Stability analysis of discrete-time infinite-horizon optimal control with discounted cost

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***Abstract*—We analyse the stability of general nonlinear discrete-time systems controlled by an optimal sequence of inputs that minimizes an infinite-horizon discounted cost. First, assumptions related to the controllability of the system and its detectability with respect to the stage cost are made. Uniform semiglobal and practical stability of the closed-loop system is then established, where the adjustable parameter is the discount factor. Stronger stability properties are thereupon guaranteed by gradually strengthening the assumptions. Next, we show that the Lyapunov function used to prove stability is continuous under additional conditions, implying that stability has a certain amount of nominal robustness. The presented approach is flexible and we show that robust stability can still be guaranteed when the sequence of inputs applied to the system is no longer optimal but near-optimal. We also analyse stability for cost functions in which the importance of the stage cost increases with time, opposite to discounting. Finally, we exploit stability to derive new relationships between the optimal value functions of the discounted and undiscounted problems, when the latter is well-defined.**

I. Introduction

Optimal control selects control inputs so as to minimize a cost incurred during the system operation [22]. In this paper, we focus on optimal control in discrete time over an infinite horizon, with general nonlinear system dynamics as well as general stage costs. In this setting, optimal control is a very powerful framework [4], able to address decision-making problems not only in control engineering, but also in artificial intelligence, operations research, economics, medicine, etc. We concentrate in particular on *discounted* optimal control, where the stage costs are weighted by an exponentially decreasing term γ^k, where $\gamma \in (0,1)$ is the discount factor and k is the time step. The discounted setting is popular in many areas, such as in dynamic programming [3], [24], reinforcement learning [8], [36], [37], and planning algorithms for optimal control [23].

A core practical question is whether the discounted optimal control law stabilizes the system. In the adaptive dynamic programming area, the analysis is usually tailored to the specific cost function considered or the specific algorithm used, see for instance [2], [7], [34]. Some results exist showing local stability in the continuous-time case e.g., [32], [35]. Recently, global asymptotic stability guarantees have been provided in [12] for continuous-time systems. However, to the best of our knowledge, the stability properties in the general discrete-time discounted case are not yet understood. Many recent works consider discounted costs but do not provide stability guarantees, see e.g., [1], [9], [10], [14]. The main issue is the impact of the value of γ on the system stability. The study of a simple linear example will show that, even in that case, γ needs to be sufficiently close to 1 to ensure stability.

Motivated by this insight, we develop a general stability analysis for discounted infinite-horizon optimal control. In contrast with the aforementioned references, we define stability using a generic measure as in [15], which allows addressing the classical equilibrium point stability as a particular case, but also set stability. We first make assumptions related to the controllability of the system and its detectability with respect to the stage cost, which are inspired by [15] where the undiscounted finite-horizon case was considered. Our main result then guarantees that the system in closed-loop with an optimal sequence of inputs is uniformly semiglobally and practically stable, where the adjustable parameter is γ. Hence, for any (arbitrarily large) basin of attraction, the system solutions initialized in this basin will converge to any (arbitrarily small) neighborhood of the target set provided γ is sufficiently close to 1. The analysis is Lyapunov-based and follows similar steps as in [15]. Nevertheless, the optimization problem is different in this paper, which leads to substantial technical differences. Afterwards, we gradually strengthen these assumptions to ensure stronger stability properties, namely uniform semiglobal asymptotic stability and uniform global exponential stability. We also separately address the case of linear systems with quadratic stage cost. An explicit lower bound on the discount factor is provided for each of these stability statements. The results are applied to two examples: a linearized model of an inverted pendulum and a nonholonomic integrator.

To endow stability with nominal robustness, it is essential to work with a Lyapunov function that is continuous, see [21]. With our construction and under our assumptions, the continuity of the Lyapunov function is equivalent to the continuity of the optimal value function. We prove that the latter is indeed continuous under additional regularity conditions. In contrast with the existing literature, we exploit stability for this purpose. This is a major difference, which allows us to derive stronger conclusions in general and to rely on weaker assumptions compared, for example, to [11], [17] where lower semicontinuity is ensured, or to [6] where concave stage costs

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are considered.

The results mentioned so far are valid when an optimal sequence of inputs is applied to the system. In practice, most algorithms generate only *near*-optimal inputs. We show that stability and the continuity of the optimal value function can still be guaranteed when a near-optimal sequence of inputs is used to control the system.

We also illustrate the generality of our approach by analysing stability for cost functions in which the stage cost is multiplied by a term which grows with time; we call this scenario *reverse-discounted* optimal control. The idea is to increase the weight of the stage cost as time grows in applications where the long-term behaviour is more important than the short-term one. We focus on two scenarios: when the stage cost is weighted by γ^k with $\gamma \geqslant 1$ or by $1-\gamma^{k+1}$ with $\gamma \in (0,1)$. Uniform global asymptotic stability is ensured in both cases.

Finally, we exploit stability to quantify the difference between the optimal value functions of the discounted and undiscounted problems, when the latter is well-defined, under appropriate conditions. This result is relevant as, in many situations, the discount factor is introduced because the optimal control sequence is harder to compute for the undiscounted problem. In such cases, γ is typically selected close to 1, hoping this will lead to an optimal value close to the one for $\gamma = 1$. We prove that this is indeed the case when the system controlled by an optimal sequence of inputs for the discounted cost is uniformly globally exponentially stable.

Discounting may be seen as complementary to relaxed dynamic programming [25] in the context of stabilization [16]. While the stage cost is weighted by a constant $\alpha \in (0,1]$ in the relaxed dynamic programming inequality, in the discounted case, it is the optimal value function at the next step which is multiplied by $\gamma \in (0,1)$ in the Bellman equation. As a result, while the two problems share similarities, the analysis in the discounted case requires different and nontrivial analytical tools to study stability, which lead to semiglobal and practical stability in general, as opposed to global and asymptotic stability in relaxed dynamic programming under similar assumptions [16].

Compared to the preliminary version of this work in [29], here we do not make any assumption on the undiscounted problem, we relax the assumptions that ensure stability, and we provide sufficient conditions for semiglobal asymptotic stability and global exponential stability. The continuity analysis of the optimal value function relies on weaker assumptions (we no longer ask the stage cost to be bounded) and uses stability. Completely novel elements include: the stability with near-optimal sequences of inputs, the analysis of reverse-discounted case and the relationship between the optimal value functions of the discounted and undiscounted problems. The case study where the stage cost is bounded in Section VI in [29] is not reported in this paper.

The paper is organised as follows. After introducing some preliminaries in Section II and stating the problem in Section III, the main stability results are provided in Section IV. The continuity of the Lyapunov function for the discounted problem is analysed in Section V. Stability using a near-optimal sequence of inputs is investigated in Section VI. Results on reverse-discounted optimal control and the relationships between the optimal value functions of the discounted and the undiscounted problems are presented in Section VII. The proofs are provided in Section VIII and the conclusion is given in Section IX. Finally, technical lemmas are reported in the appendix.

II. Preliminaries

Let $\mathbb{R} := (-\infty, \infty)$, $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, $\mathbb{R}_{\geqslant 0} := [0, \infty)$, $\mathbb{Z}_{\geqslant 0} := \{0, 1, 2, \ldots\}$, and $\mathbb{Z}_{>0} := \{1, 2, \ldots\}$. The notation (x, y) stands for $[x^{\mathrm{T}},\, y^{\mathrm{T}}]^{\mathrm{T}}$, where $(x, y) \in \mathbb{R}^{n+m}$. A function $\chi : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is of class $\mathcal{K}$ if it is continuous, zero at zero and strictly increasing, and it is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded. We say that a continuous function $\chi : \mathbb{R}^2_{\geqslant 0} \longrightarrow \mathbb{R}_{\geqslant 0}$ is of class $\mathcal{KK}$ when $\chi(s, \cdot)$ and $\chi(\cdot, s)$ are of class $\mathcal{K}$, for any $s > 0$. A continuous function $\chi : \mathbb{R}^2_{\geqslant 0} \longrightarrow \mathbb{R}_{\geqslant 0}$ is of class $\mathcal{KL}$ if for each $t \in \mathbb{R}_{\geqslant 0}$, $\chi(\cdot, t)$ is of class $\mathcal{K}$, and, for each $s > 0$, $\chi(s, \cdot)$ is decreasing to zero. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $|x|$ and the distance of $x \in \mathbb{R}^n$ to a set $\mathcal{A} \subseteq \mathbb{R}^n$ is denoted by $|x|_{\mathcal{A}} := \inf\{|x-y| \,:\, y \in \mathcal{A}\}$. Let P be a real, square, and symmetric matrix, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ are respectively the largest and the smallest eigenvalue of P. The notation $\mathbb{I}$ either stands for the identity function from $\mathbb{R}_{\geqslant 0}$ to $\mathbb{R}_{\geqslant 0}$ or for the identity matrix depending on the context.

The definitions below can be found in [31]. A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is *lower semicontinuous* at $\bar{x} \in \mathbb{R}^n$ if $\liminf\limits_{x \to \bar{x}} f(x) := \lim\limits_{\delta \searrow 0} [\inf\limits_{x \in \mathcal{B}(\bar{x}, \delta)} f(x)] = f(\bar{x})$, where $\mathcal{B}(\bar{x}, \delta)$ is the closed ball of $\mathbb{R}^n$, centered at $\bar{x}$ of radius $\delta \geqslant 0$. We say that f is lower semicontinuous on $X \subseteq \mathbb{R}^n$, when it is lower semicontinuous at any $\bar{x} \in X$. Note that when f is continuous, it is also lower semicontinuous. Let $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ denote a set-valued mapping. The outer and inner limits of S at $\bar{x} \in \mathbb{R}^n$ are respectively defined as $\limsup\limits_{x \to \bar{x}} S(x) := \left\{u \,:\, \exists\, x_n \underset{n \to \infty}{\to} \bar{x},\ \exists\, u_n \underset{n \to \infty}{\to} u \text{ with } u_n \in S(x_n)\right\}$ and $\liminf\limits_{x \to \bar{x}} S(x) := \left\{u \,:\, \forall\, x_n \underset{n \to \infty}{\to} \bar{x},\ \exists N \in \mathcal{N}_\infty,\, u_n \underset{n \in N \to \infty}{\to} u \text{ with } u_n \in S(x_n)\right\}$ where $\mathcal{N}_\infty$ is the set of subsequences of $\mathbb{Z}_{\geqslant 0}$ containing all $n \geqslant \overline{n}$ for some $\overline{n} \in \mathbb{Z}_{\geqslant 0}$. The set-valued mapping S is *continuous* at $\bar{x} \in \mathbb{R}^n$ when $\lim\limits_{x \to \bar{x}} S(x) = S(\bar{x})$, where $\lim\limits_{x \to \bar{x}} S(x) = \limsup\limits_{x \to \bar{x}} S(x) = \liminf\limits_{x \to \bar{x}} S(x)$, and it is continuous on $X \subseteq \mathbb{R}^n$ when it is continuous at any $\bar{x} \in X$. In other words, S is continuous on X when it is both *outer semicontinuous* and *inner semicontinuous* on X, see Definition 5.4 in [31]. The image of a set $\mathcal{V}$ under the mapping S is defined by $S(\mathcal{V}) = \bigcup_{x \in \mathcal{V}} S(x)$. The mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *locally bounded* when for any $\bar{x} \in \mathbb{R}^n$, for some neighborhood $\mathcal{V}$ of $\bar{x}$, the set $S(\mathcal{V}) \subset \mathbb{R}^m$ is bounded.

III. Problem statement

Consider the system

$$x(k+1) \quad = \quad f(x(k), u(k)) \tag{1}$$

with state $x \in \mathbb{R}^n$ and input $u \in \mathcal{U}(x) \subseteq \mathbb{R}^m$, where $\mathcal{U}(x)$ is a nonempty set of admissible inputs associated to state x (as in

[17], [19] for instance), and $n, m \in \mathbb{Z}_{>0}$. Let $\mathcal{W} := \{(x,u) : x \in \mathbb{R}^n \text{ and } u \in \mathcal{U}(x)\}$. Define an infinite-length sequence of control inputs $\mathbf{u} = (u_0, u_1, \ldots)$, in which the control input at time $k \in \mathbb{Z}_{\geqslant 0}$ is given by $u(k) = u_k$. We denote the solution to (1) at the k^{th}-step starting at state x and with the input sequence $\mathbf{u}$ as $\phi(k, x, \mathbf{u}|_k)$, where $\mathbf{u}|_k := (u_0, \ldots, u_{k-1})$ is the truncation of $\mathbf{u}$ to the first $k \in \mathbb{Z}_{>0}$ steps, and we use the convention $\phi(0, x, \mathbf{u}|_0) = x$ where $\mathbf{u}|_0$ is the empty set. In optimal control, the sequence of control inputs is given by the solution to an optimization problem. In this paper, we consider the *cost function*

$$J_\gamma(x, \mathbf{u}) \quad := \quad \sum_{k=0}^{\infty} \gamma^k \ell(\phi(k, x, \mathbf{u}|_k), u_k), \tag{2}$$

where $\ell : \mathcal{W} \to \mathbb{R}_{\geqslant 0}$ is the *stage cost*, which takes non-negative values, and $\gamma \in (0,1)$ is the *discount factor*.

We assume that, for any $x \in \mathbb{R}^n$, there exists (at least) one infinite-length input sequence, which minimizes (2), as formalized below.

Standing Assumption. *For any $x \in \mathbb{R}^n$ and $\gamma \in (0,1)$, there exists an infinite-length input sequence $\mathbf{u}^*_\gamma(x)$, called* optimal solution*, such that*

$$J(x, \mathbf{u}^*_\gamma(x)) \quad = \quad \inf_{\mathbf{u}} J_\gamma(x, \mathbf{u}) \quad =: \quad V_\gamma(x), \tag{3}$$

where V_γ is the optimal value function. □

Conditions on system (1) and cost function (2) to ensure the Standing Assumption are available in [19]. Note that the sequence $\mathbf{u}^*_\gamma(x)$ may be non-unique for a given $x \in \mathbb{R}^n$.

The Standing Assumption implies that the set below is nonempty for any $x \in \mathbb{R}^n$ in view of the Bellman equation

$$\mathcal{U}^*_\gamma(x) \quad := \quad \operatorname*{argmin}_{u \in \mathcal{U}(x)} \left[\ell(x,u) + \gamma V_\gamma(f(x,u))\right], \tag{4}$$

where $\mathcal{U}^*_\gamma$ is the optimal feedback law. We can then represent system (1) subject to an optimal sequence of inputs for the cost function (2) as the following difference inclusion

$$x(k+1) \quad \in \quad F^*_\gamma(x(k)) := f\big(x(k), \mathcal{U}^*_\gamma(x(k))\big) \tag{5}$$

where $f(x, \mathcal{U}^*_\gamma(x))$ is the set $\{f(x,u) : u \in \mathcal{U}^*_\gamma(x)\}$ for $x \in \mathbb{R}^n$.

The main objective of this study is to infer (robust) stability properties of system (5). We will see that γ must, in general, be appropriately selected to guarantee stability, as illustrated by the following simple example.

Example 1. *Consider the scalar system $x(k+1) = 2x(k) + u(k)$ and the discounted quadratic cost $J_\gamma(x, \mathbf{u}) = \sum_{k=0}^{\infty} \gamma^k \left(x(k)^2 + u(k)^2\right)$, where $\gamma \in (0,1)$ and $\mathbf{u}$ is an infinite-length sequence of inputs. The optimal solution is given by the feedback law $u = K^*_\gamma x$ with $K^*_\gamma = -2\Big(1 + 2\big(5\gamma - 1 + \sqrt{(5\gamma-1)^2 + 4\gamma}\big)^{-1}\Big)^{-1}$, see Section 4.2 in [3]. The origin of the closed-loop system is uniformly globally exponentially stable if and only if $2 + K^*_\gamma \in (-1, 1)$, which is equivalent to $\gamma \in (\gamma^\star, 1)$ where $\gamma^\star = \frac{1}{3}$. Hence, γ needs to be sufficiently close to 1, otherwise the optimal feedback law does not stabilize the origin of the system.* □

IV. Stability

In this section, we first impose conditions on system (1) and cost function (2), which are related to the controllability of the system and its detectability with respect to the stage cost ℓ. We then present the main stability result, which relies on Lyapunov analysis. Afterwards, we provide sufficient conditions to ensure stronger stability guarantees, and we illustrate the framework by treating the case of systems with linear dynamics and quadratic stage cost. Finally, we apply the results to two examples.

A. Controllability and detectability assumptions

We make the following assumption on system (1) and cost function (2), which is inspired by [15].

Assumption 1. *Let $\sigma : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ be continuous and positive semi-definite.*

(i) *There exists $\overline{\alpha}_V \in \mathcal{K}_\infty$ such that for any $\gamma \in (0,1)$ and $x \in \mathbb{R}^n$, $V_\gamma(x) \leqslant \overline{\alpha}_V(\sigma(x))$.*

(ii) *There exist a continuous function $W : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$, $\alpha_W, \chi_W \in \mathcal{K}_\infty$ and $\overline{\alpha}_W : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ continuous, nondecreasing and zero at zero, such that the following holds for any $(x,u) \in \mathcal{W}$*

$$\begin{aligned} W(x) &\leqslant \overline{\alpha}_W(\sigma(x)) \\ W(f(x,u)) - W(x) &\leqslant -\alpha_W(\sigma(x)) + \chi_W(\ell(x,u)). \end{aligned} \tag{6}$$

□

The generic function $\sigma : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ will serve as a state measure when investigating stability (as in [15]). It can be defined as $|\cdot|$ or $|\cdot|^2$, when studying the stability of the origin, or $|\cdot|_{\mathcal{A}}$ or $|\cdot|^2_{\mathcal{A}}$ with $\mathcal{A} \subseteq \mathbb{R}^n$, when studying the stability of set $\mathcal{A}$, for example. Item (i) of Assumption 1 is related to the controllability of system (1). This property is for instance verified when ℓ is uniformly globally exponentially controllable to zero with respect to $\sigma : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$, as stated in the next lemma.

Lemma 1. *Consider system (1) and suppose that ℓ is* uniformly globally exponentially controllable to zero with respect to $\sigma : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$*, i.e. there exist $M > 0$ and $\lambda > 0$, where λ is called the* decrease rate*, such that for any $x \in \mathbb{R}^n$ there exists an admissible infinite-length control input sequence $\mathbf{u}(x)$ verifying $\ell(\phi(k, x, \mathbf{u}|_k(x)), u_k(x)) \leqslant M\sigma(x)e^{-\lambda k}$ for any $k \in \mathbb{Z}_{>0}$. Then, item (i) of Assumption 1 holds with $\overline{\alpha}_V(s) = \frac{Ms}{1-e^{-\lambda}}$ for any $s \geqslant 0$.* □

Weaker conditions that guarantee the satisfaction of item (i) of Assumption 1 can be obtained by following similar lines as in Section III of [15].

Item (ii) of Assumption 1 states a detectability property of σ from ℓ (see Definition 1 in [15]), which is satisfied for example when $\sigma(\cdot) = |\cdot|^2$ and $\ell(x,u) = x^{\mathrm{T}}Qx + u^{\mathrm{T}}Gu$ where Q is a real, symmetric and positive definite matrix and G is a real, symmetric, positive semi-definite matrix. In that case, Assumption 1 holds with $W = 0$, $\overline{\alpha}_W = 0$, $\chi_W = \mathbb{I}$ and $\alpha_W = \lambda_{\min}(Q)\mathbb{I}$.

It is important to emphasize that the functions $\overline{\alpha}_V$, W, $\overline{\alpha}_W$, α_W and χ_W in Assumption 1 are independent of the discount factor γ.

Remark 1. *The Standing Assumption and Assumption 1 can be relaxed to hold only for any $\gamma \in (\underline{\gamma}, 1)$ where $\underline{\gamma} \in (0,1)$, instead of any $\gamma \in (0,1)$. The forthcoming results apply in this case by constraining γ to be in $(\underline{\gamma}, 1)$.* □

B. Main result

The next theorem gives Lyapunov-based properties, from which we then deduce a stability property for system (5) in Theorem 2.

Theorem 1. *Under Assumption 1, there exist $\underline{\alpha}_Y, \overline{\alpha}_Y, \alpha_Y \in \mathcal{K}_\infty$, $\Upsilon \in \mathcal{KK}$ and for any $\gamma \in (0,1)$ there exists $Y_\gamma : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ such that the following holds.*

(a) *For any $x \in \mathbb{R}^n$, $\underline{\alpha}_Y(\sigma(x)) \leqslant Y_\gamma(x) \leqslant \overline{\alpha}_Y(\sigma(x))$.*

(b) *For any $x \in \mathbb{R}^n$, $v \in F_\gamma^*(x)$, $Y_\gamma(v) - Y_\gamma(x) \leqslant -\alpha_Y(\sigma(x)) + \Upsilon(\sigma(x), \frac{1-\gamma}{\gamma})$.*

The expressions of $\underline{\alpha}_Y$, $\overline{\alpha}_Y$, α_Y, Υ and Y_γ are provided in Table I. □

Function Y_γ is either given by[1] $Y_\gamma = V_\gamma + W$ or by $Y_\gamma = \rho_V(V_\gamma) + \rho_W(W)$ where V_γ and W come from Assumption 1 and ρ_V and ρ_W are suitable $\mathcal{K}_\infty$-functions, see Table I. Item (a) of Theorem 1 means that Y_γ is positive definite and radially unbounded with respect to σ, uniformly in γ. Item (b) of Theorem 1 implies that, for any $\gamma \in (0,1)$, Y_γ strictly decreases along the solutions to system (5) up to a perturbative term $\Upsilon(\sigma(x), \frac{1-\gamma}{\gamma})$, which can be made arbitrarily small by selecting γ sufficiently close to 1 (since $\Upsilon \in \mathcal{KK}$). Because of this perturbative term, only uniform semiglobal and practical stability can be guaranteed in general, as formalized in the following theorem.

Theorem 2. *Consider system (5) and suppose Assumption 1 holds. Then, there exists $\beta \in \mathcal{KL}$ such that for any $\delta, \Delta > 0$, there exists $\gamma^\star \in (0,1)$ such that for any $\gamma \in (\gamma^\star, 1)$ and $x \in \{z \in \mathbb{R}^n : \sigma(z) \leqslant \Delta\}$, any solution $\phi(\cdot, x)$ to system (5) satisfies*

$$\sigma(\phi(k,x)) \leqslant \max\{\beta(\sigma(x), k), \delta\} \quad \forall k \in \mathbb{Z}_{\geqslant 0}. \tag{7}$$

□

This theorem means that for any set of initial conditions of the form $\{z \in \mathbb{R}^n : \sigma(z) \leqslant \Delta\}$ where $\Delta > 0$ can be arbitrarily large, and for any (arbitrarily small) $\delta > 0$, we can select γ sufficiently close to 1 such that (7) holds. An estimate of the lower bound on γ, i.e. $\gamma^\star$, is available in the proof of Theorem 2, namely, given $\delta, \Delta > 0$, $\gamma^\star$ has to be such that $\Upsilon(\underline{\alpha}_Y^{-1} \circ \overline{\alpha}_Y(\Delta), \frac{1-\gamma^\star}{\gamma^\star}) \leqslant \frac{1}{2}\alpha_Y \circ \overline{\alpha}_Y^{-1}(\underline{\alpha}_Y(\delta))$ (see Table I). Tailored estimates of $\gamma^\star$ for specific classes of systems are provided in the sequel.

[1]In principle, we can always take $Y_\gamma = \rho_V(V_\gamma) + \rho_W(W)$. We have chosen to separately treat the situation where $\chi_W \leqslant \mathbb{I}$, which leads to $Y_\gamma = V_\gamma + W$, because the construction of the Lyapunov function is easier in this case, which is consistent with [15].

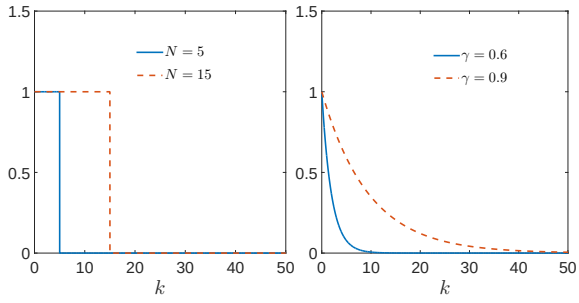

Fig. 1. Evolution of ξ for finite-horizon problems (left) and for infinite-horizon discounted problems (right) versus time.

Remark 2. *The intuition behind Theorem 2 is the following. The controllability and detectability conditions in Assumption 1 are used to ensure stability, which is consistent with results in the undiscounted case for linear systems and quadratic stage costs for instance, in which case these conditions are also necessary. The fact that $\gamma < 1$ in (2) generates extra difficulties as explained after Theorem 1. As a result, the value of the discount factor needs to be adjusted according to the desired stability property.* □

Remark 3. *It is interesting to note the analogy with the results in [15] where the finite horizon N takes the place of $\frac{\gamma}{1-\gamma}$ in the infinite-horizon discounted problem. Informally, quantity $\frac{\gamma}{1-\gamma}$ can be thought of as an 'effective horizon' of the discounted problem. Thus, while [15] shows stability for horizons N sufficiently large so that it is greater than some lower bound $N^\star$, we show it for γ sufficiently close to 1 so that $\gamma > \gamma^\star$, and thus for effective horizons larger than $\frac{\gamma^\star}{1-\gamma^\star}$. To go further with the analogy, finite-horizon costs and infinite-horizon discounted costs can be both interpreted as* infinite-horizon *costs of the general form*

$$J(x, \mathbf{u}) := \sum_{k=0}^{\infty} \xi(k)\ell(\phi(k, x, \mathbf{u}|_k), u_k), \tag{8}$$

where $\xi(k) = 1$ when $k \leqslant N$ and $\xi(k) = 0$ for $k > N$ for finite-horizon problems, and $\xi(k) = \gamma^k$ for infinite-horizon discounted problems, see Figure 1 for an illustration. This unifying viewpoint suggests that the methodology in [15] and in this paper could be extended for other cost functions of the form (8); an example is provided in Section VII-A. Note that, although our results are inspired from [15], the stability analysis exhibits important technical differences. The Bellman equation leads to inequalities in the Lyapunov analysis, which are different from those in [15] and which require new arguments to conclude about stability. Moreover, in our case, the continuity of the optimal value function, which is essential for robustness reasons, turns out to be more involved and to require different proof techniques, as we will see in Section V. □

C. Stronger stability guarantees

We strengthen below the conditions of Theorem 2 in order to derive stronger stability guarantees. The following result

	If $\chi_W \leqslant \mathbb{I}$	If $\exists s \in \mathbb{R}_{\geqslant 0},\ \chi_W(s) > s$
Y_γ	$V_\gamma + W$	$\rho_V(V_\gamma) + \rho_W(W)$
$\underline{\alpha}_Y$	α_W	$\min\left\{\rho_V\left(\chi_W^{-1}(\frac{1}{2}\alpha_W)\right), \rho_W(\frac{1}{2}\alpha_W)\right\}$
$\overline{\alpha}_Y$	$\overline{\alpha}_V + \overline{\alpha}_W$	$\rho_V(\overline{\alpha}_V) + \rho_W(\overline{\alpha}_W)$
$\alpha_Y (= \tilde{\alpha}_W)$	α_W	$q_W(\frac{1}{4}\alpha_W)\frac{1}{4}\alpha_W$
Υ	$(s_1, s_2) \mapsto \overline{\alpha}_V(s_1)s_2$	$(s_1, s_2) \mapsto q_V\big((1+s_2)\overline{\alpha}_V(s_1)\big)s_2\overline{\alpha}_V(s_1)$
$\overline{\alpha}_{\hat{Y}}$	$\overline{\alpha}_V + \hat{\alpha} + \alpha_W$	$\rho_V(2\overline{\alpha}_V + 2\hat{\alpha}) + \rho_W(\overline{\alpha}_W)$
$\hat{\vartheta}$	$\mathbb{I}$	$\rho_W(2\mathbb{I})$
$\hat{\Upsilon}$	$(s_1, s_2) \mapsto \left((\overline{\alpha}_V + \hat{\alpha}) \circ \underline{\alpha}_Y^{-1} + \eta\right) s_2$	$(s_1, s_2) \mapsto q_V\left((1+s_2)\left((\overline{\alpha}_V + \hat{\alpha}) \circ \underline{\alpha}_Y^{-1}(s_1) + \eta\right)\right) s_2 \left((\overline{\alpha}_V + \hat{\alpha}) \circ \alpha_Y^{-1}(s_1) + \eta\right)$
ρ_V, ρ_W		$\rho_V : s \mapsto \int_0^s q_V(t)\ \mathrm{d}t$, $\rho_W : s \mapsto \int_0^s q_W(t)\ \mathrm{d}t$
q_V, q_W		$q_V := 2\chi_W(2\mathbb{I})$, $q_W := \frac{1}{2}\left(\chi_W + \overline{\alpha}_W\big(\alpha_W^{-1}(2\chi_W)\big)\right)^{-1}$

TABLE I
EXPRESSIONS OF FUNCTIONS USED IN SECTIONS IV AND VI.

ensures the uniform semiglobal asymptotic stability property of system (5) with respect to a given state measure σ, i.e. that (7) can be guaranteed with $\delta = 0$. It requires Assumption 1 to hold with functions $\overline{\alpha}_V, \overline{\alpha}_W, \chi_W$, on the one hand, and α_W, on the other hand, which are respectively upper-bounded and lower-bounded by linear functions in a neighbourhood of the origin.

Corollary 1. *Suppose that Assumption 1 is satisfied and there exist $L > 0$, $\overline{a}_W \geqslant 0$, $a_W, \overline{a}_V > 0$ such that $\overline{\alpha}_V(s) \leqslant \overline{a}_V \cdot s$, $\overline{\alpha}_W(s) \leqslant \overline{a}_W \cdot s$, $\alpha_W(s) \geqslant a_W \cdot s$ and $\chi_W(s) \leqslant s$ for any $s \in [0, L]$. Then, there exists $\beta \in \mathcal{KL}$ such that for any $\Delta > 0$, $\gamma \in (\gamma^\star, 1)$ with $\gamma^\star = \max\left\{\left(1 + \frac{1}{2}\frac{a_W^2}{\overline{a}_V(\overline{a}_V + \overline{a}_W)}\right)^{-1}, \frac{1}{1 + \frac{\alpha_W((\overline{\alpha}_V + \overline{\alpha}_W)^{-1}(L))}{\overline{\alpha}_V((\alpha_W^{-1} \circ \overline{\alpha}_Y(\Delta)))}}\right\}$, and $x \in \{z \in \mathbb{R}^n : \sigma(z) \leqslant \Delta\}$, the solution $\phi(\cdot, x)$ to system (5) satisfies $\sigma(\phi(k, x)) \leqslant \beta(\sigma(x), k)$ for all $k \in \mathbb{Z}_{\geqslant 0}$.* □

When Assumption 1 is verified with linear $\mathcal{K}_\infty$-functions and linear $\overline{\alpha}_W$, the corollary below states that the stability is uniform, global and exponential.

Corollary 2. *Suppose that Assumption 1 is verified and there exist $\overline{a}_W \geqslant 0$, $a_W, \overline{a}_V > 0$ such that $\overline{\alpha}_V(s) = \overline{a}_V \cdot s$, $\overline{\alpha}_W(s) = \overline{a}_W \cdot s$, $\alpha_W(s) = a_W \cdot s$ and $\chi_W(s) = s$ for any $s \geqslant 0$. Let $\gamma^\star > \dfrac{\overline{a}_V}{\overline{a}_V + a_W}$. Then, there exist $K, \lambda > 0$ such that for any $\gamma \in (\gamma^\star, 1)$, for any $x \in \mathbb{R}^n$, the solution $\phi(\cdot, x)$ to system (5) satisfies $\sigma(\phi(k, x)) \leqslant K\sigma(x)e^{-\lambda k}$ for all $k \in \mathbb{Z}_{\geqslant 0}$.* □

It has to be emphasized that the value of $\gamma^\star$ depends on the assumptions and is therefore different for each stability result that we state. In the following, we revisit Corollary 2 in the context of linear systems with quadratic costs. Consider the system

$$x(k+1) \quad = \quad Ax(k) + Bu(k) \tag{9}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, A, B real matrices of appropriate dimensions such that the pair (A, B) is stabilizable, and the cost function is given by (2) with

$$\ell(x, u) \quad = \quad x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru, \tag{10}$$

where R is real, symmetric, and positive definite and Q is a real matrix such that $Q = C^{\mathrm{T}}C$ with (A, C) detectable. The sequence of optimal inputs is generated by a unique linear feedback law $u = K_\gamma^* x$ where $K_\gamma^* := -\gamma(\gamma B^{\mathrm{T}}P_\gamma B + R)^{-1}B^{\mathrm{T}}P_\gamma A$ and P_γ is the unique solution to $P_\gamma = A^{\mathrm{T}}\big(\gamma P_\gamma - \gamma^2 P_\gamma B(\gamma B^{\mathrm{T}}P_\gamma B + R)^{-1}B^{\mathrm{T}}P_\gamma\big)A + Q$, see Section 4.2 in [3]. While the stability of the closed-loop system can be inferred by checking whether the matrix $A + BK_\gamma^*$ is Schur for a given value of γ, it is useful to know a priori a set of values of γ under which stability is preserved. This set is usually difficult to determine based on the direct analysis of the eigenvalues of $A + BK_\gamma^*$, due to the fact that K_γ^* is obtained by solving a Riccati equation for each γ and is thus nonlinear in γ. In contrast, the result below provides an easily computable, though potentially conservative, lower bound on γ under which stability is ensured.

Corollary 3. *Consider system (9) and the cost function J_γ in (2) with ℓ defined in (10). Let S_1, S_2 be real, symmetric, positive definite matrices, and $\varpi, \alpha > 0$ such that*

$$\begin{pmatrix} A^{\mathrm{T}}S_2A - S_2 + S_1 - \varpi Q & A^{\mathrm{T}}S_2B \\ B^{\mathrm{T}}S_2A & B^{\mathrm{T}}S_2B - \varpi R \end{pmatrix} \quad \leqslant \quad 0. \tag{11}$$

and

$$\alpha P \quad \leqslant \quad S_1, \tag{12}$$

where P is the unique solution to the (undiscounted) Riccati equation $P = Q + A^{\mathrm{T}}\big(P - PB(R + B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}P\big)A$. Let $\gamma^\star > \dfrac{\varpi}{\varpi + \alpha}$, for any $\gamma \in (\gamma^\star, 1)$, the origin of the system in closed-loop with the optimal feedback law $u = K_\gamma^* x$ is uniformly globally exponentially stable, i.e. there exist $D, \mu > 0$ such that for any initial condition $x \in \mathbb{R}^n$, the corresponding solution $\phi(\cdot, x)$ verifies $|\phi(k, x)| \leqslant D|x|e^{-\mu k}$ for any $k \in \mathbb{Z}_{\geqslant 0}$. Furthermore, when Q is symmetric and positive definite, $\gamma^\star$ has to be strictly larger than $\min\left\{\gamma' \in (0, 1) : -Q + \frac{1-\gamma'}{\gamma'}P \geqslant 0\right\}$. □

It is always possible to find S_1, S_2, ϖ and α such that (11) and (12) hold, according to Lemma 4 in the appendix. The estimate of $\gamma^\star$ in Corollary 3 exclusively relies on the matrices A, B, Q and R. To minimize $\gamma^\star$, we have to maximize $\alpha > 0$

under (11), (12), which is a convex optimization problem. When P is positive definite, i.e. when the pair (A, C) is observable, the bound on $\gamma^\star$ simplifies as we can take $S_1 = P$, which leads to $\gamma^\star > \frac{\varpi}{\varpi+1}$. The minimum value of ϖ is obtained by solving: $\min \varpi$ such that $\varpi > 0$ and (11) holds. That gives 0.8090 in Example 1; recall that the true value is $\frac{1}{3}$ (see Section III). The observed conservatism notably comes from the proof of Corollary 3 (see Section VIII-F, in particular (35)).

D. Examples

1) Linearized inverted pendulum: We discretize exactly the model of a pendulum linearized at the upper position $(\pi, 0)$ with sampling period $T > 0$, which gives

$$x^+ \quad = \quad Ax + Bu, \tag{13}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, x_1 is the angle between the rod and the vertical axis, x_2 is the angular velocity, u is the control input, $A = \exp(A_c T)$, $B = \int_0^T \exp(A_c(T-s))B_c \, \mathrm{d}s$, $A_c = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{K}{m} \end{pmatrix}$ and $B_c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with $g > 0$ the acceleration due to gravity, $l > 0$ the length of the rod, $m > 0$ its mass and $k > 0$ a friction coefficient. The stage cost is given by (10) with $Q = C^{\mathrm{T}}C$, $C = [1000\ 0]$ (the pair (A, C) is observable) and $R = 1$. We apply the results of Corollary 3 for $g = 9.81$, $l = 1$, $m = 0.1$, $k = 0.1$ and $T = 0.01$, and we obtain that the origin of system (13) is uniformly globally exponentially stable for any $\gamma \in (0.9878, 1)$. A numerical study indicates that the critical value for the discount factor, which guarantees stability, is approximately 0.9063. The difference with the estimated bound is of the order of 8%. Interestingly, simulation results suggest that stability still holds for any $\gamma \geqslant 1$; we will go back to that point in Section VII-A.

2) A nonholonomic integrator: Consider the following nonholonomic integrator as in Example 2 of [15]

$$\begin{aligned} x_1^+ &= x_1 + u_1 \\ x_2^+ &= x_2 + u_2 \\ x_3^+ &= x_3 + x_1 u_2 - x_2 u_1, \end{aligned} \tag{14}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $u = (u_1, u_2) \in \mathcal{U}(x) = \mathbb{R}^2$. The stage cost is defined as $\ell(x, u) = x_1^2 + x_2^2 + 10|x_3| + |u|^2$ for $x \in \mathbb{R}^3$ and $u \in \mathbb{R}$, which differs from the stage cost considered in [15] because of the input-dependent term that is useful to prove the Standing Assumption of Section III.

We apply the results of Section V in [19] to ensure the satisfaction of the Standing Assumption. Conditions (1)-(6) in this reference are verified (with $\beta_k = 1$ for condition (5) thanks to the input-dependent term in ℓ). We now prove that condition (7) in Section V in [19] holds with $X_0 = \mathbb{R}^n$. Let $x \in \mathbb{R}^3$, we build the input sequence $\mathbf{u}(x) = (u_0(x), u_1(x), u_2(x), u_3(x), 0, \ldots)$ with $u_0(x) = (-x_1, -x_2)$, $u_1(x) = (\sqrt{|x_3|}, 0)$, $u_2(x) = (\sqrt{|x_3|}, -\mathrm{sign}(x_3)\sqrt{|x_3|})$ and $u_3(x) = (-2\sqrt{|x_3|}, \mathrm{sign}(x_3)\sqrt{|x_3|})$, where $\mathrm{sign}(0) = 0$. Then $\phi(k, x, \mathbf{u}|_k(x)) = 0$ for any $k \geqslant 4$, hence condition (7) in Section V in [19] is satisfied with $X_0 = \mathbb{R}^n$. As a result, the Standing Assumption is guaranteed. Item (ii) of Assumption 1 is ensured with $W = 0$, $\underline{\alpha}_W = 0$, and $\alpha_W = \chi_W = \mathbb{I}$. The input sequence constructed above ensures that $V_\gamma(x) \leqslant J_\gamma(x, \mathbf{u}(x)) \leqslant \frac{22}{5}\sigma(x)$ for any $\gamma \in (0, 1)$: item (i) of Assumption 1 holds with $\overline{\alpha}_V = \frac{22}{5}\mathbb{I}$. Consequently, the uniform global exponential stability property in Corollary 2 is verified when[2] $\gamma \in (\frac{22}{25}, 1)$.

V. Continuity of the Lyapunov function

The continuity of the Lyapunov function V_γ in Theorem 1 with respect to the state x would ensure that the stability properties studied in the previous section are robust to small perturbations, namely σ-perturbations according to[3] [21]. We have seen in Section IV-B that V_γ is either given by $V_\gamma + W$ or by $\rho_V(V_\gamma) + \rho_W(W)$. Since W is assumed to be continuous in Assumption 1 and so are ρ_V and ρ_W (as $\mathcal{K}_\infty$-functions), V_γ is continuous with respect to x if and only if the optimal value function V_γ is continuous with respect to x. The assumptions made so far do not a priori allow us to assert that V_γ is continuous. In the next theorem, we provide additional assumptions to guarantee the continuity of V_γ, and thus of Y_γ.

Theorem 3. *Suppose the following holds.*

(a) *Assumption 1 is verified.*
(b) *f and ℓ are continuous on $\mathcal{W}$ and $\mathcal{U} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is continuous and locally bounded on $\mathbb{R}^n$.*
(c) *For any $M \geqslant 0$, the set $\{x : \sigma(x) \leqslant M\}$ is compact.*

Then, for any $\Delta > 0$, there exists $\gamma^\star \in (0, 1)$ such that for any $\gamma \in (\gamma^\star, 1)$ the optimal value function V_γ in (3) is continuous on $\{x \in \mathbb{R}^n : \sigma(x) \leqslant \Delta\}$. □

Item (a) of Theorem 3 ensures that the uniform semiglobal practical stability property (7) holds in view of Theorem 2, which plays a crucial role in the proof. Item (b) of Theorem 3 states regularity conditions on system (1) and the stage cost ℓ. The last part of item (b) of Theorem 3 holds e.g., when $\mathcal{U}(x) = U$ with U compact. Finally, item (c) of Theorem 3 means that measure σ is radially unbounded, which is the case when it is given by the Euclidean distance, or more generally by the distance to a compact set.

The continuity of V_γ with respect to x ensured in Theorem 3 is semiglobal in γ, in the sense that $\gamma \in (\gamma^\star, 1)$ and $\gamma^\star$ depends on the considered region of the state space, which is in agreement with the stability guarantee of Theorem 2. The constant $\gamma^\star$ in Theorem 3 is the same as in Theorem 2, according to the proof of Theorem 3 in Section VIII-G. It is possible to ensure the continuity of V_γ uniformly with respect to γ in $(\gamma^\star, 1)$, that is when $\gamma^\star \in (0, 1)$ is the same for all $x \in \mathbb{R}^n$, provided we strengthen the conditions of Theorem 3.

[2]To estimate numerically the real lower bound on the discount factor would require to compute the optimal solutions, which is numerically hard for this example and is out of the scope of the paper.

[3]To apply [21], the set-valued mapping F_γ^* in (5) also has to be such that $F_\gamma^*(x)$ is non-empty and compact for any $x \in \mathbb{R}^n$, see Theorem 2.8 in [21]. Non-emptiness follows from the Standing Assumption. Compactness of $F_\gamma^*(x)$ proceeds from the compactness of $\mathcal{U}_\gamma^*(x)$ (when f is continuous, which is assumed to be the case in Theorem 3), which is a consequence of the conditions of Theorem 3 and the continuity of V_γ proved in this theorem, according to item (a) of Theorem 1.17 in [31].

This is the purpose of the corollary below, the proof of which directly proceeds from the proof of Theorem 3 and is therefore omitted in Section VIII.

Corollary 4. *Suppose the following holds.*

(a) *The conditions of Corollary 2 are verified.*
(b) *Items (b)-(c) of Theorem 3 hold.*

Then, for any $\gamma \in (\gamma^\star, 1)$ *with* $\gamma^\star > \dfrac{\overline{a}_V}{\overline{a}_V + a_W}$ *and* $\overline{a}_V$ *and* a_W *as defined in Corollary 2, the optimal value function* V_γ *in (3) is continuous on* $\mathbb{R}^n$. □

VI. Near-optimal sequence of inputs

A crucial challenge in practice is that it may be difficult to construct an optimal sequence of inputs for the discounted cost (2). An alternative is to apply a *near*-optimal sequence of inputs to system (1) instead. Many algorithms that compute near-optimal sequences in various settings are available [23], [26], while the entire fields of approximate dynamic programming and reinforcement learning deal with computing near-optimal control solutions, see [3], [5], [8], [30], [33]. In this section, we prove that robust stability can be ensured in this case under appropriate conditions.

We formalize what we mean by a near-optimal sequence of inputs in the next assumption.

Assumption 2. *The following holds.*

(i) *There exist* $\widehat{\alpha} : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ *continuous, positive semi-definite, and* $\eta \geqslant 0$ *such that for any* $x \in \mathbb{R}^n$ *and* $\gamma \in (\overline{\gamma}, 1)$ *with* $\overline{\gamma} \in [0, 1)$*, there exists an infinite-length sequence of inputs* $\widehat{\mathbf{u}}_\gamma(x)$ *such that* $\widehat{V}_\gamma(x) := J_\gamma(x, \widehat{\mathbf{u}}_\gamma(x)) \leqslant V_\gamma(x) + \widehat{\alpha}(\sigma(x)) + \eta$.
(ii) *For any* $x \in \mathbb{R}^n$, $\widehat{V}_\gamma(x) = \ell(x, \widehat{u}_{\gamma,0}(x)) + \gamma \widehat{V}_\gamma(\widehat{v})$ *where* $\widehat{u}_{\gamma,0}(x)$ *is the first element of* $\widehat{\mathbf{u}}_\gamma(x)$ *and* $\widehat{v} := f(x, \widehat{u}_{\gamma,0}(x))$. □

Item (i) of Assumption 2 means that, for any initial state $x \in \mathbb{R}^n$ and $\gamma \in (\overline{\gamma}, 1)$, we know an infinite-length sequence of inputs such that cost function in (2) evaluated along the corresponding solution to system (1) lower bounds $V_\gamma(x) + \widehat{\alpha}(\sigma(x)) + \eta$. The term $\widehat{\alpha}(\sigma(x)) + \eta$ characterizes the near-optimality of the strategy. Intuitively, it means that larger errors may be allowed far from the set we aim to stabilize, in addition to a constant error that is allowed everywhere. Item (ii) of Assumption 2 is a dynamic programming relationship, which is verified for example when $\widehat{\mathbf{u}}_\gamma(x) = (\widehat{u}_{\gamma,0}(x), \widehat{\mathbf{u}}_\gamma(f(x, \widehat{u}_{\gamma,0}(x))))$, for any $x \in \mathbb{R}^n$, that is when $\widehat{\mathbf{u}}_\gamma(f(x, \widehat{u}_0(x)))$ is the tail of $\widehat{\mathbf{u}}_\gamma(x)$ without the first term $\widehat{u}_{\gamma,0}(x)$. This is true in the common situation where the sequence of near-optimal inputs is defined by a state-feedback law.

Because the sequence $\widehat{\mathbf{u}}_\gamma(x)$ in Assumption 2 may not be unique, for $x \in \mathbb{R}^n$, consistently with Section III, we denote the set of inputs at the start of near-optimal sequences of inputs at x as $\widehat{\mathcal{U}}_\gamma(x)$. We write system (1) in closed-loop with a near-optimal sequence of inputs as

$$x(k+1) \quad \in \quad f\big(x(k), \widehat{\mathcal{U}}_\gamma(x(k))\big) =: \widehat{F}_\gamma(x(k)). \tag{15}$$

The theorem below shows that stability is guaranteed for system (15) when Assumptions 1 and 2 hold.

Theorem 4. *Consider system (15) and suppose Assumptions 1 and 2 hold. Then, there exists* $\beta \in \mathcal{KL}$ *and* $\vartheta \in \mathcal{K}_\infty$ *such that for any* $\delta, \Delta > 0$*, there exists* $\gamma^\star \in (\overline{\gamma}, 1)$ *such that for any* $\gamma \in (\gamma^\star, 1)$ *and* $x \in \{z \in \mathbb{R}^n : \sigma(z) \leqslant \Delta\}$*, any solution* $\phi(\cdot, x)$ *to system (15) satisfies*

$$\sigma(\phi(k, x)) \quad \leqslant \quad \max\{\beta(\sigma(x), k), \delta, \vartheta(\eta)\} \quad \forall k \in \mathbb{Z}_{\geqslant 0}. \tag{16}$$

□

According to Theorem 4, the uniform semiglobal practical stability property ensured in Theorem 2 is preserved when the sequence of inputs is no longer optimal but only near-optimal in the sense of Assumption 2. The only difference with Theorem 2 is the term $\vartheta(\eta)$ in (16), which is inherited from the near-optimality bound in item (i) of Assumption 2. This term vanishes when $\eta = 0$. In the proof of Theorem 4 (see Section VIII-H), the constant $\gamma^\star$ is selected such that $\widehat{\Upsilon}(\overline{\alpha}_{\widehat{Y}}(\Delta), \frac{1-\gamma^\star}{\gamma^\star}) \leqslant \frac{1}{2}\widetilde{\alpha}_W \circ \overline{\alpha}_{\widehat{Y}}^{-1}(\frac{1}{2}\underline{\alpha}_Y(\delta))$ for given $\delta, \Delta > 0$ (and $\eta \geqslant 0$ in item (i) of Assumption 2), see Table I for the expressions of these functions. It is possible to derive similar results as in Corollaries 1 and 2 for system (15) and thus tailored estimates of $\gamma^\star$; we do not do it for the sake of brevity.

The continuity of $\widehat{V}_\gamma$ with respect to x is studied in the proposition below. Its proof follows the same lines as the proof of Theorem 3 and is therefore omitted in Section VIII.

Proposition 1. *Consider system (15) and suppose the following holds.*

(a) *Assumption 1 and 2 are verified.*
(b) f *and* ℓ *are continuous on* $\mathcal{W}$ *and* $\mathcal{U}$ *is continuous and locally bounded on* $\mathbb{R}^n$.
(c) *For any* $M \geqslant 0$*, the set* $\{x : \sigma(x) \leqslant M\}$ *is compact.*

Then, for any $\Delta > 0$*, there exists* $\gamma^\star \in (\overline{\gamma}, 1)$ *such that for any* $\gamma \in (\gamma^\star, 1)$*,* $\widehat{V}_\gamma$ *is continuous on* $\{x \in \mathbb{R}^n : \sigma(x) \leqslant \Delta\}$. □

As above, a similar result as in Corollary 4 can be derived.

VII. Additional results

In this section, we present two additional results. First, we show that the approach can be used to analyse stability for time-varying cost functions for which the stage cost ℓ is multiplied by a term that, contrary to the discounted case, increases with time; we call this scenario *reverse-discounted* optimal control. Second, we provide a relationship between the optimal value functions of the discounted and the undiscounted problems (when the latter exists) using stability under the conditions of Corollary 2.

A. Reverse-discounted cost

The results presented so far concentrate on the case where $\gamma \in (0, 1)$ in (2). In that way, the discount factor γ^k in (2) penalizes the stage cost ℓ as time grows. We could think of the opposite situation where the importance of the stage cost increases with time, that is to take $\gamma > 1$ in (2). The stability

results of Section IV can be easily adapted to this case. We first suppose that the Standing Assumption holds when $\gamma \in [1, \overline{\gamma})$ for some $\overline{\gamma} \in [1, \infty]$ (the conditions in [19] may still be used in this context to verify the validity of this assumption). We can then write the system subject to an optimal sequence of inputs for the cost function (2) with $\gamma \in [1, \overline{\gamma})$ as system (5). The next result ensures stability under the same assumptions as in Theorem 2.

Theorem 5. *Consider system (5) and suppose that Assumption 1 holds for any $\gamma \in [1, \overline{\gamma})$. Then, there exists $\beta \in \mathcal{KL}$ such that for any $\gamma \in [1, \overline{\gamma})$ and $x \in \mathbb{R}^n$, any solution $\phi(\cdot, x)$ to system (5) satisfies $\sigma(\phi(k, x)) \leqslant \beta(\sigma(x), k)$ for any $k \in \mathbb{Z}_{\geqslant 0}$.* □

Regarding the conditions of Theorem 5, we first note that item (ii) of Assumption 1 is independent of γ. When ℓ is uniformly globally exponentially controllable for system (1) with respect to σ with decrease rate $\lambda > 0$ (see Lemma 1), item (i) of Assumption 1 is verified for any $\gamma \in [1, \overline{\gamma})$ with $\overline{\gamma} = e^{\lambda}$. This result directly follows from the proof of Lemma 1 in Section VIII-A. Furthermore, when ℓ is[4] *uniformly dead-beat stabilizable with respect to σ* for system (1), item (i) of Assumption 1 holds for any $\overline{\gamma} \geqslant 1$. This is the case for controllable linear systems, more generally for linear systems when dead-beat stable uncontrollable part, when the stage cost is given by (10). Conditions for uniform dead-beat controllability of a class of nonlinear systems can be found in [27], for instance.

Contrary to the case where $\gamma < 1$ (see Theorem 2), the stability property in Theorem 5 is global and asymptotic, and not semiglobal and practical. This comes from the fact that the perturbative term Υ in the Lyapunov analysis (see Theorem 1) is negative when $\gamma \geqslant 1$. On the other hand, stability does not require extra conditions on γ. The latter only needs to be in $[1, \overline{\gamma})$, which is imposed by the assumptions. Theorem 5 justifies the observations made for the linearized inverted pendulum example at the end of Section IV-D1, which suggested that stability always hold for any $\gamma \geqslant 1$: this is indeed the case as the considered linear system is controllable and the stage cost is of the form of (10).

Another way to increase the stage cost as time proceeds is to multiply it by $1 - \gamma^{k+1}$, where $\gamma \in (0, 1)$, which leads to the cost function

$$\breve{J}_\gamma(x, \mathbf{u}) \quad := \quad \sum_{k=0}^{\infty} (1 - \gamma^{k+1}) \ell(\phi(k, x, \mathbf{u}|_k), u_k). \qquad (17)$$

Contrary to the case where $\gamma > 1$ in (2), the time-varying weight is bounded here. We assume that there exists an optimal sequence of inputs for this cost, for any $x \in \mathbb{R}^n$ and $\gamma \in (0, 1)$. Hence, we can write system (1) subject to an optimal sequence of inputs for cost (17) as

$$x(k+1) \quad \in \quad \breve{F}^*_\gamma(x(k)) := f\big(x(k), \breve{\mathcal{U}}^*_\gamma(x(k))\big), \qquad (18)$$

where $\breve{\mathcal{U}}^*_\gamma(x)$ is the set of optimal inputs at x for cost (17).

The theorem below shows that the stability results derived in Section IV can also be easily adapted to this problem.

Theorem 6. *Consider system (18) and suppose Assumption 1 holds for any $\gamma \in (0, 1)$. Then, for any $\breve{\gamma} \in (0, 1)$, there exists $\beta \in \mathcal{KL}$ such that for any $\gamma \in (0, \breve{\gamma})$, $x \in \mathbb{R}^n$, any solution $\phi(\cdot, x)$ to system (18) satisfies $\sigma(\phi(k, x)) \leqslant \beta(\sigma(x), k)$ for any $k \in \mathbb{Z}_{\geqslant 0}$.* □

As in Theorem 5, the stability property of Theorem 6 is global and asymptotic, which comes from the fact that there is always a minimum weight of $1 - \breve{\gamma}$ in (17). Hence, γ has to be sufficiently small to ensure stability, and not sufficiently close to 1 as before. On the other hand, we note that we are free to select the value $\breve{\gamma}$ in Theorem 6 as we wish, and that the value of $\breve{\gamma}$ has an impact on β and therefore on the convergence of σ along the solutions to the system.

Remark 4. *The analysis of the continuity of the optimal value functions for the two reverse-discounted costs considered in this section is outside the scope of the paper and is therefore left for future work.* □

B. Relationship between the discounted and the undiscounted optimal value functions

Often, the discount factor is introduced in the cost function because the undiscounted problem is too hard to solve. In this case, γ is typically selected close to 1 in the hope of obtaining an optimal value function V_γ close to the one we would have obtained in the undiscounted case, assuming it exists. The next proposition proves that this is indeed the case under appropriate conditions and an explicit relationship between these two functions is provided.

Proposition 2. *Suppose the following holds.*

(a) *The conditions of Corollary 2 are verified.*
(b) *For any $x \in \mathbb{R}^n$, there exists an infinite-length input sequence $\overline{\mathbf{u}}^*(x)$ such that $\overline{V}(x) := \overline{J}(x, \overline{\mathbf{u}}^*(x))$, where $\overline{J}$ corresponds to cost (2) with $\gamma = 1$. In addition, $\overline{V}(x) \leqslant \overline{a}_V \sigma(x)$ for any $x \in \mathbb{R}^n$, where $\overline{a}_V$ is defined in Corollary 2.*

Let $\gamma^\star > \dfrac{\overline{a}_V}{\overline{a}_V + a_W}$, then for any $\gamma \in (\gamma^\star, 1)$ and any $x \in \mathbb{R}^n$, $V_\gamma(x) \leqslant \overline{V}(x) \leqslant V_\gamma(x) + (1 - \gamma)\theta(\gamma)(V_\gamma(x) + W(x))$ where $\theta(\gamma) := \dfrac{\gamma(\overline{a}_W - a_W) + \overline{a}_V}{\gamma a_W + \overline{a}_V(\gamma - 1)}$, $W(x)$ comes from the satisfaction of item (ii) of Assumption 1 and $\overline{a}_V$, a_W, and $\overline{a}_W$ are defined in Corollary 2. □

Item (b) of Proposition 2 means that there exists an optimal solution to the undiscounted problem and, again, conditions to ensure this property can be found in [19]. The inequalities in Proposition 2 state that, for any $x \in \mathbb{R}^n$, the undiscounted optimal value function $\overline{V}(x)$ is between $V_\gamma(x)$ and $V_\gamma(x) + (1 - \gamma)\theta(\gamma)(V_\gamma(x) + W(x))$, where the latter term vanishes as γ approaches 1. Hence, for γ close to 1, $V_\gamma(x)$ and $\overline{V}(x)$ take close values.

Remark 5. *The proof of Proposition 2 strongly relies on the exponential stability ensured by Corollary 2. It should*

[4]We say that ℓ is *uniformly dead-beat stabilizable with respect to σ* for system (1) if there exist $K \in \mathbb{Z}_{>0}$ and $\varrho \in \mathcal{K}_\infty$, such that for any $x \in \mathbb{R}^n$ there exists an admissible infinite-length control input sequence $\mathbf{u}(x)$ such that $\ell(\phi(k, x, \mathbf{u}|_k(x)), u_k(x)) \leqslant \varrho(\sigma(x))$ for any $k \in \mathbb{Z}_{>0}$ and $\ell(\phi(k, x, \mathbf{u}|_k(x)), u_k(x)) = 0$ for any $k \geqslant K$. This property implies that item (i) of Assumption 1 holds with $\overline{\alpha}_V = K\overline{\gamma}^K \varrho \in \mathcal{K}_\infty$ for any $1 \leqslant \gamma \leqslant \overline{\gamma}$.

be possible to relax these conditions to allow for semiglobal exponential stability instead. □

The next result provides a tailored relationship for linear system (9) with quadratic stage cost (10).

Corollary 5. *Consider system (9) and the cost function J_γ in (2) with ℓ defined in (10). Let $\gamma \in (0,1)$ be such that (11) holds and $S_1 > \varpi\frac{1-\gamma}{\gamma}P_\gamma$ where S_1, ϖ and P_γ are defined in Corollary 3 and in Section IV-C, respectively. Then, for any $x \in \mathbb{R}^n$, $V_\gamma(x) \leqslant \overline{V}(x) \leqslant V_\gamma(x) + (1-\gamma)\frac{1-\varsigma(\gamma)}{\varsigma(\gamma)}(V_\gamma(x) + x^{\mathrm{T}}S_2x)$ where $\varsigma(\gamma) > 0$ is such that $\frac{1}{\varpi}S_1 - \frac{1-\gamma}{\gamma}P_\gamma \geqslant \varsigma(\gamma)(\frac{1}{\varpi}S_2 + P_\gamma)$ with S_2 defined in Corollary 3. When Q is positive definite, $\gamma \in (0,1)$ has to be such that $-Q + \frac{1-\gamma}{\gamma}P_\gamma > 0$ holds. In this case, $V_\gamma(x) \leqslant \overline{V}(x) \leqslant \left(1 + (1-\gamma)\frac{1-\varsigma(\gamma)}{\varsigma(\gamma)}\right)V_\gamma(x)$ where $\varsigma(\gamma)$ is such that $Q - \frac{1-\gamma}{\gamma}P_\gamma \geqslant \varsigma(\gamma)P_\gamma$, for $x \in \mathbb{R}^n$.* □

The required conditions on γ in Corollary 5 are satisfied only for $\gamma \in (\gamma^\star, 1)$, with $\gamma^\star$ defined as in Corollary 3. We can note the difference between the upper-bound on $\overline{V}$ in Proposition 2 and Corollary 5.

Corollary 5 can be applied as follows in the general case. We first use Corollary 3 to determine $\gamma^\star$. We then select $\gamma \in (\gamma^\star, 1)$ and we maximize $\varpi > 0$ such that (11) and $S_1 > \varpi\frac{1-\gamma}{\gamma}P_\gamma$ holds (we take $S_1 = P_\gamma$ when P_γ is positive definite, i.e. when (A, C) is observable). Afterwards, we maximize $\varsigma(\gamma) > 0$ under the constraint $\frac{1}{\varpi}S_1 - \frac{1-\gamma}{\gamma}P_\gamma \geqslant \varsigma(\gamma)(\frac{1}{\varpi}S_2 + P_\gamma)$. In the case where Q is positive definite, we select $\gamma \in (\gamma^\star, 1)$ as in Corollary 3 and we maximize $\varsigma(\gamma) > 0$ under the constraint $Q - \frac{1-\gamma}{\gamma}P_\gamma \geqslant \varsigma(\gamma)P_\gamma$. We obtain in Example 1, for $\gamma = 0.9$, $V_\gamma(x) \approx 4.156 \cdot x^2 \leqslant \overline{V}(x) \approx 4.236 \cdot x^2 \leqslant 1.672 \cdot V_\gamma(x) \approx 6.950 \cdot x^2$ with $x \in \mathbb{R}$ ($\varsigma(\gamma) = 0.129$). For the example in Section IV-D1, we obtain, for $\gamma = 0.988$ and $x = (1,1)$, $V_\gamma(x) \approx 3.681 \cdot 10^4 \leqslant \overline{V}(x) \approx 4.158 \cdot 10^4 \leqslant 2.861 \cdot 10^6$ ($\varsigma(\gamma) = 1.904 \cdot 10^{-4}$). The observed conservatism is inherited from Corollary 3 on which Corollary 5 relies.

VIII. Proofs

A. Proof of Lemma 1.

Let $\gamma \in (0,1)$, $x \in \mathbb{R}^n$ and take the sequence $\mathbf{u}(x)$ that satisfies the condition of Lemma 1. It then holds that, for any $N \in \mathbb{Z}_{>0}$, as $\gamma \in (0,1)$,

$$\begin{aligned}\sum_{k=0}^{N}\gamma^k\ell(\phi(k,x,\mathbf{u}|_k(x)),u_k(x)) &\leqslant \sum_{k=0}^{N}\ell(\phi(k,x,\mathbf{u}|_k(x)),u_k(x))\\ &\leqslant \sum_{k=0}^{N}M\sigma(x)e^{-\lambda k}\\ &\leqslant \frac{M\sigma(x)}{1-e^{-\lambda}}.\end{aligned} \tag{19}$$

The inequalities above hold for any $N \in \mathbb{Z}_{>0}$ and the sequence $N \mapsto \sum_{k=0}^{N}\gamma^k\ell(\phi(k,x,\mathbf{u}|_k(x)),\ u_k(x))$ is non-decreasing, therefore the limit of the latter as $N \to \infty$ exists and $J(x,\mathbf{u}(x)) \leqslant \frac{M\sigma(x)}{1-e^{-\lambda}}$. As a result, $V_\gamma(x) \leqslant J(x,\mathbf{u}(x)) \leqslant \frac{M\sigma(x)}{1-e^{-\lambda}}$ (recall that $V_\gamma(x)$ is well-defined in view of the Standing Assumption in Section III). This implies the satisfaction of item (i) of Assumption 1 with $\overline{\alpha}_V = \frac{M}{1-e^{-\lambda}}\mathbb{I}$.

B. Proof of Theorem 1.

Let $\gamma \in (0,1)$, $x \in \mathbb{R}^n$, $\upsilon = f(x, u^*_{\gamma,0}(x))$ where $u^*_{\gamma,0}(x) \in \mathcal{U}^*_\gamma(x)$ is the first element of the optimal sequence $\mathbf{u}^*_\gamma(x)$, which exists according to the Standing Assumption. Since ℓ is nonnegative and in view of item (i) of Assumption 1,

$$\ell(x, u^*_{\gamma,0}(x)) \quad\leqslant\quad V_\gamma(x) \quad\leqslant\quad \overline{\alpha}_V(\sigma(x)), \tag{20}$$

these inequalities will be useful in the following. On the other hand, according to the Bellman equation,

$$V_\gamma(x) \quad=\quad \ell(x, u^*_{\gamma,0}(x)) + \gamma V_\gamma(\upsilon), \tag{21}$$

therefore

$$V_\gamma(\upsilon) - V_\gamma(x) \quad=\quad -\ell(x, u^*_{\gamma,0}(x)) + (1-\gamma)V_\gamma(\upsilon). \tag{22}$$

In view of (21) and since $\ell(x, u^*_{\gamma,0}(x)) \geqslant 0$, $\gamma V_\gamma(\upsilon) \leqslant V_\gamma(x)$. As a result

$$V_\gamma(\upsilon) - V_\gamma(x) \quad\leqslant\quad -\ell(x, u^*_{\gamma,0}(x)) + (1-\gamma)\gamma^{-1}V_\gamma(x), \tag{23}$$

which gives, using (20),

$$V_\gamma(\upsilon) - V_\gamma(x) \quad\leqslant\quad -\ell(x, u^*_{\gamma,0}(x)) + \tfrac{1-\gamma}{\gamma}\overline{\alpha}_V(\sigma(x)). \tag{24}$$

We now distinguish two cases like in the proof of Theorem 1 in [15].

Case 1: $\chi_W \leqslant \mathbb{I}$.
We define $Y_\gamma := V_\gamma + W$. In view of Assumption 1, $Y_\gamma(x) \leqslant \overline{\alpha}_Y(\sigma(x))$ with $\overline{\alpha}_Y = \overline{\alpha}_V + \overline{\alpha}_W \in \mathcal{K}_\infty$. From the second inequality in (6) and the fact that $\chi_W \leqslant \mathbb{I}$, $W(x) \geqslant \alpha_W(\sigma(x)) - \ell(x, u^*_{\gamma,0}(x)) + W(\upsilon) \geqslant \alpha_W(\sigma(x)) - \ell(x, \overline{u}^*_{\gamma,0}(x))$. Consequently, using (20), $Y_\gamma(x) \geqslant \alpha_W(\sigma(x)) - \ell(x, u^*_{\gamma,0}(x)) + \ell(x, u^*_{\gamma,0}(x)) = \alpha_W(\sigma(x))$ and item (a) of Theorem 1 holds with $\underline{\alpha}_Y = \alpha_W \in \mathcal{K}_\infty$.

In view of (6), (24) and the fact that $\chi_W \leqslant \mathbb{I}$,

$$\begin{aligned}Y_\gamma(\upsilon) - Y_\gamma(x) &\leqslant -\ell(x, u^*_{\gamma,0}(x)) + \tfrac{1-\gamma}{\gamma}\overline{\alpha}_V(\sigma(x))\\ &\quad -\alpha_W(\sigma(x)) + \ell(x, u^*_{\gamma,0}(x))\\ &= -\alpha_W(\sigma(x)) + \tfrac{1-\gamma}{\gamma}\overline{\alpha}_V(\sigma(x)).\end{aligned} \tag{25}$$

Item (b) of Theorem 1 is therefore verified with $\alpha_Y = \alpha_W \in \mathcal{K}_\infty$ and $\Upsilon : (s_1, s_2) \mapsto \overline{\alpha}_V(s_1)s_2 \in \mathcal{KK}$.

Case 2: There exists $s \in \mathbb{R}_{\geqslant 0}$ such that $\chi_W(s) > s$.
This case requires to modify the functions V and W. Let $q_V(s) := 2\chi_W(2s)$ and $\rho_V(s) := \int_0^s q_V(t)\,\mathrm{d}t$ for $s \geqslant 0$. We note that q_V and ρ_V are of class $\mathcal{K}_\infty$. We apply Lemma 2 in the appendix with $h(x) = \ell(x, u^*_{\gamma,0}(x))$, $\alpha_1 = \overline{\alpha}_V$ and $\alpha_2 = \frac{1-\gamma}{\gamma}\overline{\alpha}_V$ to obtain

$$\begin{aligned}\rho_V(V_\gamma(\upsilon)) - \rho_V(V_\gamma(x)) &\leqslant -\chi_W(\ell(x, u^*_{\gamma,0}(x)))\ell(x, u^*_{\gamma,0}(x)))\\ &\quad +q_V\big(\overline{\alpha}_V(\sigma(x)) + \tfrac{1-\gamma}{\gamma}\overline{\alpha}_V(\sigma(x))\big)\tfrac{1-\gamma}{\gamma}\overline{\alpha}_V(\sigma(x))\\ &= -\chi_W(\ell(x, u^*_{\gamma,0}(x)))\ell(x, u^*_{\gamma,0}(x)))\\ &\quad +q_V\big(\tfrac{1}{\gamma}\overline{\alpha}_V(\sigma(x))\big)\tfrac{1-\gamma}{\gamma}\overline{\alpha}_V(\sigma(x)).\end{aligned} \tag{26}$$

Let $q_W := \frac{1}{2}\Big(\chi_W + \overline{\alpha}_W\big(\alpha_W^{-1}(2\chi_W)\big)\Big)^{-1}$ and $\rho_W(s) := \int_0^s q_W(t)\,\mathrm{d}t$ for any $s \geqslant 0$. Both q_W and ρ_W are of class-$\mathcal{K}_\infty$. We identify $\alpha_1 = \overline{\alpha}_W$, $\alpha_2 = \chi_W$ and $\alpha_3 = \alpha_W$ in Lemma 3 in the appendix, from which we derive

$$\begin{aligned}
&\rho_W(W(\upsilon)) - \rho_W(W(x)) \leqslant \\
&\quad 2q_W\Big(\chi_W(\ell(x,u^*_{\gamma,0}(x))) + \overline{\alpha}_W\big(\alpha_W^{-1}(2\chi_W(\ell(x,u^*_{\gamma,0}(x))))\big)\Big) \\
&\quad \times\chi_W(\ell(x,u^*_{\gamma,0}(x))) - q_W(\tfrac{1}{4}\alpha_W(\sigma(x)))\tfrac{1}{4}\alpha_W(\sigma(x)) \\
&= \ell(x,u^*_{\gamma,0}(x))\chi_W(\ell(x,u^*_{\gamma,0}(x))) - \widetilde{\alpha}_W(\sigma(x))
\end{aligned} \tag{27}$$

where $\widetilde{\alpha}_W = q_W(\frac{1}{4}\alpha_W)\frac{1}{4}\alpha_W \in \mathcal{K}_\infty$. We define $Y_\gamma := \rho_V(V_\gamma) + \rho_W(W)$. By (26) and (27),

$$\begin{aligned}
Y_\gamma(\upsilon) - Y_\gamma(x) &\leqslant -\chi_W(\ell(x,u^*_{\gamma,0}(x)))\ell(x,u^*_{\gamma,0}(x))) \\
&\quad + q_V\big(\tfrac{1}{\gamma}\overline{\alpha}_V(\sigma(x))\big)\tfrac{1-\gamma}{\gamma}\overline{\alpha}_V(\sigma(x)) \\
&\quad + \ell(x,u^*_{\gamma,0}(x))\chi_W(\ell(x,u^*_{\gamma,0}(x))) - \widetilde{\alpha}_W(\sigma(x)) \\
&\leqslant -\widetilde{\alpha}_W(\sigma(x)) \\
&\quad + q_V\big(\tfrac{1}{\gamma}\overline{\alpha}_V(\sigma(x))\big)\tfrac{1-\gamma}{\gamma}\overline{\alpha}_V(\sigma(x))
\end{aligned} \tag{28}$$

with $\widetilde{\alpha}_W \in \mathcal{K}_\infty$. We have shown that item (b) of Theorem 1 holds with $\alpha_Y = \widetilde{\alpha}_W \in \mathcal{K}_\infty$ and $\Upsilon : (s_1,s_2) \mapsto q_V\big((1+s_2)\overline{\alpha}_V(s_1)\big)s_2\overline{\alpha}_V(s_1) \in \mathcal{KK}$.

We now show the satisfaction of item (a) of Theorem 1. In view of (20) and the second inequality in (6), $Y_\gamma(x) \geqslant \rho_V(\ell(x,u^*_{\gamma,0}(x))) + \rho_W\left(\max\left\{\alpha_W(\sigma(x)) - \chi_W(\ell(x,u^*_{\gamma,0}(x))),0\right\}\right)$. When $\frac{1}{2}\alpha_W(\sigma(x)) \leqslant \chi_W(\ell(x,u^*_{\gamma,0}(x)))$, $Y_\gamma(x) \geqslant \rho_V\left(\chi_W^{-1}(\frac{1}{2}\alpha_W(\sigma(x)))\right)$. When $\frac{1}{2}\alpha_W(\sigma(x)) \geqslant \chi_W(\ell(x,u^*_{\gamma,0}(x)))$, $Y_\gamma(x) \geqslant \rho_W(\frac{1}{2}\alpha_W(\sigma(x)))$. Hence $Y_\gamma(x) \geqslant \underline{\alpha}_Y(\sigma(x))$ with $\underline{\alpha}_Y = \min\left\{\rho_V\left(\chi_W^{-1}(\frac{1}{2}\alpha_W)\right), \rho_W(\frac{1}{2}\alpha_W)\right\} \in \mathcal{K}_\infty$. On the other hand, $Y_\gamma(\sigma(x)) \leqslant \overline{\alpha}_Y(\sigma(x))$ with $\overline{\alpha}_Y = \rho_V(\overline{\alpha}_V) + \rho_W(\overline{\alpha}_W) \in \mathcal{K}_\infty$ in view of Assumption 1. We have proved that item (a) of Theorem 1 holds.

C. Proof of Theorem 2.

Let $\Delta, \delta > 0$, $\gamma \in (\gamma^\star, 1)$ where $\gamma^\star$ is specified in the following, $x \in \mathbb{R}^n$ such that $\sigma(x) \leqslant \Delta$, and $\upsilon = f(x,u^*_{\gamma,0}(x))$ where $u^*_{\gamma,0}(x) \in \mathcal{U}^*_\gamma(x)$ is the first element of the optimal sequence $\mathbf{u}^*_\gamma(x)$. Define $\widetilde{\Delta} := \overline{\alpha}_Y(\Delta) > 0$ and $\widetilde{\delta} := \underline{\alpha}_Y(\delta) > 0$ where $\overline{\alpha}_Y$ and $\underline{\alpha}_Y$ come from Theorem 1. In view of Theorem 1,

$$Y_\gamma(\upsilon) - Y_\gamma(x) \quad\leqslant\quad -\widetilde{\alpha}_Y(Y_\gamma(x)) + \widetilde{\Upsilon}(Y_\gamma(x), \tfrac{1-\gamma}{\gamma}) \tag{29}$$

with $\widetilde{\alpha}_Y := \alpha_Y \circ \overline{\alpha}_Y^{-1}$ and $\widetilde{\Upsilon}(\cdot,\cdot) := \Upsilon(\underline{\alpha}_Y^{-1}(\cdot),\cdot)$. We select $\gamma^\star$ sufficiently close to 1 such that $\widetilde{\Upsilon}(\widetilde{\Delta}, \frac{1-\gamma^\star}{\gamma^\star}) \leqslant \frac{1}{2}\widetilde{\alpha}_Y(\widetilde{\delta})$, which is always possible since $\widetilde{\Upsilon} \in \mathcal{KK}$ and $\gamma \mapsto \frac{1-\gamma}{\gamma}$ strictly decreases to 0 as γ tends to 1. Consequently, as $\widetilde{\Upsilon}$ and $\widetilde{\alpha}_Y$ are increasing, for all $s \in [\widetilde{\delta}, \widetilde{\Delta}]$

$$\widetilde{\Upsilon}(s, \tfrac{1-\gamma}{\gamma}) \leqslant \widetilde{\Upsilon}(s, \tfrac{1-\gamma^\star}{\gamma^\star}) \leqslant \widetilde{\Upsilon}(\widetilde{\Delta}, \tfrac{1-\gamma^\star}{\gamma^\star}) \leqslant \tfrac{1}{2}\widetilde{\alpha}_Y(\widetilde{\delta}) \leqslant \tfrac{1}{2}\widetilde{\alpha}_Y(s). \tag{30}$$

Since $\sigma(x) \leqslant \Delta$, $Y_\gamma(x) \leqslant \widetilde{\Delta}$ in view of item (a) of Theorem 1 and the definition of $\widetilde{\Delta}$ above. Therefore, if $Y_\gamma(x) \geqslant \widetilde{\delta}$, $\widetilde{\Upsilon}(Y_\gamma(x), \frac{1-\gamma}{\gamma}) \leqslant \frac{1}{2}\widetilde{\alpha}_Y(Y_\gamma(x))$ according to (30), and we derive from (29),

$$Y_\gamma(\upsilon) - Y_\gamma(x) \quad\leqslant\quad -\tfrac{1}{2}\widetilde{\alpha}_Y(Y_\gamma(x)). \tag{31}$$

By proceeding by iteration and using Theorem 8 in [28], we deduce that there exists $\widetilde{\beta} \in \mathcal{KL}$ such that for any solution ϕ to (5) initialized at x and any $k \in \mathbb{Z}_{\geqslant 0}$,

$$Y_\gamma(\phi(k,x)) \quad\leqslant\quad \max\left\{\widetilde{\beta}(Y_\gamma(x),k), \widetilde{\delta}\right\}. \tag{32}$$

Using item (a) of Theorem 1 and the definition of $\widetilde{\delta}$, we obtain that (7) holds with $\beta(s,k) = \underline{\alpha}_Y^{-1}\left(\widetilde{\beta}(\overline{\alpha}_Y(s),k)\right)$ for any $s,k \geqslant 0$.

D. Proof of Corollary 1.

Let $\Delta > 0$, $\gamma \in (\gamma^\star, 1)$ where $\gamma^\star$ is specified in the following, and define $\widetilde{\Delta} := \overline{\alpha}_Y(\Delta)$ as in the proof of Theorem 2. First consider the case where $\widetilde{\Delta} < L$, with L defined in Corollary 1. Since $\chi_W \leqslant \mathbb{I}$ according to Corollary 1, item (b) of Theorem 1 holds with $\Upsilon(s_1,s_2) = \overline{\alpha}_V(s_1)s_2$ and $\alpha_Y(s) = \alpha_W(s)$ for $s_1,s_2,s \geqslant 0$ according to Case 1 in the proof of Theorem 1. The definitions of $\widetilde{\Upsilon}$ and $\widetilde{\alpha}_Y$ in Section VIII-C are therefore in this case $\widetilde{\Upsilon}(s_1,s_2) = \overline{\alpha}_V(\underline{\alpha}_Y^{-1}(s_1))s_2 = \overline{\alpha}_V((\alpha_W^{-1}(s_1))s_2$ and $\widetilde{\alpha}_Y(s) = \alpha_W\left((\overline{\alpha}_V + \overline{\alpha}_W)^{-1}(s)\right)$ for $s_1,s_2,s \geqslant 0$. Let $s \in [0,L]$, $\widetilde{\Upsilon}(s, \frac{1-\gamma}{\gamma}) \leqslant \frac{\overline{a}_V}{a_W}s\frac{1-\gamma}{\gamma}$. Similarly, we deduce that $\widetilde{\alpha}_Y(s) \geqslant \frac{a_W}{\overline{a}_V+\overline{a}_W}s$. Consequently, if $\frac{\overline{a}_V}{a_W}s\frac{1-\gamma}{\gamma} \leqslant \frac{1}{2}\frac{a_W}{\overline{a}_V+\overline{a}_W}s$, then $\widetilde{\Upsilon}(s, \frac{1-\gamma}{\gamma}) \leqslant \frac{1}{2}\widetilde{\alpha}_Y(s)$. For the first inequality to be true, it suffices to select $\gamma^\star \geqslant \left(1 + \frac{1}{2}\frac{a_W^2}{\overline{a}_V(\overline{a}_V+\overline{a}_W)}\right)^{-1}$. In this case, for any $\gamma \in (\gamma^\star,1)$ and any $s \in [0,\widetilde{\Delta}] \subseteq [0,L]$, $\widetilde{\Upsilon}(s, \frac{1-\gamma}{\gamma}) \leqslant \frac{1}{2}\widetilde{\alpha}_Y(s)$ and we follow the proof of Theorem 2 to obtain the desired result.

When $\widetilde{\Delta} > L$, we select $\gamma^\star \in \left[\left(1 + \frac{1}{2}\frac{a_W^2}{\overline{a}_V(\overline{a}_V+\overline{a}_W)}\right)^{-1}, 1\right)$ such that $\widetilde{\Upsilon}(\widetilde{\Delta}, \frac{1-\gamma^\star}{\gamma^\star}) \leqslant \widetilde{\alpha}_Y(L)$. The fact that $\gamma^\star \geqslant \left(1 + \frac{1}{2}\frac{a_W^2}{\overline{a}_V(\overline{a}_V+\overline{a}_W)}\right)^{-1}$ ensures that $\widetilde{\Upsilon}(s, \frac{1-\gamma}{\gamma}) \leqslant \frac{1}{2}\widetilde{\alpha}_Y(s)$ for any $s \in [0,L]$ and $\gamma \in (\gamma^\star,1)$. The fact that $\widetilde{\Upsilon}(\widetilde{\Delta}, \frac{1-\gamma^\star}{\gamma^\star}) \leqslant \widetilde{\alpha}_Y(L)$ guarantees that $\widetilde{\Upsilon}(s, \frac{1-\gamma}{\gamma}) \leqslant \frac{1}{2}\widetilde{\alpha}_Y(s)$ for $s \in [L,\widetilde{\Delta}]$. The desired result is then derived by following the proof of Theorem 2.

E. Proof of Corollary 2.

Let $x \in \mathbb{R}^n$, $\upsilon \in F^*_\gamma(x)$ and $\gamma \in (\gamma^\star,1)$. Since $\chi_W = \mathbb{I}$, we can use the developments of Case 1 in the proof of Theorem 1. In particular, in view of (25),

$$Y_\gamma(\upsilon) - Y_\gamma(x) \quad\leqslant\quad -\alpha_W(\sigma(x)) + \tfrac{1-\gamma}{\gamma}\overline{\alpha}_V(\sigma(x)). \tag{33}$$

where $Y_\gamma = V_\gamma + W$. Using the conditions of Corollary 2

$$Y_\gamma(\upsilon) - Y_\gamma(x) \quad\leqslant\quad (-a_W + \tfrac{1-\gamma}{\gamma}\overline{a}_V)\sigma(x). \tag{34}$$

Since $\gamma^\star > \dfrac{\overline{a}_V}{\overline{a}_V + a_W}$ and $\gamma \in (\gamma^\star,1)$, there exists $\varepsilon \in (0,\overline{a}_Y)$, where $\overline{a}_Y := \overline{a}_W + \overline{a}_V$, such that $-a_W + \frac{1-\gamma^\star}{\gamma^\star}\overline{a}_V < -\varepsilon$. As $Y_\gamma(x) \leqslant \overline{a}_Y\sigma(x)$ in view of item (a) of Theorem 1 and the assumptions in Corollary 2, $Y_\gamma(\upsilon) - Y_\gamma(x) \leqslant$

$-\varepsilon\sigma(x) \leqslant -\varepsilon\overline{a}_Y^{-1}Y_\gamma(x)$ and thus $Y_\gamma(\upsilon) \leqslant (1-\varepsilon\overline{a}_Y^{-1})Y_\gamma(x)$ with $1-\varepsilon\overline{a}_Y^{-1} \in (0,1)$. Let $x \in \mathbb{R}^n$ and denote $\phi(k,x)$ be a corresponding solution to (5) at time $k \in \mathbb{Z}_{\geqslant 0}$, it holds that $Y_\gamma(\phi(k,x)) \leqslant (1-\varepsilon\overline{a}_Y^{-1})^k Y_\gamma(x)$. Using the fact that $Y_\gamma(x) \geqslant \alpha_W(\sigma(x)) = a_W\sigma(x)$ (in view of Case 1 in the proof of Theorem 1), we derive that Corollary 2 holds with $K = \frac{\overline{a}_Y}{a_W} = \frac{\overline{a}_W+\overline{a}_Y}{a_W}$ and $\lambda = -\ln(1-\varepsilon\overline{a}_Y^{-1})$.

F. Proof of Corollary 3.

The existence of an optimal sequence of inputs is ensured in view of Section 4.2 in [3], hence the Standing Assumption holds. Let $\gamma \in (\gamma^\star, 1)$ and $x \in \mathbb{R}^n$. The optimal value function at x for the discounted problem, namely $V_\gamma(x)$, is less than or equal the optimal value function for the *undiscounted problem*, which corresponds to $x^{\mathrm{T}}Px$, as $\gamma \in (0,1)$ and the quadratic stage cost is non-negative. As a result, in view of (23),

$$\begin{array}{rcl} V_\gamma(\upsilon) - V_\gamma(x) & \leqslant & -\ell(x, u^*_{\gamma,0}(x)) + (1-\gamma)\gamma^{-1}V_\gamma(x) \\ & \leqslant & -\ell(x, K^*_\gamma x) + (1-\gamma)\gamma^{-1}x^{\mathrm{T}}Px. \end{array} \tag{35}$$

On the other hand, let $W(x) := \frac{1}{\varpi}x^{\mathrm{T}}S_2x$. In view of (11),

$$W(\upsilon) - W(x) \leqslant -\tfrac{1}{\varpi}x^{\mathrm{T}}S_1x + \ell(x, K^*_\gamma x), \tag{36}$$

where $\upsilon = (A+BK^*_\gamma)x$. Let $Y_\gamma = V_\gamma + W$. We derive from (35) and (36) that $Y_\gamma(\upsilon) - Y_\gamma(x) \leqslant -x^{\mathrm{T}}\left(\frac{1}{\varpi}S_1 - \frac{1-\gamma}{\gamma}P\right)x$. Since $\gamma > \gamma^\star$, $\frac{1}{\varpi}S_1 - \frac{1-\gamma}{\gamma}P > \frac{1}{\varpi}S_1 - \frac{1-\gamma^\star}{\gamma^\star}P$. As $\gamma^\star > \frac{\varpi}{\varpi+\alpha}$, $\frac{1}{\varpi}S_1 - \frac{1-\gamma^\star}{\gamma^\star}P > \frac{1}{\varpi}S_1 - \frac{\alpha}{\varpi}P$. In view of (12), $\frac{1}{\varpi}S_1 - \frac{\alpha}{\varpi}P \geqslant 0$, consequently $\frac{1}{\varpi}S_1 - \frac{1-\gamma^\star}{\gamma^\star}P > 0$. We deduce that there exists $\varepsilon > 0$ such that $Y_\gamma(\upsilon) - Y_\gamma(x) \leqslant -\varepsilon|x|^2$. Since $\frac{\lambda_{\min}(S_2)}{\varpi}|x|^2 \leqslant Y_\gamma(x) \leqslant \lambda_{\max}(P + \frac{1}{\varpi}S_2)|x|^2$ and $0 < \frac{\lambda_{\min}(S_2)}{\varpi} < \lambda_{\max}(P+\frac{1}{\varpi}S_2)$, we conclude that the origin is uniformly globally exponentially stable.

When Q is positive definite, the pair (A, C) is observable, hence P is positive definite. Then, the desired result follows by using (35) and the definition of $\gamma^\star$ in this case in Corollary 3.

G. Proof of Theorem 3

The proof follows the same steps as the part on continuity in Section 5.2 in [20]. There are several differences though, which prevent us to directly apply the results of [20]. Indeed, the stage cost we consider in (2) is different as it involves the input u (and not only the state) and it is discounted. Furthermore, we investigate stability with respect to σ, which is not necessarily the distance to a compact set as in [20]. Finally, the 'optimal' closed-loop system (5) satisfies a (uniform) semiglobal practical asymptotic stability property according to Theorem 2 (which applies in view of item (a) of Theorem 3), while the results in [20] rely on a (uniform) global asymptotic stability property. We show in the following that we can still apply similar arguments as in [20] to prove the desired property. We do it in detail for lower semicontinuity; the proof of upper semicontinuity similarly follows in view of [20], we therefore omit it[5]. The desired continuity property is a consequence of these two properties.

System (1) can be written as

$$\left\{\begin{array}{rcl} x(k+1) & = & f(x(k),u(k)) \\ u(k+1) & \in & \mathcal{U}(x(k+1)) = \mathcal{U}\big(f(x(k),u(k))\big). \end{array}\right. \tag{37}$$

We thus obtain an extended autonomous system, which we rewrite as

$$\chi(k+1) \in G(\chi(k)), \tag{38}$$

where $\chi := (x,u)$ and $G(\chi) := (f(x,u), \mathcal{U}(f(x,u)))$. Since f and $\mathcal{U}$ are continuous according to item (b) of Theorem 3, so is G. We similarly write the system in closed-loop with an optimal sequence of inputs as

$$\chi(k+1) \in G^*_\gamma(\chi(k)), \tag{39}$$

where $G^*_\gamma(\chi) := (f(x,u), \mathcal{U}^*_\gamma(f(x,u)))$ for $\chi = (x,u)$ and $\gamma \in (0,1)$. The set of solutions to (39) is included in the set of solutions to (38) as $\mathcal{U}^*_\gamma(x) \subseteq \mathcal{U}(x)$ for any $x \in \mathbb{R}^n$ and $\gamma \in (0,1)$. In the following, we respectively denote solutions to (38) and (39) at time k initialized at χ as $\psi(k,\chi) = (\phi(k,\chi), u(k,\chi))$ and $\psi^*(k,\chi) = (\phi^*(k,\chi), u^*(k,\chi))$. The notation we use in this proof for ϕ is slightly different compared to the rest of the paper, as it depends on the initial condition χ and not only on x as before.

Let $\Delta > 0$ and[6] $\delta \in (0,1)$, $x \in \{z : \sigma(z) \leqslant \Delta\}$, and consider an arbitrary sequence x_n, $n \in \mathbb{Z}_{\geqslant 0}$, such that $x_n \to x$ as $n \to \infty$. Let u_n, $n \in \mathbb{Z}_{\geqslant 0}$, be a converging sequence such that $u_n \to u$ as $n \to \infty$ and $u_n \in \mathcal{U}(x_n)$ for any $n \in \mathbb{Z}_{\geqslant 0}$. By (outer semi)continuity[7] of $\mathcal{U}$, $u \in \mathcal{U}(x)$ (see Section II). Let $\chi = (x,u)$ and $\chi_n = (x_n, u_n)$ for any $n \in \mathbb{Z}_{\geqslant 0}$.

Let $\varepsilon > 0$, and $\gamma^\star \in (0,1)$ be such that (7) holds for the set of initial conditions $\{z : \sigma(z) \leqslant \overline{\Delta}\}$ where $\overline{\Delta} > \Delta$. Let $\gamma \in (\gamma^\star, 1)$, and $j \in \mathbb{Z}_{\geqslant 0}$ be sufficiently big such that

$$(\gamma^\star)^j\overline{\alpha}_V\left(2\max\{\beta(\sigma(x)+1, j), \delta\}\right) \leqslant \frac{\varepsilon}{4}, \tag{40}$$

where β comes (7). Since $x_n \to x$ as $n \to \infty$ and $\sigma(x) \leqslant \Delta < \overline{\Delta}$, $\sigma(x_n) \leqslant \overline{\Delta}$ for n sufficiently big (recall that σ is continuous). Consequently, we apply Theorem 2 for the set of initial conditions $\{z : \sigma(z) \leqslant \overline{\Delta}\}$, noting that the required conditions hold according to item (a) of Theorem 3, to obtain, for n sufficiently big,

$$\sigma(\phi^*(k,\chi_n)) \leqslant \max\{\beta(\sigma(x_n), k), \delta\} \qquad \forall k \in \mathbb{Z}_{\geqslant 0}, \tag{41}$$

therefore $\sigma(\phi^*(k,\chi_n)) \leqslant \max\{\beta(\sigma(x_n),0),\delta\}$ for any $k \in \mathbb{Z}_{\geqslant 0}$. For $n \in \mathbb{Z}_{\geqslant 0}$ sufficiently big, by continuity of σ, $\sigma(x_n) \leqslant \sigma(x)+1$, hence

$$\sigma(\phi^*(k,\chi_n)) \leqslant \max\{\beta(\sigma(x)+1,0),\delta\} \qquad \forall k \in \mathbb{Z}_{\geqslant 0}. \tag{42}$$

[5]We can apply Lemma 18 in [20] as done in [20] since our definition of continuity is equivalent to the one in Definition 3 in [20] as we also assume $\mathcal{U}$ to be locally bounded, see Lemma 5.15 in [13] and page 193 in [31]; the fact that the distance to a set (and not a generic continuous σ) is considered in Lemma 18 in [20] is not an issue.

[6]Since the satisfaction of item (b) of Theorem 3 for $\delta \in (0,1)$ implies its satisfaction for any $\delta \geqslant 1$, there is no loss of generality in assuming $\delta \in (0,1)$.

[7]Since $\mathcal{U}$ is assumed to be continuous, it is outer semicontinuous, which allows us to write that $u \in \mathcal{U}(x)$.

Let $M(x) = \beta(\sigma(x)+1,0)+2$ and $\mathcal{M}(x) := \{z : \sigma(z) \leqslant M(x)\}$. The set $\mathcal{M}(x)$ is compact according to item (c) of Theorem 3. Consequently, $\mathcal{U}(\mathcal{M}(x))$ is also compact by continuity and local boundedness of $\mathcal{U}$; closeness follows from the fact that $\mathcal{U}$ is outer semicontinuous (see Theorem 5.25 in [31]) and boundedness proceeds from the fact that $\mathcal{U}$ is locally bounded (see Proposition 5.15 in [31]).

For $\dfrac{\varepsilon}{2j}$, on the compact set $\mathcal{M}(x)\times\mathcal{U}(\mathcal{M}(x))$, by (uniform[8]) continuity of ℓ (see item (b) of Theorem 3), there exists $\delta_1>0$ such that for any $\chi_1,\chi_2 \in \mathcal{M}(x)\times\mathcal{U}(\mathcal{M}(x))$,

$$|\chi_1-\chi_2| \leqslant \delta_1 \quad \Rightarrow \quad |\ell(\chi_1)-\ell(\chi_2)| \leqslant \frac{\varepsilon}{2j}. \tag{43}$$

On the other hand, let $\varepsilon_2 = \min\left\{\frac{1}{2}\overline{\alpha}_V^{-1}(\frac{\varepsilon}{4}),1\right\}$ (where $\overline{\alpha}_V$ comes from item (i) of Assumption 1), by (uniform) continuity of σ, there exists $\delta_2>0$ such that for any $(x_1,x_2)\in\mathcal{M}(x)\times(\mathcal{M}(x)+\delta_1\mathbb{B})$,

$$|x_1-x_2| \leqslant \delta_2 \quad \Rightarrow \quad |\sigma(x_1)-\sigma(x_2)| \leqslant \varepsilon_2. \tag{44}$$

For the triple $(j,\min\{\delta_1,\delta_2\},\chi)$, Proposition 6.14 in [13] ensures[9] that there exists $\delta_3>0$ such that for any solution $\psi^*(\cdot,\chi_n)$ to (39) with n sufficiently big so that $|\chi-\chi_n|<\delta_3$, there exists a solution to (38) initialized at χ and denoted $\widehat{\psi}_n=(\widehat{\phi}_n,\widehat{u}_n)$ such that

$$\left|\psi^*(k,\chi_n)-\widehat{\psi}_n(k,\chi)\right| \leqslant \min\{\delta_1,\delta_2\} \qquad \forall k\in\{0,\dots,j\}. \tag{45}$$

In the following, we consider $n\in\mathbb{Z}_{\geqslant 0}$ sufficiently big such that the properties obtained above hold. For any $k\in\{0,\dots,j\}$, $\psi^*(k,\chi_n)\in\mathcal{M}(x)$ (in view of (42)) and $\widehat{\psi}_n(k,\chi)\in\mathcal{M}(x)+\delta_1\mathbb{B}$ (from (42) and (45)), we thus derive from (44) and (45) that, for any $k\in\{0,\dots,j\}$,

$$\left|\sigma(\phi^*(k,\chi_n))-\sigma(\widehat{\phi}_n(k,\chi))\right| \leqslant \varepsilon_2. \tag{46}$$

We deduce from the inequality above and (42), for $k\in\{0,\dots,j\}$,

$$\begin{aligned}\sigma(\widehat{\phi}_n(k,\chi)) &\leqslant \sigma(\phi^*(k,\chi_n))+\varepsilon_2\\ &\leqslant \max\{\beta(\sigma(x)+1,0),\delta\}+\varepsilon_2\\ &\leqslant \beta(\sigma(x)+1,0)+\delta+\varepsilon_2.\end{aligned} \tag{47}$$

Since $\varepsilon_2\leqslant 1$ and $\delta<1$, in view of the definition of $M(x)$, $\sigma(\widehat{\phi}_n(k,\chi))\leqslant M(x)$ and $\widehat{\psi}_n(k,\chi)\in\mathcal{M}(x)\times\mathcal{U}(\mathcal{M}(x))$ for any $k\in\{0,\dots,j\}$. We derive from (43) and (45) that, for any $k\in\{0,\dots,j\}$,

$$\left|\ell(\psi^*(k,\chi_n))-\ell(\widehat{\psi}_n(k,\chi))\right| \leqslant \frac{\varepsilon}{2j}. \tag{48}$$

We now consider the optimal value function V_γ in (3) at x_n

$$V_\gamma(x_n) = \sum_{k=0}^{\infty}\gamma^k\ell(\psi^*(k,\chi_n)) \geqslant \sum_{k=0}^{j-1}\gamma^k\ell(\psi^*(k,\chi_n)), \tag{49}$$

from (48)

$$V_\gamma(x_n) \geqslant \sum_{k=0}^{j-1}\gamma^k\ell(\widehat{\psi}_n(k,\chi))-\frac{\varepsilon}{2}. \tag{50}$$

Adding and subtracting $\sum_{k=j}^{\infty}\gamma^k\ell(\psi^*(k-j,\widehat{\psi}_n(j,\chi)))$ (which is finite) above gives, using the definition of $V_\gamma(x)$,

$$\begin{aligned}V_\gamma(x_n) &\geqslant \sum_{k=0}^{j-1}\gamma^k\ell(\widehat{\psi}_n(k,\chi))+\sum_{k=j}^{\infty}\gamma^k\ell(\psi^*(k-j,\widehat{\psi}_n(j,\chi)))\\ &\quad -\frac{\varepsilon}{2}-\sum_{k=j}^{\infty}\gamma^k\ell(\psi^*(k-j,\widehat{\psi}_n(j,\chi)))\\ &\geqslant V_\gamma(x)-\frac{\varepsilon}{2}-\sum_{k=j}^{\infty}\gamma^k\ell(\psi^*(k-j,\widehat{\psi}_n(j,\chi))).\end{aligned} \tag{51}$$

The last term in the right hand-side of the inequality above verifies

$$\sum_{k=j}^{\infty}\gamma^k\ell(\psi^*(k-j,\widehat{\psi}_n(j,\chi))) = \gamma^j V_\gamma(\widehat{\phi}_n(j,\chi)). \tag{52}$$

On the other hand, from item (i) of Assumption 1 and (46)

$$V_\gamma(\widehat{\phi}_n(j,\chi))\leqslant\overline{\alpha}_V(\sigma(\widehat{\phi}_n(j,\chi)))\leqslant\overline{\alpha}_V(\sigma(\phi^*(j,\chi_n))+\varepsilon_2), \tag{53}$$

since $\overline{\alpha}_V\in\mathcal{K}_\infty$, $\overline{\alpha}_V(a+b)\leqslant\overline{\alpha}_V(2a)+\overline{\alpha}_V(2b)$ for any $a,b\geqslant 0$, hence

$$V_\gamma(\widehat{\phi}_n(j,\chi)) \leqslant \overline{\alpha}_V(2\sigma(\phi^*(j,\chi_n)))+\overline{\alpha}_V(2\varepsilon_2). \tag{54}$$

From (41) and the definition of ε_2,

$$V_\gamma(\widehat{\phi}_n(j,\chi)) \leqslant \overline{\alpha}_V\left(2\max\{\beta(\sigma(x_n),j),\delta\}\right)+\frac{\varepsilon}{4}. \tag{55}$$

Since n is sufficiently large such that $\sigma(x_n)\leqslant\sigma(x)+1$,

$$V_\gamma(\widehat{\phi}_n(j,\chi)) \leqslant \overline{\alpha}_V\left(2\max\{\beta(\sigma(x)+1,j),\delta\}\right)+\frac{\varepsilon}{4} \tag{56}$$

using (40) and the fact that $\gamma\in(\gamma^\star,1)$,

$$\begin{aligned}\gamma^j V_\gamma(\widehat{\phi}_n(j,\chi))&\leqslant\gamma^j\left(\overline{\alpha}_V\left(2\max\{\beta(\sigma(x)+1,j),\delta\}\right)+\frac{\varepsilon}{4}\right)\\ &\leqslant\frac{\varepsilon}{4}+\gamma^j\frac{\varepsilon}{4}\leqslant\frac{\varepsilon}{2}.\end{aligned} \tag{57}$$

Combining (51) with the inequality above and (52) leads to

$$V_\gamma(x_n) \geqslant V_\gamma(x)-\varepsilon. \tag{58}$$

We have proved that V_γ is lower semicontinuous at x. Since x has been taken arbitrarily in $\{z:\sigma(z)\leqslant\Delta\}$, for any $\gamma\in(\gamma^\star,1)$, V_γ is lower semicontinuous on $\{z:\sigma(z)\leqslant\Delta\}$.

H. Proof of Theorem 4

Let $x\in\mathbb{R}^n$, $\gamma\in(\overline{\gamma},1)$ where $\overline{\gamma}$ comes from item (i) of Assumption 2, $\widehat{v}=f(x,\widehat{u}_{\gamma,0}(x))$ where $\widehat{u}_{\gamma,0}(x)$ is the first element of the sequence $\widehat{\mathbf{u}}_\gamma(x)$ given by Assumption 2. In view of item (ii) of Assumption 2,

$$\widehat{V}_\gamma(\widehat{v})-\widehat{V}_\gamma(x) = -\ell(x,\widehat{u}_{\gamma,0}(x))+(1-\gamma)\widehat{V}_\gamma(\widehat{v}). \tag{59}$$

[8]Since we work on the compact set $\mathcal{M}(x)\times\mathcal{U}(\mathcal{M}(x))$, the continuity of ℓ is uniform on that set by Heine-Cantor theorem.

[9]The conditions of Proposition 6.14 in [13] are verified as we can embed system (38) as a hybrid system with empty flow set, which is nominally well-posed since G is continuous, the jump set is $\mathbb{R}^{n+m}$ and pre-forward complete - see Definition 6.12 in [13]. We apply it with $\varepsilon=\min\{\delta_1,\delta_2\}$, $\tau=j$ and $K=\{\chi\}$.

Moreover, item (ii) of Assumption 2 implies that $\gamma\widehat{V}_\gamma(\widehat{\upsilon}) \leqslant \widehat{V}_\gamma(x)$ since $\ell(x,\widehat{u}_{\gamma,0}(x)) \geqslant 0$. As a result

$$\widehat{V}_\gamma(\widehat{\upsilon}) - \widehat{V}_\gamma(x) \quad \leqslant \quad -\ell(x,\widehat{u}_{\gamma,0}(x)) + (1-\gamma)\gamma^{-1}\widehat{V}_\gamma(x). \tag{60}$$

Using item (i) of Assumption 2,

$$\begin{aligned}\widehat{V}_\gamma(\widehat{\upsilon}) - \widehat{V}_\gamma(x) \leqslant & -\ell(x,\widehat{u}_{\gamma,0}(x)) + (1-\gamma)\gamma^{-1} \\ & \times \left(V_\gamma(x) + \widehat{\alpha}(\sigma(x)) + \eta\right),\end{aligned} \tag{61}$$

and item (i) of Assumption 1,

$$\begin{aligned}\widehat{V}_\gamma(\widehat{\upsilon}) - \widehat{V}_\gamma(x) \leqslant & -\ell(x,\widehat{u}_{\gamma,0}(x)) + (1-\gamma)\gamma^{-1} \\ & \times \left(\overline{\alpha}_V(\sigma(x)) + \widehat{\alpha}(\sigma(x)) + \eta\right).\end{aligned} \tag{62}$$

On the other hand, we also have from item (i) of Assumption 2 that

$$\ell(x,\widehat{u}_{\gamma,0}(x)) \quad \leqslant \quad \widehat{V}_\gamma(x) \quad \leqslant \quad \overline{\alpha}_V(\sigma(x)) + \widehat{\alpha}(\sigma(x)) + \eta, \tag{63}$$

which will be useful in the following.

Like in the proof of Theorem 1, we need to distinguish two cases depending on χ_W in Assumption 1. We only treat the case where there exists $s \geqslant 0$ such that $\chi_W(s) \geqslant s$; the case where $\chi_W \leqslant \mathbb{I}$ similarly follows. Let $\widehat{Y}_\gamma := \rho_V(\widehat{V}_\gamma) + \rho_W(W)$, where $\rho_V, \rho_W \in \mathcal{K}_\infty$ are provided in Table I. We apply Lemma 2 to (62) and (63) by identifying $h(x) = \ell(x,\widehat{u}_{\gamma,0}(x))$, $\alpha_1(\sigma(x)) = \overline{\alpha}_V(x) + \widehat{\alpha}(\sigma(x)) + \eta$, and $\alpha_2(\sigma(x)) = (1-\gamma)\gamma^{-1}\left(\overline{\alpha}_V(\sigma(x)) + \widehat{\alpha}(\sigma(x)) + \eta\right)$. In that way, we obtain

$$\begin{aligned}\rho_V(\widehat{V}_\gamma(\widehat{\upsilon})) - \rho_V(\widehat{V}_\gamma(x)) \leqslant & -q_V\left(\tfrac{1}{2}\ell(x,\widehat{u}_{\gamma,0}(x))\right) \\ & \times \tfrac{1}{2}\ell(x,\widehat{u}_{\gamma,0}(x)) \\ & + q_V\left(\tfrac{1}{\gamma}\left(\overline{\alpha}_V(\sigma(x)) + \widehat{\alpha}(\sigma(x)) + \eta\right)\right) \\ & \times (1-\gamma)\gamma^{-1}\left(\overline{\alpha}_V(\sigma(x)) + \widehat{\alpha}(\sigma(x)) + \eta\right) \\ & = -\chi_W(\ell(x,\widehat{u}_{\gamma,0}(x)))\ell(x,\widehat{u}_{\gamma,0}(x)) \\ & + q_V\left(\tfrac{1}{\gamma}\left(\overline{\alpha}_V(\sigma(x)) + \widehat{\alpha}(\sigma(x)) + \eta\right)\right) \\ & \times (1-\gamma)\gamma^{-1}\left(\overline{\alpha}_V(\sigma(x)) + \widehat{\alpha}(\sigma(x)) + \eta\right)\end{aligned} \tag{64}$$

where q_V is given in Table I. We derive from the inequality above and (27) that

$$\begin{aligned}\widehat{Y}_\gamma(\widehat{\upsilon}) - \widehat{Y}_\gamma(x) \quad \leqslant \quad & -\widetilde{\alpha}_W(\sigma(x)) \\ & + q_V\left(\tfrac{1}{\gamma}\left(\overline{\alpha}_V(\sigma(x)) + \widehat{\alpha}(\sigma(x)) + \eta\right)\right) \\ & \times (1-\gamma)\gamma^{-1}\left(\overline{\alpha}_V(\sigma(x)) + \widehat{\alpha}(\sigma(x)) + \eta\right).\end{aligned} \tag{65}$$

Using that $\widehat{Y}_\gamma(x) \geqslant Y_\gamma(x) \geqslant \underline{\alpha}_Y(\sigma(x))$ where $\underline{\alpha}_Y \in \mathcal{K}_\infty$, see Table I,

$$\widehat{Y}_\gamma(\widehat{\upsilon}) - \widehat{Y}_\gamma(x) \quad \leqslant \quad -\widetilde{\alpha}_W(\sigma(x)) + \widehat{\Upsilon}(\widehat{Y}_\gamma(x), \tfrac{1-\gamma}{\gamma}), \tag{66}$$

with $\widehat{\Upsilon}(s_1,s_2) = q_V\left((1+s_2)\left((\overline{\alpha}_V + \widehat{\alpha}) \circ \underline{\alpha}_Y^{-1}(s_1) + \eta\right)\right) s_2 \times \left((\overline{\alpha}_V + \widehat{\alpha}) \circ \underline{\alpha}_Y^{-1}(s_1) + \eta\right)$ for $s_1, s_2 \geqslant 0$, note that $\widehat{\Upsilon}$ is strictly increasing in its first argument and of class $\mathcal{K}$ in its second argument. On the other hand, $\widehat{Y}_\gamma(x) \leqslant \rho_V(\overline{\alpha}_V(\sigma(x)) + \widehat{\alpha}(\sigma(x)) + \eta) + \rho_W(\overline{\alpha}_W(\sigma(x)))$ in view of (63) and item (ii) of Assumption 1. From which we derive that $\widehat{Y}_\gamma(x) \leqslant \overline{\alpha}_{\widehat{Y}}(\sigma(x)) + \widehat{\vartheta}(\eta)$ where $\overline{\alpha}_{\widehat{Y}} := \rho_V(2\overline{\alpha}_V + 2\widehat{\alpha}) + \rho_W(\overline{\alpha}_W) \in \mathcal{K}_\infty$ and $\widehat{\vartheta} := \rho_W(2\mathbb{I}) \in \mathcal{K}_\infty$. Consequently, when $\widehat{\vartheta}(\eta) \leqslant \frac{1}{2}\widehat{Y}_\gamma(x)$, $\frac{1}{2}\widehat{Y}_\gamma(x) \leqslant \overline{\alpha}_{\widehat{Y}}(\sigma(x))$. In this case, in view of (66),

$$\begin{aligned}\widehat{Y}_\gamma(\widehat{\upsilon}) - \widehat{Y}_\gamma(x) \leqslant & -\widetilde{\alpha}_W \circ \overline{\alpha}_{\widehat{Y}}^{-1}(\tfrac{1}{2}\widehat{Y}_\gamma(x)) \\ & + \widehat{\Upsilon}\left(\widehat{Y}_\gamma(x), \tfrac{1-\gamma}{\gamma}\right).\end{aligned} \tag{67}$$

Let $\Delta, \delta > 0$ and suppose $\sigma(x) \leqslant \Delta$. We now take $\gamma \in (\gamma^\star, 1)$ where $\gamma^\star \in (\overline{\gamma}, 1)$ is specified in the following. Let $\widehat{\Delta} = \overline{\alpha}_{\widehat{Y}}(\Delta) + \widehat{\vartheta}(\eta)$ and $\widehat{\delta} = \underline{\alpha}_Y(\delta)$. We select $\gamma^\star$ sufficiently close to 1 such that $\widehat{\Upsilon}(\widehat{\Delta}, \frac{1-\gamma^\star}{\gamma^\star}) \leqslant \frac{1}{2}\widetilde{\alpha}_W \circ \overline{\alpha}_{\widehat{Y}}^{-1}(\frac{1}{2}\widehat{\delta})$, which is always possible in view of the definition of $\widehat{\Upsilon}$. By following the same lines as in the proof of Theorem 2, there exists $\widehat{\beta} \in \mathcal{KL}$, which is independent of γ, such that for any solution ϕ to (15) initialized at x and any $k \in \mathbb{Z}_{\geqslant 0}$,

$$\widehat{Y}_\gamma(\phi(k,x)) \quad \leqslant \quad \max\left\{\widehat{\beta}(Y_\gamma(x),k), \widehat{\delta}, 2\widehat{\vartheta}(\eta)\right\}. \tag{68}$$

Since $\underline{\alpha}_Y(\sigma(x)) \leqslant Y_\gamma(x)$, $\sigma(\phi(k,x)) \leqslant \max\left\{\underline{\alpha}_Y^{-1}\left(\widehat{\beta}(Y_\gamma(x),k)\right), \delta, \underline{\alpha}_Y^{-1}(2\widehat{\vartheta}(\eta))\right\}$. Furthermore, $\widehat{Y}_\gamma(x) \leqslant \overline{\alpha}_{\widehat{Y}}(\sigma(x)) + \widehat{\vartheta}(\eta) \leqslant 2\max\{\overline{\alpha}_{\widehat{Y}}(\sigma(x)), \widehat{\vartheta}(\eta)\}$. Consequently, for any solution ϕ to (15) initialized at x and any $k \in \mathbb{Z}_{\geqslant 0}$, $\sigma(\phi(k,x)) \leqslant \max\left\{\underline{\alpha}_Y^{-1}\left(\widehat{\beta}(2\overline{\alpha}_{\widehat{Y}}(\sigma(x)),k)\right), \underline{\alpha}_Y^{-1}\left(\widehat{\beta}(2\widehat{\vartheta}(\eta),0)\right), \delta, \underline{\alpha}_Y^{-1}(2\widehat{\vartheta}(\eta))\right\}$. Property (16) thus holds with $\beta(\cdot) = \underline{\alpha}_Y^{-1}\left(\widehat{\beta}(2\overline{\alpha}_{\widehat{Y}}(\cdot),\cdot)\right)$, $\vartheta(\cdot) = \max\left\{\underline{\alpha}_Y^{-1}\left(\widehat{\beta}(2\widehat{\vartheta}(\cdot),0)\right), \underline{\alpha}_Y^{-1}(2\widehat{\vartheta}(\cdot))\right\}$.

I. Sketch of Proof of Theorem 5 5.

Let $\gamma \in [1,\overline{\gamma})$, $x \in \mathbb{R}^n$, $\upsilon = f(x,u^*_{\gamma,0}(x))$ where $u^*_{\gamma,0}(x) \in \mathcal{U}^*_\gamma(x)$ is the first element of the optimal sequence $\mathbf{u}^*_\gamma(x)$, which exists by assumption. According to the Bellman equation, $V_\gamma(x) = \ell(x,u^*_{\gamma,0}(x)) + \gamma V_\gamma(\upsilon)$. Hence, since $\gamma \geqslant 1$, $V_\gamma(\upsilon) - V_\gamma(x) = -\ell(x,u^*_{\gamma,0}(x)) + (1-\gamma)V_\gamma(\upsilon) \leqslant -\ell(x,u^*_{\gamma,0}(x))$. We then follow similar lines as the proof of Theorems 1 and 2 in Sections VIII-B and VIII-C, respectively, to obtain the desired result.

J. Proof of Theorem 6.

Let $\breve{\gamma} \in (0,1)$, $\gamma \in (0,\breve{\gamma})$, $x \in \mathbb{R}^n$ and $\upsilon = f(x,\breve{u}^*_{\gamma,0}(x))$ where $\breve{u}^*_{\gamma,0}(x)$ is the first element of the optimal sequence of inputs $\breve{\mathbf{u}}^*_\gamma(x)$, which exists by assumption. Let $\breve{V}_\gamma(x) := \inf_{\mathbf{u}} \breve{J}_\gamma(x,\mathbf{u})$. Since ℓ is non-negative and $1-\gamma^{k+1} \geqslant 1-\gamma \geqslant 1-\breve{\gamma}$ for any $k \in \mathbb{Z}_{\geqslant 0}$,

$$\begin{aligned}(1-\breve{\gamma})\ell(x,\breve{u}^*_{\gamma,0}(x)) &\leqslant (1-\gamma)\ell(x,\breve{u}^*_{\gamma,0}(x)) \\ &\leqslant \inf_{\mathbf{u}} \sum_{k=0}^{\infty}(1-\gamma^{k+1})\ell(\phi(k,x,\mathbf{u}|_k),u_k) \\ &= \breve{V}_\gamma(x).\end{aligned} \tag{69}$$

On the other hand, $\breve{V}_\gamma(x) \leqslant \overline{V}(x) \leqslant \overline{\alpha}_V(\sigma(x))$ in view of item (i) of Assumption 1 and since $1-\gamma \leqslant 1$.

We now study the relationship between $\breve{V}_\gamma(x)$ and $\breve{V}_\gamma(\upsilon)$. By definition

$$\breve{V}_\gamma(\upsilon) \quad = \quad \inf_{\mathbf{u}} \sum_{k=0}^{\infty}(1-\gamma^{k+1})\ell(\phi(k,\upsilon,\mathbf{u}|_k),u_k). \tag{70}$$

The sequence $\breve{\mathbf{u}}^*_\gamma(x)$ is optimal when starting from x, but if we remove its first element and apply the corresponding truncated sequence (i.e. $\breve{\mathbf{u}'}^*_\gamma(x) = (\breve{u}^*_{\gamma,1}(x), \breve{u}^*_{\gamma,2}(x), \ldots)$) starting from the initial state υ, it may no longer be optimal, therefore

$$\breve{V}_\gamma(\upsilon) \leqslant \sum_{k=0}^{\infty}(1-\gamma^{k+1})\ell(\phi(k,\upsilon,\breve{\mathbf{u}'}^*_\gamma|_k(x)),\breve{u}^*_{\gamma,k+1}(x)) \tag{71}$$

as $1-\gamma^{k+1} \leqslant 1-\gamma^{k+2}$ for any $k \in \mathbb{Z}_{\geqslant 0}$,

$$\begin{aligned}\breve{V}_\gamma(\upsilon) &\leqslant \sum_{k=0}^{\infty}(1-\gamma^{k+2})\ell(\phi(k,\upsilon,\breve{\mathbf{u}'}^*_\gamma|_k(x)),\breve{u}^*_{\gamma,k+1}(x)) \\ &= \breve{V}_\gamma(x) - (1-\gamma)\ell(x,\breve{u}^*_{\gamma,0}(x)),\end{aligned} \tag{72}$$

we note that the infinite sums above are well-defined. Consequently,

$$\breve{V}_\gamma(\upsilon) - \breve{V}_\gamma(x) \leqslant -(1-\gamma)\ell(x,\breve{u}^*_{\gamma,0}(x)). \tag{73}$$

We now distinguish two cases depending on χ_W in Assumption 1, as in the proof of Theorem 1.

Case 1: $\chi_W \leqslant \mathbb{I}$.

Let $\breve{Y}_\gamma := \breve{V}_\gamma + (1-\gamma)W$. From (73) and item (ii) of Assumption 1,

$$\begin{aligned}\breve{Y}_\gamma(\upsilon) - \breve{Y}_\gamma(x) \leqslant\ & -(1-\gamma)\ell(x,\breve{u}^*_{\gamma,0}(x)) \\ & -(1-\gamma)\alpha_W(\sigma(x)) \\ & +(1-\gamma)\chi_W(\ell(x,\breve{u}^*_{\gamma,0}(x))),\end{aligned} \tag{74}$$

since $\chi_W \leqslant \mathbb{I}$ and $1-\gamma \geqslant 1-\breve{\gamma}$,

$$\breve{Y}_\gamma(\upsilon) - \breve{Y}_\gamma(x) \leqslant -(1-\gamma)\alpha_W(\sigma(x) \leqslant -(1-\breve{\gamma})\alpha_W(\sigma(x)). \tag{75}$$

On the other hand, $\breve{Y}_\gamma(x) \leqslant \overline{\alpha}_V(\sigma(x)) + (1-\gamma)\overline{\alpha}_W(\sigma(x))$, in view of the inequalities after (69) and item (ii) of Assumption 1. Consequently, $\breve{Y}_\gamma(x) \leqslant \alpha_V(\sigma(x)) + \overline{\alpha}_W(\sigma(x)) = \overline{\alpha}_Y(\sigma(x))$ with $\overline{\alpha}_Y \in \mathcal{K}_\infty$ as in Case 1 in the proof of Theorem 1. We also have, from item (ii) of Assumption 1 and (69), that $\breve{Y}_\gamma(x) \geqslant (1-\gamma)\ell(x,\breve{u}^*_{\gamma,0}(x)) + (1-\gamma)\alpha_W(\sigma(x)) - (1-\gamma)\ell(x,\breve{u}^*_{\gamma,0}(x))$ (see Case 1 in the proof of Theorem 1 for more detail). Therefore $\breve{Y}_\gamma(x) \geqslant (1-\gamma)\alpha_W(\sigma(x)) \geqslant (1-\breve{\gamma})\alpha_W(\sigma(x)) =: \underline{\alpha}_{\breve{Y}}(\sigma(x))$ with $\underline{\alpha}_{\breve{Y}} \in \mathcal{K}_\infty$. We conclude that item (b) of Theorem 6 holds like in Case 1 in the proof of Theorem 1.

Case 2: There exists $s \in \mathbb{R}_{\geqslant 0}$ *such that* $\chi_W(s) > s$.

This case follows the same lines as Case 2 in the proof of Theorem 1 by taking $\breve{Y}_\gamma := \rho_V(\breve{V}) + \rho_W((1-\gamma)W)$. The fact that γ appears in $\rho_W((1-\gamma)W)$ is not an issue as we use the inequality $1-\breve{\gamma} \leqslant 1-\gamma \leqslant 1$ to render the obtained inequalities independent of γ as done in Case 1 above.

K. Proof of Proposition 2.

Let $\gamma \in (\gamma^\star, 1)$ and $x \in \mathbb{R}^n$. Since $\gamma \in (0,1)$ and ℓ takes non-negative values, $V_\gamma(x) \leqslant \overline{V}(x)$. We prove the other inequality of Proposition 2 in the following.

Consider the sequence of inputs $(u^*_{\gamma,0}(x), \overline{\mathbf{u}}^*(\upsilon))$ where $u^*_{\gamma,0}(x)$ is the first element of a sequence of optimal inputs for the *discounted* cost (2), i.e. $u^*_{\gamma,0}(x) \in \mathcal{U}^*(x)$, and $\overline{\mathbf{u}}^*(\upsilon)$ is the sequence of optimal inputs for the *undiscounted* cost $\overline{J}$ starting at $\upsilon = f(x, u^*_{\gamma,0}(x))$. The sequence $(u^*_{\gamma,0}(x), \overline{\mathbf{u}}^*(\upsilon))$ may not be optimal for the undiscounted cost. Therefore, by definition of $\overline{V}(x)$,

$$\overline{V}(x) \leqslant \ell(x, u^*_{\gamma,0}(x)) + \overline{V}(\upsilon), \tag{76}$$

adding and subtracting $\gamma V_\gamma(\upsilon)$ to the right-hand side leads to $\overline{V}(x) \leqslant \ell(x, u^*_{\gamma,0}(x)) + \gamma V_\gamma(\upsilon) - \gamma V_\gamma(\upsilon) + \overline{V}(\upsilon)$. According to the Bellman equation $V_\gamma(x) = \ell(x, u^*_{\gamma,0}(x)) + \gamma V_\gamma(\upsilon)$, thus

$$\overline{V}(x) \leqslant V_\gamma(x) - \gamma V_\gamma(\upsilon) + \overline{V}(\upsilon). \tag{77}$$

We add and subtract $V_\gamma(\upsilon)$ to the right-hand side above and we obtain

$$\overline{V}(x) \leqslant V_\gamma(x) + (1-\gamma)V_\gamma(\upsilon) + \overline{V}(\upsilon) - V_\gamma(\upsilon), \tag{78}$$

which we rewrite as

$$\overline{V}(x) - V_\gamma(x) \leqslant \overline{V}(\upsilon) - V_\gamma(\upsilon) + (1-\gamma)V_\gamma(\upsilon). \tag{79}$$

Let $\upsilon_1 = \upsilon$ and $\upsilon_2 = f(\upsilon_1, u^*_{\gamma,0}(\upsilon_1))$ where $u^*_{\gamma,0}(\upsilon_1)$ is the first element of the optimal sequence of inputs for the discounted cost (2) starting at υ_1. By following the same reasoning as above, we obtain $\overline{V}(\upsilon_1) - V_\gamma(\upsilon_1) \leqslant \overline{V}(\upsilon_2) - V_\gamma(\upsilon_2) + (1-\gamma)V_\gamma(\upsilon_2)$. Hence, in view of (79),

$$\begin{aligned}\overline{V}(x) - V_\gamma(x) \leqslant\ & \overline{V}(\upsilon_2) - V_\gamma(\upsilon_2) \\ & +(1-\gamma)\left(V_\gamma(\upsilon_1) + V_\gamma(\upsilon_2)\right).\end{aligned} \tag{80}$$

We proceed by iteration. Denote $\upsilon_{k+1} = f(\upsilon_k, u^*_{\gamma,0}(\upsilon_k))$ for $k \in \mathbb{Z}_{>0}\backslash\{1\}$ where $u^*_{\gamma,0}(\upsilon_k)$ is the first element of the optimal sequence of inputs for the discounted cost (2) starting at υ_k, i.e. $u^*_{\gamma,0}(\upsilon_k) \in \mathcal{U}^*_\gamma(\upsilon_k)$. From (80), we deduce that for any $k \in \mathbb{Z}_{>0}$

$$\overline{V}(x) - V_\gamma(x) \leqslant \overline{V}(\upsilon_k) - V_\gamma(\upsilon_k) + (1-\gamma)\sum_{j=1}^{k} V_\gamma(\upsilon_j). \tag{81}$$

Let $Y_\gamma := V_\gamma + W$. In view of the proof of Corollary 2, we derive that $Y_\gamma(\upsilon_k) \to 0$ as $k \to \infty$. Therefore, since $Y_\gamma(\upsilon_k) \geqslant a_W\sigma(\upsilon_k) \geqslant 0$ (still in view of the proof of Corollary 2), we deduce that $\sigma(\upsilon_k) \to 0$ as $k \to \infty$. Recall that $0 \leqslant V_\gamma(x) \leqslant \overline{V}(x) \leqslant \overline{a}_V\sigma(x)$ in view of item (i) of Assumption 1 and item (b) of Proposition 2. Consequently,

$$V_\gamma(\upsilon_k) \to 0 \quad \text{and} \quad \overline{V}(\upsilon_k) \to 0 \quad \text{as} \quad k \to \infty. \tag{82}$$

On the other hand, according to the proof of Corollary 2 (see (34)), $V_\gamma(\upsilon_k) \leqslant Y_\gamma(\upsilon_k) \leqslant \left(1 + (-a_W + \frac{1-\gamma}{\gamma}\overline{a}_V)\overline{a}_Y^{-1}\right)^k Y_\gamma(x)$ where $\overline{a}_Y = \overline{a}_V + \overline{a}_W > 0$ for $k \in \mathbb{Z}_{>0}$. As a consequence, $V_\gamma(\upsilon_k) \leqslant \left(1 + (-a_W + \frac{1-\gamma}{\gamma}\overline{a}_V)\overline{a}_Y^{-1}\right)^k Y_\gamma(x)$. Thus, since $\left(1 + (-a_W + \frac{1-\gamma}{\gamma}\overline{a}_V)\overline{a}_Y^{-1}\right) \in (0,1)$ (as $\gamma \in (\gamma^\star, 1)$ and $\gamma^\star > \frac{\overline{a}_V}{\overline{a}_V + a_W}$), we obtain after some calculations,

$$\begin{aligned}\sum_{j=1}^{k} V_\gamma(\upsilon_j) &\leqslant \sum_{j=1}^{k}\left(1 + (-a_W + \frac{1-\gamma}{\gamma}\overline{a}_V)\overline{a}_Y^{-1}\right)^j Y_\gamma(x) \\ &= \theta(\gamma)Y_\gamma(x),\end{aligned} \tag{83}$$

where $\theta(\gamma)$ is defined in Proposition 2. In view of (82) and (83), by taking the limit as $k \to \infty$ in (81), $\overline{V}(x) - V_\gamma(x) \leqslant (1-\gamma)\theta(\gamma)Y_\gamma(x)$, which corresponds to the desired bound.

L. Sketch of Proof of Corollary 5.

Let $\gamma \in (0,1)$ satisfy the conditions of Corollary 5 and $x \in \mathbb{R}^n$. We use the notation of Section VIII-F. In view of (35) and (36), $Y_\gamma(\upsilon) - Y_\gamma(x) \leqslant -x^{\mathrm{T}}\left(\frac{1}{\varpi}S_1 - \frac{1-\gamma}{\gamma}P_\gamma\right)x$. By definition of $\varsigma(\gamma)$, $Y_\gamma(\upsilon) - Y_\gamma(x) \leqslant -\varsigma(\gamma)x^{\mathrm{T}}\left(\frac{1}{\varpi}S_2 + P_\gamma\right)x = -\varsigma(\gamma)Y_\gamma(x)$. We then follow the same lines as in the proof of Proposition 2. Instead of (83), we derive $\sum_{j=1}^{k} V_\gamma(\upsilon_j) \leqslant \sum_{j=1}^{k} Y_\gamma(\upsilon_j) \leqslant \sum_{j=1}^{k}(1-\varsigma(\gamma))^j Y_\gamma(x) \leqslant \frac{1-\varsigma(\gamma)}{\varsigma(\gamma)}Y_\gamma(x)$, from which we derive the desired result. The case where Q is positive definite similarly follows.

IX. Conclusions

We have analysed the stability of general nonlinear discrete-time systems controlled by a sequence of inputs that minimizes an infinite-horizon discounted cost. In general, only uniform semiglobal practical stability is ensured, but we have also derived stronger properties under additional assumptions. Then, we have exploited stability to derive new results on the continuity of the optimal value function and thus of the Lyapunov function used to prove stability. This is fundamental to guarantee some nominal robustness for the closed-loop system. Afterwards, we have shown that the stability and the continuity results still apply when an appropriate *near*-optimal sequence of inputs is applied to the system. The approach has been shown to be general enough to address cases where the time-varying term multiplying the stage cost no longer exponentially decreases to zero but increases as time proceeds. Finally, we have investigated the relationships between the optimal value functions of the discounted and undiscounted problems, when the latter is well-defined.

Appendix

A. Technical lemmas

We report two lemmas, which are used several times in Section VIII. These results respectively correspond to Lemmas 3 and 4 in [15] but stated under slightly more general assumptions, as we explain below.

Lemma 2. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$, $\sigma : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$, $h : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ and $\alpha_1, \alpha_2 : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ such that $h(x) \leqslant V(x) \leqslant \alpha_1(\sigma(x))$ and $V(f(x)) - V(x) \leqslant \alpha_2(\sigma(x)) - h(x)$ for all $x \in \mathbb{R}^n$. Let $\rho_V \in \mathcal{K}_\infty$ be such that $q_V(s) := \frac{d\rho_V}{ds}(s)$ is well-defined, continuous, and nondecreasing. Then $\rho_V(V(f(x))) - \rho_V(V(x)) \leqslant q_V\left(\alpha_1(\sigma(x)) + \alpha_2(\sigma(x))\right)\alpha_2(\sigma(x)) - q_V\left(\frac{1}{2}h(x)\right)\frac{1}{2}h(x)$ for all $x \in \mathbb{R}^n$.* □

In Lemma 2, α_1 and α_2 are not required to be of class $\mathcal{K}_\infty$ as in Lemma 3 in [15]. Still, the proof is the same as the latter property is not exploited in [15].

Lemma 3. *Let $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$, $W : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$, $\sigma : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$, $\ell : \mathbb{R}^{n+m} \to \mathbb{R}_{\geqslant 0}$, $\alpha_1 : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$, and $\alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that $W(x) \leqslant \alpha_1(\sigma(x))$ and $W(f(x,u)) - W(x) \leqslant \alpha_2(\ell(x,u)) - \alpha_3(\sigma(x))$ for all $x \in \mathbb{R}^n$. Let $\rho_W \in \mathcal{K}_\infty$ be such that $q_W(s) = \frac{d\rho_W}{ds}(s)$ is well-defined, continuous, and nondecreasing. Then $\rho_W(W(f(x,u))) - \rho_W(W(x)) \leqslant 2q_W\left(\alpha_2(\ell(x,u)) + \alpha_1\left(\alpha_3^{-1}\left(2\alpha_2(\ell(x,u))\right)\right)\right)\alpha_2(\ell(x,u)) - q_W\left(\frac{1}{4}\alpha_3(\sigma(x))\right)\frac{1}{4}\alpha_3(\sigma(x))$ for all $(x,u) \in \mathbb{R}^{n+m}$.* □

Compared to Lemma 4 in [15], α_1 is not required to be of class $\mathcal{K}_\infty$ in Lemma 3, but the proof remains the same as this property is not used in [15].

B. Satisfaction of (11)

The next result ensures the satisfaction of (11) and (12) in Corollary 3.

Lemma 4. *The linear matrix inequalities (11) and (12) hold when R is symmetric, positive definite and $Q = C^{\mathrm{T}}C$ with (A, C) is detectable.* □

Proof. Since (A, C) is detectable, there exists a matrix L such that $A - LC$ is Schur. In view of item 4 in Theorem 8.4 in [18], given $\epsilon > 0$ and S_1 real, symmetric, positive definite, there exists S_2, real, symmetric and positive definite, such that $(A-LC)^{\mathrm{T}}S_2(A-LC) - S_2 + S_1 + A^{\mathrm{T}}A = -\epsilon\mathbb{I}$. Let $\mathfrak{M}_1 := A^{\mathrm{T}}S_2A - S_2 + S_1 - \varpi Q$. It holds that $\mathfrak{M}_1 = (A-LC+LC)^{\mathrm{T}}S_2(A-LC+LC) - S_2 + S_1 - \varpi Q$ since $Q = C^{\mathrm{T}}C$. Using that $2A^{\mathrm{T}}S_2LC \leqslant A^{\mathrm{T}}A + C^{\mathrm{T}}L^{\mathrm{T}}S_2^2LC$, we deduce that $\mathfrak{M}_1 \leqslant -\epsilon\mathbb{I} - C^{\mathrm{T}}\left(\varpi\mathbb{I} + L^{\mathrm{T}}S_2L - L^{\mathrm{T}}S_2^2L\right)C$. By taking ϖ big enough, $\mathfrak{M}_1 \leqslant -\epsilon\mathbb{I}$. Denoting $\mathfrak{M}$ the matrix in (11), we have that $\mathfrak{M} \leqslant \begin{pmatrix} -\epsilon\mathbb{I} & A^{\mathrm{T}}S_2B \\ B^{\mathrm{T}}S_2A & B^{\mathrm{T}}S_2B - \varpi R \end{pmatrix} =: \overline{\mathfrak{M}}$. By taking ϖ sufficiently big, since R is positive definite, $\overline{\mathfrak{M}}$ is negative definite, in view of Schur complement. As a consequence, (11) holds.

The satisfaction of (12) is ensured by taking $\alpha > 0$ sufficiently small as P is positive semi-definite and S_1 is positive definite. ■

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