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ANNEALED LIMIT THEOREMS FOR THE ISING MODEL ON
RANDOM REGULAR GRAPHS

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Abstract. In a recent paper [15], Giardinà, Giberti, Hofstad, Prioriello have proved a
law of large number and a central limit theorem with respect to the annealed measure
for the magnetization of the Ising model on some random graphs including the random
2-regular graph. We present a new proof of their results, which applies to all random
regular graphs. In addition, we prove the existence of annealed pressure in the case of
configuration model random graphs.

1. Introduction

The ferromagnetic Ising model is one of the most well-known models in statistical
physics describing cooperative behaviors. In this model, each vertex in a graph is assigned
by one spin that can be one of two states +1, or -1, while the configuration probability
is given by the Boltzmann-Gibbs measure. These spins cooperatively interact with each
other toward alignment: spins of vertices connected by edges tend to be at the same state.

The Ising model on regular lattices has been studied carefully by many authors, resulting
in numerous beautiful results, see e.g. [13, 18]. Recently, a lot of attention has been
draw into investigating this model on class of random graphs [1, 2, 3, 5, 6, 7, 8, 9, 10, 11,
14, 15, 19, 20]. In the new framework, the source of randomness is the combination of the
law of spin configurations and the law of random graphs. Beside of generalizing class of
graphs, some authors try to consider different types of configuration probability. Most of
previous studies focused on the quenched setting, in which a graph sample is fixed then
the configuration probability is defined according to this realization of the graph. In a
recent paper [15], Giardinà et al. consider an annealed setting, where the configuration
probability is defined by taking into account the information of all graph samples. More
precisely, they define the annealed Ising model as follows.

Let $G_n$ be a random multi-graph, that is a random graph possibly having self-loops
and multiple edges between two vertices, with $n$ vertices $v_1, \ldots, v_n$. Let $\Omega_n = \{-1, 1\}^n$
be the space of spin configurations. For any $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Omega_n$, its energy is given by
the Hamiltonian function:

$$H(\sigma) = -\beta \sum_{i \leq j} k_{i,j} \sigma_i \sigma_j - B \sum_{i=1}^{n} \sigma_i,$$

where $k_{i,j}$ is the number of edges between $v_i$ and $v_j$, where $\beta \geq 0$ is the inverse temperature
and $B \in \mathbb{R}$ is the uniform external magnetic field.

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Then the configuration probability is given by what they call the annealed measure:

$$\mu_n(\sigma) = \frac{\mathbb{E}(\exp(-H(\sigma)))}{\mathbb{E}(Z_n(\beta, B))},$$

where $\mathbb{E}$ denotes the expectation with respect to the random graph, and $Z_n(\beta, B)$ is the partition function:

$$Z_n(\beta, B) = \sum_{\sigma \in \Omega_n} \exp(-H(\sigma)).$$

In [15], the authors study this Ising model on the rank-one inhomogeneous random graph, the random 2-regular graph and the configuration model with degrees 1 and 2. After determining limits of thermodynamic quantities and the critical inverse temperature, they prove laws of large numbers (LLN) and central limit theorems (CLT) with respect to the annealed measure for the total spin $S_n = \sigma_1 + \ldots + \sigma_n$. Our main contribution in this paper is to generalize their results to the class of all random regular graphs and prove the existence of annealed pressure in the case of the configuration model - a generalization of the random regular graph, see Section 2.1 for a definition.

1.1. Main results. First, we give some definitions following [15] of the thermodynamic quantities in finite volume.

(i) The annealed pressure is given by

$$\psi_n(\beta, B) = \frac{1}{n} \log \mathbb{E}(Z_n(\beta, B)).$$

(ii) The annealed magnetization is given by

$$M_n(\beta, B) = \mathbb{E}_{\mu_n} \left( \frac{S_n}{n} \right),$$

where $S_n = \sigma_1 + \ldots + \sigma_n$. After a simple computation, we get

$$M_n(\beta, B) = \frac{\partial}{\partial B} \psi_n(\beta, B).$$

(iii) The annealed susceptibility is given by

$$\chi_n(\beta, B) = \operatorname{Var}_{\mu_n} \left( \frac{S_n}{\sqrt{n}} \right).$$

We also can prove that

$$\chi_n(\beta, B) = \frac{\partial}{\partial B} M_n(\beta, B) = \frac{\partial^2}{\partial B^2} \psi_n(\beta, B).$$

When the sequence $(M_n(\beta, B))_n$ converges to a limit, say $M(\beta, B)$, we define the spontaneous magnetization as $M(\beta, 0^+) = \lim_{B \searrow 0} M(\beta, B)$. Then the critical inverse temperature is defined as

$$\beta_c = \inf \{ \beta > 0 : M(\beta, 0^+) > 0 \}.$$

Finally, the region of the existence of the limit magnetization is defined as

$$U = \{ (\beta, B) : \beta \geq 0, B \neq 0 \text{ or } 0 < \beta < \beta_c, B = 0 \}.$$

Now, we may introduce our results in the case of random regular graphs. First, we show the limits of thermodynamic quantities when the number of vertices tends to infinity.

**Theorem 1.1.** (The thermodynamic limits). Let us consider the Ising model on the random $d$-regular graph with $d \geq 2$. Then the following assertions hold.
(i) For all $\beta \geq 0$ and $B \in \mathbb{R}$, the annealed pressure converges
\[
\lim_{n \to \infty} \psi_n(\beta, B) = \psi(\beta, B)
\]
\[
= \frac{\beta d}{2} - B + \max_{0 \leq t \leq 1} [(t-1) \log(1-t) - t \log t + 2Bt + dF(t)],
\]
where
\[
F(t) = \int_0^{u(t)} \log f(s)ds,
\]
with $u(t) = \min\{t, 1-t\}$ and
\[
f(s) = \frac{e^{-2\beta(1-2s)} + \sqrt{1 + (e^{-4\beta} - 1)(1-2s)^2}}{2(1-s)}.
\]

(ii) For all $(\beta, B) \in U$, the magnetization converges
\[
\lim_{n \to \infty} M_n(\beta, B) = M(\beta, B) = \frac{\partial}{\partial B}\psi(\beta, B).
\]
Moreover, the critical inverse temperature is
\[
\beta_c = \text{atanh}(1/(d-1)) = \begin{cases} 
\frac{1}{2} \log\left(\frac{d}{d-2}\right) & \text{if } d \geq 3 \\
\infty & \text{if } d = 2.
\end{cases}
\]

(iii) For all $(\beta, B) \in U$, the annealed susceptibility converges
\[
\lim_{n \to \infty} \chi_n(\beta, B) = \chi(\beta, B) = \frac{\partial^2}{\partial B^2}\psi(\beta, B).
\]

Based on the thermodynamic limits theorem, we obtain a law of large number and a central limit theorem for the total spin.

**Theorem 1.2. (Annealed LLN).** Suppose that $(\beta, B) \in U$. Then for any $\varepsilon > 0$, there exists a positive constant $L = L(\varepsilon)$, such that for all sufficiently large $n$
\[
\mu_n \left( \left| \frac{S_n}{n} - M(\beta, B) \right| > \varepsilon \right) \leq \exp(-nL),
\]
where $M(\beta, B)$ is defined in Theorem 1.1 (ii).

**Theorem 1.3. (Annealed CLT).** For all $(\beta, B) \in U$, the total spin under the annealed measure satisfies a central limit theorem:
\[
\frac{S_n - \mathbb{E}_{\mu_n}(S_n)}{\sqrt{n}} \xrightarrow{(D)} N(0, \chi(\beta, B)) \quad \text{w.r.t. } \mu_n,
\]
where $\chi(\beta, B)$ is defined in Theorem 1.1 (iii) and $N(0, \chi)$ denotes a centered Gaussian random variable with variance $\chi$.

In the low temperature regime and in the absence of external field, the magnetization does not converges to a constant. However, similar to Curie-Weiss model, the law of magnetization converges to a combination of two Dirac’s measures.

**Proposition 1.4.** Suppose that $\beta > \beta_c$ and $B = 0$. Then there exists a positive constant $\nu = \nu(\beta)$, such that as $n \to \infty$,
\[
\mu_n \left( \left| \frac{S_n}{n} - \nu \right| \leq n^{-1/6} \right) \to 1/2 \quad \text{and} \quad \mu_n \left( \left| \frac{S_n}{n} + \nu \right| \leq n^{-1/6} \right) \to 1/2.
\]
Our result on the existence of annealed pressure in the case of the configuration model with general degree distributions is stated in Section 7, due to its complexity.

1.2. Discussion. One challenge in the annealed setting is that we have to take into account all graph samples. There are probably some rare samples that give a non-trivial contribution. Studying them often links to a very challenging topic, the large deviation properties of random graphs. Let us give here some comments on the approach, consequence and extension of our results.

(i) On the strategy of proofs. Structure of the random $d$-regular graph strongly depends on $d$. When $d$ increases, the graph becomes more and more complicated. In the case $d = 2$, the annealed Ising model on the graph is well studied in [15]. Their approach is based on the fact that every random 2-regular graph consists of a collection of cycles and the partition function on a cycle can be computed explicitly. However, when $d \geq 3$, this particular fact does not hold anymore. On the other hand, we realize that for any spin configuration, its Hamiltonian can be expressed in terms of $\beta$, $B$ and the number of disagreeing edges (the edges whose two extremities have different spins). Moreover, by the symmetry in term of law of random regular graphs, for any pair of configurations with the same number of positive spins, these numbers of disagreeing edges have the same distribution. Thus the Halmitonians of these configurations have the same law. Hence we show that the expectation of the partition function has the form
$$\sum_{i \leq n} \binom{n}{i} \theta(i, \beta, B).$$
Furthermore,
$$\frac{1}{n} \log \sum_{i=0}^{n} \binom{n}{i} \theta(i, \beta, B) = \max_{0 \leq i \leq n} \frac{1}{n} \log \left[ \binom{n}{i} \theta(i, \beta, B) \right] + o(1).$$
This explains the form of the annealed pressure $\psi(\beta, B)$ in Theorem 1.1 (i), which somehow looks like a large deviation result.

To prove the limit theorems, we use the same general strategy as in [14, 15]. More precisely, we define the sequence of cumulant generating functions as
$$c_n(t) = \frac{1}{n} \log \mathbb{E}_{\mu_n} \left( \exp(t S_n) \right) = \psi_n(\beta, B + t) - \psi_n(\beta, B).$$
Then by Theorem 1.1, this sequence converges to
$$c(t) = \psi(\beta, B + t) - \psi(\beta, B).$$
In [14, Sections 2.1 and 2.2], the authors show that if the function $c(t)$ is differentiable at 0 then the sequence $(S_n/n)_n$ converges in probability exponentially fast to $c'(0)$ w.r.t $\mu_n$. That means, for any real number $\varepsilon > 0$, there exists a positive constant $L = L(\varepsilon)$, such that for all $n$ large enough
$$\mu_n \left( \left| \frac{S_n}{n} - c'(0) \right| > \varepsilon \right) \leq \exp(-nL).$$
We will show in Section 4 that the function $\psi(\beta, B)$ is differentiable with respect to $B$. Thus $c(t)$ is differentiable and the annealed LLN follows.

On the other hand, by using Theorem A.8.7 (a) in [12], the central limit theorem in Theorem 1.3 follows from the convergence of generating function of the normalized sum, i.e. for any fixed number $t > 0$,
$$\mathbb{E}_{\mu_n} \left( \exp \left( \frac{t(S_n - \mathbb{E}_{\mu_n}(S_n))}{\sqrt{n}} \right) \right) \longrightarrow \exp \left( \frac{\chi(\beta, B)t^2}{2} \right).$$
The authors in [15, Section 3.2] show that this convergence holds if the following condition is satisfied: For any fixed number $t > 0$ and for any sequence $(t_n)$ satisfying $t_n \in [0, t/\sqrt{n}]$, one has

$$c''_n(t_n) \to \chi(\beta, B).$$

We refer to Lemma 5.1 for the proof of this condition.

(ii) On the case $d = 2$. We show in Proposition 3.2 that with $d = 2$, the annealed pressure is exactly solved and agrees the result obtained in [15], where the limit theorems have been proved. Hence, in Sections 4, 5, 6 we only study limit theorems for the case $d \geq 3$.

(iii) On the similarity to the quenched Ising model. Theorem 1.1 (ii) shows that the annealed Ising model undergoes a phase transition at the critical inverse temperature $\beta_c = \text{atanh}(1/(d - 1))$, which is equal to the critical value of the quenched Ising model. Moreover, we will prove in Proposition 3.2 that the annealed and quenched pressures are actually the same. As a consequence, all the thermodynamic limits of the annealed and quenched models are identical, and these two models should behave alike. In fact, limit theorems similar to our results have been proved for quenched model in [14]. This similarity has been conjectured in [15, Section 1.5.1].

(iv) On the generalizations. In Section 6, we study the Ising model on the configuration model with general degree distributions. Comparing with the case of random regular graphs, we have additionally a source of randomness coming from the sequence of degrees. This randomness makes the problem much more difficult. In particular, the annealed pressure obtained in Proposition 7.3 is so complicated that we can not even prove its differentiability. Without the differentiability, we can not go further to the other thermodynamic quantities or limit theorems.

Another natural question is to generalize our result to the Potts model where the spin of vertex may take $q$ values with $q \geq 3$, and the Hamiltonian is proportional to the number of agreeing edges. Our method possibly applies for this model, but it would require much work. The symmetry property that the measures of configurations with similar structure of spins are equal will continue to hold for the Potts model. However, the Hamiltonian of configurations is more complicated than that of Ising model. Indeed, there are now $q(q - 1)/2$ types of disagreeing edges instead of 1 type as in Ising model. Hence a recursive relation between agreeing and disagreeing edges would be much harder than the one for Ising model obtained in Section 2.

(v) On the organization of the paper. In section 2, we give a definition of the configuration model and prove a key lemma for random 1-regular graph used in the proof of the existence of annealed pressure. In section 3, we study the annealed pressure and prove Theorem 1.1 (i). In Section 4, we consider the magnetization, prove Theorem 1.1 (ii) and Theorem 1.2. In Section 5, we prove Theorem 1.1 (iii) and Theorem 1.3. In Section 6, we prove Proposition 1.4. In Section 7, we prove the existence of the annealed pressure in the case of general configuration models. Appendix is devoted to prove some technical points of our proofs.

2. Preliminaries

2.1. Configuration model. Let us give a definition following [16] of the configuration model with prescribed degree sequence. For each $n$, let $V_n = \{v_1, \ldots, v_n\}$ be the vertex set of a graph $G_n$, let $D = (D_i)_{i \leq n}$ be a sequence of integers. We construct the edge set of $G_n$ as follows. First, we assume that $\ell_n = \sum_1^n D_i$ is even (if not increase one of the
For each vertex $v_i$, start with $D_i$ half-edges incident to $v_i$. Then we denote by $\mathcal{H}$ the set of all the half-edges. Select one of them $h_1$ arbitrarily and then choose a half-edge $h_2$ uniformly from $\mathcal{H} \setminus \{h_1\}$, and match $h_1$ and $h_2$ to form an edge. Next, select arbitrarily another half-edge $h_3$ from $\mathcal{H} \setminus \{h_1, h_2\}$ and match it to another $h_4$ uniformly chosen from $\mathcal{H} \setminus \{h_1, h_2, h_3\}$. Then continue this procedure until there are no more half-edges. We finally get a multiple random graph that may have self-loops and multiple edges between vertices satisfying the degree of $v_i$ is $D_i$ for all $i$. We denote the obtained graph by $G_n(D)$.

For $d \geq 1$, if $D_i = d$ for all $i = 1, \ldots, n$ we call $G_n(D)$ the random $d$-regular graph, and denote it by $G_{n,d}$. The random $1$-regular graph will be employed several times in the proofs, so we distinguish its set of vertices with that of the $G_{n,d}$. More precisely, we denote by $V_m = \{w_1, \ldots, w_m\}$ the set of vertices of $G_{n,1}$.

We now explain the role of $G_{n,1}$ in our arguments. We show in (3.1) that the Hamiltonian of a given configuration can be expressed in term of the number of disagreeing edges. By the construction of the configuration model, we have a relation between the number of disagreeing edges of $G_n(D)$ and that of $G_{\ell,m}$ with $\ell_n = D_1 + \ldots + D_n$. More concretely, for $A \subset V_n$, let us denote

$$e(A, A^c) = \#\{\text{edges between } A \text{ and } A^c \text{ in } G_n(D)\}.$$ 

On the other hand, for each integer $m$, let $\tilde{V}_m = \{w_1, \ldots, w_m\}$ be the vertex set of $G_{m,1}$. For any $1 \leq k \leq m$, we define $\tilde{U}_k = \{w_1, \ldots, w_k\}, \tilde{U}_k^c = \tilde{V}_m \setminus \tilde{U}_k$ and

$$X(k, m) = \#\{\text{edges between } \tilde{U}_k \text{ and } \tilde{U}_k^c \text{ in } G_{m,1}\}.$$ 

(2.1)

It directly follows from the construction of the configuration model that

$$e(A, A^c) \overset{(D)}{=} X(\ell_A, \ell_n),$$ 

(2.2)

where

$$\ell_A = \sum_{i=1}^{n} D_i 1(v_i \in A) \quad \text{and} \quad \ell_n = \ell_{V_n}.$$ 

The relation (2.2) allows us to reduce problems on disagreeing edges of configuration models (or Hamiltonian of Ising model) to the one of random $1$-regular graphs.

### 2.2. A key lemma on random $1$-regular graph.

We will see in (3.2) that the generating function of the number of disagreeing edges plays a central role in the display of partition function. Thanks to (2.2), we only need study the generating functions of the number of disagreeing edges in random $1$-regular graphs. For $k \leq m$, define

$$g(\beta, k, m) := \mathbb{E}\left(\exp\left(-2\beta X(k, m)\right)\right).$$ 

(2.3)

The asymptotic behavior of $g(\beta, k, m)$ is described in the following lemma.

**Lemma 2.1.** For all $\beta \geq 0$, there exists a positive constant $C = C(\beta)$, such that for all $m$ large enough the following assertions hold.

(i) For all $0 \leq k \leq \ell \leq m$,

$$\left|\log g(\beta, k, m) - mF(k/m)\right| - \left|\log g(\beta, \ell, m) - mF(\ell/m)\right| \leq \frac{C|k - \ell|}{m}.$$
Indeed, we observe that for all
\[ \max_{0 \leq k \leq m} \left| \frac{\log g(\beta, k, m)}{m} - F(k/m) \right| \leq \frac{C}{m}, \]
with \( F(t) \) as in Theorem 1.1.

Proof. We observe that \( g(\beta, 0, m) = 1 \) and \( F(0) = 0 \). Hence, (ii) is a direct consequence of (i). We first claim that to prove (i), it suffices to show
\[ (i) \text{ holds for all } 0 \leq k \leq \ell \leq [m/2]. \]  
(2.4)
Indeed, we observe that for all \( 0 \leq k \leq m \),
\[ X(k, m) \overset{(\mathcal{D})}{=} X(m - k, m). \]
Thus
\[ g(\beta, k, m) = g(\beta, m - k, m). \]  
(2.5)
Moreover, we have \( F(t) = F(1 - t) \) for all \( t \in [0, 1] \). Hence for all \( k \leq m \),
\[ F\left( \frac{k}{m} \right) = F\left( \frac{m - k}{m} \right). \]  
(2.6)
Combining (2.5) and (2.6), we get that for \( 0 \leq k \leq [m/2] < \ell \leq m \),
\[
\left| \frac{\log g(\beta, k, m) - mF(k/m)}{m} - \frac{\log g(\beta, \ell, m) - mF(\ell/m)}{m} \right|
= \left| \frac{\log g(\beta, k, m) - mF(k/m)}{m} - \frac{\log g(\beta, m - \ell, m) - mF\left( \frac{m - \ell}{m} \right)}{m} \right|
\leq \frac{C|k - (m/2)|}{m} \leq \frac{C|k - \ell|}{m},
\]
by using (2.4) for \( 0 \leq k, m - \ell \leq [m/2] \). Hence (i) holds for \( 0 \leq k \leq [m/2] < \ell \leq m \). Similarly, we can also prove that (i) holds for \( [m/2] \leq k \leq \ell \leq m \), and thus (i) follows.

We now prove (2.4). The demonstration of (2.4) is long and divided into four parts: recursive formula for \( g(k, m) \); reduced sequence of \( g(k, m) \); approximation of the reduced sequence, and conclusion.

I. Recursive formula. We claim that for all \( k \leq [m/2] \),
\[ X(k, m) \overset{(\mathcal{D})}{=} \begin{cases} X(k - 2, m - 2) & \text{with prob. } (k - 1)/(m - 1) \\ 1 + X(k - 1, m - 2) & \text{with prob. } (m - k)/(m - 1). \end{cases} \]  
(2.7)
Indeed, we remind the construction of the random 1-regular graph: to each vertex in \( \bar{V}_m \) we attach an half-edge, then we pair these half-edges uniformly. Let us denote by \( \bar{U}_k \) (resp. \( \bar{U}_k^\ell \)) the set of half-edges that incident to \( \bar{U}_k \) (resp. \( \bar{U}_k^\ell \)). Suppose that we start the procedure of pairing half-edges with an element in \( \bar{U}_k \), say \( h_1 \). Then there are two possibilities. First, with probability \( (m - k)/(m - 1) \), the half-edge \( h_1 \) is paired with an element in \( \bar{U}_k^\ell \). This paring gives an edge between \( \bar{U}_k \) and \( \bar{U}_k^\ell \). After this step, there remains \( m - 2 \) half-edges including \( k - 1 \) ones belonging to \( \bar{U}_k \). Hence \( X(k, m) \) has the same law as \( 1 + X(k - 1, m - 2) \). Secondly, with probability \( (k - 1)/(m - 1) \), the half-edge \( h_1 \) is paired with an element in \( \bar{U}_k \), and that does not give an edge between \( \bar{U}_k \) and \( \bar{U}_k^\ell \). Thus after this step, \( X(k, m) \) has the same law as \( X(k - 2, m - 2) \).
Then denote

Thus

II. Reduced sequence. Define for all $g$

Observe that

We replace $m$ in (2.10) and obtain a recursive formula

As for (2.7), starting with an half-edge in $\tilde{U}_k^c$, we get

Hence

It follows from (2.8) and (2.9) that

We replace $m-2$ by $m$ in (2.10) and obtain a recursive formula

II. Reduced sequence. Define for all $1 \leq i \leq m$,

Then we have

Moreover by (2.11),

Observe that $g(\beta,0,m) = 1$ and $g(\beta,1,m) = e^{-2\beta}$, since $X(0,m) = 0$ and $X(1,m) = 1$. Thus $h(\beta,1,m) = e^{-2\beta}$. For simplicity, we remove the notation $\beta$ in the function $h$ and denote

Then $h(1,m) = c$. Moreover, by replacing $k$ by $k+1$ in (2.13), we get

III. Approximation of $h(k,m)$. By numerical analysis, we find that $h(k+1,m)$ and $h(k,m)$ are very close when $m$ tends to infinity. Hence, it is natural to expect that $h(k,m)$ is approximated by the solution of the fixed point equation

$$
\theta_k = \frac{c(m-2k)}{m-k} + \frac{k}{(m-k)\theta_k}.
$$
Going further to approximate the sequence $h(k, m)$, we consider the following functional equation

$$\theta = \frac{c(1 - 2t)}{1 - t} + \frac{t}{\theta(1 - t)}.$$  \hfill (2.15)

The positive solution of this equation is

$$\theta = f(t) := \frac{c(1 - 2t) + \sqrt{1 + (c^2 - 1)(1 - 2t)^2}}{2(1 - t)}.$$  \hfill (2.16)

We claim the following estimates on $f(t)$ and $h(k, m)$.

- For all $t \in [0, 1/2]$,  \hfill (2.17)
  $$c \leq f(t) \leq 1.$$

- There exists a positive constant $A = A(\beta) \geq 1$, such that for all $t \in (0, 1/2)$  \hfill (2.18)
  $$1/A \leq f'(t) \leq A \quad \text{and} \quad |f''(t)| \leq A.$$

- There exists a positive constant $\kappa$, such that for all $m$ and $0 \leq k \leq [m/2]$,  \hfill (2.19)
  $$|h(k, m) - f\left(\frac{k - 1}{m}\right)| \leq \frac{\kappa}{m}.$$

Note that the bound for $f''(t)$ in (2.18) is not used in the proof of (2.4), but it is needed for the proof of (2.19). The proof of (2.17), (2.18) and (2.19) is long and complicated, so we put it in Appendix.

**IV. Conclusion.** Assuming these claims (2.17), (2.18), (2.19), we now prove (2.4). By (2.17) and (2.19), we have for all $m$ large enough and $0 \leq k \leq [m/2]$,  \hfill (2.20)

$$c/2 \leq \min\{h(k, m), f(k/m)\}.$$

Using the mean value theorem, we have for all $x, y > 0$,  \hfill (2.21)

$$|\log x - \log y| \leq \frac{|x - y|}{\min\{x, y\}}.$$

Using (2.12), (2.19), (2.20) and (2.21), we get that for all $0 \leq k \leq \ell \leq [m/2]$,  \hfill (2.22)

$$\left|\log g(k, m) - \log g(\ell, m) + \sum_{i=k+1}^{\ell} \log f\left(\frac{i - 1}{m}\right)\right| = \left|\sum_{i=k+1}^{\ell} \left[\log h(i, m) - \log f((i - 1)/m)\right]\right|$$
$$\leq \sum_{i=k+1}^{\ell} \left|\log h(i, m) - \log f((i - 1)/m)\right|$$
$$\leq \frac{2}{c} \sum_{i=k+1}^{\ell} \left|h(i, m) - f((i - 1)/m)\right|$$
$$\leq \frac{2\kappa(\ell - k)}{cm}.$$
Similarly,
\[
\left| \log \frac{f(i/m)}{m} - \int_{i/m}^{(i+1)/m} \log f(s)ds \right| \leq \int_{i/m}^{(i+1)/m} |\log f(i/m) - \log f(s)|ds
\leq 2c \int_{i/m}^{(i+1)/m} |f(i/m) - f(s)|ds
\leq 2A \int_{i/m}^{(i+1)/m} |(i/m) - s|ds
= \frac{A}{m^2c}.
\]

Here for the third inequality, we have used (2.18) and the mean value theorem. It follows from (2.22) and (2.23) that for all \(0 \leq k \leq \ell \leq [m/2]\),
\[
\left| \log g(k, m) - \log g(\ell, m) + m \int_{k/m}^{\ell/m} \log f(s)ds \right| \leq \left( \frac{2x + A}{c} \right) \left( \frac{\ell - k}{m} \right),
\]
which proves Lemma 2.1 (i). \qed

2.3. An auxiliary lemma. The following result will be used in the proof of the existence of the annealed pressure.

**Lemma 2.2.** The following assertions hold.

(i) Let \(G(t)\) be a continuous function on \([0, 1]\). Then
\[
\lim_{n \to \infty} \max_{0 \leq j \leq n} G(j/n) = \max_{0 \leq t \leq 1} G(t).
\]

(ii) Let \((G_n(t))_{n}\) be a sequence of functions on \([0, 1]\), which converges point-wise to a function \(G(t)\). Suppose that there exists a positive constant \(C\) and a sequence \((\varepsilon_n)\) tending to 0, such that for all \(0 \leq s, t \leq 1\) and \(n \geq 1\),
\[
|G_n(s) - G_n(t)| \leq C|s - t| + \varepsilon_n.
\]

Then \(G(t)\) is a Lipschitz function. Moreover, for any continuous function \(H(t)\) on \([0, 1]\), we have
\[
\lim_{n \to \infty} \max_{0 \leq j \leq n} [H(j/n) + G_n(j/n)] = \max_{0 \leq t \leq 1} [H(t) + G(t)].
\]

The results of this lemma are standard in real analysis, so we safely leave to the reader.

2.4. Notation. If \(f\) and \(g\) are two real functions, we write \(f = \mathcal{O}(g)\) if there exists a constant \(C > 0\), such that \(f(x) \leq C g(x)\) for all \(x\); \(f \asymp g\) if \(f = \mathcal{O}(g)\) and \(g = \mathcal{O}(f)\); \(f = o(g)\) if \(g(x)/f(x) \to 0\) as \(x \to \infty\).

Let \((f(j, n))_{1 \leq j \leq n}\) and \((g(j, n))_{1 \leq j \leq n}\) be two sequences of real numbers. The notion \(f(j, n) = \mathcal{O}(g(j, n))\) (or \(f(j, n) = o(g(j, n))\)) is taken uniformly in all \(j \leq n\).

For any real number \(x\), let \([x]\) denote the integer part of \(x\).
3. The annealed pressure

The first step (which is one of the most important steps) in studying the Ising model is the task of understanding the partition function and the pressure. As mentioned in the introduction, we will write the Hamiltonian in terms of the number of disagreeing edges. Then using the symmetry of random regular graphs, we can investigate the annealed pressure. Let us be more precise now.

We fix an integer $d \geq 2$. Then for any positive integer $n$, we consider the random $d$-regular graph whose vertex set is $V_n = \{v_1, \ldots, v_n\}$. For any spin configuration $\sigma \in \Omega_n$, define

$$\sigma_+ = \{v_i : \sigma_i = 1\} \quad \text{and} \quad \sigma_- = \{v_i : \sigma_i = -1\}.$$  

Then

$$\sum_{i=1}^n \sigma_i = 2|\sigma_+| - n,$$
$$\sum_{i \leq j} k_{i,j} \sigma_i \sigma_j = (dn)/2 - 2e(\sigma_+, \sigma_-),$$

where $e(\sigma_+, \sigma_-) = \#\{\text{edges between } \sigma_+ \text{ and } \sigma_-\}.$

Therefore

$$H_n(\sigma) = \left( B - \frac{\beta d}{2} \right) n + 2\beta e(\sigma_+, \sigma_-) - 2B|\sigma_+|.$$  

(3.1)

Thus

$$\mathbb{E}(e^{-H_n(\sigma)}) = e^{\left( \frac{\beta d}{2} - B \right) n} \mathbb{E}(e^{-2\beta e(\sigma_+, \sigma_-)}) e^{2B|\sigma_+|}.$$  

(3.2)

By (2.2), if $|\sigma_+| = |\sigma'_+|$ then

$$e(\sigma_+, \sigma_-) \overset{(\mathcal{D})}{=} e(\sigma'_+, \sigma'_-) \overset{(\mathcal{D})}{=} X(d|\sigma_+|, dn)).$$

Hence

$$\sum_{\sigma \in \Omega_n} \mathbb{E}\left( e^{-2\beta e(\sigma_+, \sigma_-)} \right) e^{2B|\sigma_+|} = \sum_{j=0}^n e^{2Bj} \sum_{\sigma \in \Omega_n \atop |\sigma_+| = j} \mathbb{E}(e^{-2\beta e(\sigma_+, \sigma_-)})$$

$$= \sum_{j=0}^n \binom{n}{j} e^{2Bj} \mathbb{E}(e^{-2\beta X(dj, dn)}) = \sum_{j=0}^n \binom{n}{j} e^{2Bj} g(\beta, dj, dn),$$  

(3.3)

with $g(\beta, k, m)$ defined as in (2.3) for all $k \leq m$. Therefore

$$\mathbb{E}(Z_n(\beta, B)) = e^{\left( \frac{\beta d}{2} - B \right) n} \times \sum_{j=0}^n \binom{n}{j} e^{2Bj} g(\beta, dj, dn).$$  

(3.4)

Proof of Theorem 1.1 (i). By (3.4), we have

$$\frac{1}{n} \log \mathbb{E}(Z_n(\beta, B)) = \frac{\beta d}{2} - B + \max_{0 \leq j \leq n} \left[ \log \binom{n}{j} + \frac{2Bj}{n} + \frac{\log g(\beta, dj, dn)}{n} \right] + o(1).$$
On the other hand, it follows from Stirling’s formula that
\[
\log \left(\frac{n^j}{n}\right) = \frac{j}{n} \log \left(\frac{n}{j}\right) + \frac{n-j}{n} \log \left(\frac{n}{n-j}\right) + o(1).
\]
Combining the last two equations and Lemma 2.1 (ii), we obtain
\[
\frac{1}{n} \log \mathbb{E}(Z_n(\beta, B)) = \frac{\beta d}{2} - B + \max_{0 \leq j \leq n} L(j/n) + o(1),
\]
where \(L(t)\) is a continuous function on \([0, 1]\) defined by
\[
L(t) = -t \log(t) + (t - 1) \log(1 - t) + 2Bt + dF(t).
\]
Now, the result follows from (3.5) and Lemma 2.2 (i).

An explicit formula for the function \(F(t)\) is given in the following lemma.

**Lemma 3.1.** For \(t \leq 1/2\), we have
\[
F(t) = t \log f(t) + \frac{1}{2} \log(1 - t) + \frac{1}{2} \log(1 + e^{-2\beta}) + \frac{1}{2} \log \left[1 + \frac{e^{-2\beta}(2t - 1)}{(1 - t)(f(t) + 1)}\right].
\]
For \(t \in (1/2, 1)\), we have \(F(t) = F(1 - t)\).

The quenched pressure \(\tilde{\psi}(\beta, B)\) has been determined in [5, Theorem 2.4]. The equality between the annealed and quenched pressures is established in the following proposition.

**Proposition 3.2.** For all \(\beta > 0\) and \(B \in \mathbb{R}\), we have \(\psi(\beta, B) = \tilde{\psi}(\beta, B)\). In particular, when \(d = 2\),
\[
\psi(\beta, B) = \beta + \log \left(\cosh(B) + \sqrt{\sinh^2(B) + e^{-4\beta}}\right),
\]
which agrees with the result obtained in [15].

The proof of Lemma 3.1 and Proposition 3.2 is put in Appendix.

4. THE ANNEALED MAGNETIZATION AND THE STRONG LAW OF LARGE NUMBER

In this section, we prove the existence of the annealed magnetization and Theorem 1.2 following the strategy mentioned in the introduction.

**Proof of Theorem 1.1 (ii).** We state the following claims which we prove below.

- Claim 1. For any \(\beta \geq 0\), the function \(\psi(\beta, \cdot)\) is differentiable at every point \(B \neq 0\).
- Claim 2. For any \(d \geq 3\),
\[
\beta_c = \text{atanh}(1/(d - 1)) = \frac{1}{2} \log \left(\frac{d}{d - 2}\right).
\]

Moreover, for any \(\beta \in (0, \beta_c)\), the function \(\psi(\beta, \cdot)\) is differentiable at \(B = 0\).

Assuming these claims, Theorem 1.1 (ii) follows. Indeed, using similar arguments as in the proof of Theorem 1.1 (ii) in [15], we can show that for all \((\beta, B) \in \mathcal{U}\), the annealed magnetization \((M_n(\beta, B))\) converges to
\[
\mathcal{M}(\beta, B) := \frac{\partial \psi(\beta, B)}{\partial B}.
\]
This together with the claims 1 and 2 imply Theorem 1.1 (ii).
Proof of Claim 1. We consider here the case \( B > 0 \), the other one can be handled similarly. We first define some functions on \([0, 1]\):

\[
I(t) = (t - 1) \log(1 - t) - t \log t,
\]

\[
H(t) = I(t) + d \int_0^{\alpha(t)} \log f(s) ds, \tag{4.2}
\]

\[
L(t) = H(t) + 2Bt.
\]

By Theorem 1.1 (i),

\[
\psi(\beta, B) = (\beta d)/2 - B + \max_{0 \leq t \leq 1} L(t). \tag{4.3}
\]

Observe that \( H(t) = H(1 - t) \) for all \( t \in [0, 1] \), and \( Bt \leq B(1 - t) \) for all \( t \leq 1/2 \). Hence \( L(t) = H(t) + 2Bt \) attains the maximum at a point in \([1/2, 1]\). We consider the derivative of \( L(t) \) on \([1/2, 1]\):

\[
L'(t) = H'(t) + 2B = \log \left( \frac{1 - t}{t} \right) - d \log f(1 - t) + 2B.
\]

We have \( L'(1/2) = 2B > 0 \) and \( L'(1^-) = -\infty \), so the maximum point of \( L(t) \) is a solution of the equation

\[
L'(t) = \log \left( \frac{1 - t}{t} \right) - d \log f(1 - t) + 2B = 0. \tag{4.4}
\]

Claim 1°. The equation (4.4) has a unique solution \( t_* \) in \((1/2, 1)\), and \( L''(t_*) \neq 0 \).

Assuming this claim, we can deduce from the implicit function theorem that the function \( t_* \) is differentiable with respect to \( B \). Thus the function \( \psi(\beta, \cdot) \) is also differentiable and Claim 1 follows. Moreover,

\[
\frac{\partial}{\partial B} \psi(\beta, B) = -1 + H'(t_*) \frac{\partial t_*}{\partial B} + 2t_* + 2B \frac{\partial t_*}{\partial B} = -1 + 2t_* \tag{4.5}
\]

Now we prove Claim 1°. Since \( L'(1/2) = 2B > 0 \) and \( L'(1^-) = -\infty \), the function \( L'(t) \) has at least one root in \((1/2, 1)\). Suppose that \( L'(t) \) has more than one root in \((1/2, 1)\). Then \( L''(t) \) has at least two roots in \((1/2, 1)\). We consider the following equation in \((1/2, 1)\)

\[
L''(t) = \frac{-1}{t(1 - t)} - \frac{dx'(t)}{x(t)} = 0, \tag{4.6}
\]

where \( x(t) = f(1 - t) \). Since \( f(t) \) satisfies (2.15), we have

\[
x(t) = \frac{c(2t - 1)}{t} + \frac{1 - t}{tx(t)},
\]

with \( c = e^{-2\beta} \in (0, 1) \).

After some computation, we get

\[
\frac{x'(t)}{x(t)} = \frac{c x(t) - 1}{t[c(2t - 1)x(t) + 2 - 2t]}. \tag{4.7}
\]

Using this and (4.6), we obtain

\[
L''(t) = \frac{-d(1 - t)(cx(t) - 1) - c(2t - 1)x(t) + 2 - 2t}{t(1 - t)[c(2t - 1)x(t) + 2 - 2t]}. \tag{4.8}
\]
Hence
\[ L''(t) = 0 \iff d(1 - t)(1 - cx(t)) = c(2t - 1)x(t) + 2 - 2t \]
\[ \iff x(t) = \frac{(d - 2)(1 - t)}{c(d(1 - t) + 2t - 1)} \]
\[ \iff \frac{c(2t - 1) + \sqrt{1 + (c^2 - 1)(2t - 1)^2}}{2t} = \frac{(d - 2)(1 - t)}{c(d(1 - t) + 2t - 1)} \]
\[ \iff \sqrt{1 + (c^2 - 1)(2t - 1)^2} = \frac{2t(d - 2)(1 - t)}{c(d(1 - t) + 2t - 1)} - c(2t - 1), \]
from which it follows that
\[ 1 + (c^2 - 1)(2t - 1)^2 = \frac{4t^2(d - 2)^2(1 - t)^2}{c^2(d(1 - t) + 2t - 1)^2} + c^2(2t - 1)^2 - \frac{4t(d - 2)(1 - t)(2t - 1)}{(d - 1 - t) + 2t - 1)} \]
\[ \iff 4t - 4t^2 = \frac{4t^2(d - 2)^2(1 - t)^2}{c^2(d(1 - t) + 2t - 1)^2} - \frac{4t(d - 2)(1 - t)(2t - 1)}{(d - 1 - t) + 2t - 1)} \]
\[ \iff c^2(d(1 - t) + 2t - 1)^2 = t(1 - t)(d - 2)^2 - c^2(d - 2)(2t - 1)(d(1 - t) + 2t - 1), \]
or equivalently
\[ c^2[(d - 2)^2(t - t^2) + d - 1] = t(1 - t)(d - 2)^2. \tag{4.9} \]
Since \( d \geq 3 \), the equation (4.9) is equivalent to
\[ t^2 - t + \frac{c^2(d - 1)}{(d - 2)^2(1 - c^2)} = 0. \tag{4.10} \]
Observe that the sum of two solutions of (4.10) is 1. Hence (4.10) has at most one solution in \((1/2, 1)\). Therefore \( L''(t) \) has at most one root in \((1/2, 1)\). Hence the equation \( L'(t) = 0 \) has a unique solution in \((1/2, 1)\), say \( t_* \). Now we show \( L''(t_*) \neq 0 \) by contradiction. Suppose that \( L''(t_*) = 0 \). Then \( t_* \) must be a solution of (4.10). Hence
\[ \frac{c^2(d - 1)}{(d - 2)^2(1 - c^2)} = t_* - t_*^2 < \frac{1}{4}. \]
Thus
\[ c < \frac{d - 2}{d}. \tag{4.11} \]
Since \( L'(1/2) > 0 \) and \( L'(t_*) = 0 \), there exists \( u \in (1/2, t_*) \), such that \( L''(u) < 0 \). On the other hand, by (4.8) and (4.11),
\[ L'(1/2) = -4 - 2d(c - 1) > 0. \]
Since \( L''(u)L''(1/2) < 0 \), the function \( L''(t) \) has a root in \((1/2, u)\). Hence \( L''(t) \) has at least two roots in \((1/2, 1)\), which leads to a contradiction. Therefore \( L''(t_*) \neq 0 \) and Claim 1* follows. \( \square \)

**Proof of Claim 2.** Claim 2 is a direct consequence of the following claims.

- **Claim 2a.** If \( \beta > \text{atanh}(1/(d - 1)) \) then
  \[ \lim_{B \to 0} M(\beta, B) = -1 + 2t_+ > 0, \]
  where \( t_+ \) is the unique root in \((1/2, 1)\) of the function \( H'(t) \).
- Claim 2b. If $0 < \beta < \text{atanh}(1/(d - 1))$ then $H'(t)$ is strictly decreasing on $(0, 1)$ and has a unique root $t_0 = 1/2$. Moreover, the function $\psi(\beta, \cdot)$ is differentiable at $B = 0$ and

$$\lim_{B \searrow 0} \mathcal{M}(\beta, B) = \frac{\partial}{\partial B} \psi(\beta, B) \bigg|_{B=0} = 0.$$  

We first prove Claim 2a. Observe that $H'(1/2) = 0$ and $H'(1^-) = -\infty$. Moreover, by (4.8)

$$H''(1/2) = -4 - 2d(e^{-2\beta} - 1) > 0,$$

since

$$\beta > \text{atanh}(1/(d - 1)) = \frac{1}{2} \log \left( \frac{d}{d - 2} \right).$$

Therefore $H'(t)$ has at least one root in $(1/2, 1)$. Using the same arguments as in the proof of Claim 1*, $H'(t)$ has at most one root in $(1/2, 1)$. Thus it has a unique root $t_+$ in $(1/2, 1)$. Moreover, $H'(t) > 0$ for all $t \in (1/2, t_+)$ and $H'(t) < 0$ for all $t \in (t_+, 1)$.

On the other hand, $0 = L'(t_*) = H'(t_*) + 2B$. Hence $H'(t_*) = -2B < 0$ when $B > 0$. Therefore $t_* > t_+$ for all $B > 0$. In addition, $\lim_{B \searrow 0} t_* = t_+$. Hence by (4.5), we have

$$\lim_{B \searrow 0} \mathcal{M}(\beta, B) = -1 + 2t_+ > 0,$$

which implies Claim 2a.

We now prove Claim 2b. Assume that

$$0 < \beta < \frac{1}{2} \log \left( \frac{d}{d - 2} \right).$$  \hspace{1cm} (4.12)

We first show that $H''(t) < 0$ for all $t \in (0, 1)$. We consider here the case $t \geq 1/2$, the other one is similar. Using (4.8) and the same calculation as for (4.9), we have

$$H''(t) = L''(t) < 0 \iff d(1 - t)(1 - cx(t)) < e^{-2\beta} (2t - 1)x(t) + 2 - 2t$$

$$\iff \sqrt{1 + (e^{-4\beta} - 1)(2t - 1)^2} > \frac{2t(d - 2)(1 - t)}{e^{-2\beta} (2t - 1) + 2t} - e^{-2\beta} (2t - 1),$$

from which it follows that

$$e^{-4\beta} [(d - 2)^2(t - t^2) + d - 1] > t(1 - t)(d - 2)^2.$$

Under the condition (4.12), the inequality (4.13) is a consequence of the following

$$(d - 2)^2(t - t^2) + d - 1 > t(1 - t)d^2$$

$$\iff 1 > 4t(1 - t),$$

which holds for all $t \in (0, 1) \setminus \{1/2\}$. For $t = 1/2$,

$$H''(1/2) = -4 - 2d(e^{-2\beta} - 1) < 0,$$

by (4.12). In conclusion, $H''(t) < 0$ for all $t \in (0, 1)$ and thus $H'(t)$ is strictly decreasing and has a unique zero at $t = 1/2$. Now applying the implicit function theorem for the function $L'(t)$, we get that $t_*$, the solution of the equation $L'(t) = 0$, is differentiable with respect to $B$ at 0. Thus the function $\psi(\beta, \cdot)$ is also differentiable at $B = 0$ and

$$\lim_{B \searrow 0} \mathcal{M}(\beta, B) = \frac{\partial}{\partial B} \psi(\beta, B) \bigg|_{B=0} = \left. \lim_{B \to 0} -1 + 2t_* = -1 + 2t_0 = 0 \right.$$  \hspace{1cm} (4.14)

This implies the claim 2b.
Proof of Theorem 1.2. As mentioned in the introduction, the exponentially strong law large numbers for the magnetization follows from the differentiability of the pressure $\psi(\beta, B)$ with respect to $B$, by using the same arguments in the proof of [15, Theorem 1.2].

5. The annealed susceptibility and the central limit theorem

We have shown that for all $(\beta, B) \in U$,

$$\frac{\partial}{\partial B} \psi(\beta, B) = -1 + 2t_*,$$

where $t_*$ is the solution of the equation

$$L'(t) = H'(t) + 2B = 0,$$

with $L(t)$ and $H(t)$ as in (4.2). Moreover, we showed that $t_*$ is a differentiable function with respect to $B$. Hence

$$H''(t_*) \frac{\partial t_*}{\partial B} + 2 = 0,$$

and thus

$$\frac{\partial t_*}{\partial B} = \frac{-2}{H''(t_*)}.$$

Therefore

$$\chi(\beta, B) := \frac{\partial^2}{\partial B^2} \psi(\beta, B) = 2 \frac{\partial t_*}{\partial B} = \frac{-4}{H''(t_*)}.$$ (5.1)

Let us recall the definition of the sequence of cumulant generating functions

$$c_n(t) = \psi_n(\beta, B + t) - \psi_n(\beta, B).$$

Lemma 5.1. Suppose that $(\beta, B) \in U$. Then for any positive constant $t$ and any sequence $(t_n)$ satisfying $t_n \leq t/\sqrt{n}$, we have

$$c''_n(t_n) \to \chi(\beta, B).$$

Proof of Theorem 1.1 (iii). The result is a consequence of Lemma 5.1 with $t_n \equiv 0$. □

Proof of Theorem 1.3. As mentioned in the introduction, the central limit theorem is a consequence of Lemma 5.1 by applying the same arguments as in the proof of [15, Theorem 1.6] and [12, Theorem A.8.7]. □

Proof of Lemma 5.1. We consider here the case $B \geq 0$, the other one can be handled similarly. Thanks to (5.1), we only need to show that for any positive constant $t$ and any sequence $(t_n)$ satisfying $t_n \in [0, t/\sqrt{n}]$,

$$c''_n(t_n) \to \frac{-4}{H''(t_*)}.$$ (5.2)

It follows from (3.4) that for all $s > 0$,

$$c''_n(s) = \frac{\partial^2}{\partial B^2} \log Z_n(\beta, B + s) = \frac{4}{n} \left( \frac{T_{2,n}(s)}{T_n(s)} - \left( \frac{T_{1,n}(s)}{T_n(s)} \right)^2 \right),$$ (5.3)
where

\[ T_n(s) = \sum_{j=0}^{n} \binom{n}{j} e^{2(B+s)j} g(\beta, dj, dn) \]

\[ T_{1,n}(s) = \sum_{j=0}^{n} \binom{n}{j} e^{2(B+s)j} g(\beta, dj, dn)j \]

\[ T_{2,n}(s) = \sum_{j=0}^{n} \binom{n}{j} e^{2(B+s)j} g(\beta, dj, dn)j^2. \]

Let us define

\[ j_* = \lfloor nt \rfloor. \]

We will show that the values of \( T_n(s), T_{1,n}(s), T_{2,n}(s) \) are concentrated around the \( j_* \) th term of each sum if \( s = O(1/\sqrt{n}) \). We fix a positive constant \( t \) and a sequence \( (t_n) \) satisfying \( t_n \in [0, t/\sqrt{n}] \). Define

\[ \bar{T}_n(t_n) = \sum_{|j-j_*| \geq n^{5/6}} x_j \]

and

\[ \hat{T}_n(t_n) = \sum_{|j-j_*| < n^{5/6}} x_j \]

where

\[ x_j = \binom{n}{j} e^{2(B+t_n)} g(\beta, dj, dn). \]

To prove (5.3), it suffices to show that

\[ \left| \frac{T_{1,n}(t_n)}{T_n(t_n)} - \left( \frac{T_{2,n}(t_n)}{T_n(t_n)} \right)^2 \right| - \left[ \frac{\hat{T}_{1,n}(t_n)}{\hat{T}_n(t_n)} - \left( \frac{\hat{T}_{2,n}(t_n)}{\hat{T}_n(t_n)} \right)^2 \right] \leq \frac{4}{n^2}, \tag{5.4} \]

and

\[ \frac{4}{n} \left( \frac{\hat{T}_{2,n}(t_n)}{\hat{T}_n(t_n)} - \left( \frac{\hat{T}_{1,n}(t_n)}{\hat{T}_n(t_n)} \right)^2 \right) \rightarrow -\frac{4}{H''(t_*)} \quad \text{as } n \rightarrow \infty. \tag{5.5} \]

Before proving (5.4) and (5.5), we make a comparison between \( x_j \) and the other terms. Using Stirling’s formula, we have

\[ \binom{n}{j} = \left( \frac{1}{\sqrt{2\pi}} + O(n^{-1}) \right) \sqrt{\frac{n}{j(n-j)}} \exp \left( nI \left( \frac{j}{n} \right) \right), \]
where the function \( I(t) \) is defined in (4.2). Thus

\[
\frac{x_j}{x_{j^*}} = (1 + o(1)) \sqrt{\frac{j^*(n - j^*)}{j(n - j)}} \exp \left( 2(B + t_n)(j - j^*) + nI(j/n) - nI(j^*/n) + \log g(\beta, dj, dn) - \log g(\beta, dj^*, dn) \right)
\]

\[= (1 + o(1)) \sqrt{\frac{j^*(n - j^*)}{j(n - j)}} \exp \left( 2t_n(j - j^*) + n[I(j/n) + dF(j/n) + 2Bj/n] - n[I(j^*/n) + dF(j^*/n) + 2Bj^*/n] + \log g(\beta, dj, dn) - ndF(j^*/n) \right)
\]

\[= (1 + o(1)) \sqrt{\frac{j^*(n - j^*)}{j(n - j)}} \exp \left( 2t_n(j - j^*) + n[L(j/n) - L(j^*/n)] + \log g(\beta, dj, dn) - ndF(j^*/n) \right)
\]

(5.6)

\[= (1 + o(1)) \sqrt{\frac{j^*(n - j^*)}{j(n - j)}} \exp \left( 2t_n(j - j^*) + n[L(j/n) - L(j^*/n)] - ndF(j^*/n) \right).
\]

We have some observations on the function \( L(t) \) and its derivatives. Since \( L(t) \) attains the maximum at a unique point \( t_* \in (0, 1) \),

(O1) \( L'(t_*) = 0 \) and \( L''(t_*) < 0 \),

(O2) there exists a positive constant \( \varepsilon \), such that for all \( \epsilon \leq \varepsilon \),

\[\max_{|t - t_*| \geq \varepsilon} L(t) = \max\{L(t_* - \epsilon), L(t_* + \epsilon)\}.
\]

(O3) For \( \delta = (1 - t_*)/2 \), the functions \(|L'(t)|, |L''(t)|, |L'''(t)|\) are uniformly bounded in \((t_* - \delta, t_* + \delta)\).

I. Proof of (5.4). For \( n \) large enough (such that \( n^{-1/6} \leq \varepsilon \) as in (O2)), we have for all \( |j - j^*| \geq n^{5/6}, \)

\[L(j/n) - L(j^*/n) \leq \max\{L(j_*/n + n^{-1/6}) - L(j^*/n), L(j_*/n - n^{-1/6}) - L(j^*/n)\}.
\]

(5.7)

Using (O3) and Taylor’s theorem, we get

\[L(j_*/n \pm n^{-1/6}) - L(j^*/n) = \pm n^{-1/6}L'(j_*/n) + n^{-1/3}L''(j_*/n)/2 + O(n^{-1/2}).
\]

Similarly,

\[L'(j_*/n) = L'(t_*) + O(|(j_*/n) - t_*|) = O(1/n),
\]

(5.8)

since \( L'(t_*) = 0 \) and \(|(j_*/n) - t_*| \leq 1/n \). Therefore

\[n(L(j_*/n \pm n^{-1/6}) - L(j^*/n)) = n^{2/3}L''(j_*/n)/2 + o(n^{2/3}).
\]

(5.9)

On the other hand, since \( L''(t_*) < 0 \) and the sequence \((j_*/n)\) converges to \( t_* \), for all \( n \) large enough

\[L''(j_*/n) \leq L''(t_*)/2.
\]

Combining this with (5.7), (5.9) gives that for all \( |j - j^*| \geq n^{5/6}, \)

\[n(L(j/n) - L(j^*/n)) \leq n^{2/3}L''(j_*/n)/8.
\]

(5.10)

We now turn back to the formula (5.6). Observe that for all \( j \leq n, \)

\[\sqrt{\frac{j^*(n - j^*)}{j(n - j)}} \leq \sqrt{n}.
\]

(5.11)
On the other hand, by Lemma 2.1 (ii), for all \( j \leq n \),
\[
| \log g(\beta, dj, dn) - ndF(j/n) | = \mathcal{O}(1).
\]  
(5.12)
Since \( t_n \leq t/\sqrt{n} \), we have
\[
t_n(j - j_*) = \mathcal{O}(\sqrt{n}).
\]  
(5.13)
It follows from (5.6), (5.10), (5.12), (5.13) that for \( n \) large enough and \( |j - j_*| \geq n^{5/6} \),
\[
x_j \leq x_{j_*} \exp \left( \frac{n^{2/3} L''(t_*)}{8 + \mathcal{O}(\sqrt{n})} \right) \leq x_{j, n^{-7}},
\]  
(5.14)
since \( L''(t_*) < 0 \). Therefore
\[
\frac{T_n(t_n)}{T_n(t_n)} \leq \frac{x_{j_*}}{n^4} \leq \frac{T_n(t_n)}{n^4}.
\]
On the other hand,
\[
\frac{T_1(t_n)}{T_n(t_n)} \leq n^2 T_n(t_n).
\]
Hence
\[
\frac{T_1(t_n) - T_2(t_n)}{T_n(t_n)} = \frac{T_1(t_n) + T_1(t_n) - T_n(t_n)}{T_n(t_n)} \leq \frac{T_n(t_n) T_1(t_n) + T_1(t_n) T_n(t_n)}{T_n(t_n)^2} \leq \frac{1 + n^2}{n^4}.
\]
Similarly, we also have
\[
\frac{T_2(t_n) - T_2(t_n)}{T_n(t_n)} \leq \frac{1 + n^2}{n^4}.
\]
Combining the last two inequalities, we get (5.4).

II. Proof of (5.5).
IIa. Estimate of the quotient \( x_j / x_{j_*} \). We first observe that when \( |j - j_*| < n^{5/6} \),
\[
\sqrt{\frac{j_* (n - j_*)}{j (n - j)}} = 1 + \mathcal{O}(|j - j_*|/n) = 1 + \mathcal{O}(n^{-1/6}).
\]  
(5.15)
It follows from Lemma 2.1 (i) that for all \( j \),
\[
\left| \left[ \log g(\beta, dj, dn) - ndF(j/n) \right] - \left[ \log g(\beta, dj_*, dn) - ndF(j_*/n) \right] \right| = \mathcal{O}(|j - j_*|/n). \]  
(5.16)
As for (5.9), by using (5.8) we have for all \( |j - j_*| < n^{5/6} \),
\[
n(L(j/n) - L(j_*/n)) = n \left( L' \left( \frac{j_*}{n} \right) \left( \frac{j - j_*}{n} \right) + L'' \left( \frac{j_*}{n} \right) \frac{(j - j_*)^2}{2n^2} + \mathcal{O} \left( \frac{(j - j_*)^3}{n^3} \right) \right)
\]  
\[
= L'' \left( \frac{j_*}{n} \right) \frac{(j - j_*)^2}{2n} + \mathcal{O} \left( \frac{(j - j_*)^3}{n^2} \right) + \mathcal{O} \left( \frac{(j - j_*)^3}{n^3} \right)
\]  
\[
= \left[ L'' \left( \frac{j_*}{n} \right) + \mathcal{O}(n^{-1/6}) \right] \frac{(j - j_*)^2}{2n} + \mathcal{O}(n^{-1/6}),
\]
where in the last line, we used that
\[
\mathcal{O} \left( \frac{(j - j_*)^3}{n^2} \right) = \frac{(j - j_*)^2}{2n} \mathcal{O} \left( \frac{(j - j_*)}{n} \right) = \frac{(j - j_*)^2}{2n} \mathcal{O} \left( n^{-1/6} \right).
\]
On the other hand,

\[ L''(j_*/n) = H''(j_*/n) = H''(t_*) + O(1/n). \]

Therefore

\[ n \left( L(j/n) - L(j_*/n) \right) = \left[ \frac{H''(t_*)}{2} + O(n^{-1/6}) \right] \left( \frac{j - j_*}{n} \right)^2 + O(n^{-1/6}). \tag{5.17} \]

Let us define

\[ \alpha_* := \frac{H''(t_*)}{2} = L''(t_*)/2 < 0. \]

Using (5.6), (5.15), (5.16) and (5.17), we get that for any \( \epsilon \in (0, |\alpha_*|/8) \), for all \( n \) large enough and \( |j - j_*| < n^{5/6} \),

\[ \frac{x_j}{x_*} \leq (1 + \epsilon) \exp \left( (\alpha_* + \epsilon) \left( \frac{j - j_*}{n} \right)^2 + 2t_n(j - j_*) \right), \tag{5.18} \]

and

\[ \frac{x_j}{x_*} \geq (1 - \epsilon) \exp \left( (\alpha_* - \epsilon) \left( \frac{j - j_*}{n} \right)^2 + 2t_n(j - j_*) \right). \tag{5.19} \]

**IIb. Estimate of \( \hat{T}_n(t_n) \).** Observe that

\[ \hat{T}_n(t_n) = \sum_{|j - j_*| < n^{5/6}} x_j = x_*/ \sum_{|j - j_*| < n^{5/6}} x_j. \tag{5.20} \]

Moreover, for all \( \alpha < 0 \),

\[ \sum_{|j - j_*| < n^{5/6}} \exp \left( \frac{\alpha(j - j_*)^2}{n} + 2t_n(j - j_*) \right) = \sum_{|j| < n^{5/6}} \exp \left( \frac{\alpha j^2}{n^2} + 2(t_n \sqrt{n}) \frac{j}{\sqrt{n}} \right) \]

\[ = \sum_{j = -\infty}^{\infty} \exp \left( \frac{\alpha j^2}{n} + 2(t_n \sqrt{n}) \frac{j}{\sqrt{n}} \right) + o(1) \]

\[ = \sqrt{n} \int_{-\infty}^{\infty} e^{\alpha x^2 + 2(t_n \sqrt{n})x} dx + o(1). \tag{5.21} \]

Here we used the integral approximation and the fact that \( t_n \sqrt{n} \) is uniformly bounded. Combining (5.18), (5.19), (5.21) yields that

\[ (1 - 2\epsilon) \sqrt{n} A(\alpha_* - \epsilon, t_n \sqrt{n}) \leq \sum_{|j - j_*| < n^{5/6}} \frac{x_j}{x_*} \leq (1 + 2\epsilon) \sqrt{n} A(\alpha_* + \epsilon, t_n \sqrt{n}), \tag{5.22} \]

where for \( \alpha < 0 \) and \( \gamma \in \mathbb{R} \),

\[ A(\alpha, \gamma) := \int_{-\infty}^{\infty} e^{\alpha x^2 + \gamma x} dx. \]

Using (5.20) and (5.22), we have

\[ (1 - 2\epsilon) \sqrt{n} x_j A(\alpha_* - \epsilon, t_n \sqrt{n}) \leq \hat{T}_n(t_n) \leq (1 + 2\epsilon) \sqrt{n} x_j A(\alpha_* + \epsilon, t_n \sqrt{n}). \tag{5.23} \]

**IIc. Estimate of \( \hat{T}_{1,n}(t_n) \).** We have

\[ \hat{T}_{1,n}(t_n) = \sum_{|j - j_*| < n^{5/6}} j x_j = x_*/ \sum_{|j - j_*| < n^{5/6}} j x_j \sum_{|j - j_*| < n^{5/6}} x_j. \tag{5.24} \]
As for (5.21), we have
\[ \sum_{|j-j_*|<n^{5/6}} (j - j_*) \exp \left( \frac{\alpha(j - j_*)^2}{n} + 2t_n(j - j_*) \right) = n \int_{-\infty}^{\infty} x e^{\alpha x^2 + 2t_n(\sqrt{n})x} dx + o(1). \]

Using this approximation and (5.18), (5.19), we get
\[ (1 - 2\varepsilon)nA_1(\alpha_* - \varepsilon, t_n \sqrt{n}) \leq \sum_{|j-j_*|<n^{5/6}} (j - j_*) \frac{x_j}{x_{j_*}} \leq (1 + 2\varepsilon)nA_1(\alpha_* + \varepsilon, t_n \sqrt{n}), \quad (5.25) \]
where for \( \alpha < 0 \) and \( \gamma \in \mathbb{R} \),
\[ A_1(\alpha, \gamma) := \int_{-\infty}^{\infty} x e^{\alpha x^2 + \gamma x} dx. \]

Combining (5.22), (5.24), (5.25), we get
\[ (1 - 2\varepsilon)x_j, [\sqrt{n}j_*, A(\alpha_* - \varepsilon, t_n \sqrt{n}) + nA_1(\alpha_* - \varepsilon, t_n \sqrt{n})] \leq \hat{T}_{1,n}(t_n) \leq (1 + 2\varepsilon)x_j, [\sqrt{n}j_*, A(\alpha_* + \varepsilon, t_n \sqrt{n}) + nA_1(\alpha_* + \varepsilon, t_n \sqrt{n})] \quad (5.26) \]

**IIa. Estimate of \( \hat{T}_{2,n}(t_n) \).** We observe that
\[ \hat{T}_{2,n}(t_n) = \sum_{|j-j_*|<n^{5/6}} j^2 x_j = x_{j_*} \sum_{|j-j_*|<n^{5/6}} (j - j_*)^2 \frac{x_j}{x_{j_*}} + 2j_* x_{j_*} \sum_{|j-j_*|<n^{5/6}} (j - j_*) \frac{x_j}{x_{j_*}} + j_*^2 x_{j_*} \sum_{|j-j_*|<n^{5/6}} \frac{x_j}{x_{j_*}}. \quad (5.27) \]

Similar to (5.21),
\[ \sum_{|j-j_*|<n^{5/6}} (j - j_*)^2 \exp \left( \frac{\alpha(j - j_*)^2}{n} + 2t_n(j - j_*) \right) = n \sqrt{n} \int_{-\infty}^{\infty} x^2 e^{\alpha x^2 + 2t_n(\sqrt{n})x} dx + o(1). \]

Using this equality and (5.18), (5.19), we obtain
\[ (1 - 2\varepsilon)n\sqrt{n}A_2(\alpha_* - \varepsilon, t_n \sqrt{n}) \leq \sum_{|j-j_*|<n^{5/6}} (j - j_*)^2 \frac{x_j}{x_{j_*}} \leq (1 + 2\varepsilon)n\sqrt{n}A_2(\alpha_* + \varepsilon, t_n \sqrt{n}), \]
where for \( \alpha < 0 \) and \( \gamma \in \mathbb{R} \),
\[ A_2(\alpha, \gamma) := \int_{-\infty}^{\infty} x^2 e^{\alpha x^2 + \gamma x} dx. \]

Combining this estimate with (5.24), (5.25) and (5.27), we have
\[ (1 - 2\varepsilon)x_j, [\sqrt{n}j_*, j_*^2 \sqrt{n}A(\alpha_* - \varepsilon, 2t_n \sqrt{n}) + 2j_* nA_1(\alpha_* - \varepsilon, 2t_n \sqrt{n}) + n\sqrt{n}A_2(\alpha_* - \varepsilon, 2t_n \sqrt{n})] \leq \hat{T}_{2,n}(t_n) \leq (1 + 2\varepsilon)x_j, [\sqrt{n}j_*, j_*^2 \sqrt{n}A(\alpha_* + \varepsilon, 2t_n \sqrt{n}) + 2j_* nA_1(\alpha_* + \varepsilon, 2t_n \sqrt{n}) + n\sqrt{n}A_2(\alpha_* + \varepsilon, 2t_n \sqrt{n})]. \quad (5.28) \]

**IIe. Conclusion.** We observe that the derivatives with respect to \( \alpha \) at \( \alpha_* \) of the functions \( A(\alpha, \gamma) \), \( A_1(\alpha, \gamma) \) and \( A_2(\alpha, \gamma) \) are bounded. Hence, for any \( t > 0 \), there exists a positive constant \( C = C(t) \), such that for all \(|\gamma| \leq t\),
\[ |A(\alpha_* \pm \varepsilon, \gamma) - A(\alpha_*, \gamma)| \leq CA(\alpha_*, \gamma)\varepsilon, \quad (5.29) \]
\[ |A_1(\alpha_* \pm \varepsilon, \gamma) - A_1(\alpha_*, \gamma)| \leq CA_1(\alpha_*, \gamma)\varepsilon, \quad (5.30) \]
\[ |A_2(\alpha_* \pm \varepsilon, \gamma) - A_2(\alpha_*, \gamma)| \leq CA_2(\alpha_*, \gamma)\varepsilon. \quad (5.31) \]
Using (5.23) and (5.29) we get that for any \( \varepsilon \in (0, |\alpha|/8) \) and \( n \) large enough
\[
\lambda_- B_n \leq \hat{T}_n(t_n) \leq \lambda_+ B_n,
\]

where
\[
B_n = x_j, \sqrt{n} A(\alpha, 2t_n \sqrt{n}),
\]

and
\[
\lambda_- = (1 - C\varepsilon)(1 - 2\varepsilon), \quad \lambda_+ = (1 + C\varepsilon)(1 + 2\varepsilon).
\]

Similarly, by using (5.26) and (5.30) we get
\[
\lambda_- B_{1,n} \leq \hat{T}_{1,n}(t_n) \leq \lambda_+ B_{1,n},
\]

where
\[
B_{1,n} = x_j, [j^2 \sqrt{n} A(\alpha, 2t_n \sqrt{n}) + n A_1(\alpha, 2t_n \sqrt{n})],
\]

and by using (5.28) and (5.31),
\[
\lambda_- B_{2,n} \leq \hat{T}_{2,n}(t_n) \leq \lambda_+ B_{2,n},
\]

where
\[
B_{2,n} = x_j, [j^2 \sqrt{n} A(\alpha, 2t_n \sqrt{n}) + 2j_n n A_1(\alpha, 2t_n \sqrt{n}) + n \sqrt{n} A_2(\alpha, 2t_n \sqrt{n})].
\]

Combining (5.32), (5.33), (5.34), we have
\[
\lambda_- \left[ \frac{B_{2,n}}{B_n} - \left( \frac{B_{1,n}}{B_n} \right)^2 \right] \leq \frac{\hat{T}_{2,n}(t_n)}{T_n(t_n)} - \left( \frac{\hat{T}_{1,n}(t_n)}{T_n(t_n)} \right)^2 \leq \lambda_+ \left[ \frac{B_{2,n}}{B_n} - \left( \frac{B_{1,n}}{B_n} \right)^2 \right].
\]

Moreover,
\[
\frac{B_{2,n}}{B_n} - \left( \frac{B_{1,n}}{B_n} \right)^2 = n \left[ \frac{A_2(\alpha, 2t_n \sqrt{n})}{A(\alpha, 2t_n \sqrt{n})} - \left( \frac{A_1(\alpha, 2t_n \sqrt{n})}{A(\alpha, 2t_n \sqrt{n})} \right)^2 \right].
\]

Note that \( A(\alpha, \gamma), A_1(\alpha, \gamma), A_2(\alpha, \gamma) \) are related to moments of the normal distribution with mean \( \gamma/(2\alpha) \) and variance \( 1/(2\alpha) \). By some simple calculus, we have
\[
\frac{A_2(\alpha, \gamma)}{A(\alpha, \gamma)} - \left( \frac{A_1(\alpha, \gamma)}{A(\alpha, \gamma)} \right)^2 = -\frac{1}{2\alpha}.
\]

Thus
\[
\frac{B_{2,n}}{B_n} - \left( \frac{B_{1,n}}{B_n} \right)^2 = -\frac{n}{2\alpha} = -\frac{n}{H''(\alpha)}.
\]

Combining (5.35) and (5.36) yields that
\[
\lambda_- \frac{-4}{H''(\alpha)} \leq \frac{4}{n} \left( \frac{\hat{T}_{2,n}(t_n)}{T_n(t_n)} - \left( \frac{\hat{T}_{1,n}(t_n)}{T_n(t_n)} \right)^2 \right) \leq \lambda_+ \frac{-4}{H''(\alpha)}.
\]

Letting \( \varepsilon \to 0 \) and \( n \to \infty \), we get (5.5).
6. Proof of Proposition 1.4

In this section, we assume that $\beta > \beta_c$ and $B = 0$. Then for all $\sigma \in \Omega_n$,
\[
\mu_n(\sigma) = \mu_n(-\sigma). \tag{6.1}
\]
By this symmetry of the measure $\mu_n$, we observe that Proposition 1.4 follows if there exists a positive constant $\nu$, such that as $n \to \infty$
\[
\mu_n \left( \left| \frac{S_n}{n} - \nu \right| < n^{-1/6} \right) \to 1/2. \tag{6.2}
\]
We now prove (6.2) using the same strategy as in Section 5. By (3.2) and (3.4), we have
\[
\mu_n(\sigma) = \frac{g(\beta, d\sigma_+, dn)}{\sum_{j=0}^{n} \binom{n}{j} g(\beta, dj, dn)}. \tag{6.3}
\]
We have proved in the Claim 2a in Section 4 that on $(1/2, 1)$, the function $H'(t)$ has a unique zero $t_+$, which is the maximum point of $H(t)$. Let us define
\[
\nu = 2t_+ - 1.
\]
Since $S_n = 2|\sigma_+| - n$,
\[
\mu_n \left( \left| \frac{S_n}{n} - \nu \right| < n^{-1/6} \right) = \mu_n \left( \left| \frac{|\sigma_+|}{n} - t_+ \right| < n^{-1/6} \right). \tag{6.4}
\]
Combining this with (6.3), we get
\[
\mu_n \left( \left| \frac{S_n}{n} - \nu \right| < n^{-1/6} \right) = \frac{\sum_{j=0}^{n} y_j 1(\left| (j/n) - t_+ \right| < n^{-1/6})}{\sum_{j=0}^{n} y_j},
\]
where
\[
y_j = \binom{n}{j} g(\beta, dj, dn).
\]
We note that $y_j = y_{n-j}$, so
\[
\sum_{j=0}^{n} y_j = 2 \left( \sum_{j=[n/2]+1}^{n} y_j + z/2 \right) =: 2R_n,
\]
where
\[
z = \begin{cases} y_{[n/2]} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise}. \end{cases}
\]
Let us define
\[
\hat{R}_n = \sum_{j=0}^{n} y_j 1(\left| (j/n) - t_+ \right| < n^{-1/6}) \quad \text{and} \quad \bar{R}_n = R_n - \hat{R}_n.
\]
Note that $j_+ \in (1/2, 1)$, so all the indices in the definition of $\hat{R}_n$ are in the sum $R_n$. Now we observe that by (6.4), the equation (6.2) is equivalent to
\[
\frac{\hat{R}_n}{R_n} \to 1. \tag{6.5}
\]
Using the same idea of the proof of Lemma 5.1, we define
\[ j_+ = \lfloor nt_+ \rfloor. \]

As for (5.6), for all \( j \)
\[ \frac{x_j}{x_{j+}} = (1 + o(1)) \sqrt{\frac{j_+(n - j_+)}{j(n - j)}} \exp \left( n \left[ H(j/n) - H(j_+/n) \right] + \left[ \log g(\beta, dj, dn) - ndF(j/n) \right] \right) - \left[ \log g(\beta, d(j_+), dn) - ndF(j_+/n) \right]. \]

Using the same arguments for (5.14), we can show that for all \( j \) satisfying \( j \geq \lfloor n/2 \rfloor \) and \(|(j/n) - t_+| \geq n^{-1/6} \),
\[ x_j \leq \frac{x_{j+}}{n^7}. \]
(6.6)

Note that here \( H(t) \) and \( t_+ \) play the same role of \( L(t) \) and \( t_* \) as in the proof of (5.14).

Using (6.6), we get
\[ \bar{R}_n \leq \frac{x_{j+}}{n^8} \leq \hat{R}_n. \]

Thus
\[ \frac{\hat{R}_n}{\bar{R}_n} = \frac{\hat{R}_n}{\hat{R}_n - \bar{R}_n} \to 1, \]
and (6.5) follows. \( \square \)

7. The annealed pressure of Ising model on the configuration model

Let \( G_n \) be the configuration model whose the vertex set is \( V_n = \{v_1, \ldots, v_n\} \) and the degrees of vertices \( (D_i) \) are i.i.d. integer-valued random variables with the same distribution as \( D \). Assume that
\[ \mathbb{E} \left( e^{sD} \right) < \infty \quad \text{for all} \quad s \in \mathbb{R}. \]
(7.1)
Notice that the condition (7.1) is necessary, since without it the partition function has infinite expectation when \( \beta \) is large enough.

Now we study the annealed pressure of the Ising model on \( G_n \). We use the same notation as in the proof of Theorem 1.1 (i). Observe that for all \( \sigma \in \Omega_n \),
\[ \sum_{i=1}^{n} \sigma_i = 2|\sigma_+| - n, \]
\[ \sum_{i \leq j} k_{i,j} \sigma_i \sigma_j = \ell_n/2 - 2e(\sigma_+, \sigma_-), \]
where for all \( 1 \leq j \leq n \),
\[ \ell_j = D_1 + \ldots + D_j. \]

Using (2.2) and the fact that \( (D_i)_{1 \leq i \leq n} \) are i.i.d. random variables, we have if \( |\sigma_+| = |\sigma'_+| \),
\[ \ell_n/2 - 2e(\sigma_+, \sigma_-) \overset{(D)}{=} \ell_n/2 - 2e(\sigma'_+, \sigma'_-). \]

Hence using the same arguments as for Theorem 1.1 (i), we obtain
\[ \mathbb{E}(Z_n(\beta, B)) = e^{-Bn} \sum_{j=0}^{n} \binom{n}{j} e^{2Bj} b(\beta, j, n), \]
(7.2)
where
\[ b(\beta, j, n) = \mathbb{E}\left( \exp \left[ \beta \ell_n / 2 - 2 \beta e(U_j, U_j^c) \right] \right), \]
with
\[ U_j = \{v_1, \ldots, v_j\}. \]

Using (2.2) once again, we have
\[ \mathcal{L}(e(U_j, U_j^c) \mid (D_i)_{1 \leq i \leq n}) \overset{(D)}{=} \mathcal{L}(X(\ell_j, \ell_n) \mid (D_i)_{1 \leq i \leq n}), \]
where \( X(k, m) \) is defined as in (2.1) for all \( k \leq m \). Hence
\[ \mathbb{E}_{(D_{ij})} \left( \exp \left[ -2 \beta e(U_j, U_j^c) \right] \right) = g(\beta, \ell_j, \ell_n), \]
where \( \mathbb{E}_{(D_{ij})} \) is the expectation w.r.t. configuration model conditioning on the sequence of degrees \( (D_i)_{i \leq n} \), and \( g(\beta, k, m) \) is defined as in (2.3). Thus
\[ \mathbb{E}_{(D_{ij})} \left( \exp \left[ \beta \ell_n / 2 - 2 \beta e(U_j, U_j^c) \right] \right) = \exp(\beta \ell_n / 2) g(\beta, \ell_j, \ell_n). \]

Therefore
\[ b(\beta, j, n) = \overline{\mathbb{E}} \left( \mathbb{E}_{(D_{ij})} \left( \exp \left[ \beta \ell_n / 2 - 2 \beta e(U_j, U_j^c) \right] \right) \right) = \overline{\mathbb{E}} \left( \exp(\beta \ell_n / 2) g(\beta, \ell_j, \ell_n) \right), \]
where \( \overline{\mathbb{E}} \) is the expectation w.r.t. the sequence of degrees \( (D_i)_{i \leq n} \). By Lemma 2.1 (ii), there is a positive constant \( C = C(\beta) \), such that for all \( j \leq n \)
\[ \exp \left( -C + \ell_n F(\ell_j / \ell_n) \right) \leq g(\beta, \ell_j, \ell_n) \leq \exp \left( C + \ell_n F(\ell_j / \ell_n) \right), \]
with \( F(t) \) as in Theorem 1.1 (i). Hence
\[ \left| \frac{\log b(\beta, j, n)}{n} - \frac{\log \overline{\mathbb{E}} \left( \exp \left[ \ell_n (\beta/2 + F(\ell_j / \ell_n)) \right] \right)}{n} \right| \leq \frac{C}{n}. \quad (7.3) \]

For each \( \beta \geq 0 \), we define a sequence of functions on \([0, 1]\) as follows:
\[ G_n(\beta, t) = \frac{1}{n} \log \overline{\mathbb{E}} \left( \exp \left[ \ell_n (\beta/2 + F(\ell_{[nt]} / \ell_n)) \right] \right). \]

To study the limit of the sequence of functions \( (G_n(\beta, t))_n \), we need a large deviation result for the vector \( (\ell_{[nt]}, \ell_n) \). We use the standard notion of large deviation principle (LDP) as in [4]. Let \( (X_i) \) be a sequence of i.i.d. random variables. Suppose that for all \( s \in \mathbb{R} \)
\[ \Lambda(s) = \mathbb{E}(e^{sX_1}) < \infty. \]
Let us define for \( t \in [0, 1] \),
\[ Z_n(t) = \frac{1}{n} \sum_{i=1}^{[nt]} X_i. \]

Let \( \nu_n \) be the law of \( Z_n(\cdot) \) in \( L_\infty([0, 1]) \).

**Lemma 7.1.** [4, Lemma 5.1.8] Let \( Q \) denote the collection of all ordered finite subsets of \((0, 1]\). For any \( q = \{0 < t_1 < \ldots < t_{|q|} \leq 1\} \in Q \) and \( f : [0, 1] \to \mathbb{R} \), let \( p_q(f) \) denote the vector \( (f(t_1), \ldots, f(t_{|q|})) \in \mathbb{R}^{|q|} \). Then the sequence of laws \( (\nu_n \circ p_q^{-1})_n \) satisfies the LDP in \( \mathbb{R}^{|q|} \) with the rate function
\[ R_q(z) = \sum_{i=1}^{|q|} (t_i - t_{i-1}) \Lambda^* \left( \frac{z_i - z_{i-1}}{t_i - t_{i-1}} \right), \]
where \( z = (z_1, \ldots, z_q) \), \( z_0 = t_0 = 0 \), and
\[
\Lambda^*(x) = \sup_{s \in \mathbb{R}} \{xs - \Lambda(s)\}.
\]

Using this result, we can show the convergence of the sequence \((G_n(\beta, t))_n\).

**Lemma 7.2.** For all \( \beta \geq 0 \), the following assertions hold.

(i) There exists a positive constant \( C \), such that for all \( 0 \leq s, t \leq 1 \) and \( n \geq 1 \),
\[
|G_n(\beta, t) - G_n(\beta, s)| \leq C \left( |t - s| + \frac{1}{n} \right).
\]

(ii) For all \( t \in [0, 1] \), we have
\[
\lim_{n \to \infty} G_n(\beta, t) = G(\beta, t),
\]
where
\[
G(\beta, t) = \sup_{a, b} \left\{ b(\beta/2 + F(a/b)) - t\Lambda^* \left( \frac{a}{t} \right) - (1 - t)\Lambda^* \left( \frac{b - a}{1 - t} \right) \right\},
\]
with
\[
\Lambda^*(x) = \sup_{s \in \mathbb{R}} \{xs - \Lambda(s)\},
\]
and
\[
\Lambda(s) = \log \mathbb{E}(\exp(sD)).
\]

Moreover, \( G(\beta, t) \) is a Lipschitz function.

**Proof.** We first prove (i). Observe that
\[
0 \leq F(t) \leq 1 \quad \text{and} \quad \max_{t \in [0,1]} |F'(t)| = \max_{t \in [0,1/2]} |\log f(t)| \leq 2\beta,
\]
since the function \( F(t) \) is symmetric about 1/2 and \( e^{-2\beta} \leq f(\beta, t) \leq 1 \) for all \( t \in [0, 1/2] \). Therefore
\[
\beta/2 + \max_{t \in [0,1]}(|F(t)| + |F'(t)|) \leq r := 1 + 5\beta/2. \tag{7.4}
\]
We claim that
\[
|\log \mathbb{E} (e^{\epsilon_n[\beta/2+F(\ell_j/\ell_n)]}) - \log \mathbb{E} (e^{\epsilon_n[\beta/2+F(\ell_{j-1}/\ell_n)]})| \leq C, \tag{7.5}
\]
where
\[
C = \max\{ \log \mathbb{E} (e^{3rD}) - \log \mathbb{E} (e^{-2rD}), \log \mathbb{E} (e^{2rD}) - \log \mathbb{E} (e^{-3rD}) \}.
\]
Assuming (7.5), we can easily prove (i). Indeed, by repeatedly applying (7.5), we have for all \( i \leq j \leq n \),
\[
\left| \log \mathbb{E} (e^{\epsilon_n[\beta/2+F(\ell_j/\ell_n)]}) - \log \mathbb{E} (e^{\epsilon_n[\beta/2+F(\ell_j/\ell_n)]}) \right| \leq C|i - j|. \tag{7.6}
\]
Thus
\[
|G_n(\beta, t) - G_n(\beta, s)| = \frac{1}{n} \left| \log \mathbb{E} (e^{\epsilon_n[\beta/2+F(\ell_n/\ell_n)]}) - \log \mathbb{E} (e^{\epsilon_n[\beta/2+F(\ell_n/\ell_n)]}) \right| \leq C \left( |t - s| + \frac{1}{n} \right),
\]
which implies (i).
Proof of (7.5). The idea is simple: using the mean value theorem and (7.4), we have for all \(1 \leq j \leq n\),
\[
\ell_n[F(\ell_j/\ell_n) - F(\ell_{j-1}/\ell_n)] \leq \max_{t \in [0,1]} |F'(t)|D_j \leq rD_j.
\] (7.7)
Hence (7.5) would immediately follow if \(\ell_n\) and \(D_j\) are independent. Since this fact is not true, we break \(\ell_n\) into two independent parts \(D_i\) and \(\ell_{n,i}\), with
\[
\ell_{n,j} = \ell_n - D_j.
\]
We have
\[
\ell_n(\beta/2 + F(\ell_{j-1}/\ell_n)) = \ell_{n,j}(\beta/2 + F(\ell_{j-1}/\ell_n)) + D_j(\beta/2 + F(\ell_{j-1}/\ell_n))
\]
\[
= \ell_{n,j}(\beta/2 + F(\ell_{j-1}/\ell_{n,j})) + \ell_{n,j}(F(\ell_{j-1}/\ell_n) - F(\ell_{j-1}/\ell_{n,j})) + D_j(\beta/2 + F(\ell_{j-1}/\ell_n)).
\]
Using the mean value theorem and (7.4), we get
\[
|\ell_{n,j}(F(\ell_{j-1}/\ell_n) - F(\ell_{j-1}/\ell_{n,j})) + D_j(\beta/2 + F(\ell_{j-1}/\ell_n))| \leq 2rD_j.
\] Therefore
\[
|\ell_n(\beta/2 + F(\ell_{j-1}/\ell_n)) - \ell_{n,j}(\beta/2 + F(\ell_{j-1}/\ell_{n,j}))| \leq 2rD_j.
\] (7.8)
It follows from (7.7) and (7.8) that
\[
|\ell_n(\beta/2 + F(\ell_{j-1}/\ell_n)) - \ell_{n,j}(\beta/2 + F(\ell_{j-1}/\ell_{n,j}))| \leq 3rD_j.
\] (7.9)
On the other hand,
\[
\ell_{n,j}(\beta/2 + F(\ell_{j-1}/\ell_{n,j})) \quad \text{is independent of} \quad D_j.
\] (7.10)
Using (7.9) and (7.10), we obtain
\[
\mathbb{E}(e^{-3rD_j}) \leq \frac{\mathbb{E}(e^{\ell_n[\beta/2+F(\ell_{j-1}/\ell_n)]})}{\mathbb{E}(e^{\ell_{n,j}[\beta/2+F(\ell_{j-1}/\ell_{n,j})]})} \leq \mathbb{E}(e^{3rD_j}).
\]
Similarly, using (7.8) and (7.10), we have
\[
\mathbb{E}(e^{-2rD_j}) \leq \frac{\mathbb{E}(e^{\ell_n[\beta/2+F(\ell_{j-1}/\ell_n)]})}{\mathbb{E}(e^{\ell_{n,j}[\beta/2+F(\ell_{j-1}/\ell_{n,j})]})} \leq \mathbb{E}(e^{2rD_j}).
\]
Combining the last two inequalities gives (7.5).

We now prove (ii). Applying Lemma 7.1 for \(q = \{t_1 = t < t_2 = 1\}\), we get that the law of \(\frac{1}{n}(\ell_{[nt]}, \ell_n)\) satisfies the LDP in \(\mathbb{R}^2\) with the rate function
\[
I(a,b) = t\Lambda^*(\frac{a}{7}) + (1-t)\Lambda^*(\frac{b-a}{1-t}),
\]
where \(\Lambda^*\) is defined as in the statement of (ii). Therefore, using Varadhan’s Lemma (see for example [4, Theorem 4.3.1]), the sequence of functions \((G_n(\beta, \cdot))_{n}\) converges point-wise to the function \(G(\beta, \cdot)\) defined as in the statement of (ii). Moreover, applying Lemma 2.2 (ii) and Part i, we obtain that \(G(\beta, t)\) is a Lipschitz function. □

Proposition 7.3. For all \(\beta \geq 0\) and \(B \in \mathbb{R}\), the annealed pressure \(\psi_n(\beta, B)\) converges to a limit given by
\[
\psi(\beta, B) = -B + \max_{0 \leq t \leq 1} \left[ t \log \left( \frac{1}{t} \right) + (1-t) \log \left( \frac{1}{1-t} \right) + 2Bt + G(\beta, t) \right],
\]
with \(G(\beta, t)\) as in Lemma 7.2.
Proof. Using (7.2), (7.3) and Stirling’s formula, we get
\[
\log \frac{\mathbb{E}(Z_n(\beta, B))}{n} = -B + \max_{0 \leq j \leq n} \left[ \frac{\log \binom{n}{j}}{n} + 2B \frac{j}{n} + G_n(\beta, j/n) \right] + o(1)
\]
\[
= -B + \max_{0 \leq j \leq n} \left[ S(j/n) + G_n(\beta, j/n) \right] + o(1),
\]
where \(S(t)\) is continuous function on \([0, 1]\) defined by
\[
S(t) = -t \log t + (t - 1) \log(1 - t) + 2Bt.
\]
Now it follows from (7.11), Lemmas 7.2 (ii) and 2.2 (ii) that
\[
\lim_{n \to \infty} \log \frac{\mathbb{E}(Z_n(\beta, B))}{n} = -B + \max_{0 \leq t \leq 1} \left[ S(t) + G(\beta, t) \right],
\]
which proves Proposition 7.3. □

Remark 7.4. We can slightly extend Proposition 7.3 as follows. Let \((X_n)_{n \geq 1}\) and \(X\) be integer valued random variables satisfying
\[
\sup_{s \in \mathbb{R}} \mathbb{E}(e^{sX}) < \infty \quad \text{and} \quad \mathbb{E}(e^{sX_n}) \to \mathbb{E}(e^{sX}) \forall s \in \mathbb{R}.
\]
(7.12)
For each \(n\), let \((X_{n,i})_{i \leq n}\) be a sequence of i.i.d random variables with the same distribution as \(X_n\). Let \(G_n\) be the configuration model random graph of size \(n\) with sequence of degrees given by \((X_{n,i})_{i \leq n}\). Then Proposition 7.3 still holds for the annealed Ising model on \(G_n\). A nice example of degree distribution is \(X_n = \text{Bin}(n, \gamma/n)\) and \(X = \text{Poi}(\gamma)\) for some \(\gamma > 0\). This case is of particular interest due the closeness between the configuration models and Galton-Watson trees.

8. Appendix

8.1. Complement of the proof of Lemma 2.1. We first recall the formula of \(f(t)\)
\[
f(t) = c(1 - 2t) + \frac{\sqrt{1 + (c^2 - 1)(1 - 2t)^2}}{2(1 - t)}
\]
which satisfies the fixed point equation
\[
\theta = c(1 - 2t) + \frac{t}{\theta(1 - t)},
\]
with \(c = e^{-2\beta}\).

Proof of (2.17) and (2.18). It follows from (8.1) and (8.2) that for all \(0 \leq t \leq 1/2\),
\[
f(t) \leq \frac{(1 - 2t) + 1}{2(1 - t)} = 1
\]
and
\[
f'(t) = \frac{-c}{(1 - t)^2} + \frac{1}{f(t)(1 - t)^2} = \frac{f'(t)t}{f(t)^2(1 - t)}.
\]
Hence
\[
f'(t) \left(1 + \frac{t}{f(t)^2(1 - t)}\right) = \frac{(1/f(t)) - c}{(1 - t)^2} > 0.
\]
(8.4)
Thus \(f(t)\) is increasing in \((0, 1/2)\). Therefore
\[
c = f(0) \leq f(t) \leq 1,
\]
(8.5)
which implies (2.17). It follows from (8.4) and (8.5) that for all \( t \in (0, 1/2) \),
\[
\frac{1 - c}{1 + 1/c^2} \leq f'(t) \leq \frac{4(1 - c^2)}{c}.
\] (8.6)

Similarly,
\[
f''(t) = \frac{-2c (1 - t)^3 + 2}{f(t)(1 - t)^3} - \frac{2 f'(t)}{f(t)^2 (1 - t)^2} - \frac{t}{1 - t} \left( \frac{f''(t) f(t)^2 - 2 f(t) f'(t)^2}{f(t)^4} \right).
\]

Hence
\[
f''(t) \left(1 + \frac{t}{(1 - t) f(t)^2}\right) = \frac{2(1/f(t) - c)}{(1 - t)^3} - \frac{2 f'(t)}{f(t)^2 (1 - t)^2} \left( \frac{1}{1 - t} - \frac{t f'(t)}{f(t)} \right).
\]

Using this together with (8.5) and (8.6), we can show that there is a positive constant \( A = A(c) \), such that for all \( t \in (0, 1/2) \),
\[
1/A \leq f'(t) \leq A \quad \text{and} \quad |f''(t)| \leq A.
\] (8.7)

Thus (2.18) holds. \( \square \)

**Proof of (2.19).** Let us recall the sequence \( h(k, m) \) defined in Section 2: \( h(1, m) = c \) and for \( k \leq [m/2] \),
\[
h(k + 1, m) = \frac{c(m - 2k)}{m - k} + \frac{k}{(m - k) h(k, m)}. \quad \text{(8.8)}
\]

We define
\[
K = \left[ A^2 c^2 + 2 \right] c^3,
\]
with \( A \) as in (8.7). We first claim that for \( m \geq 4K \) and \( 1 \leq k \leq k_* := [m/2] - K \),
\[
f((k - 1)/m) \leq h(k, m) \leq f(k/m). \quad \text{(8.9)}
\]

Assuming (8.9), we now prove (2.19). Let us define for \( k \leq [m/2] \),
\[
a_k = |h(k, m) - f((k - 1)/m)|.
\]

By (8.9), for \( 0 \leq k \leq k_* \) we have
\[
a_k \leq |f(k/m) - f((k - 1)/m)| \leq A/m, \quad \text{(8.10)}
\]

by using the mean value theorem and (8.7). To estimate \( a_k \) with \( k \geq k_* \), we need some bounds on \( h(k, m) \). By (8.8), we have for all \( k_* \leq k \leq [m/2] = k_* + K \),
\[
\frac{1}{2h(k, m)} \leq h(k + 1, m) \leq c + \frac{1}{h(k, m)}.
\]

Moreover, \( c \leq h(k_*, m) \leq 1 \) by (8.5) and (8.9). Thus there exists a positive constant \( \Theta = \Theta(K, c) \geq 1 \), such that for all \( k_* \leq k \leq [m/2] \)
\[
1/\Theta \leq h(k, m) \leq \Theta. \quad \text{(8.11)}
\]

By (8.1), we have for \( k \leq [m/2] \),
\[
f(k/m) = \frac{c(m - 2k)}{m - k} + \frac{k}{(m - k) f(k/m)}. \quad \text{(8.12)}
\]
Then using (8.8) and (8.12), we get that for \( k_* \leq k \leq \lfloor m/2 \rfloor \),
\[
a_{k+1} = \frac{k}{m-k} \left| \frac{1}{h(k,m)} - \frac{1}{f(k/m)} \right|
\]
(use (8.5), (8.11))
\[
\leq \frac{\Theta |h(k,m) - f(k/m)|}{c}
\]
\[
\leq \frac{\Theta |h(k,m) - f((k-1)/m)|}{c} + \frac{\Theta |f(k,m) - f((k-1)/m)|}{c}
\]
(use (8.7))
\[
\leq \frac{\Theta a_k}{c} + \frac{\Theta A}{mc}.
\]  

(8.13)

Using (8.13), we can prove by induction on \( t \) that for all \( k_* \leq k_* + t \leq \lfloor m/2 \rfloor \),
\[
a_{k_*+t} \leq \left( \frac{\Theta}{c} \right)^t a_{k_*} + \frac{\Theta A}{mc} \sum_{i=0}^{t-1} \left( \frac{\Theta}{c} \right)^i \leq \left( \frac{\Theta}{c} \right)^t a_{k_*} + \frac{A}{m} \left( \frac{\Theta}{c} \right)^{t+1}.
\]  

(8.14)

Using (8.10) and (8.14), we obtain that for all \( k \leq \lfloor m/2 \rfloor \),
\[
a_k \leq \zeta/m,
\]
with
\[
\zeta = A \left[ \left( \frac{\Theta}{c} \right)^K + \left( \frac{\Theta}{c} \right)^{K+1} \right],
\]
which implies (2.19).

We now prove (8.9) by induction on \( k \). For \( k = 1 \), we have
\[
c = h(1,m) = f(0/m) \leq f(1/m),
\]
since \( f(t) \) is increasing. Suppose that (8.9) holds for all \( k_1 \leq k_* - 1 = \lfloor m/2 \rfloor - K - 1 \). We now show that it holds for \( k + 1 \). Using (8.8) and (8.12) and \( h(k,m) \leq f(k/m) \), we get
\[
h(k+1,m) = \frac{c(m-2k)}{m-k} + \frac{k}{(m-k)h(k,m)}
\]
\[
\geq \frac{c(m-2k)}{m-k} + \frac{k}{(m-k)f(k/m)}
\]
\[
= f(k/m).
\]  

(8.15)

Similarly, using \( f((k-1)/m) \leq h(k,m) \), we obtain
\[
h(k+1,m) \leq \frac{c(m-2k)}{m-k} + \frac{k}{(m-k)f((k-1)/m)}
\]
\[
= f(k/m) + \frac{k}{m-k} \left( \frac{1}{f((k-1)/m)} - \frac{1}{f(k/m)} \right)
\]
\[
\leq f(k/m) + \frac{k}{m-k} \left( \frac{f(k/m) - f((k-1)/m)}{f((k-1)/m)^2} \right),
\]
since \( f(t) \) is increasing in \([0,1/2]\]. Let us define for \( k \leq k_* \),
\[
b_k = f(k/m) - f((k-1)/m).
\]
It follows from (8.16), (8.17), (8.18), (8.19) that by using (8.7) and the fact that

\[ f \leq \frac{c}{m - k} \]

for some \( y_k \in ((k - 1)/m, k/m) \) and \( y_{k+1} \in (k/m, (k + 1)/m) \). Using the mean value theorem, we have

\[ b_{k+1} - b_k = \frac{f'(y_{k+1}) - f'(y_k)}{m} \geq \frac{-2}{m^2} \max_{m/2 \leq t \leq 2} |f''(t)| \geq \frac{-2A}{m^2}. \]  

Using (8.12), we obtain

\[ 1 - \frac{(k-1)}{(m-k+1)f((k-1)/m)^2} = \frac{c(m-2k+2)}{(m-k+1)f((k-1)/m)} \geq \frac{c(m-2k+2)}{(m-k+1)}, \]

since \( f(t) \leq 1 \) for all \( t \leq 1/2 \). On the other hand, for \( k \leq [m/2] \)

\[ \left| \frac{k}{(m-k)f((k-1)/m)^2} - \frac{(k-1)}{(m-k+1)f((k-1)/m)^2} \right| \leq \frac{4}{mf((k-1)/m)^2} \leq \frac{4}{mc^2}, \]

since \( c \leq f(t) \) for all \( t \leq 1/2 \). Combining the last two inequalities yields that

\[ 1 - \frac{4k}{mc^2} \leq \frac{c(m-2k+2)}{m} - \frac{4}{mc^2} \geq \frac{c(m-2k+2)}{m} - \frac{4}{mc^2}. \]  

It follows from (8.16), (8.17), (8.18), (8.19) that

\[ f((k+1)/m) - h(k+1,m) \geq -\frac{2A}{m^2} + \frac{f'(y_k)}{m} \left( \frac{c(m-2k+2)}{m} - \frac{4}{mc^2} \right) \geq -\frac{2A}{m^2} + \frac{1}{Am} \left( \frac{c(m-2k+2)}{m} - \frac{4}{mc^2} \right) \geq 0, \]  

by using (8.7) and the fact that

\[ k \leq [m/2] - \left[ \frac{A^2c^2 + 2}{c^3} \right]. \]

It follows from (8.15) and (8.20) that the induction step from \( k \) to \( k + 1 \) holds. Thus the proof of (8.9) is completed. \( \square \)

8.2. Proof of Lemma 3.1. Assume that \( t \leq 1/2 \). Using integration by parts, we have

\[ F(t) = \int_0^t \log f(s)ds = t \log f(t) - \int_0^t \frac{f'(s)}{f(s)}ds. \]  

We have \( f(s) = A(s)/B(s) \), where

\[ A(s) = e^{-2\beta}(1 - 2s) + \sqrt{1 + (e^{-4\beta} - 1)(2s - 1)^2} \quad \text{and} \quad B(s) = 2(1 - s). \]

Moreover,

\[ \frac{A'(s)}{A(s)} = \frac{1}{2s(1-s)} \left[ 1 - 2s - \frac{e^{-2\beta}}{\sqrt{1 + (e^{-4\beta} - 1)(2s - 1)^2}} \right], \]
and

\[
\frac{B'(s)}{B(s)} = -\frac{1}{1-s}.
\]

Hence

\[
\frac{f'(s)}{f(s)} = s \left[ \frac{A'(s)}{A(s)} - \frac{B'(s)}{B(s)} \right] = \frac{1}{2(1-s)} - \frac{e^{-2\beta}}{2(1-s)\sqrt{1 + (e^{-4\beta} - 1)(2s - 1)^2}}.
\] (8.22)

Combining (8.21) and (8.22) gives that

\[
F(t) = t \log f(t) + \frac{1}{2} \log(1-t) + \int_0^t \frac{e^{-2\beta}}{2(1-s)\sqrt{1 + (e^{-4\beta} - 1)(2s - 1)^2}} ds.
\] (8.23)

Let \(\alpha = \sqrt{1 - e^{-4\beta}} \in (0, 1)\). Then by computation and changing variables, we have

\[
J = \int_0^t \frac{e^{-2\beta}}{(1-s)\sqrt{1 + (e^{-4\beta} - 1)(2s - 1)^2}} ds
\]

\[
(u = 1 - 2s) = \int_{1-2t}^{1} \frac{2\sqrt{1 - \alpha^2}}{(1+u)\sqrt{1 - \alpha^2}u^2} du
\]

\[
(v = \arcsin(\alpha u)) = \int_{\arcsin(\alpha(1-2t))}^{\arcsin(\alpha)} \alpha + \sin v dv
\]

\[
(w = \tan(v/2)) = \int_{w_1}^{w_2} \frac{2\sqrt{1 - \alpha^2}}{\alpha w^2 + 2w + \alpha} dw
\]

\[
= \log \left( \frac{\alpha w + 1 - \sqrt{1 - \alpha^2}}{\alpha w + 1 + \sqrt{1 - \alpha^2}} \right)^{w_2}_{w_1}.
\]

For \(x \in (0, 1)\), we have

\[
\tan \left( \frac{\arcsin(x)}{2} \right) = \frac{1 - \sqrt{1 - x^2}}{x}.
\]

Thus

\[
w_2 = \frac{1 - \sqrt{1 - \alpha^2}}{\alpha} \quad \text{and} \quad w_1 = \frac{1 - \sqrt{1 - \alpha^2(1-2t)^2}}{\alpha(1-2t)}.
\]

Therefore

\[
J = \log (1 - e^{-2\beta}) - \log \left( \frac{(1 - e^{-4\beta})(1-2t) + (1 - e^{-2\beta})[1 + \sqrt{1 + (e^{-4\beta} - 1)(1-2t)^2}]}{(1 - e^{-4\beta})(1-2t) + (1 - e^{-2\beta})[1 + \sqrt{1 + (e^{-4\beta} - 1)(1-2t)^2}]} \right)
\]

\[
= \log (1 + e^{-2\beta}) - \log \left( \frac{(1 + e^{-2\beta})(1-2t) + 1 + \sqrt{1 + (e^{-4\beta} - 1)(1-2t)^2}}{(1 - e^{-2\beta})(1-2t) + 1 + \sqrt{1 + (e^{-4\beta} - 1)(1-2t)^2}} \right)
\]

\[
= \log (1 + e^{-2\beta}) + \log \left( \frac{1 - t + tf(t-1)}{(1-t)(f(t) + 1)} \right).\]

Combining this with (8.23), we obtain

\[
F(t) = t \log f(t) + \frac{1}{2} \log(1-t) + \frac{1}{2} \log(1 + e^{-2\beta}) + \frac{1}{2} \log \left[ 1 + \frac{e^{-2\beta}(2t - 1)}{(1-t)(f(t) + 1)} \right].\] (8.24)
Thus Lemma 3.1 follows.

8.3. **Proof of Proposition 3.2.** We first recall the formula for the quenched pressure determined in [5]. Suppose that $\beta > 0$ and $B > 0$. Let $h_*$ be the positive solution of a fixed point equation:

$$h = B + (d - 1)\tanh(\tanh(\beta) \tanh(h)).$$  \hfill (8.25)

Then

$$\tilde{\psi}(\beta, B) = \frac{d}{2} \log(\cosh(\beta)) - \frac{d}{2} \log(1 + \tanh(\beta) \tanh(h_*))$$  \hfill (8.26)

$$+ \log \left[ e^B (1 + \tanh(\beta) \tanh(h_*))^d + e^{-B} (1 - \tanh(\beta) \tanh(h_*))^d \right].$$

For $B < 0$, one has $\tilde{\psi}(\beta, B) = \tilde{\psi}(\beta, -B)$, and $\tilde{\psi}(\beta, 0) = \lim_{B \to 0} \tilde{\psi}(\beta, B)$. In this subsection, we will show that $\tilde{\psi}(\beta, B) = \psi(\beta, B)$.

We prove here the case $B > 0$, then the other case follows from the fact that both of the functions $\tilde{\psi}(\beta, \cdot)$ and $\psi(\beta, \cdot)$ are even. We have proved in Sections 3 and 4 that for $B > 0$,

$$\psi(\beta, B) = \frac{\beta d}{2} - B + L(t_*),$$  \hfill (8.28)

where

$$L(t) = -t \log(t) + (t - 1) \log(1 - t) + 2Bt + dF(t),$$

and $t_* \in (1/2, 1)$ is the unique solution of the equation

$$L'(t) = \log \left( \frac{1 - t}{t} \right) - d \log f(1 - t) + 2B = 0.$$  \hfill (8.29)

We claim a relation between $h_*$ and $t_*$, which will prove later.

$$2t_* - 1 = \tanh(h_* + \tanh(\tanh(\beta) \tanh(h_*))).$$  \hfill (8.30)

Assuming (8.30), we now prove (8.27).

**Expression of $\tilde{\psi}(\beta, B)$**. Let us denote

$$u_* = \tanh(\beta) \tanh(h_*).$$

Applying the function $\tanh$ to both sides of the equation (8.25), we get

$$\tanh(h_*) = \tanh(B + (d - 1)\tanh(u_*))$$

$$= \frac{e^{2B} (1 + u_*)^{d-1} - (1 - u_*)^{d-1}}{e^{2B} (1 + u_*)^{d-1} + (1 - u_*)^{d-1}}.$$  \hfill (8.31)

Thus

$$1 + \tanh(\beta) \tanh(h_*)^2 = 1 + u_* \tanh(h_*) = \frac{e^{2B} (1 + u_*)^d + (1 - u_*)^d}{e^{2B} (1 + u_*)^{d-1} + (1 - u_*)^{d-1}}.$$  

Therefore, using (8.26) we have

$$\tilde{\psi}(\beta, B) = \frac{d}{2} \log(\cosh(\beta)) - \frac{d}{2} \log \left[ \frac{e^{2B} (1 + u_*)^d + (1 - u_*)^d}{e^{2B} (1 + u_*)^{d-1} + (1 - u_*)^{d-1}} \right]$$

$$+ \log \left[ e^B (1 + u_*)^d + e^{-B} (1 - u_*)^d \right].$$  \hfill (8.32)
Expression of $\psi(\beta, B)$. We first display $t_*$ and $e^{-2\beta}$ in term of $u_*$. Using (8.30), we have
\begin{equation}
2t_* - 1 = \tanh(B + \text{d tanh}(u_*)) = \frac{e^{2B}(1 + u_*)^d - (1 - u_*)^d}{e^{2B}(1 + u_*)^d + (1 - u_*)^d}.
\end{equation}

Thus
\begin{equation}
t_* = \frac{e^{2B}(1 + u_*)^d}{e^{2B}(1 + u_*)^d + (1 - u_*)^d} \quad \text{and} \quad 1 - t_* = \frac{(1 - u_*)^d}{e^{2B}(1 + u_*)^d + (1 - u_*)^d}.
\end{equation}

On the other hand, using (8.31) we get
\begin{equation}
e^{-2\beta} = \frac{1 - \tanh(\beta)}{1 + \tanh(\beta)} = \frac{1 - \frac{u_*}{\tanh(h_*)}}{1 + \frac{u_*}{\tanh(h_*)}} = (1 - u_*^2) \frac{e^{2B}(1 + u_*)^{d-2} - (1 - u_*)^{d-2}}{e^{2B}(1 + u_*)^d - (1 - u_*)^d}.
\end{equation}

Since $t_* > 1/2$, we have $F(t_*) = F(1 - t_*)$. Thus using (8.24),
\begin{equation}
F(t_*) = F(1 - t_*) = (1 - t_*) \log f(1 - t_*) + \frac{1}{2} \log t_* + \frac{1}{2} \log(1 + e^{-2\beta}) + \frac{1}{2} \log \left[1 + \frac{e^{-2\beta}(1 - 2t_*)}{t_*(f(1 - t_*) + 1)}\right].
\end{equation}

Since $t_*$ is the solution of (8.29), we have
\begin{equation}
f(1 - t_*) = e^{\frac{2B}{d}} \left(\frac{1 - t_*}{t_*}\right)^{\frac{1}{d}}.
\end{equation}

Using (8.37) and (8.34),
\begin{equation}
f(1 - t_*) = \frac{1 - u_*}{1 + u_*}.
\end{equation}

Combining (8.33), (8.34), (8.35) and (8.38) yields that
\begin{equation}
1 + \frac{e^{-2\beta}(1 - 2t_*)}{t_*(f(1 - t_*) + 1)} = \frac{(1 + u_*)^2}{2} \times \frac{e^{2B}(1 + u_*)^{d-1} + (1 - u_*)^{d-1}}{e^{2B}(1 + u_*)^d}.
\end{equation}

Using (8.36), (8.37) and (8.39),
\begin{align*}
dF(t_*) &= (1 - t_*) \left(2B + \log \left(\frac{1 - t_*}{t_*}\right)\right) + \frac{d}{2} \log(1 + e^{-2\beta}) + d \log(1 + u_*) - \frac{d}{2} \log 2 + \frac{d}{2} \log \left(t_* \frac{e^{2B}(1 + u_*)^{d-1} + (1 - u_*)^{d-1}}{e^{2B}(1 + u_*)^d}\right) \\
&= (2 - 2t_*)B + (1 - t_*) \log \left(\frac{1 - t_*}{t_*}\right) + \frac{d}{2} \log \left[(1 + e^{-2\beta})/2\right] \\
&\quad (\text{use (8.34)}) + d \log(1 + u_*) + \frac{d}{2} \log \left(\frac{e^{2B}(1 + u_*)^{d-1} + (1 - u_*)^{d-1}}{e^{2B}(1 + u_*)^d + (1 - u_*)^d}\right).
\end{align*}
Hence
\[
\psi(\beta, B) = \frac{\beta d}{2} - B + (t_* - 1) \log(1 - t_*) - t_* \log t_* + 2Bt_* + dF(t_*)
\]
\[
= \frac{d}{2} \left( \beta + \log \left[ (1 + e^{-2\beta})/2 \right] \right) + B - \log t_* + d \log(1 + u_*) + \frac{d}{2} \log \left( \frac{e^{2B}(1 + u_*^{d-1}) + (1 - u_*)^{d-1}}{e^{2B}(1 + u_*)^d + (1 - u_*)^d} \right)
\]
\[
= \frac{d}{2} \log(\cosh(\beta)) + \log(e^{B} t_*^{-1}(1 + u_*)^d) + \frac{d}{2} \log \left( \frac{e^{2B}(1 + u_*^{d-1}) + (1 - u_*)^{d-1}}{e^{2B}(1 + u_*)^d + (1 - u_*)^d} \right)
\]
where for the last line, we used (8.34). Using this equation with (8.32), we obtain that
\[
\psi(\beta, B) = \tilde{\psi}(\beta, B),
\]
which proves (8.27). For \(d = 2\), it has been shown in [14, 15] that
\[
\psi(\beta, B) = \tilde{\psi}(\beta, B) = \beta + \log \left( \cosh(\beta) + \sqrt{\sinh^2(\beta) + e^{-4\beta}} \right).
\]

Proof of (8.30). Let us denote
\[
v_* = \tanh(h_* + \text{atanh}(\tanh(\beta) \tanh(h_*))).
\]
We claim the following identity (E): For all \(x > 0\) and \(y \in \mathbb{R}\), if
\[
v = \tanh(y + \text{atanh}(\tanh(x) \tanh(y))),
\]
then
\[
e^{-2x}v + \sqrt{1 + (e^{-2x} - 1)v^2} = \frac{\cosh(x - y)}{\cosh(x + y)}.
\]
Assuming (E), we can prove (8.30). Indeed, using (8.41) we get
\[
e^{-2\beta}v_* + \sqrt{1 + (e^{-4\beta} - 1)v_*^2} = \frac{\cosh(\beta - h_*)}{\cosh(\beta + h_*)}.
\]
Since \(h_*\) is the solution of (8.25), we have
\[
v_* = \tanh(B + \text{atanh}(\tanh(\beta) \tanh(h_*))).
\]
Applying the function atanh to both sides of the above equation, we obtain
\[
\frac{1}{2} \log \left( \frac{1 + v_*}{1 - v_*} \right) = B + \frac{d}{2} \log \left( \frac{\cosh(\beta + h_*)}{\cosh(\beta - h_*)} \right).
\]
Combining this with (8.42), we have
\[
\log \left( \frac{1 - v_*}{1 + v_*} \right) - d \log \left( \frac{e^{-2\beta}v_* + \sqrt{1 + (e^{-4\beta} - 1)v_*^2}}{v_* + 1} \right) + 2B = 0,
\]
or equivalently, by using (8.29)
\[
L' \left( \frac{v_* + 1}{2} \right) = 0.
\]
It follows from (8.29) and (8.44) that \(t_*\) and \((v_* + 1)/2\) are solutions in \((1/2, 1)\) of the equation \(L'(x) = 0\). We have proved in Claim 1* in Section 4 that this equation has unique solution. Thus \(t_* = (v_* + 1)/2\), and (8.30) follows.
We now prove the identity (E). Applying the function atanh to the both sides of (8.40) gives that
\begin{equation}
\frac{1 + v}{1 - v} = \frac{e^{2y} \cosh(x + y)}{\cosh(x - y)}. \tag{8.45}
\end{equation}
Hence, (8.41) is equivalent to
\begin{equation}
(1 - v)e^{2y} = e^{-2x} + \sqrt{1 + (e^{-4x} - 1)v^2},
\end{equation}
or
\begin{align}
(1 - v)e^{2y} - e^{-2x}v & \geq 0 \\
((1 - v)e^{2y} - e^{-2x}v)^2 & = 1 + (e^{-4x} - 1)v^2 \tag{8.46}
\end{align}
We have
\begin{align}
(1 - v)e^{2y} - e^{-2x}v & \geq 0 \iff e^{2x+2y} \geq \frac{v}{1 - v}.
\end{align}
On the other hand
\begin{align}
\frac{v}{1 - v} & \leq \frac{1 + v}{1 - v} = \frac{e^{2y} \cosh(x + y)}{\cosh(x - y)} = \frac{e^{y-x} + e^{3y+x}}{e^{-y} + e^{y-x}} \leq e^{2x+2y},
\end{align}
since $x > 0$. Hence, the inequality in (8.46) holds. The equation in (8.46) is equivalent to
\begin{align}
(1 - v)e^{2y} - 2v(1 - v)e^{2y-2x} & = 1 - v^2 \\
\iff & \left(\frac{1 - v}{1 + v}\right) e^{4y} - 2v \left(\frac{1 - v}{1 + v}\right) e^{2y-2x} = 1 \\
\iff & \left(\frac{1 - v}{1 + v}\right) e^{4y} + \left(\frac{1 - v}{1 + v} - 1\right) e^{2y-2x} = 1 \\
\iff & \left(\frac{1 - v}{1 + v}\right) (e^{4y} + e^{2y-2x}) = 1 + e^{2y-2x}. \quad \text{(using (8.45))}
\end{align}
which holds for all $x, y$. In conclusion, (8.46) holds, and thus (8.41) follows. □

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**References**


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