where the integration on \( \partial \Omega \) is done with respect to the \((n - 1)\)-dimensional Hausdorff measure \( A(\bullet) \), where \( n \) refers to the unit normal vector to \( \partial \Omega \) pointing outwards \( \Omega \), and where \( H := \text{div}_\partial(\mathbf{n}) \) (respectively \( K := \det[D_\partial(\mathbf{n})] \)) denotes the scalar mean (resp. Gaussian) curvature of \( \partial \Omega \).

In fact, the present paper can be seen as the PDE counterpart and the continuation of a previous work [8], both coming from the original study [7]. More precisely, we have considered in [8] a wide range of purely geometric functionals (and constraints) i.e. of the form given by (6) but where the integrand does not depend on \( u_\Omega \) and \( \nabla u_\Omega \). Under some rather mild assumptions, we have proved that there always exists a \( C^{1,1} \)-regular minimizer to (1) in the following class of admissible shapes.

**Definition 1.1.** Let \( \varepsilon > 0 \) and \( B \subseteq \mathbb{R}^n \) be open, \( n \geq 2 \). We say that an open set \( \Omega \subset B \) with a non-empty boundary \( \partial \Omega := \overline{\Omega} \setminus \Omega \) satisfies the \( \varepsilon \)-ball condition and we write \( \Omega \in \mathcal{O}_\varepsilon(B) \) if for any \( x \in \partial \Omega \), there exists a unit vector \( \mathbf{d}_x \) of \( \mathbb{R}^n \) such that \( B_\varepsilon(x - \varepsilon \mathbf{d}_x) \subseteq \Omega \) and \( B_\varepsilon(x + \varepsilon \mathbf{d}_x) \subseteq B \setminus \overline{\Omega} \), where \( B_\varepsilon(z) := \{ y \in \mathbb{R}^n, |y - z| < \varepsilon \} \) denotes the open ball of \( \mathbb{R}^n \) centred at \( z \) and of radius \( \varepsilon \).

**Figure 1:** Illustration of an open set \( \Omega \subset B \) satisfying the \( \varepsilon \)-ball condition whereas \( \tilde{\Omega} \subset B \) does not.

**Remark 1.2.** The \( \varepsilon \)-ball condition of Definition 1.1 only makes sense for sets having a non-empty boundary. Hence, we always assume \( \partial \Omega \neq \emptyset \) in the sequel, or equivalently \( \Omega \notin \{\emptyset, \mathbb{R}^n\} \). We also impose \( \Omega \) to be the subset of a fixed set \( B \). However, since we only require \( B \) to be open, one can take \( B = \mathbb{R}^n \) and consider the class \( \mathcal{O}_\varepsilon(\mathbb{R}^n) \) but our results will always assume that \( B \) is bounded.
As the uniform cone property is characterizing the Lipschitz regularity of the boundary for a compact domain [3] [9, Chapter 2; 6.4 [14, §2.4], the \( \varepsilon \)-ball condition characterizes uniformly its \( C^{1,1} \)-regularity [6, 7, 8], a feature illustrated in Figure 1. Consequently, the class \( \mathcal{O}_c(\mathbb{R}^n) \) can also be described in terms of positive reach [12] and \( C^{1,1} \)-oriented distance functions [9, Chapter 7]. We refer to [8, §2] and [6] for precise statements with proofs and further references about these well-known facts. We only recall here that if \( \Omega \in \mathcal{O}_c(B) \), then we have that [8, Theorem 2.7]:

(i) \( \Omega \) satisfies the \( f^{-1}(\varepsilon) \)-cone property [14, Definition 2.4.1] with \( f : \alpha \in [0, \frac{2\alpha}{\cos \alpha}] \in [0, +\infty] \);

(ii) the \( d_x \) of Definition 1.1 is the unit outer normal vector to the hypersurface at the point \( x \);

(iii) the Gauss map \( n : x \in \partial \Omega \mapsto d_x \in \mathbb{S}^{n-1} \) is well defined and \( \frac{1}{\varepsilon} \)-Lipschitz continuous.

Hence, equipped with this class of admissible shapes, the main contribution of this article is to extend the existence results obtained in [8] for functionals of the form (6). To our knowledge, such a study has not been carried out in its generality and these new results may have some potential direct applications for engineers in the industry. We refer to Sections 1.1-1.3 for details.

We recall that in the specific case where (6) does not depend on \( u_0 \) and \( \nabla u_0 \), we only need to assume the continuity of \( J \) in order to ensure that \( J : \Omega \in \mathcal{O}_c(B) \mapsto J(\Omega) \) is well defined. However, in the general case, we also have to impose additional growth conditions on the integrand of (6). For this purpose, we introduce some more definitions. First, we say that two well-defined maps \( f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) have a quadratic growth in the first two variables if there exists two positive continuous maps \( c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) and \( \bar{c} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) such that:

\[
\forall (s, z, x, y, t) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, \quad |f(s, z, x, y, t)| \leq c(x, y, t) (1 + s^2 + |z|^2),
\]

\[
\forall (s, z, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \quad |g(s, z, x, y)| \leq \bar{c}(x, y) (1 + s^2 + |z|^2).
\]

Then, we say that \( f \) is convex in the last variable if the map \( t \mapsto f(s, z, x, y, t) \) is convex for any \( (s, z, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \). Finally, for any \( l \in [1, n-1] \), \( H^{(l)} := \sum_{1 \leq n_1 < \ldots < n_l \leq n-1} \kappa_{n_1} \ldots \kappa_{n_l} \) denotes the \( l \)-th order symmetric polynomial of the principal curvatures \( (\kappa_i)_{1 \leq i \leq n-1} \), which are the eigenvalues of the self-adjoint endomorphism \( D_{\Omega \Omega} \). In particular, we have \( H^{(1)} = H, \ H^{(n-1)} = K \), and \( H^{(l)} \) are the coefficients of the characteristic polynomial of \( D_{\Omega \Omega} \). We also mention that they also corresponds to the curvature measures associated with the \( C^{1,1} \)-hypersurface \( \partial \Omega \). They were originally introduced by Federer in the more general context of sets of positive reach [12], which is relevant here since we have Reach(\( \partial \Omega \)) = sup\{\( \varepsilon > 0 : \Omega \in \mathcal{O}_c(\mathbb{R}^n) \}) for any \( \Omega \subset \mathbb{R}^n \) such that \( V(\partial \Omega) = 0 \) [8, Theorem 2.6]. We are now in position to state our main existence result.

**Theorem 1.3.** Let \( n \geq 2, \varepsilon > 0 \), and \( B \subset \mathbb{R}^n \) be open, bounded, large enough to ensure \( \mathcal{O}_c(B) \neq \emptyset \). We consider \( (C, \tilde{C}) = \mathbb{R} \times \mathbb{R} \), some continuous maps \( j_0, j_0, j_1 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) with quadratic growth (8) in the first two variables, and continuous maps \( j_l, j_l : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) with quadratic growth (7) in the first two variables and convex in the last variable, \( l = 1, \ldots, n-1 \). Then, the following shape optimization problem has at least one solution:

\[
\inf \int_{\partial \Omega} j_0 [u_0(x), \nabla u_0(x), x, n(x)] dA(x)
+ \sum_{i=1}^{n-1} \int_{\partial \Omega} j_i [u_0(x), \nabla u_0(x), x, n(x), H^{(i)}(x)] dA(x),
\]

where \( u_0 \in H^2(\Omega, \mathbb{R}) \) is the unique solution of either (3) or (4) or (5) with \( f \in L^2(B, \mathbb{R}) \) and \( \lambda > 0 \), and where the infimum is taken among any \( \Omega \in \mathcal{O}_c(B) \) satisfying a finite number of constraints of the following form:

\[
\int_{\partial \Omega} f_0 [u_0(x), \nabla u_0(x), x, n(x)] dA(x)
+ \sum_{i=1}^{n-1} \int_{\partial \Omega} f_i [u_0(x), \nabla u_0(x), x, n(x), H^{(i)}(x)] dA(x) \leq C,
\]

\[
\int_{\partial \Omega} g_0 [u_0(x), \nabla u_0(x), x, n(x)] dA(x)
+ \sum_{i=1}^{n-1} \int_{\partial \Omega} H^{(i)}(x) g_i [u_0(x), \nabla u_0(x), x, n(x)] dA(x) = \tilde{C}.
\]
Before giving some applications, we now detail what are the main difficulties to overcome in order to prove Theorem 1.3, since an important part of the work has already been settled in [8]. First, note that under the measurability and the growth assumptions (7)–(8) of Theorem 1.3, any shape functional of the form given in (9) is well defined over the class \( \mathcal{O}_c(B) \). Indeed, considering for example the term involving \( j_1 \) in (9), we have for any \( \Omega \in \mathcal{O}_c(B) \):

\[
\left| \int_{\partial \Omega} j_1 (u_\Omega, \nabla u_\Omega, \mathbf{n}, H) \, dA \right| \leq \int_{\partial \Omega} c \left| x \cdot \mathbf{n}(x) \right| \left[ 1 + u_\Omega(x)^2 + |\nabla u_\Omega(x)|^2 \right] \, dA(x),
\]

where \( \mathbf{Id} : x \mapsto x \) is the identity map. The Gauss map \( \mathbf{n} : x \in \partial \Omega \mapsto \mathbf{n}(x) \in \mathbb{S}^{n-1} \) is \( \frac{1}{2} \)-Lipschitz continuous \((\text{iii}) \) below Figure 1\) so it is differentiable almost everywhere \([11, \text{§3.1.2}] \) and we have \( \|D\mathbf{n}\|_{L^\infty(\partial \Omega)} \leq \frac{1}{2} \). In particular, \( \|\mathbf{n}\|_{L^\infty(\partial \Omega)} \leq \frac{1}{2} \) from which we deduce \( \|H\|_{L^\infty(\partial \Omega)} \leq \frac{n-1}{2} \).

Therefore, \( x \mapsto (x, \mathbf{n}(x), H(x)) \) is always valued in the compact set \( K := B \times \mathbb{S}^{n-1} \times \left[ \frac{-n-1}{2}, \frac{n-1}{2} \right] \) and from the continuity of \( c \), we obtain:

\[
\left| \int_{\partial \Omega} j_1 (u_\Omega(x), \nabla u_\Omega(x), x, \mathbf{n}(x), H(x)) \, dA(x) \right| \leq \|c\|_{C^0(K)} \left[ A(\partial \Omega) + \|\mathbf{n}\|_{H^2(\Omega)} \right],
\]

where \( \bar{c} > 0 \) refers here to the continuity norm of the trace operator \( H^2(\Omega) \to H^1(\partial \Omega) \). Since \( \partial \Omega \) is a compact \( C^{1,1} \)-hypersurface, one can check that \( A(\partial \Omega) < +\infty \) and there always exists a unique solution \( u_\Omega \in H^2(\Omega) \) satisfying (3)–(5) if \( \lambda > 0 \) and \( f \in L^p(B) [13, \text{§2.1 Theorems 2.4.2.5–2.4.2.7}] \). In particular, the map \( \Omega \in \mathcal{O}_c(B) \mapsto u_\Omega \in H^2(\Omega) \) is well defined and similar arguments for the other terms in (9), and also in (10)–(11) yield to the proof of the following result.

**Lemma 1.4.** Under the measurability and growth assumptions (7)–(8) of Theorem 1.3, any shape functional \( J : \Omega \in \mathcal{O}_c(B) \mapsto J(\Omega) \in \mathbb{R} \) of the form given in (9)–(11) is a well-defined map.

**Remark 1.5.** If the measurability and quadratic growth are enough to define the functionals, the continuity and convexity assumptions of Theorem 1.3 will be essentially used to get their continuity.

Then, we prove Theorem 1.3 by following the classical method from Calculus of Variations, since we have checked that the shape optimization problem (9) is well defined in the class \( \mathcal{O}_c(B) \). Hence, we consider any minimizing sequence \( (\Omega_i)_{i \in \mathbb{N}} \subset \mathcal{O}_c(B) \) of (9). The sequential compactness of \( \mathcal{O}_c(B) \) has already been studied in [8, Proposition 3.2] and it states as follows.

**Proposition 1.6.** There exists \( \Omega \in \mathcal{O}_c(B) \) such that a subsequence \( (\Omega'_i)_{i \in \mathbb{N}} \) converges to \( \Omega \) for some various modes of convergence (for the Hausdorff distance of the complements in \( \mathcal{B} \), of the adherences, of the boundaries, for the \( L^p(B) \)-norm of the characteristic functions, for the \( W^{1,p}(B) \)-norm of the oriented distance functions, \( p \in [1, +\infty] \), and in the sense of compact sets [14, §2.2.4]).

Therefore, the existence result of Theorem 1.3 is achieved if we can get the lower-semicontinuity of the functional in (9) and in the inequality constraint (10) whereas we need continuity in the equality constraint (11). Let us emphasize the fact that we are only able to get continuity if the integrands in (11) are linear in \( H(\cdot) \). We can relax this hypothesis by assuming the convexity of the integrands in \( H^{(i)} \) but in this case, we only obtain the lower-semicontinuity of the functionals in (9)–(10), which is enough to prove Theorem 1.3.

Our method of proof is similar to the one used in [8] for studying purely geometric functionals. It is based on localization and the study of convergence of graphs. We proved in [8, Theorem 3.3] that if \( (\Omega_i)_{i \in \mathbb{N}} \) converges to \( \Omega \) as in Proposition 1.6, then for \( i \) sufficiently large, the boundary \( \partial \Omega_i \) can be locally parametrized by a \( C^{1,1} \)-graph in a local frame associated with \( \partial \Omega \). The key point here is that the local frame does not depend on \( i \). Moreover, we obtain the \( C^{1,1}-\delta \)-strong for any \( \delta \in (0,1] \) and \( W^{2,\infty} \)-weak-star convergence of these local graphs, where the limit graph is precisely the one associated with \( \partial \Omega \). This local result is illustrated in Figure 2 and it states as follows.

**Proposition 1.7.** Let \( (\Omega_i)_{i \in \mathbb{N}} \subset \mathcal{O}_c(B) \) converge to \( \Omega_\infty \in \mathcal{O}_c(B) \) in the sense of compact sets [14, §2.2.4] and such that \( \partial \Omega_i \to \partial \Omega_\infty \) for the Hausdorff distance. Then, for any \( x_\infty \in \partial \Omega_\infty \), there exists a direct orthonormal frame centred at \( x_\infty \), and also \( I \in \mathbb{N} \) depending on \( x_\infty, \varepsilon, \) and \( (\Omega_i)_{i \in \mathbb{N} \cup \{\infty\}} \), such that inside this frame, for any \( i \in \mathbb{N} \cup \{\infty\}, i \geq I \), there exists a continuously differentiable map \( \varphi_i : D_r(0') \to -\varepsilon, \varepsilon \) such that:

\[
\begin{align*}
\partial \Omega_i \cap (D_r(0') \times ]-\varepsilon, \varepsilon[) &= \{ (x', \varphi_i(x')) : x' \in D_r(0') \} \quad \text{and} \quad -\varepsilon < x_n < \varphi_i(x'),
\end{align*}
\]
where \( D_r(0') := \{ x' \in \mathbb{R}^{n-1}, |x'| < r \} \) denotes the open ball of \( \mathbb{R}^{n-1} \) centered at the origin \( 0' \) (identified with \( x_\infty \)) with a radius \( r > 0 \) that only depends on \( \varepsilon \). Moreover, any of the \( (\varphi_i)_{i \in [1, \infty]} \) has a unique \( C^{1,1} \)-extension on the closure \( D_r(0') \) and we have:

\[
\begin{align*}
\varphi_i &\to \varphi_\infty \text{ strongly in } C^{1,1-\delta}(D_r(0')) \text{ for any } \delta \in [0, 1] \\
\varphi_i &\to \varphi_\infty \text{ weakly star in } W^{2,\infty}(D_r(0')).
\end{align*}
\]

(12)

\[\text{Figure 2: Illustration of Proposition 1.7 stating that there exists a fixed common local frame in which a converging sequence of elements in } \mathcal{O}_i(B) \text{ can be simultaneously parametrized by } C^{1,1} \text{-graphs.}\]

With this result in mind, it remains to build a suitable partition of unity in order to study properly the continuity of a shape functional defined on \( \mathcal{O}_i(B) \). Let us recall this procedure that was already detailed in the proof of [8, Proposition 4.7]. The index \( \infty \) of Proposition 1.7 is dropped to lighten the notation. For example \( \partial \Omega \) now refers to \( \partial \Omega_\infty \) and \( \varphi \) to \( \varphi_\infty \). First, for any \( x \in \partial \Omega \), we introduce the cylinder \( C_{r,\varepsilon}(x) \) of Proposition 1.7 denoted by \( D_r(0') \times ]-\varepsilon,\varepsilon[ \) in the statement, and written here \( D_r(x) \times ]-\varepsilon,\varepsilon[ \). In particular, it is centred at \( x \) and its radius \( r > 0 \) refers precisely to the one given in Proposition 1.7 so it only depends on \( \varepsilon \). Since \( \partial \Omega \) is compact, there exists a finite number \( K \geq 1 \) of points of \( \partial \Omega \) written \( x_1, \ldots, x_K \) such that \( \partial \Omega \subset \bigcup_{k=1}^{K} C_{\frac{r}{2}, \varepsilon}(x_k) \). We set \( \delta := \frac{r}{2} \min\{r, \varepsilon\} > 0 \). From the triangle inequality, the open tubular neighbourhood \( V_{\delta}(\partial \Omega) := \{ y \in \mathbb{R}^n, d(y, \partial \Omega) < \delta \} \) has its closure embedded in \( \bigcup_{k=1}^{K} C_{r,\varepsilon}(x_k) \).

Then, we can introduce a partition associated with this covering: there exists \( K \) non-negative smooth maps \( \xi^k \) with compact in \( C_{r,\varepsilon}(x_k) \) and such that \( \sum_{k=1}^{K} \xi^k = 1 \) on \( V_{\delta}(\partial \Omega) \). Let us now apply Proposition 1.7 to the \( K \) points. Hence, there exists \( K \) integers \( I_k \in \mathbb{N} \) and some maps \( \varphi^k_i : D_r(x_k) \to ]-\varepsilon, \varepsilon[ \) such that for any \( i \geq I_k \) and \( k \in [1, K] \):

\[
\begin{align*}
\partial \Omega_i \cap C_{r,\varepsilon}(x_k) &\subseteq \{ (x', \varphi^k_i(x')) : x' \in D_r(x_k) \} \\
\Omega_i \cap C_{r,\varepsilon}(x_k) &\subseteq \{ (x', x_n) : x' \in D_r(x_k) \text{ and } -\varepsilon < x_n < \varphi^k_i(x') \}.
\end{align*}
\]

Moreover, the \( K \) sequences of functions \( (\varphi^k_i)_{i \geq I_k} \) and \( (\nabla \varphi^k_i)_{i \geq I_k} \) converge uniformly on \( D_r(x_k) \) respectively to the map \( \varphi^k \) and \( \nabla \varphi^k \) associated with \( \partial \Omega \) at each point \( x_k \), \( k = 1, \ldots, K \). Finally, from the Hausdorff convergence of the boundaries, there also exist \( I_0 \in \mathbb{N} \) such that for any \( i \geq I_0 \), we have \( \partial \Omega_i \in V_{\delta}(\partial \Omega) \). We set \( I := \max_{k \in \{0, K \}} I_k \) which thus only depends on \( \varepsilon \), \( \Omega \) and \( (\Omega_i)_{i \in \mathbb{N}} \).

We are now in position to study the continuity of (9)–(11) by expressing the functionals in our parametrization. For simplicity, we consider the term involving \( j_0 \) in (9) and we assume that \( \omega_\Omega \) refers to the solution of the Dirichlet Laplacian (3). In this specific case, note that the dependence of \( j_0 \) in \( \omega_\Omega \) can be dropped since \( \omega_\Omega = 0 \) on \( \partial \Omega \). We deduce that for any integer \( i \geq I \), we have:

\[
F(\Omega_i) := \int_{\partial \Omega_i} j_0(\nabla u_{\Omega_i}, \text{Id}, n) \, dA = \int_{\partial \Omega_i \cap \partial \Omega_0 \cap \Omega_0} j_0(\nabla u_{\Omega_0}, \text{Id}, n) \, dA
\]

\[
= \int_{\partial \Omega_i} \left( \sum_{k=1}^{K} \xi^k \right) j_0(\nabla u_{\Omega_i}, \text{Id}, n) \, dA = \sum_{k=1}^{K} \int_{\partial \Omega_i \cap C_{r,\varepsilon}(x_k)} \xi^k j_0(\nabla u_{\Omega_i}, \text{Id}, n) \, dA.
\]
Using the expression of a boundary integral parametrized by local graphs (see e.g. [16, Proposition 5.13]), we obtain that what we have defined above as \( F(\Omega) \) is equal to:

\[
\sum_{k=1}^{K} \int_{D_r(x_k)} \xi^k \left( x', (\varphi^k_0(x')) \right) j_0 \left[ \nabla u_{\Omega_i} \left( x', (\varphi^k_0(x')) \right), \left( \frac{-\nabla u_{\Omega_i}^{\ast}(x')}{\sqrt{1 + |\nabla \varphi^k_0(x')|^2}}, \frac{1}{\sqrt{1 + |\nabla \varphi^k_0(x')|^2}} \right) \right] \sqrt{1 + |\nabla \varphi^k_i(x')|^2} dx'.
\]

(13)

Our strategy is to let correctly \( i \to +\infty \) in (13). For this purpose, we aim to apply Lebesgue's Dominated Convergence Theorem. From the quadratic growth and the continuity of \( j_0 \) combined with the convergence properties of the \( K \) sequences \( (\varphi^k_i)_{i \geq 1} \), \( k = 1, \ldots, K \), one can observe that the following implication holds true.

**Lemma 1.8.** If the map \( v^k_i : x' \mapsto \nabla u_{\Omega_i}(x', \varphi^k_i(x')) \) converges to \( v^k_i : x' \mapsto \nabla u_{\Omega_i}(x', \varphi^k_i(x')) \) strongly in \( L^2(D_r(x_k)) \) for any \( k \in [1, K] \), then we can correctly let \( i \to +\infty \) in (13) and \( F(\Omega_i) \to F(\Omega) \).

**Proof.** We drop the index \( i \) to lighten the notation. If \( v^k_i \) converges to \( v \) strongly in \( L^2 \), then there exists a subsequence \( (v_{i,\ell})_{\ell \in \mathbb{N}} \) such that \( v_{i,\ell} \) converges to \( v \) almost everywhere. In addition, \( v_{i,\ell} \) is dominated by an \( L^2 \)-function. On the one hand, from the quadratic growth assumption made on \( j_0 \), we deduce the integrand of (13) is also dominated. On the other hand, from the continuity of \( j_0 \), the integrand of (13) is converging almost everywhere to the right quantity. Applying Lebesgue's Dominated Convergence Theorem, we have proved that \( F(\Omega_{i,\ell}) \to F(\Omega) \). We also have obtained that the limit is uniquely determined so the whole sequence \( F(\Omega_{i,\ell}) \to F(\Omega) \).

In fact, we can say more by introducing the map \( X^k_i : x' \in D_r(x_k) \mapsto (x', \varphi^k_i(x')) \in C_{r,\varepsilon}(x_k) \cap \partial \Omega_i \) associated with the parametrization of \( \partial \Omega_i \). Indeed, from the proof of Lemma 1.8, one can check that if \( \nabla u_{\Omega_i} \circ X^k_i \) converges \( L^2 \)-strongly to \( \nabla u_{\Omega_i} \circ X^k_i \), then the integrand of (13) converges strongly in \( L^1 \). Recalling that we have proved in [8, Section 4.3] that \( H_{\Omega_i} \circ X^k_i \) converges \( L^{\infty} \)-weakly-star to \( H_{\Omega_i} \circ X^k_i \), and more generally \( H_{\Omega_i}^{(l)} \circ X^k_i \to^{*} H_{\Omega_i}^{(l)} \circ X^k_i \) [8, Section 4.4], then we deduce that any integrand which is linear in \( H^{(l)} \) is continuous \( (L^1 \text{-strong vs } L^{\infty} \text{-weakly-star}) \). For example, this is the case for the ones appearing in the equality constraints (11). Then, from the same arguments than the ones given in the proof of [8, Corollary 4.14], the functional becomes only lower semi-continuous if we relax the linearity by convexity in \( H^{(l)} \). This is the case for the functionals of (9)-(10), which is enough for our purpose. Finally, the previous arguments also works for the Robin and Neumann boundary conditions. We only have to additionally ensure that \( u_{\Omega_i} \circ X^k_i \) converges to \( u_{\Omega_i} \circ X^k_i \) strongly in \( L^2(D_r(x_k)) \) for any \( k \in [1, K] \). Therefore, the conclusion of our discussion can be sum up as follows.

**Remark 1.9.** If \( u_{\Omega_i} \circ X^k_i \to u_{\Omega_i} \circ X^k_i \) and \( \nabla u_{\Omega_i} \circ X^k_i \to \nabla u_{\Omega_i} \circ X^k_i \) strongly in \( L^2(D_r(x_k)) \) for any \( k \in [1, K] \), then Theorem 1.3 holds true.

The remaining part of this paper consists in proving that the hypothesis of Remark 1.9 holds true. We conclude the introduction by giving three applications of Theorems 1.3 and the paper is next organized as follows. In Section 2, we establish a priori \( H^2 \)-estimates for the solutions of (3) in the class \( \mathcal{O}_r(B) \), where the constant obtained depend only on \( \varepsilon \), the diameter of \( B \), and the dimension \( n \) of the space. We essentially follow the method suggested by Grisvard [13, Sections 3.1.1-3.1.2]. Then, in Sections 3 and 4, we respectively treat the Dirichlet and the Neumann / Robin case. The method is classical. We first establish that the sequence is bounded. Then, we identify the weak limit and we prove that in fact to convergence is strong by studying the convergence in norm.

### 1.1 First application: the specific case of domain integrals

Until now, note that we only treat the case of functionals involving boundary integrals. Indeed, the case where the domain of integration corresponds to the one of (2) is standard within the framework of the uniform cone property [14, Section 4.3]. Since the \( \varepsilon \)-ball condition implies an \( \alpha(\varepsilon) \)-cone property (ii below Figure 1), we have not considered such functionals for the time being. However, the class \( \mathcal{O}_r(B) \) becomes interesting if some second-order partial derivatives of \( u_{\Omega_i} \) appear in the integrand of (2). Our result states as follows.
**Proposition 1.10.** Consider the assumption of Theorem 1.3 and a measurable map $j : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^2 \rightarrow \mathbb{R}$. We assume that $j$ has a quadratic growth in its three last variables i.e. there exists a positive continuous map $c : \mathbb{R}^n \rightarrow \mathbb{R}$ such that:

$$
\forall (x, s, z, Y) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^2, \quad |j(x, s, z, Y)| \leq c(x) \left(1 + s^2 + |z|^2 + \|Y\|^2\right),
$$

where $\|\cdot\|$ refers to the Frobenius norm over the set of $(n \times n)$-matrices i.e. $\|Y\| = \sqrt{\text{trace}([Y]^T Y)}$ for any $Y \in \mathbb{R}^n$.

Then, Theorem 1.3 still holds true if the functionals of (9)-(11) contain terms of the form:

$$
\int_\Omega j(x, u_\Omega(x), \nabla u_\Omega(x), \text{Hess } u_\Omega(x)) \, dV(x).
$$

Again, in the above theorem, if we denote by $J : \mathcal{O}_c(B) \rightarrow \mathbb{R}$ the functional to minimize, note that it is well defined since from the quadratic growths (7)-(8) and (14) of the maps and from the continuity of the trace operator $H^2(\Omega, \mathbb{R}) \rightarrow H^1(\partial \Omega, \mathbb{R})$, we have:

$$
\forall \Omega \in \mathcal{O}_c(B), \quad |J(\Omega)| \leq \tilde{c} \left[\mathcal{V}(\Omega) + A(\partial \Omega) + \|u\|^2_{H^2(\Omega, \mathbb{R})}\right] < +\infty.
$$

Also observe that the above statement treats the case where the integration is not done on the whole domain $\Omega$ but only on a measurable part $\bar{\Omega} \subseteq \Omega$. Indeed, it suffices to introduce the characteristic function $\mathbb{1}_{\bar{\Omega}}$ in the integrand $j$. Similarly, the formulation adopted above allows constraints of the form $K \subset \bar{\Omega}$ for a given compact set $K \subset B$.

### 1.2 Second application: boundary shape identification problems

Let $\varepsilon > 0$ and $B$ be a large open set. We consider $\Omega_0 \in \mathcal{O}_c(B)$, a subset $\Gamma_0 \subseteq \partial \Omega_0$, and $g_0 \in L^2(\Gamma_0, \mathbb{R})$. Imagine there is good reason to think that $g_0$ is the restriction to $\Gamma_0$ of the normal derivative with the solution $u_{\Omega_0}$ of the Dirichlet Laplacian (3) posed on an unknown domain $\Omega \in \mathcal{O}_c(B)$ such that $\Gamma_0 \subseteq \partial \Omega$. In order to find the best $\Omega \in \mathcal{O}_c(B)$ such that $\partial_n(u_{\Omega_0})|_{\partial \Omega_0} = g_0$, one possibility is to solve the following problem:

$$
\inf_{\Omega \in \mathcal{O}_c(B)} \int_{\Gamma_0} \left[\partial_n(u_{\Omega}) - g_0\right]^2 \, dA.
$$

Similarly, if we suspect that $f_0 \in L^2(\Gamma_0, \mathbb{R})$ is the restriction to $\Gamma_0$ of the solution $u_{\Omega_0}$ to the Neumann/Robin Laplacian (4)-(5) posed on an unknown domain $\Omega \in \mathcal{O}_c(B)$ such that $\Gamma_0 \subseteq \partial \Omega$, then we have to solve:

$$
\inf_{\Omega \in \mathcal{O}_c(B)} \int_{\Gamma_0} (u_{\Omega} - f_0)^2 \, dA.
$$

Of course, we can build more complicated functionals but the main difficulty here is that the domain of integration is not the whole surface. We prove the following result.

**Proposition 1.11.** Let $\Omega_0 \in \mathcal{O}_c(B)$ and $\Gamma_0$ be a measurable subset of $\partial \Omega_0$. Then, Theorem 1.3 remains true if we add the constraint $\Gamma_0 \subseteq \partial \Omega$ and if the domain of integration $\partial \Omega$ in the functional and the constraints are restricted to $\Gamma_0$. In particular, Problems (15)-(16) have a minimizer.

The identification of shape through its boundary like (15)-(16) often appear in inverse and optimal control problems. For example, let us try to detect a tumor in the brain. We put some electrodes on the head $\Gamma_0$ of a patient, measure some electric activity $g_0$, and solve Problem (16). If no tumor exists, then the infimum is zero and the optimal shape is $\Gamma_0$, otherwise it is $\Gamma_0 \cup \Gamma_1$, where $\Gamma_1$ is the boundary of the tumor.

### 1.3 Third application: the MIT-bag model in relativistic quantum mechanics

In his thesis [15], Le Treust has studied some shape optimization problems coming from relativistic quantum mechanics. In particular, bag models are introduced to study the internal structure of hadrons. The energy of these particles is given by summing the energy of the quarks and anti-quarks living in the bag.
In the MIT-bag model, the wave functions of the quarks are the eigenvectors of the Dirac operator. Hence, the fundamental state problem corresponds to the minimization with prescribed volume of the first positive eigenvalue associated with this Dirac operator among non-empty open bounded subset of \( \mathbb{R}^3 \) with \( C^2 \)-boundary. The existence of an optimal shape is actually open.

We did not study this problem but it seems that the framework of the uniform ball condition might be used again to approximate the fundamental state of the MIT-bag model:

\[
\inf_{\Omega \in \mathcal{O}_c(B)} \lambda_{\text{MIT}}^1(\Omega)
\]

with

\[
\lambda_{\text{MIT}}^1(\Omega) = \inf_{u \in H^2(\Omega, \mathbb{C})} \frac{\sqrt{m^2 + \int_\Omega \|\nabla u\|^2 + \int_{\partial \Omega} \left( m + \frac{H_\Omega}{2} \right) |u|^2 dA}}{\int_\Omega |u|^2 d\Omega}
\]

where \( m > 0 \) is a given fixed parameter (the mass of the particle) and where \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) is the vector formed by the three Pauli \((2 \times 2)\)-matrices:

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

The main difficulty comes from the non-linear constraint \(-\sigma \cdot n_{\Omega}) u = u\), which has to be understood as \(-\sigma_1 n_1 + \sigma_2 n_2 + \sigma_3 n_3) u = u\) on \(\partial \Omega\). This is related to some eigenvalue problem associated with the Dirac operator. In this paper, we only study the case of the Laplace operator but it seems clear that the method developed here can be adapted to many other operators. In particular, the class \( \mathcal{O}_c(B) \) is very well adapted to get the existence for general shape optimization problem involving the boundary of the domain and its geometry.

## 2 Controlling uniformly the \( H^2 \)-norm by the Laplacian

In this section, we want to control uniformly the constant appearing in a priori estimates related to the Dirichlet Laplacian. First, we recall some geometric definitions in the case of hypersurfaces with \( C^{1,1} \)-regularity. Then, we establish an identity for general functions, some Poincaré and trace inequalities, in order to finally prove Theorem 2.1. We follow essentially the method described in [13, Section 3.2] which treats the case of convex \( C^2 \)-domains.

**Theorem 2.1.** Let \( \varepsilon > 0 \), \( n \geq 2 \), and \( B \) be any non-empty open bounded subset of \( \mathbb{R}^n \) containing the origin. We consider the class \( \mathcal{O}_c(B) \) formed by all the non-empty open subsets of \( B \) satisfying the \( \varepsilon \)-ball condition. We assume that the diameter \( D \) of \( B \) is large enough to ensure \( \mathcal{O}_c(B) \neq \emptyset \). Then, there exists a constant \( C > 0 \), depending only on \( \varepsilon \), \( D \), and \( n \), such that:

\[
\forall \Omega \in \mathcal{O}_c(B), \forall u \in H^2(\Omega, \mathbb{R}) \cap H^1_0(\Omega, \mathbb{R}), \quad \|u\|_{H^2(\Omega, \mathbb{R})} \leq C(\varepsilon, n, D, \|\Delta u\|_{L^2(\Omega, \mathbb{R})}).
\]

### 2.1 On the geometry of hypersurfaces with \( C^{1,1} \)-regularity

Let us consider any non-empty bounded open set \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \). We assume that its boundary \( \partial \Omega \) is a \( C^{1,1} \)-hypersurface of \( \mathbb{R}^n \) i.e. for any point \( x_0 \in \partial \Omega \), there exists \( r_{x_0} > 0 \), \( a_{x_0} > 0 \), and a unit vector \( d_{x_0} \) of \( \mathbb{R}^n \) such that in the cylinder defined by:

\[
C_{r_{x_0}, a_{x_0}}(x_0) = \{ x \in \mathbb{R}^n, |(x - x_0, d_{x_0})| < a_{x_0} \text{ and } \|(x - x_0) - (x - x_0, d_{x_0})d_{x_0}\| < r_{x_0} \},
\]

the boundary \( \partial \Omega \) is the graph of a \( C^{1,1} \)-map. To be more precise, introducing the orthogonal projection on the affine hyperplane \( x_0 + d_{x_0}^\perp \):

\[
\Pi_{x_0} : \mathbb{R}^n \rightarrow x_0 + d_{x_0}^\perp, \quad x \mapsto x - (x - x_0, d_{x_0})d_{x_0}.
\]
and considering the set \( D_{\tau_0}(x_0) = \Pi_{\tau_0}(C_{\tau_0, a_{\tau_0}}(x_0)) \), this means that there exists a continuously differentiable map \( \varphi_{x_0} : \mathcal{X} \cap D_{\tau_0}(x_0) \to \mathcal{X} \) such that its gradient \( \nabla \varphi_{x_0} \) and \( \varphi_{x_0} \) are \( L_{\tau_0} \)-Lipschitz continuous maps, \( L_{\tau_0} > 0 \), and such that:

\[
\begin{align*}
\partial \Omega \cap C_{\tau_0, a_{\tau_0}}(x_0) &= \{ x' + \varphi_{x_0}(x')d_{\Omega}, \ x' \in D_{\tau_0}(x_0) \} \\
\Omega \cap C_{\tau_0, a_{\tau_0}}(x_0) &= \{ x' + x_n d_n, \ x' \in D_{\tau_0}(x_0) \text{ and } -a_n < x_n < \varphi_{x_0}(x') \}.
\end{align*}
\]

Hence, we can introduce the local parametrization:

\[ X_{x_0} : D_{\tau_0}(x_0) \to \partial \Omega \cap C_{\tau_0, a_{\tau_0}}(x_0), \]

and \( \partial \Omega \) is a \( C^{1,1} \)-hypersurface in the sense of [16, Definition 2.2]. Indeed, \( X_{x_0} \) is an homeomorphism, its inverse map is the restriction of \( \Pi_{\tau_0} \) to \( C_{\tau_0, a_{\tau_0}}(x_0) \), and \( X_{x_0} \) is an immersion of class \( C^{1,1} \).

We usually drop the dependence in \( x_0 \) to lighten the notation, and consider a direct orthonormal frame \((x_0, B_{x_0}, d_{x_0})\) where \( B_{x_0} \) is a basis of \( d_{x_0} \). In this local frame, the point \( x_0 \) is identified with the zero vector \( 0 \in \mathbb{R}^n \), the affine hyperplane \( x_0 + d_{x_0} \) with \( \mathbb{R}^{n-1} \) and \( x_0 + R d_{x_0} \) with \( \mathbb{R}^n \). Hence, the cylinder \( C_{\tau_0, a_{\tau_0}}(x_0) \) becomes \( D_r(0^{(n-1)}) \), \( \varphi_{x_0} \) is the \( C^{1,1} \)-map \( \varphi : D_r(0^{(n-1)}) \to -a, a \), the projection \( \Pi_{x_0} \) is \( X^{-1} : (x', x_n) \to x' \), and the parametrization \( X_{x_0} \) becomes the \( C^{1,1} \)-map \( X : x' \in D_r(0^{(n-1)}) \to \varphi(X'(x')), (x', x_n) \in \partial \Omega \cap X(D_r(0^{(n-1)}) \to -a, a) \).

Since \( x' \in D_r(0^{(n-1)}) \to D_rX \) is injective, the vectors \( \partial_i X, \ldots, \partial_{n-1} X \) are linearly independent. For any \( x \in \partial \Omega \cap X(D_r(0^{(n-1)}) \to -a, a) \), we define the tangent hyperplane \( T_x(\partial \Omega) \) by \( D_{X^{-1}(x)}X(\mathbb{R}^{n-1}) \). It is an \((n-1)\)-dimensional vector space so \( (\partial_1 X, \ldots, \partial_{n-1} X) \) forms a basis of \( T_x(\partial \Omega) \). However, this basis is not necessarily orthonormal. Consequently, the first fundamental form of \( \partial \Omega \) at \( x \) is defined as the restriction of the usual scalar product in \( \mathbb{R}^n \) to the tangent hyperplane \( T_x(\partial \Omega) \), i.e. as \( I(x) : (v, w) \in T_x(\partial \Omega) \times T_x(\partial \Omega) \to \langle v, w \rangle \). In the basis \((\partial_1 X, \ldots, \partial_{n-1} X)\), it is represented by a positive-definite symmetric matrix usually referred to as \((g_{ij})_{1 \leq i,j \leq n-1}\) and its inverse denoted by \((g^{ij})_{1 \leq i,j \leq n-1}\) is also explicitly given in this case:

\[
\begin{align*}
g^{ij} &= \langle \partial_i X, \partial_j X \rangle = \delta_{ij} + \partial_i \varphi \partial_j \varphi, \\
g^{ij} &= \delta_{ij} - \partial_i \varphi \partial_j \varphi \frac{1}{1 + \| \varphi \|^2}.
\end{align*}
\]

As a function of \( x' \), observe that each coefficient of these two matrices is Lipschitz continuous thus it is a \( W^{1,\infty} \)-map [11, Section 4.2.3], and from Rademacher’s Theorem [11, Section 3.1.2], its differential exists almost everywhere. Moreover, any \( v \in T_x(\partial \Omega) \) can be decomposed in the basis \((\partial_1 X, \ldots, \partial_{n-1} X)\). Denoting by \( V_i \) the component of \( \partial_i X \) and \( v_i = \langle v, \partial_i X \rangle \), we have:

\[
v = \sum_{i=1}^{n-1} V_i \partial_i X \implies v_j = \sum_{i=1}^{n} V_i g_{ij} \implies V_i = \sum_{j=1}^{n-1} g^{ij} v_j \implies v = \sum_{i=1}^{n-1} \left( \sum_{j=1}^{n-1} g^{ij} v_j \right) \partial_i X.
\]

In particular, we deduce \( I(v, w) = \sum_{i,j=1}^{n-1} g^{ij} v_i w_j \). Then, the orthogonal of the tangent hyperplane is one dimensional. Hence, there exists a unique unit vector \( n \) orthogonal to the \((n-1)\) vectors \( \partial_1 X, \ldots, \partial_{n-1} X \) and pointing outwards \( \Omega \) i.e. \( \det(\partial_1 X, \ldots, \partial_{n-1} X, n) > 0 \). It is called the unit outer normal to the hypersurface and we have its explicit expression in the parametrization:

\[
\forall x' \in D_r(0^{(n-1)}), \quad n \circ X(x') = \frac{1}{\sqrt{1 + \| \varphi(x') \|^2}} \left( -\nabla \varphi(x'), \frac{1}{1 + \| \varphi(x') \|^2} \right).
\]

It is a Lipschitz continuous map, like the coefficients of the first fundamental form. In particular, it is differentiable almost everywhere and introducing the Gauss map \( n : x \in \partial \Omega \to n(x) \in \mathbb{S}^{n-1} \), we can compute its differential almost everywhere called the Weingarten map:

\[
D_X n : T_x(\partial \Omega) = D_{X^{-1}(x)}(\mathbb{R}^2) \to D_{X^{-1}(x)}(n \circ X)(\mathbb{R}^2) \quad v = D_{X^{-1}(x)}(n \circ X)(w) \implies D_X n(v) = D_{X^{-1}(x)}(n \circ X)(w).
\]

Since \( \| n \circ X \|^2 = 1 \), note that \( D_{X^{-1}(x)}(n \circ X)(\mathbb{R}^2) \subseteq n(x)^{\perp} = T_x(\partial \Omega) \) so the map \( D_X n \) is an endomorphism of \( T_x(\partial \Omega) \). Moreover, one can prove it is self-adjoint so it can be diagonalized to
obtain \( n - 1 \) eigenvalues denoted by \( \kappa_1(x), \ldots, \kappa_{n-1}(x) \) and called the principal curvatures. Recall that the eigenvalues of an endomorphism do not depend on the chosen basis and thus are really properties of the operator. This also holds for the trace and the determinant of \( D_x n \) so we can define the scalar mean curvature \( H = \text{Trace}(D_x n) \) and the Gaussian curvature \( K = \det(D_x n) \):

\[
H(x) = \kappa_1(x) + \cdots + \kappa_{n-1}(x) \quad \text{and} \quad K(x) = \kappa_1(x)\kappa_2(x) \cdots \kappa_{n-1}(x). \tag{22}
\]

Moreover, introducing the symmetric matrix \((b_{ij})_{1 \leq i,j \leq n-1}\) defined by:

\[
b_{ij} = -\langle D n(\partial_i X) \mid \partial_j X \rangle = -\langle \partial_i(n \circ X) \mid \partial_j X \rangle = \frac{\text{Hess} \phi}{\sqrt{1 + \|\nabla \phi\|^2}} = \langle n \circ X \mid \partial_i j X \rangle, \tag{23}
\]

we get from (20) that the Weingarten map \( D n \) is represented in the local basis \((\partial_1 X, \ldots, \partial_{n-1} X)\) by the symmetric matrix \((-\sum_{k=1}^{n-1} g^{ij} b_{kj})_{1 \leq i,j \leq n-1}\) and in particular, we have:

\[
H \circ X = -\sum_{i,j=1}^{n-1} g^{ij} b_{ij} = -\sum_{i,j=1}^{n-1} \left( \delta_{ij} - \frac{\partial_i \phi \partial_j \phi}{1 + \|\nabla \phi\|^2} \right) \frac{\partial_j \phi}{\sqrt{1 + \|\nabla \phi\|^2}}. \tag{24}
\]

Finally, we introduce the symmetric bilinear form whose representation in the local basis is \((b_{ij})\). It is called the second fundamental form of the hypersurface and it is defined by:

\[
\Pi(x) : T_x(\partial \Omega) \times T_x(\partial \Omega) \longrightarrow \mathbb{R}
\]

\[
(v, w) \longmapsto \langle -D_x n(v) \mid w \rangle = \sum_{i,j,k,l=1}^{n-1} g^{ij} v_l g^{kl} b_{li}. \tag{25}
\]

Note that in local coordinates, the coefficients of the first fundamental form and the Gauss map are Lipschitz continuous functions i.e. \( n \circ X, g_{ij}, g^{ij} \in W^{1,\infty}(D_r(0')) \). Hence, the Weingarten map and the coefficients of the second fundamental form exist almost everywhere and \( b_{ij} \in L^\infty(D_r(0')) \). Henceforth, we do not indicate anymore the dependence on the point \( x \) or in the parameter \( x' \) such that \( X(x') = x \). The same notation is now used to denote a map \( f : x \in \partial \Omega \cap C_r, o(x_0) \mapsto f(x) \) and its parametrized version \( x' \in D_r(x_0) \mapsto (f \circ X)(x') \).

\section{2.2 An identity based on two integrations by parts}

In [13, Theorem 3.1.1.1], an identity based on two integration by parts is established in the case of domains with \( C^2 \)-boundary. It is the main ingredient to get a uniform control on the constant appearing in \textit{a priori} estimates associated with the Dirichlet/Neumann Laplacian. In this section, our only contribution is to show that Equality (26) remains true for domains with \( C^{1,1} \)-boundary.

**Theorem 2.2 (Grisvard [13, Theorem 3.1.1.1]):** Let us consider any non-empty bounded open set \( \Omega \subset \mathbb{R}^n \) such that its boundary is a \( C^{1,1} \)-hypersurface of \( \mathbb{R}^n \), \( n \geq 2 \). Then, for any function \( v = (v_1, \ldots, v_n) \in H^1(\Omega, \mathbb{R}^n) \), we have the following identity:

\[
\sum_{i,j=1}^{n} \int_{\Omega} \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} - \int_{\partial \Omega} |\text{div}(v)|^2 = 2 \langle \nabla v, v \rangle_{H^{-\frac{1}{2}}(\Omega, \mathbb{R}^n), H^{\frac{1}{2}}(\partial \Omega, \mathbb{R}^n)} + \int_{\partial \Omega} \left[ \Pi(v, v) - H(v_n)^2 \right] dA, \tag{26}
\]

where \( n \) is the unit outer normal to the hypersurface as in (21), where \( v_n = (v \mid n) \), \( v_{\partial \Omega} = v - v_n n \), \( \nabla_{\partial \Omega}(v_n) = \nabla(v_n) - (\nabla(v_n) \mid n) n \), where \( H \) is the scalar mean curvature as in (22) and \( \Pi \) refers to the second fundamental form defined in (25).

**Proof.** Let \( \Omega \) be a non-empty bounded open subset of \( \mathbb{R}^n \) whose boundary is a \( C^{1,1} \)-hypersurface. We consider \( v = (v_1, \ldots, v_n) \in C^\infty(\Omega, \mathbb{R}^n) \) and we get from two integrations by parts:

\[
\int_{\Omega} |\text{div}(v)|^2 = \sum_{i,j=1}^{n} \int_{\Omega} \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} = \sum_{i,j=1}^{n} \int_{\partial \Omega} \frac{\partial v_i}{\partial x_j} v_j n_i dA - \sum_{i,j=1}^{n} \int_{\Omega} \frac{\partial v_i}{\partial x_i} n_j dA + \sum_{i,j=1}^{n} \int_{\partial \Omega} \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} dA.
\]

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Consequently, introducing the notation $v_n = \langle v \mid n \rangle$, the above equality takes the following form:

$$
\sum_{i,j=1}^{n} \int_{\Omega} \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} - \int_{\Omega} |\text{div}(v)|^2 = \int_{\Omega} \left[ \langle (v \mid \nabla)(v) \mid n \rangle - v_n \text{div}(v) \right] dA. \quad (27)
$$

We now show that the right member of (27) is equal to the right one of (26) by expressing the right integrand of (27) in the local parametrization associated with $\partial \Omega$. More precisely, we set $x_0 \in \partial \Omega$. There exists a cylinder (17) simply denoted by $C(x_0)$ in which $\partial \Omega$ is the graph of a $C^1$-map $\varphi$. Hence, we can introduce the local $C^{1,1}$-parametrization $X : x' \rightarrow (x', \varphi(x')) \in \partial \Omega \cap C(x_0)$ and we first assume that the smooth map $v : \Omega \rightarrow \mathbb{R}^n$ has compact support in $\Omega \cap C(x_0)$. We decompose it locally in the basis $(\partial_1 X, \ldots, \partial_{n-1} X, n)$ which is direct but not necessarily orthonormal. There is a tangential component denoted by $v_{\partial \Omega}$ and a normal one. We set $g_{ij}$, $g^{ij}$ as in (18)-(19), $v_i = \langle v \mid \partial_i X \rangle$ for $i = 1, \ldots, n-1$, and $v_n = \langle v \mid n \rangle$. We have from (20):

$$
v = v_{\partial \Omega} + v_n n = \sum_{i=1}^{n-1} \left( \sum_{j=1}^{n-1} g^{ij} v_j \right) \partial_i X + v_n n. \quad (28)
$$

Similarly, we can decompose the action of the gradient into tangential and normal components:

$$
\nabla(\ . \ ) = \nabla_{\partial \Omega}(\ . \ ) + \partial_n(\ . \ ) n = \sum_{i=1}^{n-1} \left( \sum_{j=1}^{n-1} g^{ij} \partial_j \right) \partial_i X + \partial_n(\ . \ ) n,
$$

where $\partial_n(\ . \ ) = \langle \nabla(\ . \ ) \mid n \rangle$ and $\partial_j(\ . \ )$ are the partial derivatives in the parametrization. Observe that (28) shows that $v$ is a Lipschitz continuous map in the parametrization, since it is a product and sum of such functions. Consequently, it is differentiable almost everywhere and we can compute:

$$
\text{div}(v) = \sum_{i,j=1}^{n-1} g^{ij} \langle \partial_j(v) \mid \partial_i X \rangle + \langle \partial_n(v) \mid n \rangle = \sum_{i,j=1}^{n-1} g^{ij} \langle \partial_j(v_{\partial \Omega} + v_n n) \mid \partial_i X \rangle + \langle \partial_n \left( \sum_{i,j=1}^{n-1} g^{ij} v_j \partial_i X + v_n n \right) \mid n \rangle = \sum_{i,j=1}^{n-1} g^{ij} \langle \partial_j(v_{\partial \Omega}) \mid \partial_i X \rangle + v_n \langle \partial_j(n \mid \partial_i X) + v_j \langle \partial_n(\partial_i X) \mid n \rangle + \langle \partial_n(v_n n) \mid n \rangle.
$$

To obtain the last expression, we used $\langle \partial_i X \mid n \rangle = 0$. As we did for the gradient, we introduce the tangential component of the divergence operator $\text{div}_{\partial \Omega}(\ . \ ) = \sum_{i,j=1}^{n-1} g^{ij} \langle \partial_j(\ . \ ) \mid \partial_i X \rangle$. Moreover, note that $H = \text{div}_{\partial \Omega}(n) = \sum_{i,j=1}^{n-1} g^{ij} \langle \partial_j n \mid \partial_i X \rangle$ by using (23) and (24), so we can write:

$$
\text{div}(v) = \text{div}_{\partial \Omega}(v_{\partial \Omega}) + Hv_n + \sum_{i,j=1}^{n-1} g^{ij} v_j \langle \partial_n(\partial_i X) \mid n \rangle + \langle \partial_n(v_n n) \mid n \rangle. \quad (29)
$$

Similarly, we can express the operator $\langle v \mid \nabla(\ . \ ) \rangle$ in the basis and we obtain:

$$
\langle v \mid \nabla(\ . \ ) \rangle = \left( \sum_{i,j=1}^{n-1} g^{ij} v_j \partial_i X + v_n n \right) \langle \partial_i X + \partial_n(\ . \ ) n \rangle = \sum_{i,j=1}^{n-1} g^{ij} v_j \partial_i X + v_n n \right) \langle \partial_n(\partial_i X) \rangle + v_n \langle \partial_n(v_n n) \rangle.
$$

As we already noticed, $v$ is Lipschitz continuous hence differentiable almost everywhere so we can compute $\langle (v \mid \nabla)(v) \mid n \rangle$ and it is equal to:

$$
\left( \sum_{i,j=1}^{n-1} g^{ij} v_j \partial_i X + v_n n \right) \langle \partial_n(\partial_i X) \rangle + v_n \langle \partial_n(v_n n) \rangle.
$$

After some simplifications using $\langle \partial_i X \mid n \rangle = 0$ and $\langle \partial_n n \mid n \rangle = 0$ since $\|n\|^2 = 1$, we get that $\langle (v \mid \nabla)(v) \mid n \rangle$ is almost everywhere equal to:

$$
\sum_{i,j=1}^{n-1} g^{ij} \left( \sum_{i',j'=1}^{n-1} g^{i'j'} v_{i'j'} \langle \partial_n(\partial_i X) \mid n \rangle + v_n \langle \partial_n(v_n n) \rangle \right) + v_n \langle \partial_n(v_n n) \rangle.
$$
Observing from (23) that we have $\langle \partial_i \cdot X | n \rangle = h_{i\alpha}$, and recalling that the first fundamental form is defined as $I(v_{\partial\Omega}, w_{\partial\Omega}) := \langle v_{\partial\Omega} | w_{\partial\Omega} \rangle = \sum_{j=1}^{n-1} g^{ij} w_j$ and the second fundamental form in (25) by $II(v_{\partial\Omega}, w_{\partial\Omega}) := (-Dn(v_{\partial\Omega}) | w_{\partial\Omega}) = -\sum_{i,j,k,l=1}^{n-1} g^{ij} g^{kl} v_k v_j (\partial_i n | \partial_l X)$, then the above expression can be written as:

$$\langle (v | \nabla) (v) | n \rangle = II(v_{\partial\Omega}, v_{\partial\Omega}) + I[v_{\partial\Omega}, \nabla_{\partial\Omega}(v_n)] + v_n \left( \sum_{i, j=1}^{n-1} g^{ij} v_j (\partial_i (\partial_i X) | n) + \langle \partial_i (v_n n) | n \rangle \right).$$

Finally, we combine the above relation with (29) to deduce the following identity:

$$\langle (v | \nabla) (v) | n \rangle = v_n \text{div}(v) = II(v_{\partial\Omega}, v_{\partial\Omega}) + I[v_{\partial\Omega}, \nabla_{\partial\Omega}(v_n)] - H(v_n)^2 = v_n \text{div}_{\partial\Omega}(v_{\partial\Omega}).$$

It remains to slightly modify the last term of right hand side in (30) by observing that:

$$\text{div}_{\partial\Omega}(v_n v_{\partial\Omega}) - v_n \text{div}_{\partial\Omega}(v_{\partial\Omega}) = \sum_{i,j=1}^{n-1} g^{ij} \partial_j(v_n) \left( \sum_{j', k=1}^{n-1} g^{j'k} v_{j'} \partial_k X \right) = \sum_{i,j=1}^{n-1} g^{i\alpha} \partial_j(v_n) \left( \sum_{j', k=1}^{n-1} g^{j'k} v_{j'} \right) = \sum_{i,j=1}^{n-1} g^{i\alpha} \partial_j(v_n) = I[v_{\partial\Omega}, \nabla_{\partial\Omega}(v_n)].$$

Inserting this last relation in (30), we obtain:

$$\langle (v | \nabla) (v) | n \rangle = v_n \text{div}(v) = 2I[v_{\partial\Omega}, \nabla_{\partial\Omega}(v_n)] + II(v_{\partial\Omega}, v_{\partial\Omega}) - H(v_n)^2 - \text{div}_{\partial\Omega}(v_n v_{\partial\Omega}).$$

We can integrate over $\partial\Omega$ the above equality since $v$ has compact support in $\overline{\Omega} \cap C(x_0)$ and for any point $x_0 \in \partial\Omega$. We now extend the result globally thanks to a suitable partition of unity. Let $v \in C^\infty(\overline{\Omega}, R^n)$. Since $\partial\Omega$ is compact, there exists a finite number $K \geq 1$ of points denoted by $x_1, \ldots, x_K$ such that $\partial\Omega \subset \bigcup_{k=1}^{K} C(x_k)$. We build a partition of unity on this set. There exists $K$ smooth maps $\xi_k : R^n \rightarrow [0, 1]$ with compact support in $C(x_k)$, and such that $\sum_{k=1}^{K} \xi_k = 1$ on $\partial\Omega$. Then, we have for $k = 1, \ldots, K$:

$$\left(\sqrt{\xi_k} v\right)_n \text{div}(\sqrt{\xi_k} v) = \sqrt{\xi_k} v_n \left(\nabla \left(\sqrt{\xi_k} v\right) | n\right) + \xi_k v_n \text{div}(v) = \frac{1}{2} \left[ v_n \text{div}(\xi_k v) + (\xi_k v_n) \text{div}(v) \right].$$

Integrating the above relations on $\partial\Omega$ and summing them from $k = 1$ to $K$, we deduce that:

$$\sum_{k=1}^{K} \int_{\partial\Omega} \left(\sqrt{\xi_k} v\right)_n \text{div}(\sqrt{\xi_k} v) dA = \frac{1}{2} \sum_{k=1}^{K} \int_{\partial\Omega} [v_n \text{div}(\xi_k v) + (\xi_k v_n) \text{div}(v)] dA = \int_{\partial\Omega} v_n \text{div}(v) dA.$$

Similarly, one can prove that the following relation holds:

$$\sum_{k=1}^{K} \int_{\partial\Omega} \langle \left(\sqrt{\xi_k} v \ | \ n\rangle \right) = \int_{\partial\Omega} \langle (v | \nabla) (v) | n \rangle dA.$$

Combining the last two equalities and applying (31) since $\sqrt{\xi_k} v$ has compact support in $\overline{\Omega} \cap C(x_k)$, we obtain that $\int_{\partial\Omega} \langle (v | \nabla) (v) | n \rangle - v_n \text{div}(v) dA$ is equal to:

$$2 \sum_{k=1}^{K} \int_{\partial\Omega} \left[ \left(\sqrt{\xi_k} v\right)_n \right] \right) \left. \left(\sqrt{\xi_k} v\right)_{\partial\Omega} \right] dA + \sum_{k=1}^{K} \int_{\partial\Omega} II \left[ \left(\sqrt{\xi_k} v\right)_{\partial\Omega}, \left(\sqrt{\xi_k} v\right)_{\partial\Omega} \right] dA - \sum_{k=1}^{K} \int_{\partial\Omega} H \left(\left(\sqrt{\xi_k} v\right)_n \left(\sqrt{\xi_k} v\right)_{\partial\Omega} \right)^2 dA - \sum_{k=1}^{K} \int_{\partial\Omega} \text{div}_{\partial\Omega} \left[ \left(\sqrt{\xi_k} v\right)_n \left(\sqrt{\xi_k} v\right)_{\partial\Omega} \right] dA,$$
from which we deduce that $\int_{\Omega}(\langle v \mid \nabla v \rangle \mid n) - v_n \text{div}(v)\,dA$ is equal to:

$$
\sum_{i=1}^{K} \int_{\partial \Omega} (\nabla_{\partial \Omega} (\xi_k v_n) \mid v_{\partial \Omega})\,dA + \sum_{i=1}^{K} \int_{\partial \Omega} (\nabla_{\partial \Omega} (v_n) \mid \xi_k v_{\partial \Omega})\,dA + \sum_{i=1}^{K} \int_{\partial \Omega} \xi_k \Pi (v_{\partial \Omega}, v_{\partial \Omega})\,dA

- \sum_{i=1}^{K} \int_{\partial \Omega} H \xi_k (v_n)^2\,dA - \sum_{i=1}^{K} \int_{\partial \Omega} \text{div}_{\partial \Omega} (\xi_k v_n v_{\partial \Omega})\,dA.
$$

Since $\sum_{i=1}^{K} \xi_k = 1$ on $\partial \Omega$, we have proved that (31) holds for any map $v \in C^\infty(\bar{\Omega}, \mathbb{R}^n)$. Combining (27) and (31), then observing that $\int_{\partial \Omega} \text{div}_{\partial \Omega} (v_n v_{\partial \Omega})\,dA = 0$ (we refer to the next result, namely Proposition 2.3, for a proof), we deduce that for any map $v \in C^\infty(\bar{\Omega}, \mathbb{R}^n)$, we have:

$$
\sum_{i,j=1}^{n} \int_{\Omega} \frac{\partial v_m}{\partial x_j} \frac{\partial v_m}{\partial x_i} - \int_{\Omega} \text{div}(v)^2 = \int_{\partial \Omega} \left[ 2 \langle \nabla_{\partial \Omega} (v_n) \mid v_{\partial \Omega} \rangle + \Pi (v_{\partial \Omega}, v_{\partial \Omega}) - H (v_n)^2 \right]\,dA \quad (32)
$$

It remains to prove that (26) holds for $v \in H^1(\Omega, \mathbb{R}^n)$ by a density argument. Let $v \in H^1(\Omega, \mathbb{R}^n)$. Since $\partial \Omega$ has $C^{1,1}$-regularity, the domain $\Omega$ is Lipschitz and there exists a sequence of smooth maps $(v^m)_{m \in \mathbb{N}} \subset C^\infty(\bar{\Omega}, \mathbb{R}^n)$ converging to $v$ in $H^1(\Omega, \mathbb{R}^n)$. From the foregoing, (32) holds for any $v^m$ and we now prove that we can let $m \to +\infty$. This is the case for the first term in left-hand side of (32) because we have from the Cauchy-Schwarz inequality for any $i, j, k, l = 1, \ldots, n$:

$$
\left| \int_{\Omega} \frac{\partial v^m}{\partial x_j} \frac{\partial v^m}{\partial x_l} - \int_{\Omega} \frac{\partial v_i}{\partial x_j} \frac{\partial v_k}{\partial x_l} \right| \leq \left\| \frac{\partial v^m}{\partial x_j} - \frac{\partial v^m}{\partial x_l} \right\|_{L^2(\Omega, \mathbb{R}^n)} \left( \left\| \frac{\partial v_i}{\partial x_j} \right\|_{L^2(\Omega, \mathbb{R}^n)} + \left\| \frac{\partial v_k}{\partial x_l} \right\|_{L^2(\Omega, \mathbb{R}^n)} \right)
$$

$$
+ \left\| \frac{\partial v^m}{\partial x_j} \right\|_{L^2(\Omega, \mathbb{R}^n)} \left\| \frac{\partial v^m}{\partial x_l} \right\|_{L^2(\Omega, \mathbb{R}^n)}.
$$

Similarly, the convergence holds for the second term in the left-hand side of (32) because we have

$$
\int_{\partial \Omega} \langle \nabla_{\partial \Omega} (v^m) \mid (v^m)_{\partial \Omega} \rangle\,dA \quad \text{to deduce:}
$$

$$
\int_{\partial \Omega} \langle \nabla_{\partial \Omega} (v^m) \mid (v^m)_{\partial \Omega} \rangle\,dA \rightarrow (\nabla v_n)_{H^{-1}(\partial \Omega, \mathbb{R}^n)}\,dA
$$

$$
\rightarrow (\nabla v_{\partial \Omega})_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^n)}\,dA
$$

Secondly, $\partial \Omega$ is a compact $C^{1,1}$-hypersurface hence there exists $\varepsilon > 0$ such that $\partial \Omega$ satisfies the $\varepsilon$-ball condition and in particular, the Gauss map $n : x \in \partial \Omega \rightarrow S^{n-1}$ is $\frac{1}{2}$-Lipschitz continuous. We deduce that the eigenvalues of its differential i.e. the principal curvatures $(\kappa_i)_{i \leq i \leq n-1}$ exists almost everywhere and belongs to $L^\infty(\partial \Omega, \mathbb{R})$. Considering the principal directions $(e_i)_{1 \leq i \leq n}$ associated with the principal curvatures, $(e_1, \ldots, e_{n-1})$ forms an orthonormal basis of the tangent hyperplane so we deduce that:

$$
\int_{\partial \Omega} \Pi [(v^m)_{\partial \Omega}, (v^m)_{\partial \Omega}]\,dA = - \int_{\partial \Omega} \left\langle D_x n \left( \sum_{i=1}^{n-1} \langle (v^m)_{\partial \Omega} \mid e_i (x) \rangle e_i (x) \right), (v^m)_{\partial \Omega} (x) \right\rangle\,dA (x)
$$

Since $D_x n (e_i (x)) = \kappa_i (x) e_i (x)$, we obtain from the linearity:

$$
\int_{\partial \Omega} \Pi [(v^m)_{\partial \Omega}, (v^m)_{\partial \Omega}]\,dA = - \sum_{i=1}^{n-1} \int_{\partial \Omega} \kappa_i (x) \left| \langle (v^m)_{\partial \Omega} \mid e_i (x) \rangle \right|^2\,dA (x),
$$

from which we deduce with the Cauchy-Schwarz inequality:

$$
\left| \int_{\partial \Omega} \Pi [(v^m)_{\partial \Omega}, (v^m)_{\partial \Omega}] - \Pi (v_{\partial \Omega}, v_{\partial \Omega})\,dA \right| \leq \left( \sum_{i=1}^{n-1} \left\| \kappa_i \right\|_{L^\infty(\partial \Omega, \mathbb{R})} \right) \int_{\partial \Omega} \left\| (v^m - v)_{\partial \Omega} \right\|^2\,dA.
$$

Using the continuity of $(\cdot)_{\partial \Omega} : H^1(\Omega, \mathbb{R}^n) \rightarrow L^2(\partial \Omega, \mathbb{R}^n)$, we get the convergence of the second term in the right-hand side of (32). Concerning the third one, the arguments are similar because (22) gives $H = \kappa_1 + \ldots + \kappa_{n-1} \in L^\infty(\partial \Omega, \mathbb{R})$ and $(\cdot)_n : H^1(\Omega, \mathbb{R}^n) \rightarrow L^2(\partial \Omega, \mathbb{R})$ is continuous. To conclude, we can apply (32) on each $v^m$ and let $m \to +\infty$ to obtain that (26) holds for any $v \in H^1(\Omega, \mathbb{R})$ as required. \qed

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Proposition 2.3. Let $\Sigma$ be a compact $C^{1,1}$-hypersurface of $\mathbb{R}^n$. Then, for any $v \in W^{1,1}(\Sigma, \mathbb{R}^n)$ such that $(v \mid n) = 0$, we have:

$$\int_{\Sigma} \text{div}_\Sigma(v) dA = 0.$$ 

Proof. Consider any compact $C^{1,1}$-hypersurface $\Sigma \subset \mathbb{R}^n$. Let $x_0 \in \Sigma$. There exists a cylinder $C(x_0)$ in which $\partial \Omega$ is the graph of a $C^{1,1}$-map $\varphi$. We thus introduce the local $C^{1,1}$-parametrization $X : x' \in D(x_0) \mapsto (x', \varphi(x')) \in \Sigma \cap C(x_0)$ and we first assume that $v : \Sigma \to \mathbb{R}^n$ is a smooth map with compact support in $\Sigma \cap C(x_0)$. We use the same notation than in the proof of Theorem 2.2. Hence, we can decompose $v$ in the basis $(\partial_1 X, \ldots, \partial_{n-1} X, n)$. Since $(v \mid n) = 0$, we have:

$$v = \sum_{i,j=1}^{n-1} g^{ij} \langle v \mid \partial_j X \rangle \partial_i X + \langle v \mid n \rangle n = \sum_{i,j=1}^{n-1} g^{ij} v_j \partial_i X.$$ 

In this decomposition, note that $v$ is a Lipschitz continuous map so it is differentiable almost everywhere and we can compute:

$$\text{div}_\Sigma(v) = \sum_{k,l=1}^{n-1} g^{kl}(\partial_l(v) \mid \partial_k X) = \sum_{k,l=1}^{n-1} g^{kl} \left( \partial_l \left( \sum_{i,j=1}^{n-1} g^{ij} \partial_i \partial_j X \right) \right) \mid \partial_k X.$$ 

$$= \sum_{k,l=1}^{n-1} g^{kl} \left( \sum_{i,j=1}^{n-1} \partial_i \left( g^{ij} \partial_j X \right) \right) + \sum_{i=1}^{n-1} \left( \sum_{j=1}^{n-1} g^{ij} \partial_j \partial_i X \right) \right) \mid \partial_k X).$$ 

Since $X$ is a $C^{1,1}$-map, it is twice-differentiable almost everywhere and at the point where it is the case, we have $\partial_i(\partial_i X) = \partial_i(\partial_j X)$. Moreover, the matrix $(g^{ij})$ is symmetric so we deduce that:

$$\text{div}_\Sigma(v) = \sum_{k,l=1}^{n-1} g^{kl} \left( \sum_{i,j=1}^{n-1} \partial_i \left( g^{ij} \partial_j X \right) \right) + \sum_{i=1}^{n-1} \left( \sum_{j=1}^{n-1} g^{ij} \partial_j \partial_i X \right) \right) \mid \partial_k X) \right),$n-1} \partial_i \left( \sum_{j=1}^{n-1} g^{ij} \partial_j X \right) \right) + \sum_{i=1}^{n-1} \left( \sum_{j=1}^{n-1} g^{ij} \partial_j \partial_i X \right) \right) \right) \left( \frac{1}{2} \sum_{k,l=1}^{n-1} \partial_k \partial_l (g^{kl}) \right).$$ 

Then, we observe that the first term has a simplification since $(g^{ij})$ is the inverse matrix of $(g_{ij})$ and the second term is the differential of a determinant. Hence, we obtain:

$$\text{div}_\Sigma(v) = \sum_{i,l=1}^{n-1} \partial_i \left( \sum_{j=1}^{n-1} g^{ij} \partial_j X \right) \right) \langle \partial_i (\partial_i (g^{ij})) \rangle + \sum_{i=1}^{n-1} \left( \sum_{j=1}^{n-1} g^{ij} \partial_j \partial_i X \right) \right) \left( \frac{1}{2} \text{Trace} (\partial_i (g^{ij})) \right)$$ 

$$= \sum_{i=1}^{n-1} \partial_i \left( \sum_{j=1}^{n-1} g^{ij} \partial_j X \right) + \sum_{i=1}^{n-1} \left( \sum_{j=1}^{n-1} g^{ij} \partial_j \partial_i X \right) \right) \left( \frac{1}{2} \text{Tr} (\partial_i (g^{ij})) \right)$$ 

$$= \sum_{i=1}^{n-1} \partial_i \left( \sum_{j=1}^{n-1} g^{ij} \partial_j X \right) + \sum_{i=1}^{n-1} \left( \sum_{j=1}^{n-1} g^{ij} \partial_j \partial_i X \right) \right) \left( \frac{1}{2} \text{Tr} (\partial_i (g^{ij})) \right)$$ 

Since $v$ has compact support in $\Sigma \cap C(x_0)$, so does $v_j = \langle v \mid \partial_j X \rangle$ on $D(x_0)$ and we get:

$$\int_{\Sigma} \text{div}_\Sigma(v) dA = \int_{D(x_0)} \text{div}_\Sigma(v \circ X) \sqrt{\text{det}(g)} \sum_{i=1}^{n-1} \partial_i \left( \sqrt{\text{det}(g)} \sum_{j=1}^{n-1} g^{ij} \partial_j \partial_i X \right) \right) = 0.$$ 

The result of Proposition 2.3 is thus established if $v : \Sigma \to \mathbb{R}^n$ is a smooth map with compact support in $\Sigma \cap C(x_0)$ for any $x_0 \in \Sigma$. Then, we assume that $v \in C^\infty(\Sigma, \mathbb{R}^n)$. Since $\Sigma$ is compact, there exists a finite number $K \geq 1$ of points denoted by $x_1, \ldots, x_K$ such that $\Sigma \subset \bigcup_{k=1}^{K} C(x_k)$.
We can build a partition of unity on this set. There exists $K$ smooth maps $\xi_k : \mathbb{R}^n \to [0, 1]$ with compact support in $C(x_k)$, and such that $\sum_{k=1}^{K} \xi_k = 1$ on $\Sigma$. Hence, we have successively:

$$\int_{\Sigma} \text{div}_\Sigma (v) dA = \int_{\Sigma} \text{div}_\Sigma \left( \sum_{k=1}^{K} \xi_k v \right) dA = \sum_{k=1}^{K} \int_{\Sigma} \text{div}_\Sigma (\xi_k v) dA = 0,$$

where the last equality comes from the previous case because $\xi_k v$ is a smooth map with compact support in $C(x_k)$ for any $k = 1, \ldots, K$. The result of Proposition 2.3 holds for any $v \in C^\infty(\Sigma, \mathbb{R}^n)$.

Finally, we assume that $v \in W^{1,1}(\Sigma, \mathbb{R}^n)$. By density, there exists a sequence of smooth maps $v^m \in C^\infty(\Sigma, \mathbb{R}^n)$ such that $v^m - v$ tends to zero in $W^{1,1}(\Sigma, \mathbb{R}^n)$. We can apply the previous case on each $v^m$ and we get:

$$\left| \int_{\Sigma} \text{div}_\Sigma (v) dA \right| = \left| \int_{\Sigma} \text{div}_\Sigma (v - v^m) dA \right| \leq \sum_{i=1}^{n} \int_{\Sigma} \| \nabla_\Sigma (v_i) - \nabla_\Sigma (v_i^m) \| dA \to 0 \quad m \to +\infty.$$

To conclude, we proved $\int_{\Sigma} \text{div}_\Sigma (v) dA = 0$ for any map $v \in W^{1,1}(\Sigma, \mathbb{R}^n)$ such that $(v \mid n) = 0$. □

### 2.3 Some Poincaré inequalities

We quickly recall here the well-known Poincaré inequality and deduce some of its consequences.

**Proposition 2.4 (Poincaré Inequality).** Let $\Omega$ be any non-empty open subset of $\mathbb{R}^n$ which is bounded in a direction i.e. there exists a constant $D > 0$, a point $x_0 \in \mathbb{R}^n$, and a unit vector $d_{x_0}$ of $\mathbb{R}^n$ such that $|(x - x_0) | d_{x_0} | \leq D$ for any point $x \in \Omega$. Then, we have:

$$\forall u \in H^1_{0}(\Omega, \mathbb{R}), \quad \int_{\Omega} u^2 \leq 4D^2 \int_{\Omega} \| \nabla u \|^2.$$

**Proof.** We consider a basis $B_{x_0}$ of the orthogonal space $d_{x_0}^\perp$ such that $(x_0, B_{x_0}, d_{x_0})$ is a direct orthonormal frame centred at $x_0$. Henceforth, the position of any point is determined in this frame. In particular, any point $x = (x', x_n) \in \Omega$ must satisfy $|x_n| \leq D$. First, we assume $u \in C^\infty(\Omega, \mathbb{R})$. Then, the map $u$ can be extended by zero to $\tilde{u} \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and in particular, for any $x' \in \mathbb{R}^{n-1}$, we have $\tilde{u}(x', -D) = \lim_{x_n \to -D} \tilde{u}(x', x_n) = 0$ since $(x', x_n) \notin \Omega$. Combining this observation with the Cauchy-Schwarz inequality, we have for any $(x', x_n) \in \mathbb{R}^{n-1} \times [-D, D]$:

$$\tilde{u}(x', x_n) = \left( \int_{-D}^{x_n} \frac{\partial \tilde{u}}{\partial x_n} (x', t) dt \right)^2 \leq (x_n + D) \int_{-D}^{x_n} \left( \frac{\partial \tilde{u}}{\partial x_n} (x', t) \right)^2 dt \leq 2D \int_{-D}^{x_n} \left( \frac{\partial \tilde{u}}{\partial x_n} (x', t) \right)^2 dt$$

Integrating this inequality in the $x_n$-variable on $[-D, D]$, and in the $x'$-variable on $\mathbb{R}^{n-1}$, we obtain:

$$\int_{\mathbb{R}^{n-1}} \left( \int_{-D}^{D} \tilde{u}(x', x_n)^2 dx_n \right) dx' \leq 4D^2 \int_{\mathbb{R}^{n-1}} \left( \int_{-D}^{D} \left( \frac{\partial \tilde{u}}{\partial x_n} (x', t) \right)^2 dt \right) dx'$$

Then, we use again the observation $\tilde{u}(x', x_n) = 0$ for any $x' \in \mathbb{R}^{n-1}$ and $x_n \notin [-D, D]$. Thanks to the Fubini-Tonelli Theorem, we get:

$$\int_{\Omega} u^2 = \int_{\mathbb{R}^n} \tilde{u}^2 = \int_{\mathbb{R}^{n-1}} \left( \int_{-\infty}^{+\infty} \tilde{u}^2 \right) = \int_{\mathbb{R}^{n-1}} \left( \int_{-D}^{D} \tilde{u}^2 \right) \leq 4D^2 \int_{\mathbb{R}^{n-1}} \left( \int_{-D}^{D} \left( \frac{\partial \tilde{u}}{\partial x_n} \right)^2 \right)$$

$$\leq 4D^2 \int_{\mathbb{R}^{n-1}} \left( \int_{-\infty}^{+\infty} \left( \frac{\partial \tilde{u}}{\partial x_n} \right)^2 \right) = 4D^2 \int_{\mathbb{R}^{n-1}} \left( \frac{\partial \tilde{u}}{\partial x_n} \right)^2 = 4D^2 \int_{\Omega} \left( \frac{\partial u}{\partial x_n} \right)^2$$

$$\leq 4D^2 \sum_{i=1}^{n} \int_{\Omega} \left( \frac{\partial u}{\partial x_i} \right)^2 = 4D^2 \int_{\Omega} \| \nabla u \|^2$$

Consequently, Proposition 2.4 is established for any $u \in C^\infty(\Omega, \mathbb{R})$. Finally, we assume that $u \in H^1_{0}(\Omega, \mathbb{R})$. There exists a sequence $(u_i)_{i \in \mathbb{N}} \subset C^\infty(\Omega, \mathbb{R})$ converging strongly to $u$ in $H^1(\Omega, \mathbb{R})$.\hfill 15
From the foregoing, we deduce that:
\[
\|u\|_{L^2(\Omega, \mathbb{R})} \leq \|u - u_i\|_{L^2(\Omega, \mathbb{R})} + \|u_i\|_{L^2(\Omega, \mathbb{R})} \leq \|u - u_i\|_{L^2(\Omega, \mathbb{R})} + 2D\|\nabla u_i\|_{L^2(\Omega, \mathbb{R}^n)}
\]
\[
\leq \|u - u_i\|_{L^2(\Omega, \mathbb{R})} + 2D\|\nabla u_i - \nabla u\|_{L^2(\Omega, \mathbb{R}^n)} + 2D\|\nabla u\|_{L^2(\Omega, \mathbb{R}^n)}
\]
\[
\leq \max(1, 2D)\|u - u_i\|_{H^1(\Omega, \mathbb{R})} + 2D\|\nabla u\|_{L^2(\Omega, \mathbb{R}^n)}
\]
To conclude, we let \( i \to +\infty \) to obtain the required inequality: \( \|u\|_{L^2(\Omega, \mathbb{R})} \leq 2D\|\nabla u\|_{L^2(\Omega, \mathbb{R}^n)}. \)

**Corollary 2.5.** Let \( \Omega \) be a non-empty bounded open subset of \( \mathbb{R}^n \). If \( D = \max_{(x,y) \in \Omega \times \Omega} \|x - y\| \), then we have:
\[
\forall u \in H^2(\Omega, \mathbb{R}), \quad \int_{\Omega} u^2 \leq 4D^2 \int_{\Omega} \|\nabla u\|^2.
\]

**Proof.** Since \( \Omega \) is bounded, \( \Omega \) is compact so the diameter \( D \) is finite and attained by two points \( x_0 \) and \( y_0 \). Moreover, it is positive because \( \Omega \) is not empty and open. We get \( \Omega \subseteq B_D(x_0) \) and applying Proposition 2.4 for the point \( x_0 \) and the unit vector \( \frac{1}{D}(y_0 - x_0) \), the inequality follows. \( \square \)

**Corollary 2.6.** Let \( \Omega \) be a non-empty bounded open subset of \( \mathbb{R}^n \). If \( D = \max_{(x,y) \in \Omega \times \Omega} \|x - y\| \), then we have:
\[
\forall u \in H^2(\Omega, \mathbb{R}) \cap H^1_0(\Omega, \mathbb{R}), \quad \int_{\Omega} \|
abla u\|^2 \leq 4D^2 \int_{\Omega} (\Delta u)^2.
\]

**Proof.** Let any \( u \in H^2(\Omega, \mathbb{R}) \cap H^1_0(\Omega, \mathbb{R}). \) We get successively from an integration by parts, the inequality \( xy \leq \frac{x}{2}^2 + \frac{1}{20}y^2 \), and Corollary 2.5:
\[
\int_{\Omega} \|
abla u\|^2 = -\int_{\Omega} u \Delta u \leq \int_{\Omega} |u \Delta u| \leq 2D^2 \int_{\Omega} (\Delta u)^2 + \frac{1}{8D^2} \int_{\Omega} u^2 \leq 2D^2 \int_{\Omega} (\Delta u)^2 + \frac{1}{2} \int_{\Omega} \|
abla u\|^2.
\]
After simplification, we obtain the required inequality: \( \|\nabla u\|_{L^2(\Omega, \mathbb{R})} \leq 2D\|\Delta u\|_{L^2(\Omega, \mathbb{R})}. \)

**2.4 Some Trace inequalities**

The constant appearing in the trace inequality depends on the \( C^1 \)-norm of any partition of unity associated with an finite open covering of the hypersurface. Therefore, we need to build a partition of unity for which we can control uniformly the number of maps and the \( C^0 \)-norm of their gradient.

**Proposition 2.7.** Let \( h > 0 \), \( n \geq 1 \), and \( B \) be any non-empty open subset of \( \mathbb{R}^n \) of diameter \( D \), large enough to contain the origin. Then, there exists \( N \in \mathbb{N} \) and a constant \( C > 0 \), both depending only on \( h \), \( D \) and \( n \), such that for any non-empty open set \( \Omega \subseteq B \), there exists \( K \) distinct points \( (x_k)_{1 \leq k \leq K} \) of \( \partial \Omega \), \( 1 \leq K \leq N(h, D, n) \), such that the tubular neighborhood \( V_{\frac{h}{2}}(\partial \Omega) \) has its closure embedded in \( \cup_{k=1}^K B_{\frac{h}{2}}(x_k) \), and there exists \( K \) smooth maps \( \xi_k : \mathbb{R}^n \to [0, 1] \) with compact support in \( B_h(x_k) \), such that \( \sum_{k=1}^K \xi_k = 1 \) on \( V_{\frac{h}{2}}(\partial \Omega) \) and \( \sum_{k=1}^K \sum_{i=1}^n \left\|\frac{\partial \xi_k}{\partial x_i}\right\|_{C^0(\mathbb{R}^n, \mathbb{R})} \leq C(h, D, n) \).

**Proof.** Let \( h > 0 \), \( n \geq 1 \), \( B \subseteq \mathbb{R}^n \) be a non-empty open subset of diameter \( D \) containing the origin \( 0 \), and \( \Omega \) be a non-empty open subset of \( B \). Since \( 0 \in B \), we have \( \Omega \subseteq B_D(0) \) so it is included in the cube of length \( D \) centred at the origin. We set:
\[
a := \frac{h}{2\sqrt{n}} \quad \text{ and } \quad N(h, D, n) := \left(1 + \left\lfloor \frac{D}{a} \right\rfloor \right)^n,
\]
where \( \lfloor \cdot \rfloor \) denotes here the integer part. Hence, the larger cube of length \( a(1 + \left\lfloor \frac{D}{a} \right\rfloor) > D \) centred at the origin can be divided into \( N(h, D, n) \) small cubes of length \( a \). We denote by \( (y_k)_{1 \leq k \leq N} \) the centres of these small cubes. Note that with our choice of \( a \), their diameter is \( \frac{D}{a} \) thus they are themselves contained in balls of radius \( \frac{D}{2} \) centred at \( y_k \). In other words, \( B_D(0) \subseteq \bigcup_{k=1}^N B_{\frac{D}{2}}(y_k) \).

Then, we deduce that:
\[
\partial \Omega \subseteq \bigcup_{1 \leq k \leq N} B_{\frac{D}{2}}(y_k).
\]
Therefore, we can relabel the points \( (y_k)_{1 \leq k \leq N} \) such that there exists an integer \( 1 \leq K \leq N \) satisfying \( \partial \Omega \subseteq \bigcup_{k=1}^K B_{\frac{D}{2}}(x_k) \) and \( \partial \Omega \cap B_{\frac{D}{2}}(y_k) \neq \emptyset \) for \( k = 1, \ldots, K \). In particular, \( d(y_k, \partial \Omega) \leq \frac{h}{4} \)
so there exists $K$ points $(x_k)_{1 \leq k \leq K}$ of $\partial \Omega$ such that $\|x_k - y_k\| \leq \frac{h}{4}$. From the triangle inequality, we successively deduce $\partial \Omega \subseteq \bigcup_{k=1}^{K} B_{\frac{h}{4}}(x_k)$ and $\nabla u(\partial \Omega) \subseteq \bigcup_{k=1}^{K} B_{\frac{h}{4}}(x_k)$. Finally, it remains to build the partition of unity. This is a standard procedure. We introduce the following function:

$$w : \mathbb{R}^n \rightarrow \mathbb{R}$$
$$x \mapsto w(x) = \begin{cases} e^{1 - \frac{h^2}{2r^2 - (\|x\|^2)}} & \text{if } \|x\| < \frac{h}{4} \\ 0 & \text{otherwise} \end{cases}$$

One can check $w \in C^\infty([\mathbb{R}^n, [0,1] \])$ and its support is $B_{\frac{h}{4}}(0)$. Then, we set $c(h,n) = \int_{\mathbb{R}^n} w(x)dx$, depending only on $n$ and $h$. We consider the following maps:

$$\Psi_k : \mathbb{R}^n \rightarrow \mathbb{R}$$
$$x \mapsto \Psi_k(x) = c(h,n) \int_{B_{\frac{h}{4}} + \frac{h}{8}(x_k)} w(x - y)dy.$$

Similarly, one can show that $\Psi_k \in C^\infty([\mathbb{R}^n, [0,1] \])$, $\Psi_k = 1$ on $B_{\frac{h}{4}}(x_k)$ and $\Psi_k \subseteq B_{h - \frac{3h}{16}}(x_k)$ so it has compact support in $B_r(x_k)$. Moreover, we have for $i = 1, \ldots, n$ and for $k = 1, \ldots, K$:

$$\left| \frac{\partial \Psi_k}{\partial x_i}(x) \right| \leq c(h,n) \|\frac{\partial w}{\partial x_i}\|_{C^0(\mathbb{R}^n, \mathbb{R})} \|\nabla w\|_{C^0(\mathbb{R}^n, \mathbb{R})} \leq c(h,n) \frac{2 \exp(-1) 4\pi}{3} \frac{3h + h}{16}.$$  

To conclude, we set $\xi_1 = \Psi_1$ and $\xi_k = \Psi_k \prod_{i=1}^{k-1}(1 - \Psi_i)$ for any $2 \leq k \leq K$. We get that $\xi_k \in C^\infty([\mathbb{R}^n, [0,1] \])$ has compact support in $B_r(x_k)$, and $\sum_{k=1}^{K} \xi_k = 1$ on $\bigcup_{k=1}^{K} B_{\frac{h}{4}}(x_k)$ thus on the closure of $V_{\frac{h}{4}}(\partial \Omega)$. Furthermore, we have:

$$\sum_{k=1}^{K} \sum_{i=1}^{n} \frac{\partial \xi_k}{\partial x_i} \|\nabla w\|_{C^0(\mathbb{R}^n, \mathbb{R})} \leq \sum_{k=1}^{K} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \Psi_k}{\partial x_i} \|\nabla w\|_{C^0(\mathbb{R}^n, \mathbb{R})} \leq n c(h,n) N(h, D, n)^2,$$

and the constant $C(h, D, n) := n c(h,n) N(h, D, n)^2$ is the one required in the statement. 

**Proposition 2.8.** Let $\alpha \in [0, \frac{\pi}{2}]$, $n \geq 2$ and $B$ be a non-empty open bounded subset of $\mathbb{R}^n$ containing the origin. We consider the class $\mathcal{D}_\alpha(B)$ formed by all the non-empty open subsets of $B$ satisfying the $\alpha$-cone property. We assume that the diameter $D$ of $B$ is large enough to ensure $\mathcal{D}_\alpha(B) \neq \emptyset$. Then, there exists a constant $C > 0$, depending only on $\alpha$, $n$, and $D$ such that:

$$\forall \Omega \in \mathcal{D}_\alpha(B), \forall \eta \in [0, 1], \forall u \in H^1(\Omega, \mathbb{R}), \int_{\partial \Omega} u^2 dA \leq C(\alpha, D, n) \left( \eta \int_{\Omega} \|\nabla u\|^2 + \frac{1}{\eta} \int_{\Omega} u^2 \right).$$

**Proof.** Let $\alpha \in [0, \frac{\pi}{2}]$, $n \geq 2$ and $B$ be a non-empty open bounded subset of $\mathbb{R}^n$ containing the origin. Introducing the class $\mathcal{D}_\alpha(B)$ formed by all the non-empty open subsets of $B$ satisfying the $\alpha$-cone property, we consider $\Omega \in \mathcal{D}_\alpha(B)$. Hence, from the uniform cone property, $\partial \Omega$ has a Lipschitz boundary i.e. for any point $x_0 \in \partial \Omega$, there exists a cylinder $C_r,\alpha(x_0)$ as in (17) of direction a unit vector $d_{x_0}$ of $\mathbb{R}^n$ in which $\partial \Omega$ is the graph of a $L$-Lipschitz continuous map $\varphi_{x_0}$, and in which $\Omega$ is the area below this graph. Moreover, the constants $r > 0$, $a > 0$, and $L > 0$ only depend on $\alpha$. Consequently, Proposition 2.7 is applied to $B$ with $h(\alpha) = \min(r, a)$ depending only on $\alpha$. There exists $K$ distinct points $(x_k)_{1 \leq k \leq K}$ of $\partial \Omega$, such that $\bigcup_{k=1}^{K} C_{r,\alpha}(x_k)$, and there exists $K$ smooth maps $\xi_k : \mathbb{R}^n \rightarrow [0,1]$ with compact support in $C_{r,\alpha}(x_k)$ such that $\sum_{k=1}^{K} \xi_k = 1$ on $\bigcup_{k=1}^{K} C_{r,\alpha}(x_k)$. Furthermore, we have $K \leq N(\alpha, D, n)$ and $\sum_{k=1}^{K} \sum_{i=1}^{n} \frac{\partial \xi_k}{\partial x_i} \|\nabla w\|_{C^0(\mathbb{R}^n, \mathbb{R})} \leq C(\alpha, D, n)$, where $N \in \mathbb{N}$ and $C > 0$ depending only on $\alpha$, $D$, and $n$. We set $m = \sum_{k=1}^{K} \xi_k d_{x_k} \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and we show that $m \neq 0$ almost everywhere on $\partial \Omega$. Indeed, since $\varphi_{x_k}$ is $L$-Lipschitz continuous, it is differentiable almost everywhere. Considering $x \in \partial \Omega$ for which the normal exists, we have:

$$\langle m(x) \mid n(x) \rangle = \sum_{k=1}^{K} \xi_k(x) \langle n(x) \mid d_{x_k} \rangle = \sum_{k=1}^{K} \xi_k(x) \langle \varphi_{x_k}(x') \mid \frac{\partial \varphi_{x_k}(x')}{|\nabla \varphi_{x_k}(x')|^2} \rangle \geq \sum_{k=1}^{K} \xi_k(x) \frac{1}{1 + L^2} = \frac{1}{1 + L^2}.$$
Let $u \in H^1(\Omega, \mathbb{R})$ and $\eta \in ]0, 1[$. We use successively the previous inequality, the Stokes Theorem, the Cauchy-Schwarz inequality, the one $2xy \leq \eta x^2 + \frac{1}{\eta} y^2$ and the fact that $\eta \in ]0, 1[$ to get:

$$\int_{\partial \Omega} u^2 dA \leq \sqrt{1 + L^2} \int_{\Omega} u^2 (m \mid n) dA = \sqrt{1 + L^2} \int_{\Omega} \text{div}(u^2 m)$$

$$= \sqrt{1 + L^2} \sum_{k=1}^{K} \int_{\Omega} 2 \xi_k u \langle \nabla u \mid d_{x_k} \rangle + \sqrt{1 + L^2} \sum_{k=1}^{K} \int_{\Omega} u^2 \langle \nabla \xi_k \mid d_{x_k} \rangle$$

$$\leq K \sqrt{1 + L^2} \int_{\Omega} 2u \| \nabla u \| + \sqrt{1 + L^2} \left( \sum_{k=1}^{K} \sum_{i=1}^{n} \left| \frac{\partial \xi_k}{\partial x_i} \right|_{C^{0}(\mathbb{R}^n, \mathbb{R})} \right) \int_{\Omega} u^2$$

$$\leq N \sqrt{1 + L^2} \left( \eta \int_{\Omega} \| \nabla u \|^2 + \frac{1}{\eta} \int_{\Omega} u^2 \right) + C \sqrt{1 + L^2} \int_{\Omega} u^2$$

$$\leq (N + C) \sqrt{1 + L^2} \left( \eta \int_{\Omega} \| \nabla u \|^2 + \frac{1}{\eta} \int_{\Omega} u^2 \right)$$

To conclude, observe that the constant only depends on $\alpha$, $D$ and $n$ as required.

**Corollary 2.9.** Using the assumptions and notation of Proposition 2.8, we get for any $\Omega \in \mathcal{O}_{\alpha}(B)$:

$$\forall \eta \in ]0, 1[, \ \forall u \in H^2(\Omega, \mathbb{R}), \ \int_{\partial \Omega} \| \nabla u \|^2 dA \leq C(\alpha, D, n) \left( \eta \sum_{i,j=1}^{n} \int_{\Omega} \left( \frac{\partial^2 u_i}{\partial x_i \partial x_j} \right)^2 + \frac{1}{\eta} \int_{\Omega} \| \nabla u \|^2 \right).$$

**Proof.** Apply Proposition 2.8 to each $\frac{\partial u}{\partial \nu} \in H^1(\Omega, \mathbb{R})$ and sum the $n$ inequalities obtained.

**Proof of Theorem 2.1.** Let $\varepsilon > 0$, $n \geq 2$, and $B$ be any non-empty open bounded subset of $\mathbb{R}^n$ containing the origin. Introducing the class $\mathcal{O}_{\varepsilon}(B)$ formed by all the non-empty open subsets of $B$ satisfying the $\varepsilon$-ball condition, we consider $\Omega \in \mathcal{O}_{\varepsilon}(B)$ and $u \in H^2(\Omega, \mathbb{R}) \cap H^1_{\alpha}(\Omega, \mathbb{R})$. First, since $u = 0$ on $\partial \Omega$, we deduce $\nabla u = \partial_n (u n)$ i.e. $\nabla_{\partial \Omega}(u) = 0$. Applying Theorem 2.2 to $v = \nabla u$, we get from (26):

$$\sum_{i,j=1}^{n} \int_{\Omega} \left( \frac{\partial^2 v_i}{\partial x_i \partial x_j} \right)^2 = \int_{\Omega} |\Delta u|^2 - \int_{\partial \Omega} H \| \nabla u \|^2 dA.$$

Then, recall that $\Omega$ satisfies the $\varepsilon$-ball condition so $n : x \in \partial \Omega \to S^{n-1}$ is $\frac{1}{\varepsilon}$-Lipschitz continuous. We deduce that the eigenvalues of its differential i.e. the principal curvatures $(\kappa_i)_{1 \leq i \leq n-1}$ exists almost everywhere and are essentially bounded by $\frac{1}{\varepsilon}$. Combining this observation with (22), we get $\|H\|_{L^\infty(\partial \Omega, \mathbb{R})} \leq \frac{n-1}{\varepsilon}$. Moreover, there exists $\alpha \in ]0, \frac{n}{2}[$ depending only on $\varepsilon$ such that $\Omega$ satisfies the $\alpha(\varepsilon)$-cone property so we deduce from Corollary 2.9 and the above equality:

$$\sum_{i,j=1}^{n} \int_{\Omega} \left( \frac{\partial^2 v_i}{\partial x_i \partial x_j} \right)^2 \leq \int_{\Omega} |\Delta u|^2 + \frac{(n-1)}{\varepsilon} C(\alpha, D, n) \left( \eta \sum_{i,j=1}^{n} \int_{\Omega} \left( \frac{\partial^2 u_i}{\partial x_i \partial x_j} \right)^2 + \frac{1}{\eta} \int_{\Omega} \| \nabla u \|^2 \right).$$

Finally, we use Corollaries 2.5 and 2.6 to obtain:

$$\left\{ \begin{array}{l}
\left( \frac{1 - \eta(n-1)C(\alpha, D, n)}{\varepsilon} \right) \sum_{i,j=1}^{n} \int_{\Omega} \left( \frac{\partial^2 v_i}{\partial x_i \partial x_j} \right)^2 \leq \left( 1 + \frac{4D^2(n-1)C(\alpha, D, n)}{\varepsilon \eta} \right) \int_{\Omega} |\Delta u|^2.
\end{array} \right.$$

$$\int_{\Omega} u^2 + \int_{\Omega} |\nabla u|^2 \leq 4D^2 (1 + 4D^2) \int_{\Omega} |\Delta u|^2.$$

If we set $\eta(\varepsilon, \alpha, D, n) = \frac{1}{2} \min(1, \frac{\varepsilon}{(n-1)C(\alpha, D, n)})$, then we get the required estimation:

$$\|u\|_{L^2(\Omega, \mathbb{R})} \leq 2 \left( 1 + 2D^2 (1 + 4D^2) + \frac{4D^2(n-1)C(\alpha(\varepsilon), D, n)}{\varepsilon \eta(\varepsilon, \alpha, D, n)} \right) \|\Delta u\|_{L^2(\Omega, \mathbb{R})}.$$

To conclude, the constant appearing in the above inequality only depends on $\varepsilon$, $D$ and $n$. □
3 Continuity of some geometric functionals based on PDE: the Dirichlet boundary condition

In this section, we want to extend the existence results obtained in $O_\varepsilon(B)$ for general geometric functionals by allowing a dependence through the solutions of some partial differential equations.

First, let us prove the sequential continuity in $O_\varepsilon(B)$ of the following functional:

$$\forall \Omega \in O_\varepsilon(B), \quad J(\Omega) := \int_{\partial \Omega} j(x, n_{\partial \Omega}(x), \nabla u_{\Omega}(x)) \, dA(x),$$

where $u_{\Omega} \in H^1_0(\Omega) \cap H^2(\Omega)$ is the unique solution of the Dirichlet Laplace posed on a domain $\Omega$ with $C^{1,1}$-boundary [13, Section 2.1]:

$$\begin{cases}
\Delta u_{\Omega} = f & \text{in } \Omega \\
u_{\Omega} = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $f \in L^2(\Omega)$ and where $j : B \times \mathbb{S}^{n-1} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuous functional satisfying an inequality of the form:

$$\exists C > 0, \quad \forall (x, y, z) \in B \times \mathbb{S}^{n-1} \times \mathbb{R}^{n-1}, \quad |j(x, y, z)| \leq C \left(1 + \|z\|^2\right). \quad (33)$$

First, note that the functional $J : O_\varepsilon(B) \rightarrow \mathbb{R}$ is well defined. Indeed, we have from (33):

$$\forall \Omega \in O_\varepsilon(B), \quad |J(\Omega)| \leq C \left(A(\partial \Omega) + \|\nabla u_{\Omega}\|_{L^2(\Omega)}^2\right) < +\infty.$$ 

Then, we recall that we managed to parametrize simultaneously by local graphs the boundaries associated with a converging sequence of domains in $O_\varepsilon(B)$. More precisely, let $(\Omega^k)_{k \in \mathbb{N}} \subset O_\varepsilon(B)$ be a sequence converging to $\Omega \in O_\varepsilon(B)$ (in various senses: Hausdorff, characteristic functions, compact sets) whose boundaries $(\partial \Omega^k)_{k \in \mathbb{N}}$ also converges to $\partial \Omega$ for the Hausdorff distance.

For any $x \in \partial \Omega$, this parametrization is made inside a cylinder $C_{r, \varepsilon}(x)$ whose base is a disk $D_{r, \varepsilon}(x)$ of radius $r > 0$ depending only on $\varepsilon$, through some $C^{1,1}$-maps $\varphi^k_\varepsilon : D_{r, \varepsilon}(x) \rightarrow [0, 1]$. We consider the uniform partition of unity defined in Proposition 2.7 with $h(\varepsilon) = \min(r, \varepsilon) > 0$.

Hence, there exists $K \geq 1$ distinct points $(x_k)_{1 \leq k \leq K}$ on $\partial \Omega$ such that $\overline{\Omega}_k(\partial \Omega) \subseteq \cup_{k=1}^K C_{r, \varepsilon}(x_k)$ and there exists $K$ associated maps $\xi_k \in C^\infty_c(C_{r, \varepsilon}(x_k), [0, 1])$ such that $\sum_{k=1}^K \xi_k = 1$ on $\overline{\Omega}$.

Considering the common parametrizations associated with $(x_k)_{1 \leq k \leq K}$, there exists $K$ integers $(I_k)_{1 \leq k \leq K}$ such that for any $i \geq I_k$, there exists $C^{1,1}$-maps $\varphi^k_\varepsilon : D_{r, \varepsilon}(x_k) \rightarrow [0, 1]$ such that:

$$\partial \Omega_i \cap C_{r, \varepsilon}(x_k) = \{(x', \varphi_k^\varepsilon(x')) \mid x' \in D_{r, \varepsilon}(x_k)\}.$$ 

Moreover, $\varphi_k^\varepsilon$ converges in $C^1(D_{r, \varepsilon}(x_k)) \cap W^{2, \infty}(D_{r, \varepsilon}(x_k))$ to the map $\varphi : D_{r, \varepsilon}(x_k) \rightarrow [0, 1]$ associated locally with the piece of boundary $\partial \Omega_i \cap C_{r, \varepsilon}(x_k)$. Furthermore, there exists $I_0 \in \mathbb{N}$ such that for any integer $i > I_0$, we have $\partial \Omega_i \in \mathcal{V}_{\varepsilon}(\partial \Omega)$.

We set $I = \max_{0 \leq k \leq K} I_k$ and consider any integer $i \geq I$. We can now write the functional in terms of local graphs associated with the common partition of unity we built. We get that the functional $J(\Omega)$ can be written into the form:

$$\sum_{k=1}^K \int_{D_k} \xi_k \left(x' \varphi^k_\varepsilon(x')\right) \left[ \left(\frac{x'}{\varphi^k_\varepsilon(x')}\right) \cdot \left(\frac{x'}{\varphi^k_\varepsilon(x')}\right) \cdot \nabla u_{\Omega_i} \left(\frac{x'}{\varphi^k_\varepsilon(x')}\right) \right] \sqrt{1 + \|\nabla \varphi^k_\varepsilon(x')\|^2} \, dx',$$

where we set $D_k := D_{r, \varepsilon}(x_k)$. To let $i \rightarrow +\infty$, we want to apply Lebesgue Dominance Convergence Theon on each integral so we need the almost-everywhere convergence and a uniform bound of each integrand. Finally, due to the hypothesis (33) made on the $j$, note that this is case if the following proposition holds, which is the main task of this section.

**Proposition 3.1.** The map $x' \in D_k \mapsto \nabla u_{\Omega_i}(x', \varphi^k_\varepsilon(x'))$ strongly converges in $L^2(D_k)$ to the map $x' \in D_k \mapsto \nabla u_{\Omega}(x', \varphi^k(x'))$, where we set $D_k := D_{r, \varepsilon}(x_k)$.

First, we show the sequence of maps is uniformly bounded in $L^2(D_k)$. Then, we show that the weak limit is the right one. Finally, we prove that the strong convergence holds. Note that this proposition can be used with similar arguments to extend the continuity result of the second part to functional depending on $\nabla u_{\Omega}$. 


3.1 A uniform $L^2$-bound for the sequence

**Proposition 3.2.** Let $\Omega$ be any non-empty bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary. Then, we have for any $u \in L^1(\partial \Omega, \mathbb{R})$:

$$\frac{1}{\sqrt{1+L^2}} \int_{\partial \Omega} |u(x)|dA(x) \leq \sum_{k=1}^{K} \int_{\Pi_k(\partial \Omega \cap C_k)} |u(x', \varphi_k(x'))|dx' \leq n2^n \alpha L^n \int_{\partial \Omega} |u(x)|dA(x),$$

where $L > 0$ is the maximum of the Lipschitz modulus of the maps $(\varphi_k)_{1 \leq k \leq K}$ associated with any points $(x_k)_{1 \leq k \leq K}$ such that $\partial \Omega \subset \bigcup_{k=1}^{K} C_k$ with $C_k$ a local cylinder centred at $x_k$.

**Proof.** Since $\Omega$ is a non-empty bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary, for any point $x_0 \in \partial \Omega$, there exists a direct orthonormal frame centred at $x_0$ such that in this local frame, there exists a $L$-Lipschitz continuous map $\varphi_{x_0} : D_{x_0}(0') \rightarrow] -a_{x_0}, a_{x_0}[\subset \mathbb{R}$ such that $\varphi_{x_0}(0') = 0$ and:

$$\begin{cases}
\partial \Omega \cap (D_{x_0}(0') \times] -a_{x_0}, a_{x_0}[, \ x' \in D_{x_0}(0')) \\
\Omega \cap (D_{x_0}(0') \times] -a_{x_0}, a_{x_0}[, \ x' \in D_{x_0}(0') \text{ and } -a_{x_0} < x_n < \varphi_{x_0}(x'(n))].
\end{cases}$$

We denote by $C_{x_0, a_{x_0}}$ the cylinder represented by $(D_{x_0}(0') \times] -a_{x_0}, a_{x_0}[, \ x' \in D_{x_0}(0')$, and more generally we have:

$$C_{x_0, a_{x_0}} = \{ x \in \mathbb{R}^n, \ |(x-x_0 \ | d_{x_0})| < a_{x_0} \text{ and } \|x-x_0 - (x-x_0 \ | d_{x_0})d_{x_0}\| < r_{x_0}\},$$

where $d_{x_0}$ refers to the last vector of the basis associated with $x_0$. Since $\partial \Omega$ is compact, we get from $\partial \Omega \subset \bigcup_{k=1}^{K} C_{x_k, a_{x_k}}(x_k)$ the existence of a finite number $K \geq 1$ of points such that the inclusion $\partial \Omega \subset \bigcup_{k=1}^{K} C_{x_k, a_{x_k}}(x_k)$ holds. Then, there exists $K$ positive smooth maps $\xi_k : \mathbb{R}^n \rightarrow \mathbb{R}$ with compact support in $C_k := C_{x_k, a_{x_k}}(x_k)$ such that $\sum_{k=1}^{K} \xi_k(x) = 1$ for any $x \in \partial \Omega$. We set $L = \max_{0 \leq k \leq K} \lambda_{x_k}$ and introduce the Lipschitz continuous parametrization:

$$X_k : \Pi_k \mapsto \partial \Omega \cap C_k, \ x' \mapsto (x', \varphi_k(x')),$$

whose inverse is the restriction of the projection $\Pi_k : x \mapsto x - (x-x_k \ | d_{x_k})d_{x_k}$ and where $D_k = \Pi_k(\partial \Omega \cap C_k)$. Let us choose $u \in L^1(\partial \Omega)$. We have:

$$\int_{\partial \Omega} |u| = \int_{\partial \Omega} \left( \sum_{k=1}^{K} \xi_k \right) |u| = \sum_{k=1}^{K} \int_{\Pi_k(\partial \Omega \cap C_k)} \xi_k |u| = \sum_{k=1}^{K} \int_{D_k} \left( \xi_k |u| \right) \circ X_k \sqrt{1 + |\nabla \varphi_k|^2} \leq \sum_{k=1}^{K} \int_{D_k} |u \circ X_k| \leq \sqrt{1 + L^2} \sum_{k=1}^{K} \int_{D_k} |u \circ X_k|.$$
Moreover, the Jacobian of $T_{kl}$ is $L^\infty$-bounded. Indeed, we have for any $x \in \partial \Omega \cap C_k \cap C_l$:
\[
J(T_{kl})(\Pi_k(x)) = |\det D\Pi_k(x)(\Pi_l \circ X_k)|
\]
\[
\leq \sum_{\sigma \in S_n} \prod_{m=1}^n |D\Pi_k(x)(\Pi_l \circ X_k)_{m\sigma(m)}| \leq \sum_{\sigma \in S_n} \prod_{m=1}^n \|D\Pi_k(x)(\Pi_l \circ X_k)^T(e_m)\|
\]
\[
\leq \sum_{\sigma \in S_n} \prod_{m=1}^n \|D\Pi_k(x)(\Pi_l \circ X_k)^T\| \leq \sum_{\sigma \in S_n} \prod_{m=1}^n \|D\Pi_k(x)(\Pi_l \circ X_k)\|
\]
\[
\leq \sum_{\sigma \in S_n} \prod_{m=1}^n \sup_{x \in \partial \Omega \cap C_k \cap C_l} \|D\Pi_k(x)(\Pi_l \circ X_k)\| \leq \sum_{\sigma \in S_n} \prod_{m=1}^n (1 + L)
\]
\[
\leq 2^n (n!)(1 + L)^n.
\]

Consequently, we can make a change of variables and we obtain:
\[
\int_{D_k} |u \circ X_k| = \sum_{1 \leq l \leq K} \int_{\partial \Omega \cap C_k \cap C_l \neq \emptyset} \int_{\Pi_k(\partial \Omega \cap C_k \cap C_l)} (\xi_l |u|) \circ X_l \circ [\Pi_l \circ X_k] J(T_{kl})|J(T_{kl})|
\]
\[
\leq 2^n (n!)(1 + L)^n \sum_{1 \leq l \leq K} \int_{\partial \Omega \cap C_k \cap C_l \neq \emptyset} \int_{\Pi_k(\partial \Omega \cap C_k \cap C_l)} (\xi_l |u|) \circ X_l J(T_{kl})
\]
\[
= 2^n (n!)(1 + L)^n \sum_{1 \leq l \leq K} \int_{\partial \Omega \cap C_k \cap C_l \neq \emptyset} (\xi_l |u|) \circ X_l
\]
\[
\leq 2^n (n!)(1 + L)^n \sum_{l=1}^K \int_{D_l} (\xi_l |u|) \circ X_l = 2^n (n!)(1 + L)^n \int_{\partial \Omega} |u|.
\]

To conclude, we get the required inequality by summing the one above from $k = 1$ to $K$. 

\begin{proposition}
Let $1 \leq k \leq K$. Considering the maps $v^k_l : x' \mapsto \partial_n(u_{\Omega_l})(x', \varphi^k(x'))$, the sequence $(v^k_l)_{l \in \mathbb{N}}$ is uniformly bounded in $L^2(D_k)$.
\end{proposition}

\textbf{Proof.} First, we apply Proposition 3.2 on $\partial \Omega_l$ to get:
\[
\int_{D_k} (v^k_l)^2 \leq \sum_{k=1}^K \int_{D_k} \partial_n(u_{\Omega_l})^2 \circ X_k^l \leq n2^n n!(1 + L)^n \int_{\partial \Omega_l} |\partial_n(u_{\Omega_l})|^2.
\]

Then, $u_{\Omega_l} \in H^3_0(\partial \Omega)$ and taking the partial derivatives in the relation $u_{\Omega_l} \circ X_k^l = 0$, we obtain that $\nabla u_{\Omega_l} = \partial_n(u_{\Omega_l})|_{\partial \Omega_l}$ on $\partial \Omega_l$. Combined with Corollary 2.9, we obtain:
\[
\int_{D_k} (v^k_l)^2 \leq n2^n n!(1 + L)^n \int_{\partial \Omega_l} \|\nabla u_{\Omega_l}\|^2 \leq C(\varepsilon, n, D) \left(\|u_{\Omega_l}\|^2_{H^2(\Omega_l)} + \|f\|^2_{L^2(\Omega_l)}\right).
\]

Finally, we can use the uniform bound proved in Theorem 2.1 to deduce the existence of a positive constant, which depends on $D, \varepsilon, n$, and $\xi$ such that:
\[
\int_{D_k} (v^k_l)^2 \leq C(\varepsilon, n, D) \int_B f^2.
\]

\section{The weak convergence in $L^2$-norm of the sequence}

\begin{proposition}
The sequence of maps $v^k_l : x' \mapsto \partial_n(u_{\Omega_l})(x', \varphi^k(x'))$ converges weakly in $L^2(D_k)$ to the map $v_k : x' \mapsto \partial_n(u_{\Omega})(x', \varphi(x'))$, where $u_\Omega \in H^3_0(\Omega) \cap H^2(\Omega)$ is the unique solution of the Dirichlet Laplacian on $\Omega \in \mathcal{C}(B)$ and where $\Omega_\varepsilon$ converges to $\Omega$ in the various sense of $\mathcal{C}(B)$. 
\end{proposition}
\textbf{Proof.} Proposition 3.3 ensures we can bound uniformly the $L^2$-norm of $(v^n_k)_{n \in \mathbb{N}}$. Consequently, there exists $v^*_k \in L^2(D_k)$ such that, up to a subsequence, $(v^n_k)_{n \in \mathbb{N}}$ weakly converges to $v^*_k$ in $L^2(D_k)$. It remains to prove that for any weakly converging subsequence, the limit is unique i.e. $v^*_k = v_k$ in order to get the weak convergence of the full sequence to $v_k$. Let $w : B \to \mathbb{R}$ be any Lipschitz continuous map. From Rademacher’s Theorem [11, Section 4.2.3], $w$ is differentiable almost everywhere and $w \in W^{1,\infty}(B)$ [11, Section 4.2.3]. Then, we have:

\[
\int_{\partial \Omega} \partial_n (u^n_\Omega) w = \int_{\Omega} \text{div} (w \nabla u^n_\Omega) = \int_{\Omega} \langle \nabla w, \nabla u^n_\Omega \rangle + \int_{\Omega} w \Delta u^n_\Omega \\
= \int_B \langle \nabla w, 1 \nabla u^n_\Omega \rangle + \int_B 1 \nabla w f \\
= \int_{\partial \Omega} \partial_n (u^n_\Omega) w + \int_B \langle \nabla w, 1 \nabla u^n_\Omega - 1 \nabla u^n_\Omega \rangle + \int_B (1 \nabla - 1) \nabla w f \\
= \int_{\partial \Omega} \partial_n (u^n_\Omega) w + \int_B \langle \nabla w, 1 \nabla u^n_\Omega - 1 \nabla u^n_\Omega \rangle + \int_B (1 \nabla - 1) \nabla w f \\
\]

The second term is bounded by $\|\nabla w\|_{L^\infty(B)} \sqrt{V(B)} \|1 \nabla u^n_\Omega - 1 \nabla u^n_\Omega\|_{L^2(B)}$ while the third one is bounded by $\|w\|_{L^\infty(B)} \|f\|_{L^2(B)} \sqrt{\|1 \nabla - 1\|_{L^2(B)}}$. Using the convergence of $(\Omega_n)_{n \in \mathbb{N}}$ to $\Omega$ in the sense of characteristic functions and [14, Theorem 2.3.13], we can let $i \to +\infty$ in order to obtain:

\[
\forall w \in W^{1,\infty}(B, \mathbb{R}), \quad \lim_{i \to +\infty} \int_{\partial \Omega} \partial_n (u^n_\Omega) w = \int_{\partial \Omega} \partial_n (u_\Omega) w. \tag{34} \]

We now consider $w : B \to \mathbb{R}$ a Lipschitz continuous map with compact support in $C_k$. Then, we have:

\[
\int_{\partial \Omega} \partial_n (u^n_\Omega) w = \int_{D_k} \partial_n (u^n_\Omega) w \circ X_k \sqrt{1 + \|\nabla \varphi_k\|^2} = \int_{D_k} v^n_k (w \circ X_k) \sqrt{1 + \|\nabla \varphi_k\|^2},
\]

and we decompose the above expression into the following terms:

\[
\int_{\partial \Omega} \partial_n (u^n_\Omega) w = \int_{D_k} v^n_k (w \circ X_k) \sqrt{1 + \|\nabla \varphi_k\|^2} + \int_{D_k} (v^n_k - v^n_k) (w \circ X_k) \sqrt{1 + \|\nabla \varphi_k\|^2} + \int_{D_k} v^n_k w \circ X_k \frac{\|\nabla \varphi_k\|^2}{\sqrt{(1 + \|\nabla \varphi_k\|^2)(1 + \|\nabla \varphi_k\|^2)}}
\]

From the $L^2(D_k)$-weak convergence of $(v^n_k)_{n \in \mathbb{N}}$ to $v^n_k$, the second term tends to zero as $i \to +\infty$ since $(w \circ X_k) \sqrt{1 + \|\nabla \varphi_k\|^2}$ is a Lipschitz continuous map and $A(D_k) = \pi \sigma_{D_k}^2$ thus it is an element of $L^2(D_k)$. The third term is bounded by $\|\nabla \varphi_k\|_{L^2(D_k)} \sqrt{1 + \|\nabla \varphi_k\|^2} \|w\|_{L^\infty(B)} \|\varphi_k - \varphi_k\|_{L^\infty(D_k)} \sqrt{A(D_k)}$. We proved that $\|v^n_k\|_{L^2(D_k)} \leq C \|f\|_{L^2(B)}$ so the third term tends to zero as $i \to +\infty$. Concerning the fourth one, it is bounded by $\|v^n_k\|_{L^2(D_k)} \|w\|_{L^\infty(B)} 2L \|\nabla \varphi_k - \nabla \varphi_k\|_{L^\infty(D_k)} \sqrt{A(D_k)}$ so the fourth term converges to zero. Hence, we can let $i \to +\infty$ in the previous equality and we obtain:

\[
\lim_{i \to +\infty} \int_{\partial \Omega} \partial_n (u^n_\Omega) w = \int_{D_k} v^n_k (w \circ X_k) \sqrt{1 + \|\nabla \varphi_k\|^2}.
\]

But from (34), we also get:

\[
\lim_{i \to +\infty} \int_{\partial \Omega} \partial_n (u^n_\Omega) w = \int_{\partial \Omega} \partial_n (u_\Omega) w = \int_{D_k} v_k (w \circ X_k) \sqrt{1 + \|\nabla \varphi_k\|^2}.
\]

Consequently, we proved that for any Lipschitz continuous map $w : B \to \mathbb{R}$ with compact support in $C_k$, we have:

\[
\int_{D_k} (v_k - v^n_k) (w \circ X_k) \sqrt{1 + \|\nabla \varphi_k\|^2} = 0.
\]
Let \( \tilde{w} \in C_c^\infty(D_k, \mathbb{R}) \) and we show that we can replace \( w \circ X_k \) by \( \tilde{w} \) in the above expression. For this purpose, we introduce the map:

\[
\begin{align*}
    w : & \quad B \rightarrow \mathbb{R} \\
    & (x', x_n) \quad \mapsto \begin{cases} \\
        \frac{2a_{x_n} - x_n^2}{a_{x_k}^2 - \varphi_k(x')^2} & \text{if } (x', x_n) \in C_k := D_k \times ] - a_{x_k}, a_{x_k}] \\
        0 & \text{otherwise}
    \end{cases}
\end{align*}
\]

One can check that \( w \) is Lipschitz continuous with compact support in \( D_k \). Hence, we can insert \( w \) in the previous equality to get:

\[
\forall \tilde{w} \in C_c^\infty(D_k, \mathbb{R}), \quad \int_{D_k} (v_k - v_k^i) \tilde{w} \sqrt{1 + \| \nabla \varphi_k \|^2} = 0.
\]

What has been done for \( v_k^i \) is also true for \( v_k \in L^2(D_k) \). Moreover, we know that \( v_k^i \in L^2(D_k) \) and \( \sqrt{1 + \| \nabla \varphi_k \|^2} \) is continuous. Hence, we deduce that \( v_k = v_k^i \) for almost every \( x' \in D_k \) as required.

To conclude, we proved that any weakly converging subsequence of \( (v_k^i)_{i \in \mathbb{N}} \) converges to \( v_k \) so the results holds for the whole sequence.

\[\square\]

### 3.3 The strong convergence in \( L^2 \)-norm of the sequence

First, we prove the result locally and then we establish the global strong convergence.

**Proposition 3.5.** Let \( k \in \{1, \ldots, n\} \). For any Lipschitz continuous map \( w : B \rightarrow \mathbb{R} \) with compact support in \( C(x_k) \), we have, up to a subsequence:

\[
\lim_{i \rightarrow +\infty} \int_{D_k} (w \circ X_k^i)(v_k^i - v_k)^2 = 0.
\]

**Proof.** Let \( w \in W^{1, \infty}(B, \mathbb{R}) \) with compact support in \( C(x_k) \). We have:

\[
\int_{D_k} (w \circ X_k^i)(v_k^i - v_k)^2 = \int_{D_k} (w \circ X_k^i)(v_k^i)^2 - 2 \int_{D_k} v_k(w \circ X_k^i)v_k^i + \int_{D_k} (w \circ X_k^i)(v_k)^2. \tag{35}
\]

First, considering the Lipschitz modulus \( L > 0 \) of \( w \), the sequence \( (w \circ X_k^i)_{i \in \mathbb{N}} \) uniformly converges to \( w \circ X_k \). Indeed, we have:

\[
\| (w \circ X_k^i) - (w \circ X_k) \|_{L^\infty(B, \mathbb{R})} \leq L \| X_k^i - X_k \|_{L^\infty(D_k)} = L \| \varphi_k^i - \varphi_k \|_{L^\infty(D_k)} \xrightarrow{i \rightarrow +\infty} 0.
\]

On the one hand, we deduce:

\[
\int_{D_k} (w \circ X_k^i)(v_k)^2 = \int_{D_k} (w \circ X_k^i)(v_k^i)^2 + \int_{D_k} [ (w \circ X_k^i) - (w \circ X_k) ](v_k^i)^2,
\]

where the (absolute value of the) last term is bounded by \( \| (w \circ X_k^i) - (w \circ X_k) \|_{L^\infty(B, \mathbb{R})} \| v_k^i \|_{L^2(D_k, \mathbb{R})} \) thus converges to zero as \( i \rightarrow \infty \). On the other hand, we have:

\[
\int_{D_k} v_k(w \circ X_k^i)v_k^i = \int_{D_k} (v_k^i)^2(w \circ X_k) + \int_{D_k} v_k(w \circ X_k)(v_k^i - v_k) + \int_{D_k} v_kv_k^i [(w \circ X_k^i) - (w \circ X_k)].
\]

The last term is bounded by \( \| (w \circ X_k^i) - (w \circ X_k) \|_{L^\infty(B, \mathbb{R})} \| v_k \|_{L^2(D_k, \mathbb{R})} \| v_k^i \|_{L^2(D_k, \mathbb{R})} \) and from Proposition 3.3, it is converging to zero as \( i \rightarrow +\infty \). Moreover, the same holds for the second term according to Proposition 3.4. Therefore, to conclude the proof, it remains to show that the first term in the right-hand side of (35) \( \int_{D_k} (w \circ X_k^i)(v_k^i)^2 \) converges to \( \int_{D_k} (w \circ X_k)(v_k)^2 \) as \( i \rightarrow +\infty \).

Let us prove this last assertion. First, we get from \( \nabla u_{\Omega_i} = \partial_n (u_{\Omega_i}) \mathbf{n}_{\partial \Omega_i} \) and Stokes' Theorem:

\[
\int_{\partial \Omega_i} [\partial_n (u_{\Omega_i})]^2 w(\mathbf{d}_{x_k} | \mathbf{n}_{\partial \Omega_i}) dA = \int_{\partial \Omega_i} \| \nabla u_{\Omega_i} \|^2 w(\mathbf{d}_{x_k} | \mathbf{n}_{\partial \Omega_i}) dA = \int_{\Omega_i} \text{div} (\| \nabla u_{\Omega_i} \|^2 w \mathbf{d}_{x_k})
\]

\[
= \int_{\Omega_i} 2w ((\mathbf{d}_{x_k} | \nabla) (\nabla u_{\Omega_i}) | \nabla u_{\Omega_i}) + \int_{\Omega_i} \| \nabla u_{\Omega_i} \|^2 (\nabla w | \mathbf{d}_{x_k})
\]

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We denote by $A_i$ and $B_i$ respectively the first and second term in the right-hand side of the last equality above. We have:

$$
\left| A_i - \int_{\Omega} \|\nabla u_{\Omega}\|^2 \langle \nabla w, \nabla d_{\Omega} \rangle \right| \leq \|\nabla w\|_{L^\infty(B,\mathbb{R}^n)} \|1_{\Omega} \nabla u_{\Omega} - 1_{\Omega} \nabla u_{\Omega}\|^2_{L^2(B,\mathbb{R}^n)} + 2 \|\nabla w\|_{L^\infty(B,\mathbb{R}^n)} \|1_{\Omega} \nabla u_{\Omega} - 1_{\Omega} \nabla u_{\Omega}\|_{L^1(\Omega,\mathbb{R}^n)} \|\nabla u_{\Omega}\|_{L^2(\Omega,\mathbb{R}^n)}
$$

Since the $\varepsilon$-ball condition implies the $\alpha(\varepsilon)$-cone property, the sequence $(1_{\Omega} \nabla u_{\Omega})_{\varepsilon \in \Omega}$ converges strongly in $L^2(B,\mathbb{R}^n)$ to the map $1_{\Omega} \nabla u_{\Omega}$ [14, Theorem 2.3.13 and Proposition 3.2.4], from which we deduce that $(A_i)_{\varepsilon \in \Omega}$ converges to $\int_{\Omega} \|\nabla u_{\Omega}\|^2 \langle \nabla w, \nabla d_{\Omega} \rangle$ as $i \to +\infty$. Concerning $B_i$, since $\Omega_i \in C(\varepsilon)$ thus satisfies the $(\alpha(\varepsilon))$-cone property, we get from [14, Proposition 3.7.2], a result due to Chenais [5], that $\nabla u_{\Omega_i} \in H^1(\Omega_i,\mathbb{R}^n)$ has a uniform extension $v^i = (v^i_1, \ldots, v^i_n) \in H^1(B,\mathbb{R}^n)$ i.e. $v^i|_{\Omega_i} = \nabla u_{\Omega_i}$ and $\|v^i\|_{H^1(\Omega_i,\mathbb{R}^n)} \leq C(n, D, \varepsilon) \|\nabla u_{\Omega_i}\|_{H^1(\Omega_i,\mathbb{R}^n)}$, where $C(n, D, \varepsilon) > 0$ is a constant depending only on $D, n$ and $\varepsilon$. Applying Theorem 2.1, we get that $(v^i)_{\varepsilon \in \Omega}$ is uniformly bounded in $H^1(B,\mathbb{R}^n)$. Hence, up to a subsequence, it is converging to $v \in H^1(B,\mathbb{R}^n)$, weakly in $H^1(B,\mathbb{R}^n)$ and strongly in $L^2(B,\mathbb{R}^n)$. We now show that $v$ is an extension of $\Omega$. Let $\varphi \in C^\infty_c(B,\mathbb{R})$ and $l \in \{1, \ldots, n\}$. We have successively:

$$
\int_{\Omega_i} v^i_l \varphi = \int_{\Omega} \frac{\partial u_{\Omega}}{\partial x_l} \varphi = \int_{\partial \Omega} \frac{u_{\Omega}}{|\partial \Omega|} \varphi - \int_{\Omega} \frac{\partial \varphi}{\partial x_l} = - \int_{\Omega} \frac{\partial \varphi}{\partial x_l}.
$$

But we also have:

$$
\int_{\Omega_i} v^i_l \varphi = \int_{\Omega} v_l \varphi + \int_{B} (1_{\Omega_i} - 1_{\Omega}) v_l \varphi \longrightarrow_{i \to +\infty} \int_{\Omega} v_l \varphi.
$$

Consequently, the uniqueness of the limit gives $v_i = \frac{\partial u_{\Omega}}{\partial x_l}$ in the sense of distributions on $\Omega$ hence almost everywhere on $\Omega$ and thus $v_i$ is an extension of $\partial \Omega$. In particular, we have the following property. Let $\varphi \in C^\infty_c(\Omega,\mathbb{R})$. From the convergence in the Hausdorff sense, for $i$ large enough, we have $\varphi \in C^\infty_c(\Omega_i,\mathbb{R})$. We deduce that:

$$
\int_{\Omega_i} \frac{\partial (v^i_l)}{\partial x_m} \varphi = - \int_{\Omega_i} v^i_l \frac{\partial \varphi}{\partial x_m} = \int_{B} 1_{\Omega_i} \frac{\partial u_{\Omega}}{\partial x_l} \frac{\partial \varphi}{\partial x_m} \longrightarrow_{i \to +\infty} \int_{B} 1_{\Omega} \frac{\partial u_{\Omega}}{\partial x_l} \frac{\partial \varphi}{\partial x_m} = \int_{\Omega} \frac{\partial^2 u_{\Omega}}{\partial x_l \partial x_m} \varphi.
$$

but we have from the convergence in the sense of characteristic functions and the weak convergence of $v^i$ in $H^1(B,\mathbb{R}^n)$ that the limit is also equal to $\int_{B} 1_{\Omega} \frac{\partial (v^i_l)}{\partial x_m} \varphi$. Therefore, we obtain that $\frac{\partial (v^i_l)}{\partial x_m} = \frac{\partial^2 u_{\Omega}}{\partial x_l \partial x_m}$ in the sense of distribution in $\Omega$ thus almost everywhere on $\Omega$. Finally, getting back to the convergence of $B_i$, we are going to use this property. We have:

$$
\left| \int_{\Omega} \left( \frac{\partial (u_{\Omega})}{\partial x_m} \frac{\partial u_{\Omega}}{\partial x_l} - \frac{\partial (u_{\Omega})}{\partial x_m} \frac{\partial^2 (u_{\Omega})}{\partial x_l \partial x_m} \right) \cdot \langle d_{\Omega}, \varphi \rangle \right| \leq \int_{B} 1_{\Omega} \frac{\partial u_{\Omega}}{\partial x_l} \frac{\partial \varphi}{\partial x_m} \left( \frac{\partial v^i_l}{\partial x_m} - \frac{\partial v_l}{\partial x_m} \right)
$$

$$
+ \|w\|_{L^\infty(B,\mathbb{R}^n)} \|\frac{\partial v^i_l}{\partial x_m}\|_{L^2(B,\mathbb{R}^n)} \|1_{\Omega}, \frac{\partial u_{\Omega}}{\partial x_m} - 1_{\Omega} \frac{\partial u_{\Omega}}{\partial x_m}\|_{L^2(B,\mathbb{R}^n)}
$$

Since the right-hand side of the above inequality converges to zero as $i \to +\infty$, so does the left-hand side. Summing from $m, l = 1$ to $n$ gives:

$$
B_i := \int_{\Omega} 2 \langle d_{\Omega}, \nabla u_{\Omega} \rangle \rightarrow_{i \to +\infty} \int_{\Omega} 2 \langle d_{\Omega}, \nabla u_{\Omega} \rangle
$$

Combining the convergence result of $A_i$ and $B_i$, we deduce that:

$$
\int_{\partial \Omega} [\partial_n (u_{\Omega})]^2 w(d_{\Omega}) \cdot n_{\partial \Omega}) \, dA \rightarrow_{i \to +\infty} \int_{\partial \Omega} [\partial_n (u_{\Omega})]^2 w(d_{\Omega}) \cdot n_{\partial \Omega}) \, dA
$$
Since \( w \) has compact support in \( C(x_k) \), it remains to look at the local expression of the integrals to obtain the required result:

\[
\int_{\partial \Omega} |\partial_n(u_{\Omega})|^2 w(d_{x_k} | n_{\partial \Omega}) dA = \int_{D_k} (w \circ X_k^i)(v_k^i)^2 \to_{i \to +\infty} \int_{D_k} (w \circ X_k)(v_k)^2.
\]

To conclude, we have proved that the right-hand side of (35) converges to zero as \( i \to +\infty \).

**Proof of Proposition 3.1.** Considering Proposition 3.5, it remains to delete the local map \( w \). This is done in a similar way than in the proof of Proposition 3.2 and the same notation are used. We have:

\[
\int_{D_k} (v_k^i - v_k)^2 = \int_{D_k} \left( \sum_{l=1}^K (\xi_l \circ X_k^i) \right) (v_k^i - v_k)^2 = \sum_{l=1}^L \int_{\Pi_k(\partial \Omega \cap C_l \cap \mathbb{R}^n \setminus C_l)} (\xi_l \circ X_k^i) (v_k^i - v_k)^2.
\]

Then, observe that \( \xi_l(x) = 0 \) for any \( x \notin C_l \) so \( \xi_l \circ X_k(x') = 0 \) for any \( x' \in \Pi_k(\partial \Omega \cap C_l \cap \mathbb{R}^n \setminus C_l) \). Hence, we deduce that:

\[
\int_{D_k} (v_k^i - v_k)^2 = \sum_{1 \leq l \leq K \atop \partial \Omega \cap C_l \cap \mathbb{R}^n \setminus C_l \neq \emptyset} \int_{\Pi_k(\partial \Omega \cap C_l \cap \mathbb{R}^n \setminus C_l)} (\xi_l \circ X_k^i) \left[ (\partial_n(u_{\Omega}) - \partial_n(u_{\Omega}))^2 \circ X_k^i \right].
\]

If \( \partial \Omega \cap C_l \cap \mathbb{R}^n \setminus C_l \neq \emptyset \), we introduce the map \( T_k^i := \Pi_l \circ X_k^i : \Pi_k(\partial \Omega \cap C_l \cap \mathbb{R}^n \setminus C_l) \to \Pi_l(\partial \Omega \cap C_l \cap \mathbb{R}^n \setminus C_l) \) which is a uniform bi-Lipschitz change of coordinates. We make a change of variable and we obtain:

\[
\int_{D_k} (v_k^i - v_k)^2 = \sum_{1 \leq l \leq K \atop \partial \Omega \cap C_l \cap \mathbb{R}^n \setminus C_l \neq \emptyset} \int_{\Pi_l(\partial \Omega \cap C_l \cap \mathbb{R}^n \setminus C_l)} (\xi_l \circ X_k^i) \left[ (\partial_n(u_{\Omega}) - \partial_n(u_{\Omega}))^2 \circ X_k^i \right] \left| \det(D_l T_k^i) \right|.
\]

Then, \( \xi_l \) has compact support in \( D_l = \Pi_l(\partial \Omega \cap C_l) \). Applying Proposition 3.5, up to a subsequence, the quantity \( (\xi_l \circ X_k^i) ((\partial_n(u_{\Omega}) - \partial_n(u_{\Omega}))^2 \circ X_k^i) \) strongly converges to zero in \( L^1(\Omega) \). Hence, up to a subsequence, this quantity is uniformly bounded by \( L^1(D_k) \) function and converges almost everywhere to zero. Similarly, in the proof of Proposition 3.2, we proved that the Jacobian of \( T_k^i \) is uniformly bounded and from the continuity of \( \Pi_l \) and the determinant, since \( X_k^i \) converges uniformly to \( X_l \), we get that the Jacobian of \( \Pi_l \circ T_k^i \) converges almost everywhere. Applying Lebesgue Dominated Convergence Theorem, we deduce that we can remove the \( i \) in each term of the sum of the above relation. We deduce that, up to a subsequence, \( (v_k^i)_{i \in \mathbb{N}} \) strongly converge to \( v_k \) in \( L^2(D_k) \). Since the limit is unique, we deduce that the convergence of the whole sequence, which concludes the proof.

**4 Continuity of some geometric functionals based on PDE: the Neumann/Robin boundary condition**

In this section, we assume that there exists a unique solution \( u_{\Omega} \in H^2(\Omega) \) associated with the \( C^{1,1} \)-domain \( \Omega \) and satisfying:

\[
\begin{cases}
-\Delta u_{\Omega} + \lambda u_{\Omega} = f & \text{in } \Omega \\
-\partial_n(u) = \beta(u) & \text{on } \partial \Omega,
\end{cases}
\tag{36}
\]

where \( \lambda > 0, f \in L^2(\Omega), \) and \( \beta : \mathbb{R} \to \mathbb{R} \) is a non-decreasing Lipschitz continuous map satisfying \( \beta(0) = 0 \). Note that if \( \beta \) is identically zero, then the above problem is the Laplacian with Neumann boundary condition, and if \( \beta \) is linear, then it is the Robin boundary condition. At least for these two cases, we know there exists a unique solution \( u_{\Omega} \in H^2(\Omega) \) [13, Theorems 2.4.2.6 and 2.4.2.7]. We now establish an *a priori* \( H^2 \)-estimate for this problem, where the constant is controlled. We essentially follow [13, Theorem 3.1.2.3] which treat the case of convex domain with \( C^2 \)-boundary. Our only contribution is to treat the \( C^{1,1} \)-case with the \( \varepsilon \)-ball condition.
4.1 A uniform \textit{a priori} $H^2$-estimate for the Neuman/Robin Laplacian

\textbf{Theorem 4.1.} Let $\varepsilon > 0$, $n \geq 2$, and $B$ be any non-empty open bounded subset of $\mathbb{R}^n$ containing the origin. We consider the class $\mathcal{O}_\varepsilon(B)$ formed by all the non-empty open subsets of $B$ satisfying the $\varepsilon$-ball condition. We assume that the diameter of $D$ is large enough to ensure $\mathcal{O}_\varepsilon(B) \neq \emptyset$.

Then, there exists a constant $C > 0$, depending only on $\varepsilon$, $D$, and $n$, such that for any $\Omega \in \mathcal{O}_\varepsilon(B)$, we have:

$$\forall u \in \{ v \in H^2(\Omega, \mathbb{R}), -\partial_n(u) = \beta(u) \text{ on } \partial \Omega \}, \quad \| u \|_{H^2(\Omega, \mathbb{R})} \leq C (\lambda, \varepsilon, n, D) - \Delta u + \lambda u \|_{L^2(\Omega, \mathbb{R})},$$

where $\partial_n(u) := \langle \nabla u \mid n_{\partial \Omega} \rangle$, $\lambda > 0$, and $\beta : \mathbb{R} \to \mathbb{R}$ is a non-decreasing Lipschitz continuous map satisfying $\beta(0) = 0$.

\textbf{Proof.} We apply (26) with $v = \nabla u$. We obtain:

$$\sum_{i,j=1}^n \int_\Omega \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 - \int_\Omega |\Delta u|^2 \leq \int_\Omega [ -2\beta'(u)\|\nabla \partial u\|^2 + II (\nabla \partial u, \nabla \partial u) - H \partial_n(u)^2 ] dA$$

Note that in (26), we can rewrite the bracket as an integral because $v_n = \partial_n(u) = -\beta(u)$. Indeed, since $u \in H^2(\Omega)$, we have $u \in H^1(\partial \Omega)$ and since $\beta$ is Lipschitz continuous, we get $\beta(u) \in L^1(\partial \Omega)$ so $\nabla \partial(\nu_n) = \beta'(u)\nabla \partial(\nu_n) \in L^2(\partial \Omega)$. Then, observe that the first term in the expression above is non-negative since $\beta$ is non-decreasing. We deduce that:

$$\sum_{i,j=1}^n \int_\Omega \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 \leq \int_\Omega |\Delta u|^2 + \int_\partial \Omega \left[ II (\nabla \partial u, \nabla \partial u) - H \partial_n(u)^2 \right] dA$$

(37)

Next, we can rewrite the last term of the above expression. Considering the orthonormal basis of $\mathbb{R}^n$ denoted $(e_1, e_{n-1}, n_{\partial \Omega})$ associated with the principal curvature $(\kappa_i)_{1 \leq i \leq n}$, we have:

$$II (\nabla \partial u, \nabla \partial u) = -Dn_{\partial \Omega} (\nabla \partial u) | \nabla \partial u)$$

$$= \left\langle -Dn_{\partial \Omega} \left( \sum_{i=1}^{n-1} (\nabla \partial u \mid e_i) e_i \right) \mid \nabla \partial u \right\rangle$$

$$= -\sum_{i=1}^{n-1} \langle \nabla \partial u \mid e_i \rangle \left( Dn_{\partial \Omega} (e_i) \mid \nabla \partial u \right)$$

$$= -\sum_{i=1}^{n-1} \kappa_i |\nabla \partial(u) \mid e_i|^2$$

Recalling that $H = \sum_{i=1}^{n-1} \kappa_i$, and inserting the above relation in the right member of (37), it comes:

$$\sum_{i,j=1}^n \int_\Omega \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 \leq \int_\Omega |\Delta u|^2 - \sum_{i=1}^{n-1} \kappa_i \|\nabla u\|^2 dA$$

$$\leq 2 \int_\Omega | -\Delta u + \lambda u |^2 + 2\lambda \int_\Omega u^2 + \frac{n-1}{\varepsilon} \int_\partial \Omega \|\nabla u\|^2 dA.$$  

In the last inequality, we use the fact that $\Omega \in \mathcal{O}_\varepsilon(B)$ hence its Gauss map $n_{\partial \Omega} : \partial \Omega \to S^2$ is $\frac{1}{\varepsilon}$-Lipschitz continuous (ii) below Figure 1 so it is differentiable almost everywhere and its principal curvature are essentially bounded on $\partial \Omega$ by $\frac{1}{\varepsilon}$. Finally, we get from Point (i) below Figure 1 that $\Omega$ satisfies the $\alpha(\varepsilon)$-cone condition so we can apply Corollary 2.9 to deduce:

$$\left(1 - \frac{\eta(n-1)C(\alpha, D, n)}{\varepsilon}\right) \sum_{i,j=1}^n \int_\Omega \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 \leq 2 \int_\Omega | -\Delta u + \lambda u |^2 + 2\lambda \int_\Omega u^2 + \frac{(n-1)C(\alpha, D, n)}{\varepsilon \eta} \int_\Omega \|\nabla u\|^2.$$
It remains to obtain an \textit{a priori} estimate for the $H^1$-norm. We have:

$$\int_{\Omega} (-\Delta u + \lambda u) u = -\int_{\partial \Omega} u \partial_n(u) dA + \int_{\Omega} \|\nabla u\|^2 + \lambda \int_{\Omega} u^2 = \int_{\partial \Omega} u \beta(u) dA + \int_{\Omega} \|\nabla u\|^2 + \lambda \int_{\Omega} u^2.$$  

Since $\beta(0) = 0$ and $\beta$ is non-decreasing, we deduce that $\beta(u) u \geq 0$. Combining this observation with the Cauchy-Schwarz inequality, we get:

$$\lambda \int_{\Omega} u^2 + \int_{\Omega} \|\nabla u\|^2 \leq \int_{\Omega} (-\Delta u + \lambda u) u \leq \| -\Delta u + \lambda u\|_{L^2(\Omega, \mathbb{R})} \|u\|_{L^2(\Omega, \mathbb{R})}.$$  

We deduce that $\|u\|_{L^2(\Omega, \mathbb{R})} \leq \frac{1}{\lambda} \| -\Delta u + \lambda u\|_{L^2(\Omega, \mathbb{R})}$ and $\|\nabla u\|_{L^2(\Omega, \mathbb{R})} \leq \frac{1}{\lambda} \| -\Delta u + \lambda u\|_{L^2(\Omega, \mathbb{R})}$.  

which yields to:

$$\left\{ \begin{aligned} &\left( 1 - \frac{(n-1)C(\alpha, D, n)}{\varepsilon} \right) \sum_{i,j=1}^n \int_{\Omega} \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 \leq \left( 4 + \frac{(n-1)C(\alpha, D, n)}{\varepsilon \eta \lambda} \right) \int_{\Omega} | -\Delta u + \lambda u|^2 \\
&\int_{\Omega} u^2 + \int_{\Omega} \|\nabla u\|^2 \leq \left( \frac{1}{\lambda^2} + \frac{1}{\lambda} \right) \int_{\Omega} | -\Delta u + \lambda u|^2 \\
\end{aligned} \right.$$  

Finally, we set $\eta = \frac{1}{2} \min(1, \frac{\varepsilon}{(n-1)C(\alpha, D, n)})$, which depends only on $\varepsilon$, $D$, and $n$, in order to obtain the required result:

$$\|u\|^2_{H^2(\Omega, \mathbb{R})} \leq \left( 8 + \frac{1}{\lambda^2} + \frac{1}{\lambda} + \frac{2(n-1)C(\alpha, D, n)}{\varepsilon \eta \lambda} \right) \| -\Delta u + \lambda u\|^2_{L^2(\Omega, \mathbb{R})}.$$  

To conclude, observe that the constant above only depends on $\varepsilon$, $D$, $\lambda$, and $n$. $\square$

\subsection*{4.2 Extending the continuity result to the Neuman/Robin case}

We only sketch the procedure to obtain similar results in this case. Indeed, note that all the results and arguments used in Chapter 3 are only based on the $H^2$-estimation, which also holds for the solution $u_\Omega$ of the Neumann/Robin boundary condition. Therefore, we can proceed exactly in the same way than we did for the Dirichlet boundary condition. Considering a minimizing sequence of domains $(\Omega_i)_{i \in \mathbb{N}}$, this uniform bound ensures the the local maps $x' \mapsto u_\Omega(x', \varphi_i(x'))$ is uniformly bounded in $H^1$. Considering a weakly converging subsequence, we can prove it is converging in $H^1$ to the map $x' \mapsto u_\Omega(x', \varphi(x'))$. We obtain

\textbf{Proposition 4.2.} The map $x' \in D_k \mapsto \nabla u_\Omega(x', \varphi_i(x'))$ strongly converges in $H^1(D_k)$ to the map $x' \in D_k \mapsto \nabla u_\Omega(x', \varphi(x'))$, where we set $D_k := D(x_k)$.

Therefore, all the continuity results of the previous part can be extended to the Robin/Neuman case. In the three-dimensional case, there is a simpler way to get Proposition 4.2 for functional depending only on $u_\Omega$ and not on $\nabla u_\Omega$. Indeed, we can combine the uniform $H^2$-bound we establish in the previous section with the Morrey embedding.

\textbf{Proposition 4.3.} Let $n = 3$, $\varepsilon > 0$, and $B$ be any non-empty open bounded subset of $\mathbb{R}^n$ containing the origin. We assume that for any $\Omega \in \mathcal{O}(B)$, there exists a unique solution $u_\Omega \in H^2(\Omega, \mathbb{R})$ to (36). Then, for any $(\Omega_i)_{i \in \mathbb{N}} \subset \mathcal{O}(B)$ converging to $\Omega \in \mathcal{O}(B)$ in the sense of Proposition 1.6, the sequence of maps $u_i : x' \in D_{r_i}(x_0) \mapsto u_\Omega(x', \varphi_{i}(x'))$ converges uniformly on $D_{r_i}(x_0)$ to the map $u : x' \in D_{r}(x_0) \mapsto u_\Omega(x', \varphi_{i}(x'))$, where $D_r(x_0)$ is the disk of Proposition 1.7 associated with any $x_0 \in \partial \Omega$.

\textbf{Proof.} Let $\Omega \in \mathcal{O}(B)$ and $u \in H^2(\Omega, \mathbb{R})$. First, from Point (i) below Figure 1, any $\Omega \in \mathcal{O}(B)$ satisfies the $(\alpha, \varepsilon)$-cone property in the sense of Chenais [5]. Hence, we can apply [5, Theorem II.1]: there exists a map $\tilde{u} \in H^2(\mathbb{R}^3, \mathbb{R})$ such that $\|\tilde{u}\|_{H^2(\mathbb{R}^3, \mathbb{R})} \leq c(\varepsilon)\|u\|_{H^2(\Omega, \mathbb{R})}$, where the constant $c > 0$ only depends on $\varepsilon$ (maybe also on $D$ and $n$). Then, we want to use Morrey’s embeddings and we have to be careful with the constants. First, since $\tilde{u} \in H^2(\mathbb{R}^n, \mathbb{R})$, we deduce from the Gagliardo-Nirenberg-Sobolev inequality [11, Section 4.5.1 Theorem 1] that there exists a constant $c_1(p, n) > 0$ depending only on $p = 2$ and $n = 3$ such that $\|\tilde{u}\|^p_{W^{1, p}(\mathbb{R}^3, \mathbb{R})} \leq c_1\|\tilde{u}\|^p_{H^2(\mathbb{R}^3, \mathbb{R})}$. Next, we use Morrey’s inequality [10, Section 5.6.2 Theorem 4]: there exists a constant $c_2(p, n) > 0$...
Finally, we assume that $u$ is the unique solution $u_0 \in H^2(\Omega, \mathbb{R})$ to (36). Applying Theorem 4.1, there exists a constant $C(\varepsilon, D, n)$ depending only on $\varepsilon$, $D$ and $n = 3$ such that:

$$\| \tilde{u} \|_{C^{0,1/2}(\mathbb{R}^3, \mathbb{R})} \leq C_2 \| u \|_{W^{1,1}(\mathbb{R}^3, \mathbb{R})} \leq C_2 c_1 \| u \|_{H^2(\mathbb{R}^3, \mathbb{R})} \leq C_2 c_1 c(\varepsilon) \| u \|_{H^2(\Omega, \mathbb{R})}.$$ 

In particular, if we consider the maps $(u_i)_{i \in \mathbb{N}}$ and $u$ of the statement, we obtain:

$$|u_i(x') - u(x')| = |u_{\Omega_i}(x', \varphi(x')) - u_0(x', \varphi(x'))| = |\tilde{u}_{\Omega_i}(x', \varphi(x')) - \tilde{u}_0(x', \varphi(x'))|$$

$$\leq \| \tilde{u}_{\Omega_i}(x', \varphi(x')) - \tilde{u}_0(x', \varphi(x')) \| + \| \tilde{u}_0(x', \varphi(x')) - \tilde{u}_0(x', \varphi(x')) \|$$

$$\leq \| \tilde{u}_0 \|_{C^{0,1/2}(\mathbb{R}^3, \mathbb{R})} \| \varphi(x') - \varphi(x') \| + \| \tilde{u}_0 - \tilde{u}_0 \|_{C^0(\mathbb{R}^3, \mathbb{R})}$$

$$\leq C(\varepsilon, D) \| f \|_{L^2(B, \mathbb{R})} \| \varphi - \varphi \|_{C^0(\overline{B}(\rho_0))} + c_0(\varepsilon, D) \| \tilde{u}_0 - \tilde{u}_0 \|_{H^2(\Omega, \mathbb{R})}$$

To conclude, we can let $i \to +\infty$ only if $\tilde{u}_i$ converge strongly to $\tilde{u}$. Using relation (26) with $v = \nabla u_{\Omega_i}$, we can express the $L^2$ norm of the second derivative of $\tilde{u}$ as boundary term and show these terms tend to zero as we did in the previous section.

\begin{proof}[Proof of Theorem 1.10] From the foregoing, we only need to prove that for any converging sequence of domains $(\Omega_i)_{i \in \mathbb{N}} \subset \mathcal{O}_c(B)$ we have that $\mathbf{1}_{\Omega_i} u_{\Omega_i}$ converges to $\mathbf{1}_{\Omega} u_0$ in $H^2(B, \mathbb{R})$, where $\Omega \in \mathcal{O}_c(B)$ is the limit domain. The $H^1(B, \mathbb{R})$ convergence is standard in the framework of the uniform cone property. Since the uniform ball condition implies a uniform cone property, to prove the assertion, we only have to express the second-order terms as boundary terms and apply the previous results.

This can be done using the estimation we proved in Theorem 2.2 with $v = \nabla u_{\Omega_i}$.
\end{proof}

\begin{proof}[Proof of Proposition 1.11] The local parametrization we use is made on the limit boundary $\partial \Omega$. Hence, since the Hausdorff convergence is stable for the inclusion, the constraint $\Gamma_0 \subseteq \partial \Omega$ pass to the limit and we have $\Gamma_0 \subseteq \partial \Omega$. Then, we can proceed as before with a partition of unity only made on $\Gamma_0$ and the result follows.
\end{proof}

\section*{References}


