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To Chuck Newman, Friend, Colleague, and Mentor

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Abstract

We review a (constructive) approach first introduced in [6] and further developed in [7, 8, 38, 9] for hydrodynamic limits of asymmetric attractive particle systems, in a weak or in a strong (that is, almost sure) sense, in an homogeneous or in a quenched disordered setting.

1 Introduction

Among the most studied conservative interacting particle systems (*IPS*) are the simple exclusion and the zero-range processes. They are *attractive pro-*

cesses, and possess a one-parameter family of product extremal invariant and translation invariant probability measures, that we denote by $\{\nu_\alpha\}_\alpha$, where α represents the mean density of particles per site: for simple exclusion $\alpha \in [0,1]$, and for zero-range $\alpha \in [0,+\infty)$ (see [36, Chapter VIII], and [1]). Both belong to a more general class of systems with similar properties, called *misanthropes* processes ([19]).

Hydrodynamic limit ([21, 45, 33]) is a law of large numbers for the time evolution (usually described by a limiting PDE, called the hydrodynamic equation) of empirical density fields in interacting particle systems. Most usual IPS can be divided into two groups, diffusive and hyperbolic. In the first group, which contains for instance the symmetric or mean-zero asymmetric simple exclusion process, the macroscopic→microscopic space-time scaling is $(x,t) \mapsto (Nx, N^2t)$ with $N \to \infty$, and the limiting PDE is a diffusive equation. In the second group, which contains for instance the nonzero mean asymmetric simple exclusion process, the scaling is $(x,t) \mapsto (Nx, Nt)$, and the limiting PDE is of Euler type. In both groups the PDE often exhibits nonlinearity, either via the diffusion coefficient in the first group, or via the flux function in the second one. This raises special difficulties in the hyperbolic case, due to shocks and non-uniqueness for the solution of the PDE, in which case the natural problem is to establish convergence to the so-called *entropy solution* ([44]).

In most known results, only a weak law of large numbers is established. In this description one need not have an explicit construction of the dynamics: the limit is shown in probability with respect to the law of the process, which is characterized in an abstract way by its Markov generator and Hille-Yosida's theorem ([36]). Nevertheless, when simulating particle systems, one naturally uses a pathwise construction of the process on a Poisson space-time random graph (the so-called *graphical construction*). In this description the dynamics is deterministically driven by a random space-time measure which tells when and where the configuration has a chance of being modified. It is of special interest to show that the hydrodynamic limit holds for almost every realization of the space-time measure, as this means a single simulation is enough to approximate solutions of the limiting PDE.

We are interested here in the hydrodynamic behavior of a class of asymmetric particle systems of $\mathbb{Z}$, which arise as a natural generalization of the asymmetric exclusion process. For such processes, hydrodynamic limit is given by the entropy solutions to a scalar conservation law of the form

$$\partial_t u(x,t) + \partial_x G(u(x,t)) = 0 \tag{1}$$

where $u(.,.)$ is the density field and G is the *macroscopic flux*. The latter is given for the asymmetric exclusion process by $G(u) = \gamma u(1-u)$, where γ is the mean drift of a particle. Because there is a single conserved quantity (*i.e.* mass) for the particle system, and an ergodic equilibrium measure for each density value, (1) can be guessed through heuristic arguments if one takes for granted that the system is in *local equilibrium*. The macroscopic flux G is obtained by an equilibrium expectation of a microscopic flux which can be written down explicitly from the dynamics. A rigorous proof of the hydrodynamic limit turns out to be a difficult problem, mainly because of the non-existence of strong solutions for (1) and the non-uniqueness of weak solutions. Since the conservation law is not sufficient to pick a single solution, the so-called entropy weak solution must be characterized by additional properties; one must then look for related properties of the particle system to establish its convergence to the entropy solution.

The derivation of hyperbolic equations of the form (1) as hydrodynamic limits began with the seminal paper [41], which established a strong law of large numbers for the totally asymmetric simple exclusion process on $\mathbb{Z}$, starting with 1's to the left of the origin and 0's to the right. This result was extended by [15] and [2] to nonzero mean exclusion process starting from product Bernoulli distributions with arbitrary densities λ to the left and ρ to the right (the so-called *Riemann initial condition*). The Bernoulli distribution at time 0 is related to the fact that uniform Bernoulli measures are invariant for the process. For the one-dimensional totally asymmetric (nearest-neighbor) K-exclusion process, a particular misanthropes process without explicit invariant measures, a strong hydrodynamic limit was established in [43], starting from arbitrary initial profiles, by means of the so-called *variational coupling*, that is a microscopic version of the Lax-Hopf formula. These were the only strong laws available before the series of works reviewed here. A common feature of these works is the use of subadditive ergodic theorem to exhibit some a.s. limit, which is then identified by additional arguments.

On the other hand, many *weak* laws of large numbers were established for attractive particle systems. A first series of results treated systems with product invariant measures and product initial distributions. In [3], for a particular zero-range model, a weak law was deduced from conservation of local equilibrium under Riemann initial condition. It was then extended in [4] to the misanthropes process of [19] under an additional convexity assumption on the flux function. These were substantially generalized (using Kružkov's entropy inequalities, see [34]) in [39] to multidimensional attractive systems with product invariant measures for arbitrary Cauchy data, without any convexity requirement on the flux. In [40], using an abstract characterization of the evolution semigroup associated with the limiting equation, hydrodynamic limit was established for the one-dimensional nearest-neighbor K-exclusion process.

The above results are concerned with translation-invariant particle dynamics. We are also interested in hydrodynamic limits of particle systems in random environment, leading to homogenization effects, where an effective diffusion matrix or flux function is expected to capture the effect of inhomogeneity. Hydrodynamic limit in random environment has been widely addressed and robust methods have been developed in the diffusive case. In the hyperbolic setting, the few available results in random environment (prior to [9]) depended on particular features of the investigated models. In [16], the authors prove, for the asymmetric zero-range process with site disorder on $\mathbb{Z}^d$, a quenched hydrodynamic limit given by a hyperbolic conservation law with an effective homogenized flux function. To this end, they use in particular the existence of explicit product invariant measures for the disordered zero-range process below some critical value of the density value. In [42], extension to the supercritical case is carried out in the totally asymmetric case with constant jump rate. In [43], the author establishes a quenched hydrodynamic limit for the totally asymmetric nearest-neighbor K-exclusion process on $\mathbb{Z}$ with i.i.d site disorder, for which explicit invariant measures are not known. The last two results rely on variational coupling. However, the simple exclusion process beyond the totally asymmetric nearest-neighbor case, or more complex models with state-dependent jump rates, remain outside the scope of this approach.

In this paper, we review successive stages ([6, 7, 8, 38, 9]) of a *constructive* approach to hydrodynamic limits given by equations of the type (1), which ultimately led us in [9] to a very general hydrodynamic limit result for attractive particle systems in one dimension in ergodic random environment. We shall detail our method in the setting of [9]. However, we will first explain our approach and advances along the progression of papers, and quote results for each one, since they are interesting in their own. We hope this could be helpful for a reader looking for hydrodynamics of a specific model: according to the available knowledge on this model, this reader could derive either a weak, or a strong (without disorder or with a quenched disorder) hydrodynamic limit.

Our motivation for [6] was to prove with a constructive method hydrodynamics of one-dimensional attractive dynamics with product invariant measures, but without a concave/convex flux, in view of examples of k-step exclusion processes and misanthropes processes. We initiated for that a "resurrection" of the approach of [4], and we introduced a variational formula for the entropy solution of the hydrodynamic equation in the Riemann case, and an approximation scheme to go from a Riemann to a general initial profile. Our method is based on an interplay of macroscopic properties for the conservation law and analogous microscopic properties for the particle system. The next stage, achieved in [7], was to derive hydrodynamics (which was still a weak law) for attractive processes without explicit invariant measures. In the same setting, we then obtained almost sure hydrodynamics in [8], relying on a graphical representation of the dynamics. The latter result, apart from its own interest, proved to be an essential step to obtain quenched hydrodynamics in the disordered case ([9]). For this last paper, we also relied on [38], which improves an essential property we use, macroscopic stability.

Let us mention that we are now working ([12, 13]) on the hydrodynamic behavior of the disordered asymmetric zero-range process. This model falls outside the scope of the present paper because it exhibits a phase transition with a critical density above which no invariant measure exists. In the supercritical regime, the hydrodynamic limit cannot be established by local equilibrium arguments, and condensation may occur locally on a finer scale than the hydrodynamic one. Other issues related to this model have been studied recently in [10, 11].

This review paper is organized as follows. In Section 2, after giving general notation and definitions, we introduce the two basic models we originally worked with, the misanthropes process and the k-step exclusion process.

Then we describe informally the results, and the main ideas involved in [4] which was our starting point, and in each of the papers [6, 7, 8, 38, 9]. Section 3 contains our main results, stated (for convenience) for the misanthropes process. We then aim at explaining how these results are proved. In Section 4, we first give a self-contained introduction to scalar conservation laws, with the main definitions and results important for our purposes; then we explain the derivation of our variational formula in the Riemann case (illustrated by an example of 2-step exclusion process), and finally our approximation scheme to solve the general Cauchy problem. In Section 5, we outline the most important steps of our proof of hydrodynamic limit in a quenched disordered setting: again, we first deal with the Riemann problem, then with the Cauchy problem. Finally, in Section 6, we define a general framework which enables to describe a class of models possessing the necessary properties to derive hydrodynamics, and study a few examples.

2 Notation and preliminaries

Throughout this paper $\mathbb{N} = \{1, 2, ...\}$ will denote the set of natural numbers, $\mathbb{Z}^+ = \{0, 1, 2, ...\}$ the set of non-negative integers, and $\mathbb{R}^{+*} = \mathbb{R}^+ \setminus \{0\}$ the set of positive real numbers. The integer part $\lfloor x \rfloor \in \mathbb{Z}$ of $x \in \mathbb{R}$ is uniquely defined by $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$.

The set of environments (or disorder) is a probability space $(\mathbf{A}, \mathcal{F}_{\mathbf{A}}, Q)$, where $\mathbf{A}$ is a compact metric space and $\mathcal{F}_{\mathbf{A}}$ its Borel σ-field. On $\mathbf{A}$ we have a group of space shifts $(\tau_x : x \in \mathbb{Z})$, with respect to which Q is ergodic.

We consider particle configurations (denoted by greek letters $\eta, \xi \ldots$) on $\mathbb{Z}$ with at most K (but always finitely many) particles per site, for some given $K \in \mathbb{N} \cup \{+\infty\}$. Thus the state space, which will be denoted by $\mathbf{X}$, is either $\mathbb{N}^{\mathbb{Z}}$ in the case $K = +\infty$, or $\{0, \cdots, K\}^{\mathbb{Z}}$ for $K \in \mathbb{N}$. For $x \in \mathbb{Z}$ and $\eta \in \mathbf{X}$, $\eta(x)$ denotes the number of particles on site x. This state space is endowed with the product topology, which makes it a metrisable space, compact when $\mathbf{X} = \{0, \cdots, K\}^{\mathbb{Z}}$.

A function f defined on $\mathbf{A} \times \mathbf{X}$ (resp. g on $\mathbf{A} \times \mathbf{X}^2$, h on $\mathbf{X}$) is called *local* if there is a finite subset Λ of $\mathbb{Z}$ such that $f(\alpha, \eta)$ depends only on α and $(\eta(x), x \in \Lambda)$ (resp. $g(\alpha, \eta, \xi)$ depends only on α and $(\eta(x), \xi(x), x \in \Lambda)$,

$h(\eta)$ depends only on $(\eta(x), x \in \Lambda))$. We denote again by τ_x either the spatial translation operator on the real line for $x \in \mathbb{R}$, defined by $\tau_x y = x + y$, or its restriction to $x \in \mathbb{Z}$. By extension, if f is a function defined on $\mathbb{Z}$ (resp. $\mathbb{R}$), we set $\tau_x f = f \circ \tau_x$ for $x \in \mathbb{Z}$ (resp. $\mathbb{R}$). In the sequel this will be applied to different types of functions: particle configurations $\eta \in \mathbf{X}$, disorder configurations $\alpha \in \mathbf{A}$, or joint disorder-particle configurations $(\alpha, \eta) \in \mathbf{A} \times \mathbf{X}$. In the latter case, unless mentioned explicitely, τ_x applies simultaneously to both components.

If τ_x acts on some set and μ is a measure on this set, $\tau_x \mu = \mu \circ \tau_x^{-1}$. We let $\mathcal{M}^+(\mathbb{R})$ denote the set of nonnegative measures on $\mathbb{R}$ equipped with the metrizable topology of vague convergence, defined by convergence on continuous test functions with compact support. The set of probability measures on $\mathbf{X}$ is denoted by $\mathcal{P}(\mathbf{X})$. If η is an $\mathbf{X}$-valued random variable and $\nu \in \mathcal{P}(\mathbf{X})$, we write $\eta \sim \nu$ to specify that η has distribution ν. Similarly, for $\alpha \in \mathbf{A}, Q \in \mathcal{P}(\mathbf{A})$, $\alpha \sim Q$ means that α has distribution Q.

2.1 Preliminary definitions

Let us introduce briefly the various notions we shall use in this review, in view of the next section, where we informally tell the content of each of our papers. We shall be more precise in the following sections. Reference books are [36, 33].

The process. We work with a conservative (i.e. involving only particle jumps but no creation/annihilation), attractive (see (2) for its definition below) Feller process $(\eta_t)_{t \geq 0}$ with state space $\mathbf{X}$. When this process evolves in a random environment $\alpha \in \mathbf{A}$, we denote its generator by L_α and its semigroup by $(S_\alpha(t), t \geq 0)$. Otherwise we denote them by L and $S(t)$. In the absence of disorder, we denote by $\mathcal{S}$ the set of translation invariant probability measures on $\mathbf{X}$, by $\mathcal{I}$ the set of invariant probability measures for the process $(\eta_t)_{t \geq 0}$, and by $(\mathcal{I} \cap \mathcal{S})_e$ the set of extremal invariant and translation invariant probability measures for $(\eta_t)_{t \geq 0}$. In the disordered case, $\mathcal{S}$ will denote the set of translation invariant probability measures on $\mathbf{A} \times \mathbf{X}$, see Proposition 3.1.

A sequence $(\nu_n, n \in \mathbb{N})$ of probability measures on $\mathbf{X}$ converges weakly to some $\nu \in \mathcal{P}(\mathbf{X})$, if and only if $\lim_{n \to \infty} \int f \, d\nu_n = \int f \, d\nu$ for every continuous function f on $\mathbf{X}$. The topology of weak convergence is metrizable and makes

$\mathcal{P}(\mathbf{X})$ compact when $\mathbf{X} = \{0, \cdots, K\}^{\mathbb{Z}}$.

We equip $\mathbf{X}$ with the *coordinatewise order*, defined for $\eta, \xi \in \mathbf{X}$ by $\eta \leq \xi$ if and only if $\eta(x) \leq \xi(x)$ for all $x \in \mathbb{Z}$. A partial stochastic order is defined on $\mathcal{P}(\mathbf{X})$; namely, for $\mu_1, \mu_2 \in \mathcal{P}(\mathbf{X})$, we write $\mu_1 \leq \mu_2$ if the following equivalent conditions hold (see *e.g.* [36, 46]):
(i) For every non-decreasing nonnegative function f on $\mathbf{X}$, $\int f\, d\mu_1 \leq \int f\, d\mu_2$.
(ii) There exists a coupling measure $\overline{\mu}$ on $\mathbf{X} \times \mathbf{X}$ with marginals μ_1 and μ_2, such that $\overline{\mu}\{(\eta, \xi) :\ \eta \leq \xi\} = 1$.

The process $(\eta_t)_{t\geq 0}$ is *attractive* if its semigroup acts monotonically on probability measures, that is: for any $\mu_1, \mu_2 \in \mathcal{P}(\mathbf{X})$,

$$\mu_1 \leq \mu_2 \Rightarrow \forall t \in \mathbb{R}^+,\ \mu_1 S_\alpha(t) \leq \mu_2 S_\alpha(t) \tag{2}$$

Hydrodynamic limits. Let $N \in \mathbb{N}$ be the *scaling parameter* for the hydrodynamic limit, that is, the inverse of the macroscopic distance between two consecutive sites. The empirical measure of a configuration η viewed on scale N is given by

$$\pi^N(\eta)(dx) = N^{-1} \sum_{y \in \mathbb{Z}} \eta(y) \delta_{y/N}(dx) \in \mathcal{M}^+(\mathbb{R})$$

where, for $x \in \mathbb{R}$, δ_x denotes the Dirac measure at x, and $\mathcal{M}^+(\mathbb{R})$ denotes the space of Radon measures on $\mathbb{R}$. This space will be endowed with the metrizable topology of vague convergence, defined by convergence against the set $C_K^0(\mathbb{R})$ of continuous test functions on $\mathbb{R}$ with compact support. Let d_v be a distance associated with this topology, and $\pi_.$, $\pi'_.$ be two mappings from $[0, +\infty)$ to $\mathcal{M}^+(\mathbb{R})$. We set

$$\begin{aligned} D_T(\pi_., \pi'_.) &:= \operatorname*{ess\ sup}_{t \in [0,T]} d_v(\pi_t, \pi'_t) \\ D(\pi_., \pi'_.) &:= \sum_{n=0}^{+\infty} 2^{-n} \min[1, D_n(\pi_., \pi'_.)] \end{aligned}$$

A sequence $(\pi^n_.)_{n\in\mathbb{N}}$ of random $\mathcal{M}^+(\mathbb{R})$-valued paths is said to converge locally uniformly in probability to a random $\mathcal{M}^+(\mathbb{R})$-valued path $\pi_.$ if, for every $\varepsilon > 0$,

$$\lim_{n \to +\infty} \mu^n \left(D(\pi^n_., \pi_.) > \varepsilon \right) = 0$$

where μ^n denotes the law of $\pi^n_.$.

Let us now recall, in the context of scalar conservation laws, standard definitions in hydrodynamic limit theory. Recall that $K \in \mathbb{Z}^+ \cup \{+\infty\}$ bounds the number of particles per site. Macroscopically, the set of possible particle densities will be $[0, K] \cap \mathbb{R}$. Let $G : [0, K] \cap \mathbb{R} \to \mathbb{R}$ be a Lipschitz-continuous function, called the *flux*. It is a.e. differentiable, and its derivative G' is an (essentially) uniformly bounded function. We consider the scalar conservation law

$$\partial_t u + \partial_x [G(u)] = 0 \tag{3}$$

where $u = u(x, t)$ is some $[0, K] \cap \mathbb{R}$-valued density field defined on $\mathbb{R} \times \mathbb{R}^+$. We denote by $L^{\infty,K}(\mathbb{R})$ the set of bounded Borel functions from $\mathbb{R}$ to $[0, K] \cap \mathbb{R}$.

Definition 2.1 *Let $(\eta^N)_{N \geq 0}$ be a sequence of $\mathbf{X}$-valued random variables, and $u_0 \in L^{\infty,K}(\mathbb{R})$. We say that the sequence $(\eta^N)_{N \geq 0}$ has:*

(i) weak density profile $u_0(.)$, *if $\pi^N(\eta^N) \to u_0(.)$ in probability with respect to the topology of vague convergence, that is equivalent to: for all $\varepsilon > 0$ and test function $\psi \in C^0_K(\mathbb{R})$,*

$$\lim_{N \to \infty} \mu^N \left(\left| \int_{\mathbb{R}} \psi(x) \pi^N(\eta^N)(dx) - \int_{\mathbb{R}} \psi(x) u_0(x) dx \right| > \varepsilon \right) = 0$$

where μ^N denotes the law of η^N.

(ii) strong density profile $u_0(.)$, *if the random variables are defined on a common probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$, and $\pi^N(\eta^N) \to u_0(.)$ $\mathbb{P}_0$-almost surely with respect to the topology of vague convergence, that is equivalent to: for all test function $\psi \in C^0_K(\mathbb{R})$,*

$$\mathbb{P}_0 \left(\lim_{N \to \infty} \int_{\mathbb{R}} \psi(x) \pi^N(\eta^N)(dx) = \int_{\mathbb{R}} \psi(x) u_0(x) dx \right) = 1,$$

We consider hydrodynamic limits under hyperbolic time scaling, that is Nt, since we work with asymmetric dynamics.

Definition 2.2 *The sequence $(\eta^N_t, t \geq 0)_{N \geq 0}$ has* hydrodynamic limit *(resp.* a.s. hydrodynamic limit*) $u(.,.)$ if: for all $t \geq 0$, $(\eta^N_{Nt})_N$ has weak (resp.*

strong) density profile $u(.,t)$ where $u(.,t)$ is the weak entropy solution *of (3) with initial condition $u_0(.)$, for an appropriately defined* macroscopic flux function G, *where u_0 is the density profile of the sequence $(\eta_0^N)_N$ in the sense of Definition 2.1.*

2.2 Our motivations and approach

Most results on hydrodynamics deal with dynamics with product invariant measures; in the most familiar cases, the flux function appearing in the hydrodynamic equation is convex/concave ([33]). But for many usual examples, the first or the second statement is not true.

Reference examples. We present the attractive misanthropes process on one hand, and the k-step exclusion process on the other hand: these two classical examples will illustrate our purposes along this review. In these basic examples, we take $\alpha \in \mathbf{A} = [c, 1/c]^{\mathbb{Z}}$ (for a constant $0 < c < 1$) as the space of environments; this corresponds to site disorder. We also consider those models without disorder, which corresponds to $\alpha(x) \equiv 1$. However, our approach applies to a much broader class of models and environments, as will be explained in Section 6.

The misanthropes process was introduced in [19] (without disorder). It has state space either $\mathbf{X} = \mathbb{N}^{\mathbb{Z}}$ or $\mathbf{X} = \{0, \cdots, K\}^{\mathbb{Z}}$ ($K \in \mathbb{N}$), and $b\ : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{R}^+$ is the jump rate function. A particle present on $x \in \mathbb{Z}$ chooses $y \in \mathbb{Z}$ with probability $p(y-x)$, where $p(.)$ (the particles' jump kernel) is an asymmetric probability measure on $\mathbb{Z}$, and jumps to y at rate $\alpha(x)b(\eta(x), \eta(y))$. We assume the following:

(M1) $b(0, .) = 0$, with a K-exclusion rule when $\mathbf{X} = \{0, \cdots, K\}^{\mathbb{Z}}$: $b(., K) = 0$;
(M2) Attractiveness: b is nondecreasing (nonincreasing) in its first (second) argument.
(M3) b is a bounded function.
(M4) p has a finite first moment, that is, $\sum_{z\in\mathbb{Z}} |z|\, p(z) < +\infty$.

The quenched disordered process has generator

$$L_\alpha f(\eta) = \sum_{x,y\in\mathbb{Z}} \alpha(x)p(y-x)b(\eta(x), \eta(y)) \left[f\left(\eta^{x,y}\right) - f(\eta)\right] \tag{4}$$

where $\eta^{x,y}$ denotes the new state after a particle has jumped from x to y (that is $\eta^{x,y}(x) = \eta(x) - 1$, $\eta^{x,y}(y) = \eta(y) + 1$, $\eta^{x,y}(z) = \eta(z)$ otherwise). There are two well-known particular cases of attractive misanthropes processes: the *simple exclusion process* ([36]) corresponds to

$$\mathbf{X} = \{0,1\}^{\mathbb{Z}} \quad \text{with} \quad b(\eta(x), \eta(y)) = \eta(x)(1 - \eta(y));$$

the *zero-range process* ([1]) corresponds to

$$\mathbf{X} = \mathbb{N}^{\mathbb{Z}} \quad \text{with} \quad b(\eta(x), \eta(y)) = g(\eta(x)),$$

for a non-decreasing function $g \ : \mathbb{Z}^+ \to \mathbb{R}^+$ (in [1] it is not necessarily bounded).

Let us now restrict ourselves to the model without disorder. For the simple exclusion and zero-range processes, $(\mathcal{I} \cap \mathcal{S})_e$ is a one-parameter family of product probability measures. The flux function is convex/concave for simple exclusion, but not necessarily for zero-range. However, in the general set-up of misanthropes processes, unless the rate function b satisfies additional algebraic conditions (see [19, 23]), the model does not have product invariant measures; Even when this is the case, the flux function is not necessarily convex/concave. We refer the reader to [6, 23] for examples of misanthropes processes with product invariant measures. Note also that a misanthropes process with product invariant measures generally loses this property if disorder is introduced, with the sole known exception of the zero-range process ([16, 22]).

The *k-step exclusion process* ($k \in \mathbb{N}$) was introduced in [29] (without disorder). Its state space is $\mathbf{X} := \{0,1\}^{\mathbb{Z}}$. Let $p(.)$ be a probability distribution on $\mathbb{Z}$, and $\{X_n\}_{n\in\mathbb{N}}$ denote a random walk on $\mathbb{Z}$ with jump distribution $p(.)$. We denote by $\mathbf{P}^x$ the law of the random walk starting from x; expectation with respect to this law is denoted by $\mathbf{E}^x$. The k-step exclusion process with jump distribution $p(.)$ has generator

$$L_\alpha f(\eta) = \sum_{x,y\in\mathbb{Z}} \alpha(x) c(x,y,\eta) \left[f\left(\eta^{x,y}\right) - f(\eta) \right] \qquad \text{with} \tag{5}$$

$$c(x,y,\eta) = \eta(x)(1-\eta(y))\mathbf{E}^x \left[\prod_{i=1}^{\sigma_y - 1} \eta(X_i), \sigma_y \leq \sigma_x, \sigma_y \leq k \right]$$

where $\sigma_y = \inf\{n \geq 1 : X_n = y\}$ is the first (non zero) arrival time to site y of the walk starting at site x. In words if a particle at site x wants to jump it may go to the first empty site encountered before returning to site x following the walk $\{X_n\}_{n\in\mathbb{N}}$ (starting at x) provided it takes less than k attempts; otherwise the movement is cancelled. When $k = 1$, we recover the simple exclusion process. The k-step exclusion is an attractive process.

Let us now restrict ourselves to the model without disorder. Then $(\mathcal{I}\cap\mathcal{S})_e$ is a one-parameter family of product Bernoulli measures. In the totally asymmetric nearest-neighbor case, $c(x, y, \eta) = 1$ if $\eta(x) = 1$, $y-x \in \{1, \dots, k\}$ and y is the first nonoccupied site to the right of x; otherwise $c(x, y, \eta) = 0$. The flux function belongs to $\mathcal{C}^2(\mathbb{R})$, it has one inflexion point, thus it is neither convex nor concave. Besides, flux functions with arbitrarily many inflexion points can be constructed by superposition of different k-step exclusion processes with different kernels and different values of k ([6]).

A constructive approach to hydrodynamics. To overcome the difficulties to derive hydrodynamics raised by the above examples, our starting point was the constructive approach introduced in [4]. There, the authors proved the *conservation of local equilibrium* for the one-dimensional zero-range process with a concave macroscopic flux function G *in the Riemann case* (in a translation invariant setting, G is the mean flux of particles through the origin), that is

$$\forall t > 0, \quad \eta^N_{Nt} \xrightarrow{\mathcal{L}} \nu_{u(t,x)} \tag{6}$$

where ν_ρ is the product invariant measure of the zero-range process with mean density ρ, and $u(.,.)$ is the *entropy solution* of the conservation law

$$\partial_t u + \partial_x[G(u)] = 0; \quad u(x, 0) = R_{\lambda,\rho}(x) = \lambda\mathbf{1}_{\{x<0\}} + \rho\mathbf{1}_{\{x\geq 0\}} \tag{7}$$

One can show (see [33]) that (6) implies the hydrodynamic limit in the sense of Definition 2.2. Let us begin by explaining (informally) their method. They first show in [4, Lemma 3.1] that a weak Cesaro limit of (the measure of) the process is an invariant and translation invariant measure, thus a convex combination of elements of $(\mathcal{I}\cap\mathcal{S})_e$, the one-parameter family of extremal invariant and translation invariant probability measures for the dynamics. Then they compute in [4, Lemma 3.2] the (Cesaro) limiting density inside a macroscopic box, thanks to the explicit knowledge of the product measures elements of $(\mathcal{I}\cap\mathcal{S})_e$. They prove next in [4, Lemma 3.3 and Theorem 2.10] that the above convex combination is in fact the Dirac measure concentrated on the solution of the hydrodynamic equation, thanks to the concavity of their flux function. They conclude by proving that the Cesaro limit implies the weak limit via monotonicity arguments, in [4, Propositions 3.4 and 3.5]. Their proof is valid for misanthropes processes with product invariant measures and a concave macroscopic flux.

In [6], we derive by a constructive method the hydrodynamic behavior of attractive processes with finite range irreducible jumps, and for which the set $(\mathcal{I} \cap \mathcal{S})_e$ consists in a one-parameter family of explicit product measures but the flux is not necessarily convex or concave. Our approach relies on (i) an explicit construction of Riemann solutions without assuming convexity of the macroscopic flux, and (ii) a general result which proves that the hydrodynamic limit for Riemann initial profiles implies the same for general initial profiles.
For point (i), we rely on the (parts of) the proofs in [4] based only on attractiveness and on the knowledge of the product measures composing $(\mathcal{I} \cap \mathcal{S})_e$, and we provide a new approach otherwise. Instead of the convexity assumption on the flux, which belongs here to $\mathcal{C}^2(\mathbb{R})$, we prove that the solution of the hydrodynamic equation is given by a variational formula, whose index set is an interval, namely the set of values of the parameter of the elements of $(\mathcal{I} \cap \mathcal{S})_e$. Knowing $(\mathcal{I} \cap \mathcal{S})_e$ explicitly enables us to deal with dynamics with the non compact state space $\mathbb{N}^{\mathbb{Z}}$.
Point (ii) is based on an approximation scheme inspired by Glimm's scheme for hyperbolic systems of conservation laws (see [44]). Among our tools are the *finite propagation property* and the *macroscopic stability* of the dynamics. The latter property is due to [17]; both require finite range transitions. We illustrate our results on variations of our above reference examples.

While the results and examples of [6] include the case $K = +\infty$, in our subsequent works, for reasons explained below, we considered $K < +\infty$, thus $\mathbf{X} = \{0, \cdots, K\}^{\mathbb{Z}}$, which will be assumed from now on. Under this additional assumption, in [7], we extend the hydrodynamics result of [6] to dynamics without explicit invariant measures. Indeed, thanks to monotonicity, we prove that $(\mathcal{I} \cap \mathcal{S})_e$ is still a one-parameter family of probability measures, for which the set $\mathcal{R}$ of values of the parameter is a priori not an interval anymore, but a closed subset of $[0, K]$:

Proposition 2.1 *([7, Proposition 3.1]). Assume p satisfies the irreducibility assumption*

$$\forall z \in \mathbb{Z}, \quad \sum_{n=1}^{+\infty} [p^{*n}(z) + p^{*n}(-z)] > 0 \tag{8}$$

where p^{*n} denotes the n-th convolution power of the kernel p, that is the law of the sum of n independent p-distributed random variables. Then there exists a closed subset $\mathcal{R}$ of $[0, K]$, containing 0 and K, such that

$$(\mathcal{I} \cap \mathcal{S})_e = \{\nu^\rho : \rho \in \mathcal{R}\} \tag{9}$$

where the probability measures ν^ρ on $\mathbf{X}$ satisfy the following properties:

$$\lim_{l \to +\infty} (2l+1)^{-1} \sum_{x=-l}^{l} \eta(x) = \rho, \quad \nu^\rho\text{- a.s.} \tag{10}$$

and

$$\rho \leq \rho' \Rightarrow \nu^\rho \leq \nu^{\rho'} \tag{11}$$

Following the same general scheme as in [6], but with additional difficulties, we then obtain the following main result.

Theorem 2.1 *([7, Theorem 2.2]). Assume $p(.)$ satisfies the irreducibility assumption (8). Then there exists a Lipschitz-continuous function $G : [0, K] \to \mathbb{R}^+$ such that the following holds. Let $u_0 \in L^{\infty,K}(\mathbb{R})$, and $(\eta_.^N)_N$ be any sequence of processes with generator (4), such that the sequence $(\eta_0^N)_N$ has density profile $u_0(.)$. Then, the sequence $(\eta_{N.}^N)_N$ has hydrodynamic limit given by $u(.,.)$, the entropy solution to (3) with initial condition $u_0(.)$.*

The drawback is that we have (and it will also be the case in the following papers) to restrict ourselves to dynamics with compact state space to prove hydrodynamics with general initial data. This is necessary to define the macroscopic flux outside $\mathcal{R}$, by a linear interpolation; this makes this flux Lipschitz continuous, a minimal requirement to define entropy solutions. We have to consider a $\mathcal{R}$-valued Riemann problem, for which we prove conservation of local equilibrium. Then we use an averaging argument to prove hydrodynamics (in the absence of product invariant measures, the passage from local equilibrium to hydrodynamics is no longer a consequence of [33]). For general initial profiles, we have to refine the approximation procedure of [6]: we go first to $\mathcal{R}$-valued entropy solutions, then to arbitrary entropy solutions.

In [8], by a refinement of our method, we obtain a *strong* (that is an almost sure) hydrodynamic limit, when starting from an arbitrary initial profile. By almost sure, we mean that we construct the process with generator (4) on an explicit probability space defined as the product $(\Omega_0 \times \Omega, \mathcal{F}_0 \otimes \mathcal{F}, \mathbb{P}_0 \otimes \mathbb{P})$, where $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$ is a probability space used to construct random initial states, and $(\Omega, \mathcal{F}, \mathbb{P})$ is a Poisson space used to construct the evolution from a given state.

Theorem 2.2 *([8, Theorem 2.1]) Assume $p(.)$ has finite first moment and satisfies the irreducibility assumption (8). Then the following holds, where G is the same function as in Theorem 2.1. Let $(\eta_0^N,\ N \in \mathbb{N})$ be any sequence of $\mathbf{X}$-valued random variables on a probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$ such that*

$$\lim_{N\to\infty} \pi^N(\eta_0^N)(dx) = u_0(.)dx \quad \mathbb{P}_0\text{-}a.s.$$

for some measurable $[0, K]$-valued profile $u_0(.)$. Then the $\mathbb{P}_0 \otimes \mathbb{P}$-a.s. convergence

$$\lim_{N\to\infty} \pi^N(\eta_{Nt}^N)(dx) = u(., t)dx$$

holds uniformly on all bounded time intervals, where $(x, t) \mapsto u(x, t)$ denotes the unique entropy solution to (3) with initial condition $u_0(.)$.

Our constructive approach requires new ideas since the sub-additive ergodic theorem (central to the few previous existing proofs for strong hydrodynamics) is no longer effective in our setting. We work with the graphical representation of the dynamics, on which we couple an arbitrary number of processes, thanks to the *complete monotonicity property* of the dynamics. To solve the $\mathcal{R}$-valued Riemann problem, we combine proofs of almost sure analogues of the results of [4], rephrased for currents which become our centerpiece, with a space-time ergodic theorem for particle systems and large deviation results for the empirical measure. In the approximation steps, new error analysis is necessary: In particular, we have to do an explicit time discretization (vs. the "instantaneous limit" of [6, 7]), we need estimates uniform in time, and each approximation step requires a control with exponential bounds.

In [38] we derive the macroscopic stability property when the particles' jump kernel $p(.)$ has a finite first moment and a positive mean. We also extend under those hypotheses the ergodic theorem for densities due to [40] that we use in [8]. Finally, we prove the finite propagation property when $p(.)$ has a finite third moment. This enables us to get rid of the finite range assumption on p required so far, and to extend the strong hydrodynamic result of [8] when the particles' jump kernel has a finite third moment and a positive mean.

In [9], we derive, thanks to the tools introduced in [8], a quenched strong hydrodynamic limit for a bounded attractive particle system on $\mathbb{Z}$ evolving in a random ergodic environment. (This result, which contains Theorems 2.1 and 2.2 above, is stated later on in this paper as Theorem 3.1). Our method is robust with respect to the model and disorder (we are not restricted to site or bond disorder). We introduce a general framework to describe the rates of the dynamics, which applies to a large class of models. To overcome the difficulty of the simultaneous loss of translation invariance and lack of knowledge of explicit invariant measures for the disordered system, we study a joint disorder-particle process, which is translation invariant. We characterize its extremal invariant and translation invariant measures, and prove its strong hydrodynamic limit. This implies the quenched hydrodynamic result we look for.
We illustrate our results on various examples.

3 Main results

The construction of interacting particle systems is done either analytically, through generators and semi-groups (we refer to [36] for systems with compact state space, and to [37, 1, 23] otherwise), or through a graphical representation. Whereas the former is sufficient to derive hydrodynamic limits in a weak sense, which is done in [6, 7], the latter is necessary to derive strong hydrodynamic limits, which is done in [8, 9]. First, we explain in Subsection 3.1 the graphical construction, then in Subsection 3.2 we detail our results from [9] on invariant measures for the dynamics and hydrodynamic limits.
For simplicity, we restrict ourselves in this section to the misanthropes process with site disorder, which corresponds to the generator (4). However, considering only the necessary properties of the misanthropes process required to prove our hydrodynamic results, it is possible to deal with more

general models including the k-step exclusion process, by embedding them in a global framework, in which the dynamics is viewed as a random transformation of the configuration; the latter simultaneously defines the graphical construction *and* generator. More general forms of random environments than site disorder can also be considered. In Subsection 3.3 we list the above required properties of misanthropes processes, and we defer the study of the k-step exclusion process and more general models to Section 6.

3.1 Graphical construction

This subsection is based on [8, Section 2.1]. We now describe the graphical construction (that is the pathwise construction on a Poisson space) of the system given by (4), which uses a Harris-like representation ([30, 31]; see for instance [2, 25, 11, 47] for details and justifications). This enables us to define the evolution from arbitrarily many different initial configurations simultaneously on the same probability space, in a way that depends monotonically on these initial configurations.

We consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of measures ω on $\mathbb{R}^+ \times \mathbb{Z}^2 \times [0,1]$ of the form

$$\omega(dt, dx, dz, du) = \sum_{m \in \mathbb{N}} \delta_{(t_m, x_m, z_m, u_m)}$$

where $\delta_{(\cdot)}$ denotes Dirac measure, and $(t_m, x_m, z_m, u_m)_{m \geq 0}$ are pairwise distinct and form a locally finite set. The σ-field $\mathcal{F}$ is generated by the mappings $\omega \mapsto \omega(S)$ for Borel sets S. The probability measure $\mathbb{P}$ on Ω is the one that makes ω a Poisson process with intensity

$$m(dt, dx, dz, du) = ||b||_\infty \lambda_{\mathbb{R}^+}(dt) \times \lambda_{\mathbb{Z}}(dx) \times p(dz) \times \lambda_{[0,1]}(du)$$

where λ denotes either the Lebesgue or the counting measure. We denote by $\mathbb{E}$ the corresponding expectation. Thanks to assumption *(M4)* on page 10, we can proceed as in [2, 25] (for a construction with a weaker assumption we refer to [11, 47]): for $\mathbb{P}$-a.e. ω, there exists a unique mapping

$$(\alpha, \eta_0, t) \in \mathbf{A} \times \mathbf{X} \times \mathbb{R}^+ \mapsto \eta_t = \eta_t(\alpha, \eta_0, \omega) \in \mathbf{X} \tag{12}$$

satisfying: (a) $t \mapsto \eta_t(\alpha, \eta_0, \omega)$ is right-continuous; (b) $\eta_0(\alpha, \eta_0, \omega) = \eta_0$; (c) for $t \in \mathbb{R}^+$, $(x,z) \in \mathbb{Z}^2$, $\eta_t = \eta_{t^-}^{x,x+z}$ if

$$\exists u \in [0,1] : \omega\{(t,x,z,u)\} = 1 \text{ and } u \leq \alpha(x) \frac{b(\eta_{t^-}(x), \eta_{t^-}(x+z))}{||b||_\infty} \tag{13}$$

and (d) for all $s,t \in \mathbb{R}^{+*}$ and $x \in \mathbb{Z}$,

$$\omega\{[s,t] \times Z_x \times (0,1)\} = 0 \Rightarrow \forall v \in [s,t], \eta_v(x) = \eta_s(x) \tag{14}$$

where

$$Z_x := \left\{(y,z) \in \mathbb{Z}^2 : y = x \text{ or } y+z = x\right\}$$

In short, (13) tells how the state of the system can be modified by an "ω-event", and (14) says that the system cannot be modified outside ω-events.

Thanks to assumption *(M2)* on page 10, we have that

$$(\alpha, \eta_0, t) \mapsto \eta_t(\alpha, \eta_0, \omega) \text{ is nondecreasing w.r.t. } \eta_0 \tag{15}$$

Property (15) implies (2), that is, attractiveness. But it is more powerful: it implies the *complete monotonicity* property ([24, 20]), that is, existence of a monotone Markov coupling for an *arbitrary* number of processes with generator (4), which is necessary in our proof of strong hydrodynamics for general initial profiles. The coupled process can be defined by a Markov generator, as in [19] for two components, that is

$$\begin{aligned}
&\overline{L}_\alpha f(\eta,\xi) = \\
&\sum_{x,y\in\mathbb{Z}:\, x\neq y} \Big\{\alpha(x)p(y-x)[b(\eta(x),\eta(y)) \wedge b(\xi(x),\xi(y))]\,[f(\eta^{x,y},\xi^{x,y}) - f(\eta,\xi)] \\
&+\alpha(x)p(y-x)[b(\eta(x),\eta(y)) - b(\xi(x),\xi(y))]^+\,[f(\eta^{x,y},\xi) - f(\eta,\xi)] \\
&+\alpha(x)p(y-x)[b(\xi(x),\xi(y)) - b(\eta(x),\eta(y))]^+\,[f(\eta,\xi^{x,y}) - f(\eta,\xi)]\Big\}
\end{aligned} \tag{16}$$

One may further introduce an "initial" probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$, large enough to construct random initial configurations $\eta_0 = \eta_0(\omega_0)$ for $\omega_0 \in \Omega_0$. The general process with random initial configurations is constructed on the enlarged space $(\widetilde{\Omega} = \Omega_0 \times \Omega, \widetilde{\mathcal{F}} = \sigma(\mathcal{F}_0 \times \mathcal{F}), \widetilde{\mathbb{P}} = \mathbb{P}_0 \otimes \mathbb{P})$ by setting

$$\eta_t(\alpha, \widetilde{\omega}) = \eta_t(\alpha, \eta_0(\omega_0), \omega)$$

for $\widetilde{\omega} = (\omega_0, \omega) \in \widetilde{\Omega}$. One can show (see for instance [11, 25, 47]) that this defines a Feller process with generator (4): that is for any $t \in \mathbb{R}^+$ and $f \in C(\mathbf{X})$ (the set of continuous functions on $\mathbf{X}$), $S_\alpha(t)f \in C(\mathbf{X})$ where $S_\alpha(t)f(\eta_0) = \mathbb{E}[f(\eta_t(\alpha,\eta_0,\omega))]$. If η_0 has distribution μ_0, then the process thus constructed is Feller with generator (4) and initial distribution μ_0.

We define on Ω the *space-time shift* θ_{x_0,t_0}: for any $\omega \in \Omega$, for any (t, x, z, u)

$$(t, x, z, u) \in \theta_{x_0,t_0}\omega \text{ if and only if } (t_0 + t, x_0 + x, z, u) \in \omega \tag{17}$$

where $(t, x, z, u) \in \omega$ means $\omega\{(t, x, z, u)\} = 1$. By its very definition, the mapping introduced in (12) enjoys the following properties, for all $s, t \geq 0$, $x \in \mathbb{Z}$ and $(\eta, \omega) \in \mathbf{X} \times \Omega$:

$$\eta_s(\alpha, \eta_t(\alpha, \eta, \omega), \theta_{0,t}\omega) = \eta_{t+s}(\alpha, \eta, \omega) \tag{18}$$

which implies Markov property, and

$$\tau_x \eta_t(\alpha, \eta, \omega) = \eta_t(\tau_x\alpha, \tau_x\eta, \theta_{x,0}\omega) \tag{19}$$

which yields the *commutation property*

$$L_\alpha \tau_x = \tau_x L_{\tau_x \alpha} \tag{20}$$

3.2 Hydrodynamic limit and invariant measures

This section is based on [9, Sections 2, 3]. A central issue in interacting particle systems, and more generally in the theory of Markov processes, is the characterization of invariant measures ([36]). Besides, this characterization plays a crucial role in the derivation of hydrodynamic limits ([33]). We detail here our results on these two questions.

Hydrodynamic limit. We first state the strong hydrodynamic behavior of the process with quenched site disorder.

Theorem 3.1 *([9, Theorem 2.1]). Assume $K < +\infty$, $p(.)$ has finite third moment, and satisfies the irreducibility assumption (8). Let Q be an ergodic probability distribution on $\mathbf{A}$. Then there exists a Lipschitz-continuous function G^Q on $[0, K]$ defined in (35) and (36)–(37) below (depending only on $p(.)$, $b(.,.)$ and Q) such that the following holds. Let $(\eta_0^N,\ N \in \mathbb{N})$ be a sequence of $\mathbf{X}$-valued random variables on a probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$ such that*

$$\lim_{N\to\infty} \pi^N(\eta_0^N)(dx) = u_0(.)dx \quad \mathbb{P}_0\text{-}a.s.$$

for some measurable $[0,K]$*-valued profile* $u_0(.)$*. Then for* Q*-a.e.* $\alpha \in \mathbf{A}$*, the* $\mathbb{P}_0 \otimes \mathbb{P}$*-a.s. convergence*

$$\lim_{N\to\infty} \pi^N(\eta_{Nt}(\alpha, \eta_0^N(\omega_0), \omega))(dx) = u(., t)dx$$

holds uniformly on all bounded time intervals, where $(x,t) \mapsto u(x,t)$ *denotes the unique entropy solution with initial condition* u_0 *to the conservation law*

$$\partial_t u + \partial_x[G^Q(u)] = 0 \tag{21}$$

The $\mathbb{P}$-almost sure convergence in Theorem 3.1 refers to the graphical construction of Subsection 3.1, and is stronger than the usual notion of hydrodynamic limit, which is a convergence in probability (cf. Definition 2.2). The strong hydrodynamic limit implies the weak one ([8]). This leads to the following weaker but more usual statement, which also has the advantage of not depending on a particular construction of the process.

Theorem 3.2 *Under assumptions and notations of Theorem 3.1, there exists a subset* $\mathbf{A}'$ *of* $\mathbf{A}$*, with* Q*-probability* 1*, such that the following holds for every* $\alpha \in \mathbf{A}'$*. For any* $u_0 \in L^\infty(\mathbb{R})$*, and any sequence* $\eta_.^N = (\eta_t^N)_{t\geq 0}$ *of processes with generator (4) satisfying the convergence in probability*

$$\lim_{N\to\infty} \pi^N(\eta_0^N)(dx) = u_0(.)dx, \tag{22}$$

one has the locally uniform convergence in probability

$$\lim_{N\to\infty} \pi^N\left(\eta_{Nt}^N\right)(dx) = u(., t)dx \tag{23}$$

Remark 3.1 *Theorem 2.1 (resp. Theorem 2.2) is a special case of Theorem 3.2 (resp. Theorem 3.1). Indeed, it suffices to consider the "disorder" distribution* Q *that is the Dirac measure supported on the single homogeneous environment* α_{hom} *defined by* $\alpha_{\text{hom}}(x) = 1$ *for all* $x \in \mathbb{Z}$ *(see also Remark 3.2).*

Note that the statement of Theorem 3.2 is stronger than hydrodynamic limit in the sense of Definition 2.2, because it states convergence of the empirical measure *process* rather then convergence at every fixed time. To define the *macroscopic flux* G^Q, we first define the *microscopic flux* as follows. The generator (4) can be decomposed as a sum of translates of a "seed" generator centered around the origin:

$$L_\alpha = \sum_{x\in\mathbb{Z}} L_\alpha^x \tag{24}$$

Note that such a decomposition is not unique. A natural choice of component L_α^x at $x \in \mathbb{Z}$ is given in the case of (4) by

$$L_\alpha^x f(\eta) = \alpha(x) \sum_{z\in\mathbb{Z}} p(z) \left[f\left(\eta^{x,x+z}\right) - f(\eta) \right] \tag{25}$$

We then define j to be either of the functions j_1, j_2 defined below:

$$j_1(\alpha,\eta) := L_\alpha \left[\sum_{x>0} \eta(x) \right], \quad j_2(\alpha,\eta) := L_\alpha^0 \left[\sum_{x\in\mathbb{Z}} x\eta(x) \right] \tag{26}$$

The definition of j_1 is partly formal, because the function $\sum_{x>0} \eta(x)$ does not belong to the domain of the generator L_α. Nevertheless, the formal computation gives rise to a well-defined function j_1, because the rate b is a local function. Rigorously, one defines j_1 by difference, as the unique function such that

$$j_1 - \tau_x j_1 = L_\alpha \left[\sum_{y=1}^{x} \eta(y) \right] \tag{27}$$

for every $x \in \mathbb{N}$. The action of the generator in (27) is now well defined, because we have a local function that belongs to its domain.

In the case of (4), we obtain the following microscopic flux functions:

$$\begin{aligned} j_1(\alpha,\eta) &= \sum_{(x,z)\in\mathbb{Z}^2, x\le 0<x+z} \alpha(x) b(\eta(x),\eta(x+z)) \\ &\quad - \sum_{(x,z)\in\mathbb{Z}^2, x+z\le 0<x} \alpha(x) b(\eta(x),\eta(x+z)) \qquad (28) \\ j_2(\alpha,\eta) &= \alpha(0) \sum_{z\in\mathbb{Z}} z p(z) b(\eta(0),\eta(z)) \qquad (29) \end{aligned}$$

Once a microscopic flux function j is defined, the macroscopic flux function is obtained by averaging with respect to a suitable family of measures, that we now introduce.

Invariant measures. We define the markovian *joint disorder-particle process* $(\alpha_t, \eta_t)_{t\geq 0}$ on $\mathbf{A}\times\mathbf{X}$ with generator given by, for any local function f on $\mathbf{A}\times\mathbf{X}$,

$$\mathfrak{L}f(\alpha,\eta)=\sum_{x,y\in\mathbb{Z}}\alpha(x)p(y-x)b(\eta(x),\eta(y))\left[f\left(\alpha,\eta^{x,y}\right)-f(\alpha,\eta)\right] \tag{30}$$

Given $\alpha_0=\alpha$, this dynamics simply means that $\alpha_t=\alpha$ for all $t\geq 0$, while $(\eta_t)_{t\geq 0}$ is a Markov process with generator L_α given by (4). Note that $\mathfrak{L}$ is *translation invariant*, that is

$$\tau_x\mathfrak{L}=\mathfrak{L}\tau_x \tag{31}$$

where τ_x acts jointly on (α,η). This is equivalent to the commutation relation (20) for the quenched dynamics.

Let $\mathcal{I}_{\mathfrak{L}}$, $\mathcal{S}$ and $\mathcal{S}^{\mathbf{A}}$ denote the sets of probability measures that are respectively invariant for $\mathfrak{L}$, shift-invariant on $\mathbf{A}\times\mathbf{X}$ and shift-invariant on $\mathbf{A}$.

Proposition 3.1 *([9, Proposition 3.1]). For every $Q\in\mathcal{S}^{\mathbf{A}}_e$, there exists a closed subset $\mathcal{R}^Q$ of $[0,K]$ containing 0 and K, depending on $p(.)$ and $b(.,.)$, such that*

$$\left(\mathcal{I}_{\mathfrak{L}}\cap\mathcal{S}\right)_e \quad=\quad \left\{\nu^{Q,\rho},\, Q\in\mathcal{S}^{\mathbf{A}}_e,\, \rho\in\mathcal{R}^Q\right\}$$

where index e denotes the set of extremal elements, and $(\nu^{Q,\rho}:\,\rho\in\mathcal{R}^Q)$ is a family of shift-invariant measures on $\mathbf{A}\times\mathbf{X}$, weakly continuous with respect to ρ, such that

$$\int\eta(0)\nu^{Q,\rho}(d\alpha,d\eta) \quad=\quad \rho \tag{32}$$

$$\lim_{l\to\infty}(2l+1)^{-1}\sum_{x\in\mathbb{Z}:|x|\leq l}\eta(x) \quad=\quad \rho,\quad \nu^{Q,\rho}-a.s. \tag{33}$$

$$\rho\leq\rho' \quad\Rightarrow\quad \nu^{Q,\rho}\ll\nu^{Q,\rho'} \tag{34}$$

Here, $\ll$ denotes the conditional stochastic order defined in Lemma 3.1 below.

From the family of invariant measures in the above proposition for the joint disorder-particle process, one may deduce a family of invariant measures for the quenched particle process.

Corollary 3.1 *([9, Corollary 3.1]). There exists a subset $\widetilde{\mathbf{A}}^Q$ of $\mathbf{A}$ with Q-probability 1 (depending on $p(.)$ and $b(.,.)$), such that the family of probability measures $(\nu_\alpha^{Q,\rho}:\ \alpha\in\widetilde{\mathbf{A}}^Q,\rho\in\mathcal{R}^Q)$ on $\mathbf{X}$, defined by $\nu_\alpha^{Q,\rho}(.):=\nu^{Q,\rho}(.|\alpha)$ satisfies the following properties, for every $\rho\in\mathcal{R}^Q$:*

(B1) For every $\alpha\in\widetilde{\mathbf{A}}^Q$, $\nu_\alpha^{Q,\rho}$ is an invariant measure for L_α.

(B2) For every $\alpha\in\widetilde{\mathbf{A}}^Q$, $\nu_\alpha^{Q,\rho}$-a.s.,

$$\lim_{l\to\infty}(2l+1)^{-1}\sum_{x\in\mathbb{Z}:\,|x|\leq l}\eta(x)=\rho$$

(B3) The quantity

$$G_\alpha^Q(\rho):=\int j(\alpha,\eta)\nu_\alpha^{Q,\rho}(d\eta)=:G^Q(\rho),\quad \rho\in\mathcal{R}^Q \tag{35}$$

does not depend on $\alpha\in\widetilde{\mathbf{A}}^Q$.

Remark 3.2 *As already observed in Remark 3.1 above, we can view the non-disordered model as a special "disordered" model by taking Q to be the Dirac measure on the homogeneous environment α_{hom} with constant value 1. Hence, Proposition 2.1 is a special case of Proposition 3.1. Note that we then have $\nu^\rho=\nu_\alpha^{Q,\rho}$ and $\nu^{Q,\rho}=\delta_{\alpha_{\mathrm{hom}}}\otimes\nu^\rho$ for Q-a.e. $\alpha\in\mathbf{A}$.*

We now come back to the macroscopic flux function $G^Q(\rho)$. We define it as (35) for $\rho\in\mathcal{R}^Q$ and we extend it by linear interpolation on the complement of $\mathcal{R}^Q$, which is a finite or countably infinite union of disjoint open intervals: that is, we set

$$G^Q(\rho):=\frac{\rho-\rho^-}{\rho^+-\rho^-}G(\rho^+)+\frac{\rho^+-\rho}{\rho^+-\rho^-}G(\rho^-),\quad \rho\notin\mathcal{R}^Q \tag{36}$$

where

$$\rho^-:=\sup[0,\rho]\cap\mathcal{R}^Q,\quad \rho^+:=\inf[\rho,+\infty)\cap\mathcal{R}^Q \tag{37}$$

By definition of $\nu_\alpha^{Q,\rho}$ and statement *(B3)* of Corollary 3.1, we also have

$$G^Q(\rho):=\int j(\alpha,\eta)\nu^{Q,\rho}(d\alpha,d\eta),\quad \rho\in\mathcal{R}^Q \tag{38}$$

We point out that (38) yields the same macroscopic flux function, whether $j = j_1$ or $j = j_2$ defined in (26) is plugged into it. It does not depend on the choice of a particular decomposition (24) either. These invariance properties follow from translation invariance of $\nu^{Q,\rho}$ and translation invariance (31) of the joint dynamics. Definitions (26) of the microscopic flux, and (35)–(38) of the macroscopic flux are model-independent, and can thus be used for other models, such as the k-exclusion process, or the models reviewed in Section 6. Of course, the invariant measures involved in (35)–(38) depend on the model and disorder.

The function G^Q can be shown to be Lipschitz continuous ([9, Remark 3.3]). This is the minimum regularity required for the classical theory of entropy solutions, see Section 4. We cannot say more about G^Q in general, because the measures $\nu_\alpha^{Q,\rho}$ are most often not explicit.

Note that in the special case of the non-disordered model investigated in [7] (see Proposition 2.1, Theorem 2.1, Remarks 3.1 and 3.2 above), the microscopic flux functions do not depend on α, and (35) or (38) both reduce to

$$G(\rho) := \int j(\eta) d\nu^\rho(\eta), \quad \rho \in \mathcal{R} \tag{39}$$

Even in the absence of disorder, only a few models have explicit invariant measures, and thus an explicit flux function. In Section 6, we define a variant of the k-step exclusion process, that we call the exclusion process with overtaking. It is possible to tune the microscopic parameters of this model so as to obtain any prescribed polynomial flux function (constrained to vanish at density values 0 and 1 due to the exclusion rule).
In contrast, a most natural and seemingly simple generalization of the asymmetric exclusion process, the asymmetric K-exclusion process, for which $K \geq 2$ and $b(n,m) = \mathbf{1}_{\{n>0\}}\mathbf{1}_{\{m<K\}}$ in (4), does not have explicit invariant measures. Thus, nothing more than the Lipschitz property can be said about its flux function in general. In the special case of the totally asymmetric K-exclusion process, that is for $p(1) = 1$, the flux function is shown to be concave in [43], as a consequence of the variational approach used there to derive hydrodynamic limit. But this approach does not apply to the models we consider in the present paper.

An important open question is whether the set $\mathcal{R}^Q$ (or its analogue $\mathcal{R}$ in the absence of disorder) covers the whole range of possible densities, or if it contains gaps corresponding to phase transitions. The only partial answer to this question so far was given by the following result from [7] for the totally asymmetric K-exclusion process without disorder.

Theorem 3.3 *([7, Corollary 2.1]). For the totally asymmetric K-exclusion process without disorder,* 0 *and* K *are limit points of* $\mathcal{R}$*, and* $\mathcal{R}$ *contains at least one point in* $[1/3, K-1/3]$.

We end this section with a skeleton of proof for Proposition 3.1. We combine the steps done to prove [7, Proposition 3.1] (without disorder) and [9, Proposition 3.1].

Proof of proposition 3.1. The proof has two parts.
Part 1. It is an extension to the joint particle-disorder process of a classical scheme in a non-disordered setting due to [35], which is also the basis for similar results in [1, 19, 29, 27]. It relies on couplings.
a) We first need to couple measures, through the following lemma, analogous to Strassen's Theorem ([46]).

Lemma 3.1 *([9, Lemma 3.1]). For two probability measures* μ^1, μ^2 *on* $\mathbf{A}\times\mathbf{X}$*, the following properties (denoted by* $\mu^1 \ll \mu^2$*) are equivalent:*
(i) For every bounded measurable local function f *on* $\mathbf{A}\times\mathbf{X}$*, such that* $f(\alpha,.)$ *is nondecreasing for all* $\alpha\in\mathbf{A}$*, we have* $\int f\,d\mu^1 \leq \int f\,d\mu^2$.
(ii) The measures μ^1 *and* μ^2 *have a common* α*-marginal* Q*, and* $\mu^1(d\eta|\alpha)\leq\mu^2(d\eta|\alpha)$ *for* Q*-a.e.* $\alpha\in\mathbf{A}$.
(iii) There exists a coupling measure $\overline{\mu}(d\alpha,d\eta,d\xi)$ *supported on* $\{(\alpha,\eta,\xi)\in\mathbf{A}\times\mathbf{X}^2:\eta\leq\xi\}$ *under which* $(\alpha,\eta)\sim\mu^1$ *and* $(\alpha,\xi)\sim\mu^2$.

b) Then, for the dynamics, we denote by $\overline{\mathfrak{L}}$ the coupled generator for the joint process $(\alpha_t,\eta_t,\xi_t)_{t\geq 0}$ on $\mathbf{A}\times\mathbf{X}^2$ defined by

$$\overline{\mathfrak{L}}f(\alpha,\eta,\xi)=(\overline{L}_\alpha f(\alpha,.))(\eta,\xi) \tag{40}$$

for any local function f on $\mathbf{A}\times\mathbf{X}^2$, where $\overline{L}_\alpha$ was defined in (16). Given $\alpha_0=\alpha$, this means that $\alpha_t=\alpha$ for all $t\geq 0$, while $(\eta_t,\xi_t)_{t\geq 0}$ is a Markov process with generator $\overline{L}_\alpha$. We denote by $\overline{\mathcal{S}}$ the set of probability measures on $\mathbf{A}\times\mathbf{X}^2$ that are invariant by space shift $\tau_x(\alpha,\eta,\xi)=(\tau_x\alpha,\tau_x\eta,\tau_x\xi)$. We prove successively (next lemma combines [9, Lemmas 3.2, 3.4 and Proposition 3.2]):

Lemma 3.2 *(i) Let $\mu', \mu'' \in (\mathcal{I}_{\mathfrak{L}} \cap \mathcal{S})_e$ with a common α-marginal Q. Then there exists $\overline{\nu} \in \left(\mathcal{I}_{\overline{\mathfrak{L}}} \cap \overline{\mathcal{S}}\right)_e$ such that the respective marginal distributions of (α, η) and (α, ξ) under $\overline{\nu}$ are μ' and μ''.*
(ii) Let $\overline{\nu} \in \left(\mathcal{I}_{\overline{\mathfrak{L}}} \cap \overline{\mathcal{S}}\right)_e$. Then $\overline{\nu}\{(\alpha, \eta, \xi) \in \mathbf{A} \times \mathbf{X}^2 : \eta \leq \xi\}$ and $\overline{\nu}\{(\alpha, \eta, \xi) \in \mathbf{A} \times \mathbf{X}^2 : \xi \leq \eta\}$ belong to $\{0, 1\}$.
(iii) Every $\overline{\nu} \in \left(\mathcal{I}_{\overline{\mathfrak{L}}} \cap \overline{\mathcal{S}}\right)_e$ is supported on $\{(\alpha, \eta, \xi) \in \mathbf{A} \times \mathbf{X}^2 : \eta \leq \xi \text{ or } \xi \leq \eta\}$.

c) This last point *(iii)* is the core of the proof of Proposition 3.1: Attractiveness assumption ensures that an initially ordered pair of coupled configurations remains ordered at later times. We say that there is a positive (resp. negative) discrepancy between two coupled configurations ξ, ζ at some site x if $\xi(x) > \zeta(x)$ (resp. $\xi(x) < \zeta(x)$). Irreducibility assumption (8) induces a stronger property: pairs of discrepancies of opposite signs between two coupled configurations eventually get killed, so that the two configurations become ordered.

Part 2. We define

$$\mathcal{R}^Q := \left\{ \int \eta(0)\nu(d\alpha, d\eta) : \nu \in (\mathcal{I}_{\mathfrak{L}} \cap \mathcal{S})_e, \, \nu \text{ has } \alpha\text{-marginal } Q \right\}$$

Let $\nu^i \in (\mathcal{I}_{\mathfrak{L}} \cap \mathcal{S})_e$ with α-marginal Q and $\rho^i := \int \eta(0)\nu^i(d\alpha, d\eta) \in \mathcal{R}^Q$ for $i \in \{1, 2\}$. Assume $\rho^1 \leq \rho^2$. Using Lemma 3.1,*(iii)*, then Lemma 3.2, we obtain $\nu^1 \ll \nu^2$, that is (34). Existence (33) of an asymptotic particle density can be obtained by a proof analogous to [38, Lemma 14], where the space-time ergodic theorem is applied to the joint disorder-particle process. Then, closedness of $\mathcal{R}^Q$ is established as in [7, Proposition 3.1]: it uses (34), (33). Given the rest of the proposition, the weak continuity statement comes from a coupling argument, using (34) and Lemma 3.1. $\square$

3.3 Required properties of the model

For the proofs of Theorem 3.1 and Proposition 3.1, we have not used the particular form of L_α in (4), but the following properties.

1) The set of environments is a probability space $(\mathbf{A}, \mathcal{F}_{\mathbf{A}}, Q)$, where $\mathbf{A}$ is a compact metric space and $\mathcal{F}_{\mathbf{A}}$ its Borel σ-field. On $\mathbf{A}$ we assume given a group of space shifts $(\tau_x : x \in \mathbb{Z})$, with respect to which Q is ergodic.

For each $\alpha \in \mathbf{A}$, L_α is the generator of a Feller process on $\mathbf{X}$ that satisfies (20). The latter should be viewed as the assumption on "how the disorder enters the dynamics". It is equivalent to $\mathfrak{L}$ satisfying (31), that is being a translation-invariant generator on $\mathbf{A} \times \mathbf{X}$.

2) For L_α we can define a graphical construction on a space-time Poisson space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the complete monotonicity property (15).

3) Irreducibility assumption (8), combined with attractiveness assumption *(M2)* on page 10, are responsible for Lemma 3.2,*(iii)*.

Indeed, given the generator (4) of the process, there is not a unique graphical construction. The strong convergence in Theorem 3.1 would hold for *any* graphical construction satisfying (15) *plus* the existence of a sequence of Poissonian events killing any remaining pair of discrepancies of opposite signs (see Part 1,c) of the proof of Proposition 3.1 for this last point). The latter property follows in the case of the misanthropes process from irreducibility assumption (8).

In Section 6 we shall therefore introduce a general framework to consider other models satisfying 1) and 2), with appropriate assumptions replacing (8) to imply Proposition 3.1. We refer to Lemma 6.1 for a statement and proof of Property (15) in the context of a general model, including the k-step exclusion process. The coupled process linked to this property can be tedious to write in the usual form of explicit coupling rates for more than two components, or for complex models. We shall see that it can be written in a simple model-independent way using the framework of Section 6.1.

4 Scalar conservation laws and entropy solutions

In Subsection 4.1, we recall the definition and characterizations of entropy solutions to scalar conservation laws, which will appear as hydrodynamic limits of the above models. Then in Subsection 4.2, we explain our variational formula for the entropy solution in the Riemann case, first when the flux function G is Lipschitz continuous, then when $G \in \mathcal{C}^2(\mathbb{R})$ has a single inflexion point, so that the entropy solution has a more explicit form. Finally, in Subsection 4.3, we explain the approximation schemes to go from a Riemann initial profile to a general initial profile.

4.1 Definition and properties of entropy solutions

This section is taken from [7, Section 2.2] and [8, Section 4.1]. For more details, we refer to the textbooks [28, 44], or [18]. Equation (3) has no strong solutions in general: even starting from a smooth Cauchy datum $u(.,0) = u_0$, discontinuities (called shocks in this context) appear in finite time. Therefore it is necessary to consider weak solutions, but then uniqueness is lost for the Cauchy problem. To recover uniqueness, we need to define *entropy solutions*.

Let $\phi : [0, K] \cap \mathbb{R} \to \mathbb{R}$ be a convex function. In the context of hyperbolic systems, such a function is called an *entropy*. We define the associated *entropy flux* ψ on $[0, K]$ as

$$\psi(u) := \int_0^u \phi'(v) G'(v) dv$$

(ϕ, ψ) is called an *entropy-flux pair*. A Borel function $u : \mathbb{R} \times \mathbb{R}^{+*} \to [0, K]$ is called an *entropy solution* to (3) if and only if it is entropy-dissipative, *i.e.*

$$\partial_t \phi(u) + \partial_x \psi(u) \leq 0 \tag{41}$$

in the sense of distributions on $\mathbb{R} \times \mathbb{R}^{+*}$ for any entropy-flux pair (ϕ, ψ). Note that, by taking $\phi(u) = \pm u$ and hence $\psi(u) = \pm G(u)$, we see that an entropy solution is indeed a weak solution to (3). This definition can be motivated by the following points: *i)* when G and ϕ are continuously differentiable, (3) implies equality in strong sense in (41) (this follows from the chain rule for differentiation); *ii)* this no longer holds in general if u is only a weak solution to (3); *iii)* the inequality (41) can be seen as a macroscopic version of the second law of thermodynamics that selects physically relevant solutions. Indeed, one should think of the *concave* function $h = -\phi$ as a thermodynamic entropy, and spatial integration of (41) shows that the total thermodynamic entropy may not decrease during the evolution (this is rigorously true for periodic boundary conditions, in which case the total entropy is well defined).

Kružkov proved the following fundamental existence and uniqueness result:

Theorem 4.1 *([34, Theorem 2 and Theorem 5]). Let $u_0 : \mathbb{R} \to [0, K]$ be a Borel measurable initial datum. Then there exists a unique (up to a Lebesgue-null subset of $\mathbb{R}\times\mathbb{R}^{+*}$) entropy solution u to (3) subject to the initial condition*

$$\lim_{t\to 0^+} u(.,t) = u_0(.) \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}) \tag{42}$$

This solution (has a representative in its $L^\infty(\mathbb{R}\times\mathbb{R}^{+*})$ equivalence class that) is continuous as a mapping $t \mapsto u(.,t)$ from $\mathbb{R}^{+*}$ to $L^1_{\mathrm{loc}}(\mathbb{R})$.

We recall here that a sequence $(u_n, n \in \mathbb{N})$ of Borel measurable functions on $\mathbb{R}$ is said to converge to u in $L^1_{\mathrm{loc}}(\mathbb{R})$ if and only if

$$\lim_{n\to\infty} \int_I |u_n(x) - u(x)|\, dx = 0$$

for every bounded interval $I \subset \mathbb{R}$.

Remark 4.1 *Kružkov's theorems are stated for a continuously differentiable G. However the proof of the uniqueness result ([34, Theorem 2]) uses only Lipschitz continuity. In the Lipschitz-continuous case, existence could be derived from Kružkov's result by a flux approximation argument. However a different, self-contained (and constructive) proof of existence in this case can be found in [18, Chapter 6].*

The following proposition is a collection of results on entropy solutions. We first recall the following definition. Let TV_I denote the variation of a function defined on some bounded closed interval $I = [a, b] \subset \mathbb{R}$, *i.e.*

$$\mathrm{TV}_I[u(.)] := \sup_{x_0=a<x_1\cdots<x_n=b} \sum_{i=0}^{n-1} |u(x_{i+1}) - u(x_i)|$$

The total variation of u is defined by

$$\mathrm{TV}[u(.)] := \sup_{I\subset\mathbb{R}} \mathrm{TV}_I[u(.)]$$

Let us say that $u = u(.,.)$ defined on $\mathbb{R} \times \mathbb{R}^{+*}$ has locally bounded space variation if

$$\sup_{t\in J} \mathrm{TV}_I[u(.,t)] < +\infty \tag{43}$$

for every bounded closed space interval $I \subset \mathbb{R}$ and bounded time interval $J \subset \mathbb{R}^{+*}$.

For two measures $\alpha, \beta \in \mathcal{M}^+(\mathbb{R})$ with compact support, we define

$$\Delta(\alpha, \beta) := \sup_{x \in \mathbb{R}} |\alpha((-\infty, x]) - \beta((-\infty, x])| \tag{44}$$

When α or β is of the form $u(.)dx$ for $u(.) \in L^\infty(\mathbb{R})$ with compact support, we simply write u in (44) instead of $u(.)dx$. For a sequence $(\mu_n)_{n\geq 0}$ of measures with uniformly bounded support, the following equivalence holds:

$$\mu_n \to \mu \text{ vaguely if and only if } \lim_{n\to\infty} \Delta(\mu_n, \mu) = 0 \tag{45}$$

Proposition 4.1 *([8, Proposition 4.1])*

i) Let $u(.,.)$ be the entropy solution to (3) with Cauchy datum $u_0 \in L^\infty(\mathbb{R})$. Then the mapping $t \mapsto u_t = u(.,t)$ lies in $C^0([0,+\infty), L^1_{\text{loc}}(\mathbb{R}))$.

ii) If u_0 has constant value c, then for all $t > 0$, u_t has constant value c.

iii) If $u_0^i(.)$ has finite variation, that is $\text{TV}u_0^i(.) < +\infty$, then so does $u^i(.,t)$ for every $t > 0$, and $\text{TV}u^i(.,t) \leq \text{TV}u_0^i(.)$.

iv) Finite propagation property: Assume $u^i(.,.)$ ($i \in \{1,2\}$) is the entropy solution to (3) with Cauchy data $u_0^i(.)$. Let

$$V = ||G'||_\infty := \sup_\rho |G'(\rho)| \tag{46}$$

Then, for every $x < y$ and $0 \leq t < (y-x)/2V$,

$$\int_{x+Vt}^{y-Vt} \left[u^1(z,t) - u^2(z,t)\right]^\pm dz \leq \int_x^y \left[u_0^1(z) - u_0^2(z)\right]^\pm dz \tag{47}$$

In particular, assume $u_0^1 = u_0^2$ (resp. $u_0^1 \leq u_0^2$) on $[a,b]$ for some $a, b \in \mathbb{R}$ such that $a < b$. Then, for all $t \leq (b-a)/(2V)$, $u_t^1 = u_t^2$ (resp. $u_t^1 \leq u_t^2$) on $[a+Vt, b-Vt]$.

(v) If $\int_{\mathbb{R}} u_0^i(z)dz < +\infty$*, then*

$$\Delta(u^1(.,t), u^2(.,t)) \leq \Delta(u_0^1(.), u_0^2(.)) \tag{48}$$

Properties *(o)–(iv)* are standard. Property *(v)* can be deduced from the correspondence between entropy solutions of (3) and viscosity solutions of the Hamilton-Jacobi equation

$$\partial_t h(x,t) + G[\partial_x h(x,t)] = 0 \tag{49}$$

Namely, h is a viscosity solution of (49) if and only if $u = \partial_x h$ is an entropy solution of (3). Then *(v)* follows from the monotonicity of the solution semigroup for (49). Properties *(iv)* and *(v)* have microscopic analogues (respectively Lemma 5.5 and Proposition 5.5) in the class of particle systems we consider, which play an important role in the proof of the hydrodynamic limit, as will be sketched in Section 5.

We next recall a possibly more familiar definition of entropy solutions based on shock admissibility conditions, but valid only for solutions with bounded variation. This point of view selects the relevant weak solutions by specifying what kind of discontinuities are permitted. First, in particular, the following two conditions are necessary and sufficient for a piecewise smooth function $u(x,t)$ to be a weak solution to equation (3) with initial condition (52) (see [14]):
1. $u(x,t)$ solves equation (3) at points of smoothness.
2. If $x(t)$ is a curve of discontinuity of the solution then the *Rankine-Hugoniot condition*

$$\frac{d}{dt}x(t) = S[u^+; u^-] := \frac{G(u^-) - G(u^+)}{u^- - u^+} \tag{50}$$

holds along $x(t)$.
Moreover, to ensure uniqueness, the following geometric condition, known as *Oleĭnik's entropy condition* (see *e.g.* [28] or [44]), is sufficient. A discontinuity (u^-, u^+), with $u^\pm := u(x\pm 0, t)$, is called an entropy shock, if and only if:

The chord of the graph of G between u^- and u^+ lies:
below the graph if $u^- < u^+$, above the graph if $u^+ < u^-$. (51)

In the above condition, "below" or "above" is meant in wide sense, *i.e.* does not exclude that the graph and chord coincide at some points between u^-

and u^+. In particular, when G is strictly convex (resp. concave), one recovers the fact that only (and all) decreasing (resp. increasing) jumps are admitted (as detailed in subsection 4.2 below). Note that, if the graph of G is linear on some nontrivial interval, condition (51) implies that any increasing or decreasing jump within this interval is an entropy shock.

Indeed, condition (51) can be used to select entropy solutions among weak solutions. The following result is a consequence of [48].

Proposition 4.2 *([7, Proposition 2.2]). Let u be a weak solution to (3) with locally bounded space variation. Then u is an entropy solution to (3) if and only if, for a.e. $t > 0$, all discontinuities of $u(.,t)$ are entropy shocks.*

One can show that, if the Cauchy datum u_0 has locally bounded variation, the unique entropy solution given by Theorem 4.1 has locally bounded space variation. Hence Proposition 4.2 extends into an existence and uniqueness theorem within functions of locally bounded space variation, where entropy solutions may be defined as weak solutions satisfying (51), without reference to (41).

4.2 The Riemann problem

This subsection is based first on [7, Section 4.1], then on [6, Section 2.1]. Of special importance among entropy solutions are the solutions of the Riemann problem, *i.e.* the Cauchy problem for particular initial data of the form

$$R_{\lambda,\rho}(x) = \lambda \mathbf{1}_{\{x<0\}} + \rho \mathbf{1}_{\{x\geq 0\}} \tag{52}$$

Indeed: *(i)* as developed in the sequel of this subsection, these solutions can be computed explicitly and have a variational representation; *(ii)* as will be seen in Subsection 4.3, one can construct approximations to the solution of the general Cauchy problem by using only Riemann solutions. This has inspired our belief that one could derive general hydrodynamics from Riemann hydrodynamics.

In connection with Theorem 3.1 and Proposition 3.1, it will be important in the sequel to consider flux functions G with possible linear degeneracy on some density intervals. Therefore, in the sequel of this section, $\mathcal{R}$ will denote a closed subset of $[0,K]\cap\mathbb{R}$ such that G is affine on each of the countably many disjoint open intervals whose union is the complement of $\mathcal{R}$. Such a subset exists (for instance one can take $\mathcal{R} = [0, K] \cap \mathbb{R}$) and is not necessarily unique.

From now on we assume $\lambda < \rho$; adapting to the case $\lambda > \rho$ is straightforward, by replacing in the sequel lower with upper convex hulls, and minima/minimizers with maxima/maximizers. Consider G_c, the lower convex envelope of G on $[\lambda, \rho]$. There exists a nondecreasing function H_c (hence with left/right limits) such that G_c has left/right hand derivative denoted by $H_c(\alpha \pm 0)$ at every α. The function H_c is defined uniquely outside the at most countable set of non-differentiability points of G_c

$$\Theta = \{\alpha \in [\lambda, \rho] : H_c(\alpha - 0) < H_c(\alpha + 0)\} \tag{53}$$

As will appear below, the particular choice of H_c on Θ does not matter. Let $v_* = v_*(\lambda, \rho) := H_c(\lambda + 0)$ and $v^* = v^*(\lambda, \rho) := H_c(\rho - 0)$. Since H_c is nondecreasing, there is a nondecreasing function h_c on $[v_*, v^*]$ such that, for every $v \in [v_*, v^*]$,

$$\begin{array}{l} \alpha < h_c(v) \Rightarrow H_c(\alpha) \leq v \\ \alpha > h_c(v) \Rightarrow H_c(\alpha) \geq v \end{array} \tag{54}$$

Any such h_c satisfies

$$\begin{array}{l} h_c(v - 0) = \inf\{\alpha \in \mathbb{R} : \ H_c(\alpha) \geq v\} = \sup\{\alpha \in \mathbb{R} : \ H_c(\alpha) < v\} \\ h_c(v + 0) = \inf\{\alpha \in \mathbb{R} : \ H_c(\alpha) > v\} = \sup\{\alpha \in \mathbb{R} : \ H_c(\alpha) \leq v\} \end{array} \tag{55}$$

We have that, anywhere in (55), $H_c(\alpha)$ may be replaced with $H_c(\alpha \pm 0)$. The following properties can be derived from (54) and (55):

1. Given G, h_c is defined uniquely, and is continuous, outside the at most countable set

$$\begin{aligned} \Sigma_{low}(G) \quad = \quad & \{v \in [v_*, v^*] : G_c \text{ is differentiable with derivative } v \\ & \text{in a nonempty open subinterval of } [\lambda, \rho]\} \end{aligned} \tag{56}$$

By "defined uniquely" we mean that for such v's, there is a unique $h_c(v)$ satisfying (54), which does not depend on the choice of H_c on Θ.

2. Given G, $h_c(v \pm 0)$ is uniquely defined, *i.e.* independent of the choice of H_c on Θ, for any $v \in [v_*, v^*]$. For $v \in \Sigma_{low}(G)$, $(h_c(v - 0), h_c(v + 0))$ is the

maximal open interval over which H_c has constant value v.

3. For every $\alpha \in \Theta$ and $v \in (H_c(\alpha - 0), H_c(\alpha + 0))$, $h_c(v)$ is uniquely defined and equal to α.

In the sequel we extend h_c outside $[v_*, v^*]$ in a natural way by setting

$$h_c(v) = \lambda \text{ for } v < v_*, \quad h_c(v) = \rho \text{ for } v > v^* \tag{57}$$

Next proposition extends [6, Proposition 2.1], where we assumed $G \in \mathcal{C}^2(\mathbb{R})$.

Proposition 4.3 *([7, Proposition 4.1]). Let $\lambda, \rho \in [0, K] \cap \mathbb{R}$, $\lambda < \rho$. For $v \in \mathbb{R}$, we set*

$$\mathcal{G}_v(\lambda, \rho) := \inf \{G(r) - vr : r \in [\lambda, \rho] \cap \mathcal{R}\} \tag{58}$$

Then i) For every $v \in \mathbb{R} \setminus \Sigma_{low}(G)$, The minimum in (58) is achieved at the unique point $h_c(v)$, and $u(x, t) := h_c(x/t)$ is the weak entropy solution to (3) with Riemann initial condition (52), denoted by $R_{\lambda,\rho}(x, t)$.

ii) If $\lambda \in \mathcal{R}$ and $\rho \in \mathcal{R}$, the previous minimum is unchanged if restricted to $[\lambda, \rho] \cap \mathcal{R}$. As a result, the Riemann entropy solution is a.e. $\mathcal{R}$-valued.

iii) For every $v, w \in \mathbb{R}$,

$$\int_v^w R_{\lambda,\rho}(x, t)dx = t[\mathcal{G}_{v/t}(\lambda, \rho) - \mathcal{G}_{w/t}(\lambda, \rho)] \tag{59}$$

In the case when $G \in \mathcal{C}^2(\mathbb{R})$ is such that G'' vanishes only finitely many times, the expression of $u(x, t)$ is more explicit. Let us detail, as in [6, Section 2.1], following [14], the case where G has a single inflexion point $a \in (0, K)$ and $G(u)$ is strictly convex in $0 \leq u < a$ and strictly concave in $a < u \leq 1$. We denote $H = G'$, h_1 the inverse of H restricted to $(-\infty, a)$, and h_2 the inverse of H restricted to $(a, +\infty)$. We have

Lemma 4.1 *([14, Lemma 2.2, Lemma 2.4]). Let $w < a$ be given, and define*

$$w^* := \sup\{u > w : S[w; u] > S[v; w], \forall v \in (w, u)\}$$

Suppose that $w^* < \infty$. Then

a) $S[w;w^*] = H(w^*)$*;*

b) w^* *is the only zero of* $S[u;w] - H(u)$*,* $u > w$*.*

If $\rho < \lambda < \rho^*$ ($\rho < a$), then $H(\lambda) > H(\rho)$: the characteristics starting from $x \leq 0$ have a speed (given by H) greater than the speed of those starting from $x > 0$. If the characteristics intersect along a curve $x(t)$, then Rankine-Hugoniot condition will be satisfied if

$$x'(t) = S[u^+;u^-] = \frac{G(\lambda) - G(\rho)}{\lambda - \rho} = S[\lambda;\rho].$$

The convexity of G implies that Oleĭnik's Condition is satisfied across $x(t)$. Therefore the unique entropy weak solution is the *shock*:

$$u(x,t) = \begin{cases} \lambda, & x \leq S[\lambda;\rho]t; \\ \rho, & x > S[\lambda;\rho]t; \end{cases} \tag{60}$$

If $\lambda < \rho < a$, the relevant part of the flux function is convex. The characteristics starting respectively from $x \leq 0$ and $x > 0$ never meet, and they never enter the space-time wedge between lines $x = H(\lambda)t$ and $x = H(\rho)t$. It is possible to define piecewise smooth weak solutions with a jump occurring in the wedge satisfying the Rankine-Hugoniot condition. But the convexity of G prevents such solutions from satisfying Oleĭnik's Condition. Thus the unique entropy weak solution is the *continuous solution with a rarefaction fan*:

$$u(x,t) = \begin{cases} \lambda, & x \leq H(\lambda)t; \\ h_1(x/t), & H(\lambda)t < x \leq H(\rho)t; \\ \rho, & H(\rho)t < x; \end{cases} \tag{61}$$

Let $\rho < \rho^* < \lambda$ ($\rho < a$): Lemma 4.1 applied to ρ suggests that a jump from ρ^* to ρ along the line $x = H(\rho^*)t$ will satisfy the Rankine-Hugoniot condition. Due to the definition of ρ^*, a solution with such a jump will also satisfy Oleĭnik's Condition, therefore it would be the unique entropic weak solution. Notice that since $H(\lambda) < H(\rho^*)$, no characteristics intersect along the line of discontinuity $x = H(\rho^*)t$. This case is called a *contact discontinuity* in [14]. The solution is defined by

$$u(x,t) = \begin{cases} \lambda, & x \leq H(\lambda)t; \\ h_2(x/t), & H(\lambda)t < x \leq H(\rho^*)t; \\ \rho, & H(\rho^*)t < x; \end{cases} \tag{62}$$

Corresponding cases on the concave side of G are treated similarly.

Let us illustrate this description on a reference example, *the totally asymmetric 2-step exclusion* (what follows is taken from [6, Section 4.1.1]):
Its flux function $G_2(u) = u + u^2 - 2u^3$ is strictly convex in $0 \leq u < 1/6$ and strictly concave in $1/6 < u \leq 1$. For $w < 1/6$, $w^* = (1-2w)/4$, and for $w > 1/6$, $w_* = (1-2w)/4$; $h_1(x) = (1/6)(1-\sqrt{7-6x})$ for $x \in (-\infty, 7/6)$, and $h_2(x) = (1/6)(1+\sqrt{7-6x})$ for $x \in (7/6, +\infty)$. We reproduce here [6, Figure 1], which shows the six possible behaviors of the (self-similar) solution $u(v,1)$, namely a rarefaction fan with either an increasing or a decreasing initial condition, a decreasing shock, an increasing shock, and a contact discontinuity with either an increasing or a decreasing initial condition. Cases (a) and (b) present respectively a rarefaction fan with increasing initial condition and a preserved decreasing shock. These situations as well as cases (c) and (f) cannot occur for simple exclusion. Observe also that $\rho \geq 1/2$ implies $\rho_* \leq 0$, which leads only to cases (d),(e), and excludes case (f) (going back to a simple exclusion behavior).

4.3 From Riemann to Cauchy problem

The beginning of this subsection is based on [7, Section 2.4]. We will briefly explain here the principle of approximation schemes based on Riemann solutions, the most important of which is probably Glimm's scheme, introduced in [26]. Consider as initial datum a piecewise constant profile with finitely many jumps. The key observation is that, for small enough times, this can be viewed as a succession of noninteracting Riemann problems. To formalize this, we recall part of [6, Lemma 3.4], which is a consequence of the finite propagation property for (3), see statement iv) of Proposition 4.1. We denote by $R_{\lambda,\rho}(x,t)$ the entropy solution to the Riemann problem with initial datum (52).

Lemma 4.2 *([7, Lemma 2.1]). Let* $x_0 = -\infty < x_1 < \cdots < x_n < x_{n+1} = +\infty$*, and* $\varepsilon := \min_k(x_{k+1} - x_k)$*. Consider the Cauchy datum*

$$u_0 := \sum_{k=0}^{n} r_k \mathbf{1}_{(x_k, x_{k+1})}$$

where $r_k \in [0, K]$*. Then for* $t < \varepsilon/(2V)$*, with* V *given by (46), the entropy* *solution $u(.,t)$ at time t coincides with $R_{r_{k-1},r_k}(.-x_k,t)$ on $(x_{k-1}+Vt, x_{k+1}-Vt)$. In particular, $u(.,t)$ has constant value r_k on $(x_k+Vt, x_{k+1}-Vt)$.*

Given some Cauchy datum u_0, we construct an approximate solution $\tilde{u}(.,.)$ for the corresponding entropy solution $u(.,.)$. To this end we define an approximation scheme based on a time discretization step $\Delta t > 0$ and a space discretization step $\Delta x > 0$. In the limit we let $\Delta x \to 0$ with the ratio $R := \Delta t/\Delta x$ kept constant, under the condition

$$R \leq 1/(2V) \tag{63}$$

known as the *Courant-Friedrichs-Lewy (CFL) condition.* Let $t_k := k\Delta t$ denote discretization times. We start with $k = 0$, setting $\tilde{u}_0^- := u_0$.

Step one (approximation step): Approximate $\tilde{u}_k^-$ with a piecewise constant profile $\tilde{u}_k^+$ whose step lengths are bounded below by Δx.

Step two (evolution step): For $t \in [t_k, t_{k+1})$, denote by $\tilde{u}_k(.,t)$ the entropy solution at time t with initial datum $\tilde{u}_k^+$ at time t_k. By (63) and Lemma 4.2, $\tilde{u}_k(.,t)$ can be computed solving only Riemann problems. Set $\tilde{u}_{k+1}^- = \tilde{u}_k(.,t_{k+1})$.

Step three (iteration): increment k and go back to step one.

The approximate entropy solution is then defined by

$$\tilde{u}(.,t) := \sum_{k\in\mathbb{N}} \tilde{u}_k(.,t)\mathbf{1}_{[t_k,t_{k+1})}(t) \tag{64}$$

The efficiency of the scheme depends on how the approximation step is performed. In Glimm's scheme, the approximation $\tilde{u}_k^+$ is defined as

$$\tilde{u}_k^+ := \sum_{j\in k/2+\mathbb{Z}} \tilde{u}_k^-((j+a_k/2)\Delta x)\mathbf{1}_{((j-1/2)\Delta x,(j+1/2)\Delta x)} \tag{65}$$

where $a_k \in (-1,1)$. Then we have the following convergence result.

Theorem 4.2 *([7, Theorem 2.3]). Let u_0 be a given measurable initial datum. Then every sequence $\varepsilon_n \downarrow 0$ as $n \to \infty$ has a subsequence $\delta_n \downarrow 0$ such that, for a.e. sequence (a_k) w.r.t. product uniform measure on $(-1,1)^{\mathbb{Z}^+}$, the Glimm approximation defined by (64) and (65) converges to u in $L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^{+*})$ as $\Delta x = \delta_n \downarrow 0$.*

When u_0 has locally bounded variation, the above result is a specialization to scalar conservation laws of a more general result for systems of conservation laws: see Theorems 5.2.1, 5.2.2, 5.4.1 and comments following Theorem 5.2.2 in [44]. In [7, Appendix B], we prove that it is enough to assume u_0 measurable.

Due to the nature of the approximation step (65), the proof of Theorem 4.2 does not proceed by direct estimation of the error between $\tilde{u}_k^{\pm}$ and $u(.,t_k)$, but indirectly, by showing that limits of the scheme satisfy (41).

We will now present a different Riemann-based approximation procedure, introduced first in [6, Lemma 3.6], and refined in [7, 8]. This approximation allows direct control of the error by using the distance Δ defined in (44). Intuitively, errors accumulate during approximation steps, but might be amplified by the resolution steps. The key properties of our approximation are that the total error accumulated during the approximation step is negligible as $\varepsilon \to 0$, and the error is not amplified by the resolution step, because Δ does not increase along entropy solutions, see Proposition 4.1*(v)*.

Theorem 4.3 *([6, Theorem 3.1]). Assume $(T_t)_{t\geq 0}$ is a semigroup on the set of bounded $\mathcal{R}$-valued functions, with the following properties:*

(1) For any Riemann initial condition u_0, $t \mapsto u_t = T_t u_0$ is the entropy solution to (3) with Cauchy datum u_0.

(2) (Finite speed of propagation). There is a constant v such that, for any $a, b \in \mathbb{R}$, any two initial conditions u_0 and u_1 coinciding on $[a; b]$, and any $t < (b-a)/(2V)$, $u_t = T_t u_0$ and $v_t = T_t v_0$ coincide on $[a + Vt; b - Vt]$.

(3) (Time continuity). For every bounded initial condition u_0 with bounded support and every $t \geq 0$, $lim_{\varepsilon\to 0+}\Delta(T_t u_0, T_{t+\varepsilon}u_0) = 0$.

(4) (Stability). For any bounded initial conditions u_0 and v_0, with bounded support, $\Delta(T_t u_0, T_t v_0) \leq \Delta(u_0, v_0)$.

Then, for any bounded u_0, $t \mapsto T_t u_0$ is the entropy solution to (3) with Cauchy datum u_0.

A crucial point is that properties (1)–(4) in the above Theorem 4.3 hold at particle level, where this will allow us to mimic the scheme. The proof of Theorem 4.3 relies on the following uniform approximation (in the sense of distance Δ) by step functions, which is also important at particle level.

Lemma 4.3 *([8, Lemma 4.2]). Assume $u_0(.)$ is a.e. $\mathcal{R}$-valued, has bounded support and finite variation, and let $(x,t)\mapsto u(x,t)$ be the entropy solution to (3) with Cauchy datum $u_0(.)$. For every $\varepsilon>0$, let $\mathcal{P}_\varepsilon$ be the set of piecewise constant $\mathcal{R}$-valued functions on $\mathbb{R}$ with compact support and step lengths at least ε, and set*

$$\delta_\varepsilon(t):=\varepsilon^{-1}\inf\{\Delta(u(.),u(.,t)):\ u(.)\in\mathcal{P}_\varepsilon\}$$

Then there is a sequence $\varepsilon_n\downarrow 0$ as $n\to\infty$ such that δ_{ε_n} converges to 0 uniformly on any bounded subset of $\mathbb{R}^+$.

5 Proof of hydrodynamics

In this section, based on [9, Section 4], we prove the hydrodynamic limit in the quenched disordered setting, that is Theorem 3.1, following the strategy introduced in [6, 7] and significantly strengthened in [8] and [9]. First, we prove the hydrodynamic limit for $\mathcal{R}^Q$-valued Riemann initial conditions (the so-called Riemann problem), and then use a constructive scheme to mimic the proof of Theorem 4.3 at microscopic level.

5.1 Riemann problem

Let $\lambda,\rho\in\mathcal{R}^Q$ with $\lambda<\rho$ (for $\lambda>\rho$ replace infimum with supremum below). We first need to derive hydrodynamics for the Riemann initial condition $R_{\lambda,\rho}$ defined in (52). Microscopic Riemann states with profile (52) can be constructed using the following lemma.

Lemma 5.1 *([9, Lemma 4.1]). There exist random variables α and $(\eta^\rho:\rho\in\mathcal{R}^Q)$ on a probability space $(\Omega_{\mathbf{A}},\mathcal{F}_{\mathbf{A}},\mathbb{P}_{\mathbf{A}})$ such that*

$$(\alpha,\eta^\rho)\sim\nu^{Q,\rho},\qquad \alpha\sim Q \tag{66}$$

$$\mathbb{P}_{\mathbf{A}}-a.s.,\qquad \rho\mapsto\eta^\rho \text{ is nondecreasing} \tag{67}$$

Let $\overline{\nu}^{Q,\lambda,\rho}$ denote the distribution of $(\alpha,\eta^\lambda,\eta^\rho)$, and $\overline{\nu}^{\lambda,\rho}_\alpha$ the conditional distribution of $(\alpha,\eta^\lambda,\eta^\rho)$ given α. Recall the definition (17) of the space-time shift θ_{x_0,t_0} on Ω for $(x_0,t_0)\in\mathbb{Z}\times\mathbb{R}^+$. We now introduce an extended shift θ' on $\Omega'=\mathbf{A}\times\mathbf{X}^2\times\Omega$. If $\omega'=(\alpha,\eta,\xi,\omega)$ denotes a generic element of Ω', we set

$$\theta'_{x,t}\omega'=(\tau_x\alpha,\tau_x\eta_t(\alpha,\eta,\omega),\tau_x\eta_t(\alpha,\xi,\omega),\theta_{x,t}\omega) \tag{68}$$

It is important to note that this shift incorporates disorder. Let $T:\mathbf{X}^2\to\mathbf{X}$ be given by

$$T(\eta,\xi)(x)=\eta(x)\mathbf{1}_{\{x<0\}}+\xi(x)\mathbf{1}_{\{x\geq 0\}} \tag{69}$$

A strong (that is almost sure with respect to the Poisson space) form of hydrodynamic limit for Riemann data can now be stated as follows.

Proposition 5.1 *([9, Proposition 4.1]). Set, for $t\geq 0$,*

$$\beta_t^N(\omega')(dx):=\pi^N(\eta_t(\alpha,T(\eta,\xi),\omega))(dx) \tag{70}$$

For all $t>0$, $s_0\geq 0$ and $x_0\in\mathbb{R}$, we have that, for Q-a.e. $\alpha\in\mathbf{A}$,

$$\lim_{N\to\infty}\beta_{Nt}^N(\theta'_{\lfloor Nx_0\rfloor,Ns_0}\omega')(dx)=R_{\lambda,\rho}(.,t)dx,\quad \overline{\nu}_\alpha^{\lambda,\rho}\otimes\mathbb{P}\text{-}a.s.$$

Proposition 5.1 will follow from a law of large numbers for currents. Let $x_.=(x_t,\,t\geq 0)$ be a $\mathbb{Z}$-valued *cadlag* random path, with $|x_t-x_{t^-}|\leq 1$, independent of the Poisson measure ω. We define the particle current seen by an observer travelling along this path by

$$\varphi_t^{x_.}(\alpha,\eta_0,\omega)=\varphi_t^{x_.,+}(\alpha,\eta_0,\omega)-\varphi_t^{x_.,-}(\alpha,\eta_0,\omega)+\widetilde{\varphi}_t^{x_.}(\alpha,\eta_0,\omega) \tag{71}$$

where $\varphi_t^{x_.,\pm}(\alpha,\eta_0,\omega)$ count the number of rightward/leftward crossings of $x_.$ due to particle jumps, and $\widetilde{\varphi}_t^{x_.}(\alpha,\eta_0,\omega)$ is the current due to the self-motion of the observer. We shall write φ_t^v in the particular case $x_t=\lfloor vt\rfloor$. Set $\phi_t^v(\omega'):=\varphi_t^v(\alpha,T(\eta,\xi),\omega)$. Note that for $(v,w)\in\mathbb{R}^2$,

$$\beta_{Nt}^N(\omega')([v,w])=t(Nt)^{-1}(\phi_{Nt}^{v/t}(\omega')-\phi_{Nt}^{w/t}(\omega')) \tag{72}$$

We view (72) as a microscopic analogue of (59). Thus, Proposition 5.1 boils down to showing that each term of (72) converges to its counterpart in (59).

Proposition 5.2 *([9, Proposition 4.2]). For all $t>0$, $a\in\mathbb{R}^+,b\in\mathbb{R}$ and $v\in\mathbb{R}$,*

$$\lim_{N\to\infty}(Nt)^{-1}\phi_{Nt}^v(\theta'_{\lfloor bN\rfloor,aN}\omega')=\mathcal{G}_v(\lambda,\rho)\qquad\overline{\nu}^{Q,\lambda,\rho}\otimes\mathbb{P}-a.s. \tag{73}$$

where $\mathcal{G}_v(\lambda,\rho)$ is defined by (58).

To prove Proposition 5.2, we introduce a probability space Ω^+, whose generic element is denoted by ω^+, on which is defined a Poisson process $(N_t(\omega^+))_{t\geq 0}$ with intensity $|v|$ ($v \in \mathbb{R}$). Denote by $\mathbb{P}^+$ the associated probability. Set

$$\begin{aligned} x_s^N(\omega^+) &:= (\mathrm{sgn}(v))\left[N_{aN+s}(\omega^+) - N_{aN}(\omega^+)\right] && (74)\\ \widetilde{\eta}_s^N(\alpha,\eta_0,\omega,\omega^+) &:= \tau_{x_s^N(\omega^+)}\eta_s(\alpha,\eta_0,\omega) && (75)\\ \widetilde{\alpha}_s^N(\alpha,\omega^+) &:= \tau_{x_s^N(\omega^+)}\alpha && (76)\end{aligned}$$

Thus $(\widetilde{\alpha}_s^N,\widetilde{\eta}_s^N)_{s\geq 0}$ is a Feller process with generator

$$L^v = \mathfrak{L} + S^v, \quad S^v f(\alpha,\zeta) = |v|\left[f(\tau_{\mathrm{sgn}(v)}\alpha, \tau_{\mathrm{sgn}(v)}\zeta) - f(\alpha,\zeta)\right]$$

for f local and $\alpha \in \mathbf{A}$, $\zeta \in \mathbf{X}$. Since any translation invariant measure on $\mathbf{A}\times\mathbf{X}$ is stationary for the pure shift generator S^v, we have $\mathcal{I}_{\mathfrak{L}}\cap\mathcal{S} = \mathcal{I}_{L^v}\cap\mathcal{S}$. Define the time and space-time empirical measures (where $\varepsilon > 0$) by

$$\begin{aligned} m_{tN}(\omega',\omega^+) &:= (Nt)^{-1}\int_0^{tN}\delta_{(\widetilde{\alpha}_s^N(\alpha,\omega^+),\widetilde{\eta}_s^N(\alpha,T(\eta,\xi),\omega,\omega^+))}ds && (77)\\ m_{tN,\varepsilon}(\omega',\omega^+) &:= |\mathbb{Z}\cap[-\varepsilon N,\varepsilon N]|^{-1}\sum_{x\in\mathbb{Z}:\,|x|\leq\varepsilon N}\tau_x m_{tN}(\omega',\omega^+) && (78)\end{aligned}$$

Notice that there is a disorder component we cannot omit in the empirical measure, although ultimately we are only interested in the behavior of the η-component. Let $\mathcal{M}^Q_{\lambda,\rho}$ denote the compact set of probability measures $\mu(d\alpha,d\eta)\in\mathcal{I}_{\mathfrak{L}}\cap\mathcal{S}$ such that μ has α-marginal Q, and $\nu^{Q,\lambda}\ll\mu\ll\nu^{Q,\rho}$. By Proposition 3.1,

$$\mathcal{M}^Q_{\lambda,\rho} = \left\{\nu(d\alpha,d\eta) = \int\nu^{Q,r}(d\alpha,d\eta)\gamma(dr) :\, \gamma\in\mathcal{P}([\lambda,\rho]\cap\mathcal{R}^Q)\right\} \quad (79)$$

The key ingredients for Proposition 5.2 are the following lemmas.

Lemma 5.2 *([9, Lemma 4.2]). The function $\phi_t^v(\alpha,\eta,\xi,\omega)$ is increasing in η, decreasing in ξ.*

Lemma 5.3 *([9, Lemma 4.3]). With $\overline{\nu}^{Q,\lambda,\rho}\otimes\mathbb{P}\otimes\mathbb{P}^+$-probability one, every subsequential limit as $N\to\infty$ of $m_{tN,\varepsilon}(\theta'_{\lfloor bN\rfloor,aN}\omega',\omega^+)$ lies in $\mathcal{M}^Q_{\lambda,\rho}$.*

Lemma 5.2 is a consequence of the monotonicity property (15). Lemma 5.3 relies in addition on a space-time ergodic theorem and on a general uniform large deviation upper bound for space-time empirical measures of Markov processes. We state these two results before proving Proposition 5.2.

Proposition 5.3 *([8, Proposition 2.3]). Let $(\eta_t)_{t\geq 0}$ be a Feller process on* $\mathbf{X}$ *with a translation invariant generator L, that is*

$$\tau_1 L \tau_{-1} = L \tag{80}$$

Assume further that

$$\mu \in (\mathcal{I}_L \cap \mathcal{S})_e$$

where $\mathcal{I}_L$ denotes the set of invariant measures for L. Then, for any local function f on $\mathbf{X}$, and any $a > 0$

$$\lim_{\ell\to\infty} \frac{1}{a\ell^2} \int_0^{a\ell} \sum_{i=0}^{\ell} \tau_i f(\eta_t) dt = \int f d\mu = \lim_{\ell\to\infty} \frac{1}{a\ell^2} \int_0^{a\ell} \sum_{i=-\ell}^{-1} \tau_i f(\eta_t) dt \tag{81}$$

a.s. with respect to the law of the process with initial distribution μ.

Lemma 5.4 *([8, Lemma 3.4]) Let $\mathbf{P}^v_\nu$ denote the law of a Markov process $(\widetilde{\alpha}_., \widetilde{\xi}_.)$ with generator L^v and initial distribution ν. For $\varepsilon > 0$, let*

$$\pi_{t,\varepsilon} := |\mathbb{Z} \cap [-\varepsilon t, \varepsilon t]|^{-1} \sum_{x\in\mathbb{Z}\cap[-\varepsilon t,\varepsilon t]} t^{-1} \int_0^t \delta_{(\tau_x\widetilde{\alpha}_s, \tau_x\widetilde{\xi}_s)} ds \tag{82}$$

Then, there exists a functional $\mathcal{D}_v$ which is nonnegative, l.s.c., and satisfies $\mathcal{D}_v^{-1}(0) = \mathcal{I}_{L^v}$, such that, for every closed subset F of $\mathcal{P}(\mathbf{A}\times\mathbf{X})$,

$$\limsup_{t\to\infty} t^{-1} \log \sup_{\nu\in\mathcal{P}(\mathbf{A}\times\mathbf{X})} \mathbf{P}^v_\nu \left(\pi_{t,\varepsilon}(\widetilde{\xi}_.) \in F\right) \leq -\inf_{\mu\in F} \mathcal{D}_v(\mu) \tag{83}$$

Proof of Proposition 5.2. We will show that

$$\liminf_{N\to\infty} (Nt)^{-1} \phi^v_{tN} \circ \theta'_{\lfloor bN\rfloor, aN}(\omega') \geq \mathcal{G}_v(\lambda,\rho), \quad \overline{\nu}^{Q,\lambda,\rho} \otimes \mathbb{P}\text{-a.s.} \tag{84}$$

$$\limsup_{N\to\infty} (Nt)^{-1} \phi^v_{tN} \circ \theta'_{\lfloor bN\rfloor, aN}(\omega') \leq \mathcal{G}_v(\lambda,\rho), \quad \overline{\nu}^{Q,\lambda,\rho} \otimes \mathbb{P}\text{-a.s.} \tag{85}$$

Step one: proof of (84).
Setting $\varpi_{aN} = \varpi_{aN}(\omega') := T\left(\tau_{\lfloor bN\rfloor}\eta_{aN}(\alpha,\eta,\omega), \tau_{\lfloor bN\rfloor}\eta_{aN}(\alpha,\xi,\omega)\right)$, we have

$$(Nt)^{-1} \phi^v_{tN} \circ \theta'_{\lfloor bN\rfloor, aN}(\omega') = (Nt)^{-1} \varphi^v_{tN}(\tau_{\lfloor bN\rfloor}\alpha, \varpi_{aN}, \theta_{\lfloor bN\rfloor, aN}\omega) \tag{86}$$

Let, for every $(\alpha,\zeta,\omega,\omega^+)\in\mathbf{A}\times\mathbf{X}\times\Omega\times\Omega^+$ and $x^N_{\cdot}(\omega^+)$ given by (74),

$$\psi^{v,\varepsilon}_{tN}(\alpha,\zeta,\omega,\omega^+):=|\mathbb{Z}\cap[-\varepsilon N,\varepsilon N]|^{-1}\sum_{y\in\mathbb{Z}:\,|y|\leq\varepsilon N}\varphi^{x^N_{\cdot}(\omega^+)+y}_{tN}(\alpha,\zeta,\omega)\qquad(87)$$

Note that $\lim_{N\to\infty}(Nt)^{-1}x^N_{tN}(\omega^+)=v$, $\mathbb{P}^+$-a.s., and that for two paths $y_{\cdot},z_{\cdot}$ (see (71)),

$$|\varphi^{y_{\cdot}}_{tN}(\alpha,\eta_0,\omega)-\varphi^{z_{\cdot}}_{tN}(\alpha,\eta_0,\omega)|\leq K\left(|y_{tN}-z_{tN}|+|y_0-z_0|\right)$$

Hence the proof of (84) reduces to that of the same inequality where we replace $(Nt)^{-1}\phi^v_{tN}\circ\theta'_{\lfloor bN\rfloor,aN}(\omega')$ by $(Nt)^{-1}\psi^{v,\varepsilon}_{tN}(\tau_{\lfloor bN\rfloor}\alpha,\varpi_{aN},\theta_{\lfloor bN\rfloor,aN}\omega,\omega^+)$ and $\overline{\nu}^{Q,\lambda,\rho}\otimes\mathbb{P}$ by $\overline{\nu}^{Q,\lambda,\rho}\otimes\mathbb{P}\otimes\mathbb{P}^+$. By definitions (26), (71) of flux and current, for any $\alpha\in\mathbf{A}$, $\zeta\in\mathbf{X}$,

$$\begin{aligned}&M^{x,v}_{tN}(\alpha,\zeta,\omega,\omega^+):=\varphi^{x^N_{\cdot}(\omega^+)+x}_{tN}(\alpha,\zeta,\omega)-\\&\int_0^{tN}\tau_x\left\{j(\widetilde{\alpha}^N_s(\alpha,\omega^+),\widetilde{\eta}^N_s(\alpha,\zeta,\omega,\omega^+))-v(\widetilde{\eta}^N_s(\alpha,\zeta,\omega,\omega^+))(\mathbf{1}_{\{v>0\}})\right\}ds\end{aligned}$$

is a mean 0 martingale under $\mathbb{P}\otimes\mathbb{P}^+$. Let

$$\begin{aligned}R^{\varepsilon,v}_{tN}&:=&&(Nt\,|\mathbb{Z}\cap[-\varepsilon N,\varepsilon N]|)^{-1}\sum_{x\in\mathbb{Z}:\,|x|\leq\varepsilon N}M^{x,v}_{tN}(\tau_{\lfloor bN\rfloor}\alpha,\varpi_{aN},\theta_{\lfloor bN\rfloor,aN}\omega,\omega^+)\\&=&&(Nt)^{-1}\psi^{v,\varepsilon}_{tN}(\tau_{\lfloor bN\rfloor}\alpha,\varpi_{aN},\theta_{\lfloor bN\rfloor,aN}\omega,\omega^+)\\&&&-\int[j(\alpha,\eta)-v\eta(\mathbf{1}_{\{v>0\}})]m_{tN,\varepsilon}(\theta'_{\lfloor bN\rfloor,aN}\omega',\omega^+)(d\alpha,d\eta)\end{aligned}\qquad(88)$$

where the last equality comes from (78), (87). The exponential martingale associated with $M^{x,v}_{tN}$ yields a Poissonian bound, uniform in (α,ζ), for the exponential moment of $M^{x,v}_{tN}$ with respect to $\mathbb{P}\otimes\mathbb{P}^+$. Since ϖ_{aN} is independent of $(\theta_{\lfloor bN\rfloor,aN}\omega,\omega^+)$ under $\overline{\nu}^{Q,\lambda,\rho}\otimes\mathbb{P}\otimes\mathbb{P}^+$, the bound is also valid under this measure, and Borel-Cantelli's lemma implies $\lim_{N\to\infty}R^{\varepsilon,v}_{tN}=0$. From (88), Lemma 5.3 and Corollary 3.1, *(B2)* imply (84), as well as

$$\limsup_{N\to\infty}(Nt)^{-1}\phi^v_{tN}\circ\theta'_{\lfloor bN\rfloor,aN}(\omega')\leq\sup_{r\in[\lambda,\rho]\cap\mathcal{R}^Q}[G^Q(r)-vr],\quad\overline{\nu}^{Q,\lambda,\rho}\otimes\mathbb{P}\text{-a.s.}\qquad(89)$$

Step two: proof of (85). Let $r\in[\lambda,\rho]\cap\mathcal{R}^Q$. We define $\overline{\nu}^{Q,\lambda,r,\rho}$ as the distribution of $(\alpha,\eta^\lambda,\eta^r,\eta^\rho)$. With respect to this measure, by (84) and (89), we have the almost sure limit

$$\lim_{N\to\infty}(Nt)^{-1}\phi^v_{tN}\circ\theta'_{\lfloor bN\rfloor,aN}(\alpha,\eta^r,\eta^r,\omega)=G^Q(r)-vr$$

By Lemma 5.2,

$$\phi^v_{tN} \circ \theta'_{\lfloor bN \rfloor, aN}(\omega') \leq \phi^v_{tN} \circ \theta'_{\lfloor bN \rfloor, aN}(\alpha, \eta^r, \eta^r, \omega)$$

The result follows by continuity of G^Q and minimizing over r. $\square$

5.2 Cauchy problem

Using (48) and the fact that an arbitrary function can be approximated by a $\mathcal{R}^Q$-valued function with respect to the distance Δ defined by (44), the proof of Theorem 3.1 for general initial data u_0 can be reduced (see [7]) to the case of $\mathcal{R}^Q$-valued initial data by coupling and approximation arguments (see [8, Section 4.2.2]).

Proposition 5.4 *([9, Proposition 4.3]). Assume (η_0^N) is a sequence of configurations such that: (i) there exists $C > 0$ such that for all $N \in \mathbb{N}$, η_0^N is supported on $\mathbb{Z} \cap [-CN, CN]$;*
(ii) $\pi^N(\eta_0^N) \to u_0(.)dx$ as $N \to \infty$, where u_0 has compact support, is a.e. $\mathcal{R}^Q$-valued and has finite space variation.
Let $u(.,t)$ denote the unique entropy solution to (21) with Cauchy datum $u_0(.)$. Then, $Q \otimes \mathbb{P}$-a.s. as $N \to \infty$,

$$\Delta^N(t) := \Delta(\pi^N(\eta^N_{Nt}(\alpha, \eta_0^N, \omega)), u(.,t)dx)$$

converges uniformly to 0 on $[0,T]$ for every $T > 0$.

Before proving this proposition, we state two crucial tools in their most complete form (see [38]), the macroscopic stability and the finite propagation property for the particle system. Macroscopic stability yields that the distance Δ defined in (44) is an "almost" nonincreasing functional for two coupled particle systems. It is thus a microscopic analogue of property (48) in Proposition 4.1. The finite propagation property is a microscopic analogue of Proposition 4.1, *iv)*. For the misanthropes process, it follows essentially from the finite mean assumption (M4) on page 10.

Proposition 5.5 *([8, Proposition 4.2] with [38, Theorem 2]) Assume $p(.)$ has a finite first moment and a positive mean. Then there exist constants $C > 0$ and $c > 0$, depending only on $b(.,.)$ and $p(.)$, such that the following holds.*

For every $N \in \mathbb{N}$, $(\eta_0, \xi_0) \in \mathbf{X}^2$ with $|\eta_0|+|\xi_0| := \sum_{x\in\mathbb{Z}}[\eta_0(x)+\xi_0(x)] < +\infty$, and every $\gamma > 0$, the event

$$\forall t > 0 : \ \Delta(\pi^N(\eta_t(\alpha, \eta_0, \omega)), \pi^N(\eta_t(\alpha, \xi_0, \omega))) \leq \Delta(\pi^N(\alpha, \eta_0), \pi^N(\alpha, \xi_0)) + \gamma \tag{90}$$

has $\mathbb{P}$-probability at least $1 - C(|\eta_0| + |\xi_0|)e^{-cN\gamma}$.

Lemma 5.5 *([38, Lemma 15]) Assume $p(.)$ has a finite third moment. There exist constant v, and function $A(.)$ (satisfying $\sum_n A(n) < \infty$), depending only on $b(.,.)$ and $p(.)$, such that the following holds. For any $x, y \in \mathbb{Z}$, any $(\eta_0, \xi_0) \in \mathbf{X}^2$, and any $0 < t < (y-x)/(2v)$: if η_0 and ξ_0 coincide on the site interval $[x, y]$, then with $\mathbb{P}$-probability at least $1 - A(t)$, $\eta_s(\alpha, \eta_0, \omega)$ and $\eta_s(\alpha, \xi_0, \omega)$ coincide on the site interval $[x+vt, y-vt] \cap \mathbb{Z}$ for every $s \in [0, t]$.*

To prove Proposition 5.5, we work with a coupled process, and reduce the problem to analysing the evolution of labeled positive and negative discrepancies to control their coalescences. For this we order the discrepancies, we possibly relabel them according to their movements to favor coalescences, and define *windows*, which are space intervals on which coalescences are favored, in the same spirit as in [17]. The irreducibility assumption (that is (8) in the case of the misanthropes process) plays an essential role there. To take advantage of [17], we treat separately the movements of discrepancies corresponding to big jumps, which can be controlled thanks to the finiteness of the first moment (that is, assumption *(M4)* on page 10).

Proof of proposition 5.4. By assumption (ii) of the proposition, $\lim_{N\to\infty} \Delta^N(0) = 0$. Let $\varepsilon > 0$, and $\varepsilon' = \varepsilon/(2V)$, for V given by (46). Set $t_k = k\varepsilon'$ for $k \leq \kappa := \lfloor T/\varepsilon' \rfloor$, $t_{\kappa+1} = T$. Since the number of steps is proportional to ε^{-1}, if we want to bound the total error, the main step is to prove

$$\limsup_{N\to\infty} \sup_{k=0,\ldots,\mathcal{K}-1} \left[\Delta^N(t_{k+1}) - \Delta^N(t_k)\right] \leq 3\delta\varepsilon, \quad Q \otimes \mathbb{P}\text{-a.s.} \tag{91}$$

where $\delta := \delta(\varepsilon)$ goes to 0 as ε goes to 0; the gaps between discrete times are filled by an estimate for the time modulus of continuity of $\Delta^N(t)$ (see [8, Lemma 4.5]).

Proof of (91). Since $u(., t_k)$ has locally finite variation, by [8, Lemma 4.2], for all $\varepsilon > 0$ we can find functions

$$v_k = \sum_{l=0}^{l_k} r_{k,l} \mathbf{1}_{[x_{k,l}, x_{k,l+1})} \tag{92}$$

with $-\infty = x_{k,0} < x_{k,1} < \ldots < x_{k,l_k} < x_{k,l_k+1} = +\infty$, $r_{k,l} \in \mathcal{R}^Q$, $r_{k,0} = r_{k,l_k} = 0$, such that $x_{k,l} - x_{k,l-1} \geq \varepsilon$, and

$$\Delta(u(.,t_k)dx, v_k dx) \leq \delta\varepsilon \tag{93}$$

For $t_k \leq t < t_{k+1}$, we denote by $v_k(.,t)$ the entropy solution to (21) at time t with Cauchy datum $v_k(.)$. The configuration $\xi^{N,k}$ defined on $(\Omega_{\mathbf{A}} \otimes \Omega, \mathcal{F}_{\mathbf{A}} \otimes \mathcal{F}, \mathbb{P}_{\mathbf{A}} \otimes \mathbb{P})$ (see Lemma 5.1) by

$$\xi^{N,k}(\omega_{\mathbf{A}}, \omega)(x) := \eta_{Nt_k}(\alpha(\omega_{\mathbf{A}}), \eta^{r_{k,l}}(\omega_{\mathbf{A}}), \omega)(x), \text{ if } \lfloor N x_{k,l} \rfloor \leq x < \lfloor N x_{k,l+1} \rfloor$$

is a microscopic version of $v_k(.)$, since by Proposition 5.1 with $\lambda = \rho = r^{k,l}$,

$$\lim_{N\to\infty} \pi^N(\xi^{N,k}(\omega_{\mathbf{A}}, \omega))(dx) = v_k(.)dx, \quad \mathbb{P}_{\mathbf{A}} \otimes \mathbb{P}\text{-a.s.} \tag{94}$$

We denote by $\xi_t^{N,k}(\omega_{\mathbf{A}}, \omega) = \eta_t(\alpha(\omega_{\mathbf{A}}), \xi^{N,k}(\omega_{\mathbf{A}}, \omega), \theta_{0,Nt_k}\omega)$ the evolved configuration starting from $\xi^{N,k}$. By triangle inequality,

$$\begin{aligned} \Delta^N(t_{k+1}) - \Delta^N(t_k) \quad &\leq \quad \Delta\left[\pi^N(\eta^N_{Nt_{k+1}}), \pi^N(\xi^{N,k}_{N\varepsilon'})\right] - \Delta^N(t_k) && (95)\\ &+ \quad \Delta\left[\pi^N(\xi^{N,k}_{N\varepsilon'}), v_k(.,\varepsilon')dx\right] && (96)\\ &+ \quad \Delta(v_k(.,\varepsilon')dx, u(.,t_{k+1})dx) && (97) \end{aligned}$$

To conclude, we rely on Properties (45), (90) and (48) of Δ: Since $\varepsilon' = \varepsilon/(2V)$, finite propagation property for (21) and for the particle system (see Proposition 4.1, *iv)* and Lemma 5.5) and Proposition 5.1 imply

$$\lim_{N\to\infty} \pi^N(\xi^{N,k}_{N\varepsilon'}(\omega_{\mathbf{A}}, \omega)) = v_k(.,\varepsilon')dx, \qquad \mathbb{P}_A \otimes \mathbb{P}\text{-a.s.}$$

Hence, the term (96) converges a.s. to 0 as $N \to \infty$. By Δ-stability for (21), the term (97) is bounded by $\Delta(v_k(.)dx, u(.,t_k)dx) \leq \delta\varepsilon$. We now consider the term (95). By macroscopic stability (Proposition 5.5), outside probability $e^{-CN\delta\varepsilon}$,

$$\Delta\left[\pi^N(\eta^N_{Nt_{k+1}}), \pi^N(\xi^{N,k}_{N\varepsilon'})\right] \leq \Delta\left[\pi^N(\eta^N_{Nt_k}), \pi^N(\xi^{N,k})\right] + \delta\varepsilon \tag{98}$$

Thus the event (98) holds a.s. for N large enough. By triangle inequality,

$$\begin{aligned} & \Delta\left[\pi^N(\eta^N_{Nt_k}),\pi^N(\xi^{N,k})\right]-\Delta^N(t_k) \\ \leq\ & \Delta\left(u(.,t_k)dx,v_k(.)dx\right)+\Delta\left[v_k(.)dx,\pi^N(\xi^{N,k})\right] \end{aligned}$$

for which (93), (94) yield as $N\to\infty$ an upper bound $2\delta\varepsilon$, hence $3\delta\varepsilon$ for the term (95). $\square$

6 Other models under a general framework

As announced in Section 3, we first define in Subsection 6.1, as in [5, 9, 47] a general framework (which encompasses all known examples), that we illustrate with our reference examples, the misanthropes process (with generator (4)), and the k-exclusion process (with generator (5)). Next in Subsection 6.2, we study examples of more complex models thanks to this new framework.

6.1 Framework

This section is based on parts of [9, Sections 2, 5]. The interest of an abstract description is to summarize all details of the microscopic dynamics in a single mapping, hereafter denoted by $\mathcal{T}$. This mapping contains both the generator description of the dynamics and its graphical construction. Once a given model is written in this framework, all proofs can be done without any model-specific computations, only relying on the properties of $\mathcal{T}$.

Monotone transformations. Given an environment $\alpha\in\mathbf{A}$, we are going to define a Markov generator whose associated dynamics can be generically understood as follows: we pick a location on the lattice and around this location, apply a random monotone transformation to the current configuration. Let $(\mathcal{V},\mathcal{F}_{\mathcal{V}},m)$ be a measure space, where m is a nonnegative finite measure. This space will be used to generate a monotone conservative transformation, that is a mapping $\mathcal{T}:\mathbf{X}\to\mathbf{X}$ such that:
(i) $\mathcal{T}$ is nondecreasing: that is, for every $\eta\in\mathbf{X}$ and $\xi\in\mathbf{X}$, $\eta\leq\xi$ implies $\mathcal{T}\eta\leq\mathcal{T}\xi$;
(ii) $\mathcal{T}$ acts on finitely many sites, that is, there exists a finite subset S of $\mathbb{Z}$ such that, for all $\eta\in\mathbf{X}$, $\mathcal{T}\eta$ only depends on the restriction of η to S, and

coincides with η outside S;
(iii) $\mathcal{T}\eta$ is conservative, that is, for every $\eta \in \mathbf{X}$,

$$\sum_{x\in S}\mathcal{T}\eta(x)=\sum_{x\in S}\eta(x) \tag{99}$$

We denote by $\mathfrak{T}$ the set of monotone conservative transformations, endowed with the σ-field $\mathcal{F}_{\mathfrak{T}}$ generated by the evaluation mappings $\mathcal{T}\mapsto\mathcal{T}\eta$ for all $\eta\in\mathbf{X}$.
Definition of the dynamics. In order to define the process, we specify a mapping

$$\mathbf{A}\times\mathcal{V}\to\mathfrak{T},\quad (\alpha,v)\mapsto\mathcal{T}^{\alpha,v}$$

such that for every $\alpha\in\mathbf{A}$ and $\eta\in\mathbf{X}$, the mapping $v\mapsto\mathcal{T}^{\alpha,v}\eta$ is measurable from $(\mathcal{V},\mathcal{F}_{\mathcal{V}},m)$ to $(\mathfrak{T},\mathcal{F}_{\mathfrak{T}})$. When m is a probability measure, this amounts to saying that for each $\alpha\in\mathbf{A}$, the mapping $v\mapsto\mathcal{T}^{\alpha,v}$ is a $\mathfrak{T}$-valued random variable. The transformation $\mathcal{T}^{\alpha,v}$ must be understood as applying a certain update rule around 0 to the current configuration, depending on the environment around 0. If $x\in\mathbb{Z}\setminus\{0\}$, we define

$$\mathcal{T}^{\alpha,x,v}:=\tau_{-x}T^{\tau_x\alpha,v}\tau_x \tag{100}$$

This definition can be understood as applying the same update rule around site x, which involves simultaneous shifts of the initial environment and transformation.

We now define the Markov generator

$$L_\alpha f(\eta)=\sum_{x\in\mathbb{Z}}\int_{\mathcal{V}}\left[f\left(\mathcal{T}^{\alpha,x,v}\eta\right)-f(\eta)\right]m(dv) \tag{101}$$

As a result of (100), the generator (101) satisfies the commutation property (20).

Basic examples. To illustrate the above framework, we come back to our reference examples of Section 2.2.

The misanthropes process. Let

$$\mathcal{V}:=\mathbb{Z}\times[0,1],\quad v=(z,u)\in\mathcal{V},\quad m(dv)=c^{-1}||b||_\infty p(dz)\lambda_{[0,1]}(du) \tag{102}$$

For $v=(z,u)\in\mathcal{V}$, $\mathcal{T}^{\alpha,v}$ is defined by

$$\mathcal{T}^{\alpha,v}\eta=\left\{\begin{array}{lll}\eta^{0,z} & \text{if} & u<\alpha(0)\dfrac{b(\eta(0),\eta(z))}{c^{-1}||b||_\infty}\\ \eta & & \text{otherwise}\end{array}\right. \tag{103}$$

Once given $\mathcal{T}^{\alpha,v}$ in (103), we deduce $\mathcal{T}^{\alpha,x,v}$ from (100):

$$\mathcal{T}^{\alpha,x,v}\eta=\left\{\begin{array}{lll}\eta^{x,x+z} & \text{if} & u<\alpha(x)\dfrac{b(\eta(x),\eta(x+z))}{c^{-1}||b||_\infty}\\ \eta & & \text{otherwise}\end{array}\right. \tag{104}$$

Though $\mathcal{T}^{\alpha,v}$ is the actual input of the model, from which $\mathcal{T}^{\alpha,x,v}$ follows, in the forthcoming examples, for the sake of readability, we will directly define $\mathcal{T}^{\alpha,x,v}$.

Monotonicity of the transformation $\mathcal{T}^{\alpha,x,z}$ given in (104) follows from assumption *(M2)* on page 10. One can deduce from (101) and (104) that L_α is indeed given by (4).

The k-step exclusion process. Here we let $\mathcal{V}=\mathbb{Z}^k$ and m denote the distribution of the first k steps of a random walk on $\mathbb{Z}$ with increment distribution $p(.)$ absorbed at 0. We define, for $(x,v,\eta)\in\mathbb{Z}\times\mathcal{V}\times\mathbf{X}$, with $v=(z_1,\ldots,z_k)$,

$$N(x,v,\eta)=\inf\{i\in\{1,\ldots,k\}:\,\eta(x+z_i)=0\}$$

with the convention that $\inf\emptyset=+\infty$. We then set

$$\mathcal{T}^{\alpha,x,v}\eta=\left\{\begin{array}{lll}\eta^{x,x+N(x,v,\eta)} & \text{if} & N(x,v,\eta)<+\infty\\ \eta & \text{if} & N(x,v,\eta)=+\infty\end{array}\right. \tag{105}$$

One can show that this transformation is monotone, either directly, or by application of Lemma 6.1, since the k-step exclusion is a particular k-step misanthropes process (see special case 2a below). Plugging (105) into (101) yields (5).

We now describe the so-called *graphical construction* of the system given by (101), that is its pathwise construction on a Poisson space. We consider the probability space $(\Omega,\mathcal{F},\mathbb{P})$ of locally finite point measures $\omega(dt,dx,dv)$

on $\mathbb{R}^+ \times \mathbb{Z} \times \mathcal{V}$, where $\mathcal{F}$ is generated by the mappings $\omega \mapsto \omega(S)$ for Borel sets S of $\mathbb{R}^+ \times \mathbb{Z} \times \mathcal{V}$, and $\mathbb{P}$ makes ω a Poisson process with intensity

$$M(dt, dx, dv) = \lambda_{\mathbb{R}^+}(dt)\lambda_{\mathbb{Z}}(dx)m(dv)$$

denoting by λ either the Lebesgue or the counting measure. We write $\mathbb{E}$ for expectation with respect to $\mathbb{P}$. There exists a unique mapping

$$(\alpha, \eta_0, t) \in \mathbf{A} \times \mathbf{X} \times \mathbb{R}^+ \mapsto \eta_t = \eta_t(\alpha, \eta_0, \omega) \in \mathbf{X} \tag{106}$$

satisfying: *(a)* $t \mapsto \eta_t(\alpha, \eta_0, \omega)$ is right-continuous; *(b)* $\eta_0(\alpha, \eta_0, \omega) = \eta_0$; *(c)* the particle configuration is updated at points $(t, x, v) \in \omega$ (and only at such points; by $(t, x, v) \in \omega$ we mean $\omega\{(t, x, v)\} = 1$) according to the rule

$$\eta_t(\alpha, \eta_0, \omega) = \mathcal{T}^{\alpha,x,v}\eta_{t^-}(\alpha, \eta_0, \omega) \tag{107}$$

The processes defined by (101) and (106)–(107) exist and are equal in law under general conditions given in [47], see also [11] for a summary of this construction).

Coupling and monotonicity. The monotonocity of $\mathcal{T}^{\alpha,x,u}$ implies monotone dependence (15) with respect to the initial state. Thus, an arbitrary number of processes can be coupled via the graphical construction. This implies complete monotonicity and thus attractiveness. It is also possible to define the coupling of any number of processes using the tranformation $\mathcal{T}$. For instance, in order to couple two processes, we define the coupled generator $\overline{L}_\alpha$ on $\mathbf{X}^2$ by

$$\overline{L}_\alpha f(\eta, \xi) := \sum_{x \in \mathbb{Z}} \int_{\mathcal{V}} \left[f\left(\mathcal{T}^{\alpha,x,v}\eta, \mathcal{T}^{\alpha,x,v}\xi\right) - f(\eta, \xi)\right] m(dv) \tag{108}$$

for any local function f on $\mathbf{X}^2$.

6.2 Examples

We refer the reader to [9, Section 5] for various examples of completely monotone models defined using this framework. We now review two of the models introduced in [9, Section 5], then present a new model containing all the other models in this paper, *the k-step misanthropes process.*

The generalized misanthropes process. ([9, Section 5.1]). Let $K \in \mathbb{N}$. Let $c \in (0,1)$, and $p(.)$ (resp. $P(.)$), be a probability distribution on $\mathbb{Z}$. Define $\mathbf{A}$ to be the set of functions $B : \mathbb{Z}^2 \times \{0, \ldots, K\}^2 \to \mathbb{R}^+$ such that:

(GM1) For all $(x,z) \in \mathbb{Z}^2$, $B(x,z,.,.)$ satisfies assumptions *(M1)–(M3)* on page 10;
(GM2) There exists a constant $C > 0$ and a probability measure $P(.)$ on $\mathbb{Z}$ such that $B(x,z,K,1) \le CP(z)$ for all $x \in \mathbb{Z}$.

Assumption *(GM2)* is a natural sufficient assumption for the existence of the process and graphical construction below. The shift operator τ_y on $\mathbf{A}$ is defined by $(\tau_y B)(x,z,n,m) = B(x+y,z,n,m)$ We generalize (4) by setting

$$L_B f(\eta) = \sum_{x,y \in \mathbb{Z}} B(x, y-x, \eta(x), \eta(y)) \left[f\left(\eta^{x,y}\right) - f(\eta) \right] \tag{109}$$

Thus, the environment at site x is given here by the jump rate function $B(x,.,.)$ with which jump rates from site x are computed.

For $v = (z,u)$, set $m(dv) = CP(dz)\lambda_{[0,1]}(du)$ in (102), and replace (103) with

$$\mathcal{T}^{B,x,v}\eta = \begin{cases} \eta^{x,x+z} & \text{if} \quad u < \dfrac{B(x,z,\eta(x),\eta(x+z))}{CP(z)} \\ \eta & \qquad \text{otherwise} \end{cases} \tag{110}$$

A natural irreducibility assumption generalizing (8) is the existence of a constant $c > 0$ and a probability measure $p(.)$ on $\mathbb{Z}$ satisfying (8), such that

$$\forall z \in \mathbb{Z}, \quad \inf_{x \in \mathbb{Z}} b(x,z,1,K-1) \ge cp(z) \tag{111}$$

The basic model (4) is recovered for $B(x,z,n,m) = \alpha(x)p(z)b(n,m)$. Another natural example is the misanthropes process with bond disorder. Here $\mathbf{A} = [c, 1/c]^{\mathbb{Z}^2}$ with the space shift defined by $\tau_z\alpha = \alpha(.+z,.+z)$. We set $B(x,z,n,m) = \alpha(x,x+z)b(n,m)$, where $\alpha \in \mathbf{A}$. Assumption *(GM2)* is now equivalent to existence of a constant $C > 0$ and a probability measure $P(.)$ on $\mathbb{Z}$ such that $\alpha(x,y) \le CP(y-x)$.

The microscopic flux function j_2 in (26) is given here by

$$j_2(\alpha, \eta) = \sum_{z \in \mathbb{Z}} zB(0,z,\eta(0),\eta(z)) \tag{112}$$

that is well defined under the assumption that $P(.)$ has a finite first moment.

Asymmetric exclusion process with overtaking. This example is a particular case of the generalized k-step K-exclusion studied in [9, Section 5.2, Example 5.4], see also the traffic flow model in [9, Section 5.3]. The former model is itself a special case of the k-step misanthropes process defined below.

Let $K = 1$, $k \in \mathbb{N}$, and $\mathfrak{K}$ denote the set of $(2k)$-tuples $(\beta^j)_{j\in\{-k,\dots,k\}\setminus\{0\}}$ such that $\beta^{j+1} \leq \beta^j$ for every $j = 1,\dots,k-1$, $\beta^{j-1} \leq \beta^j$ for every $j = -1,\dots,-k+1$, and $\beta^1 + \beta^{-1} > 0$. We define $\mathbf{A} = \mathfrak{K}^{\mathbb{Z}}$. An element of $\mathbf{A}$ is denoted by $\beta = (\beta^j_x)_{j\in\{-k,\dots,k\},x\in\mathbb{Z}}$. The dynamics of this model is defined informally as follows. A site $x \in \mathbb{Z}$ is chosen as the initial site, then a jump direction (right or left) is chosen, and in this direction, the particle jumps to the first available site if it is no more than k sites ahead. The jump occurs at rate β^j_x if the first available site is $x + j$. Let $\mathcal{V} = [0,1] \times \{-1,1\}$ and $m = \delta_1 + \delta_{-1}$. For $x \in \mathbb{Z}$ and $v \in \{-1,1\}$, we set

$$N(x,v,\eta) := \inf\{i \in \{1,\dots,k\} : \eta(x+iv) = 0\}$$

with the usual convention $\inf\emptyset = +\infty$. The corresponding monotone transformation is defined for $(u,v) \in \mathcal{V}$ by

$$\mathcal{T}^{\alpha,x,v}\eta = \begin{cases} \eta^{x,x+N(x,v,\eta)v} & \text{if} \quad N(x,v,\eta) < +\infty \text{ and } u \leq \beta_x^{vN(x,v,\eta)} \\ \eta & \text{otherwise} \end{cases} \tag{113}$$

Monotonicity of the transformation $\mathcal{T}^{\alpha,x,z}$ is given by [9, Lemma 5.1], and is also a particular case of Lemma 6.1 below, which states the same property for the k-step misanthropes process. It follows from (101) and (113) that the generator of this process is given for $\beta \in \mathbf{A}$ by

$$\begin{aligned} L_\beta f(\eta) &= \sum_{x\in\mathbb{Z}} \eta(x) \sum_{j=1}^{k} \left\{ \beta^j_x [1-\eta(x+j)] \prod_{i=1}^{i=j-1} \eta(x+i) \right. \\ &\quad + \left. \beta^{-j}_x [1-\eta(x-j)] \prod_{i=1}^{i=j-1} \eta(x-i) \right\} \end{aligned} \tag{114}$$

A sufficient irreducibility property replacing (8) is the existence of a constant $c > 0$ such that

$$\inf_{x\in\mathbb{Z}} \left(\beta^1_x + \beta^{-1}_x\right) > 0 \tag{115}$$

The microscopic flux function j_2 in (26) is given here by

$$\begin{aligned} j_2(\beta,\eta) &= \eta(0)\sum_{j=1}^{k} j\beta_0^j[1-\eta(j)]\prod_{i=1}^{j-1}\eta(i) \\ &- \eta(0)\sum_{j=1}^{k} j\beta_0^{-j}[1-\eta(j)]\prod_{i=1}^{j-1}\eta(i) \end{aligned} \tag{116}$$

with the convention that an empty product is equal to 1. For $\rho \in [0,1]$, let $\mathcal{B}_\rho$ denote the Bernoulli distribution on $\{0,1\}$. In the absence of disorder, that is when β_x^i does not depend on x, the measure ν_ρ defined by

$$\nu_\rho(d\eta) = \bigotimes_{x\in\mathbb{Z}} \mathcal{B}_\rho[d\eta(x)]$$

is invariant for this process. It follows from (116) that the macroscopic flux function for the model without disorder is given by

$$G(u) = (1-u)\sum_{j=1}^{k} j[\beta^j - \beta^{-j}]u^j \tag{117}$$

The k-step misanthropes process. In the sequel, an element of $\mathbb{Z}^k$ is denoted by $\underline{z} = (z_1,\dots,z_k)$. Let $K \geq 1$, $k \geq 1$, $c \in (0,1)$.

Define $\mathcal{D}_0$ to be the set of functions $b : \{0,\dots,K\}^2 \to \mathbb{R}^+$ such that $b(0,.) = b(.,K) = 0$, $b(n,m) > 0$ for $n > 0$ and $m < K$, and b is nondecreasing (resp. nonincreasing) w.r.t. its first (resp. second) argument. Let $\mathcal{D}$ denote the set of functions $b = (b^1,\dots,b^k)$ from $\mathbb{Z}^k \times \{0,\dots,K\}^2 \to (\mathbb{R}^+)^k$ such that $b^j(\underline{z},.,.) \in \mathcal{D}_0$ for each $j = 1,\dots,k$, and

$$\forall j = 2,\dots,k, \quad b^j(.,K,0) \leq b^{j-1}(.,1,K-1) \tag{118}$$

Let q be a probability distribution on $\mathbb{Z}^k$, and $b \in \mathcal{D}$. We define the (q,b) k-step misanthrope process as follows. A particle at x (if some) picks a q-distributed random vector $\underline{Z} = (Z_1,\dots,Z_k)$, and jumps to the first site $x + Z_i$ ($i \in \{1,\dots,k\}$) with strictly less than K particles along the path $(x+Z_1,\dots,x+Z_k)$, if such a site exists, with rate $b^i(\underline{Z},\eta(x),\eta(x+Z_i))$. Otherwise, it stays at x.

Next, disorder is introduced: the environment is a field $\alpha = ((q_x, b_x) : x \in \mathbb{Z}) \in \mathbf{A} := (\mathcal{P}(\mathbb{Z}^k) \times \mathcal{D})^{\mathbb{Z}}$. For a given realization of the environment, the distribution of the path $\underline{Z}$ picked by a particle at x is q_x, and the rate at which it jumps to $x + Z_i$ is $b_x^i(\underline{Z}, \eta(x), \eta(x + Z_i))$. The corresponding generator is given by

$$L_\alpha f(\eta) = \sum_{i=1}^{k} \sum_{x,y\in\mathbb{Z}} c_\alpha^i(x, y, \eta) \left[f(\eta^{x,y}) - f(\eta)\right] \tag{119}$$

for a local function f on $\mathbf{X}$, where (with the convention that an empty product is equal to 1)

$$c_\alpha^i(x, y, \eta) = \int \left[b_x^i(\underline{z}, \eta(x), \eta(y)) \mathbf{1}_{\{x+z_i=y\}} \prod_{j=1}^{i-1} \mathbf{1}_{\{\eta(x+z_j)=K\}} \right] dq_x(\underline{z})$$

The distribution Q of the environment on $\mathbf{A}$ is assumed ergodic with respect to the space shift τ_y, where $\tau_y\alpha = ((q_{x+y}, b_{x+y}) : x \in \mathbb{Z})$.

A sufficient condition for the existence of the process and graphical construction below is the existence of a probability measure $P(.)$ on $\mathbb{Z}$ and a constant $C > 0$ such that

$$\sup_{i=1,\ldots,k} \sup_{x\in\mathbb{Z}} q_x^i(.) \leq C^{-1} P(.) \tag{120}$$

where q_x^i denotes the i-th marginal of q_x. On the other hand, a natural irreducibility assumption sufficient for Proposition 3.1 and Theorem 3.1, is the existence of a constant $c > 0$, and a probability measure $p(.)$ on $\mathbb{Z}$ satisfying (120), such that

$$\inf_{x\in\mathbb{Z}} q_x^1(.) \geq cp(.) \tag{121}$$

To define a graphical construction, we set, for $(x, \underline{z}, \eta) \in \mathbb{Z} \times \mathbb{Z}^k \times \mathbf{X}$, $b \in \mathcal{D}_0$ and $u \in [0, 1]$,

$$N(x, \underline{z}, \eta) = \inf \{i \in \{1, \ldots, k\} : \eta(x + z_i) < K\} \text{ with} \inf \emptyset = +\infty \tag{122}$$

$$Y(x, \underline{z}, \eta) = \begin{cases} x + z_{N(x,\underline{z},\eta)} & \text{if } N(x, \underline{z}, \eta) < +\infty \\ x & \text{if } N(x, \underline{z}, \eta) = +\infty \end{cases} \tag{123}$$

$$\mathcal{T}_0^{x,\underline{z},b,u}\eta = \begin{cases} \eta^{x,Y(x,\underline{z},\eta)} & \text{if } u < b^{N(x,\underline{z},\eta)}(\underline{z}, \eta(x), \eta(Y(x, \underline{z}, \eta))) \\ \eta & \text{otherwise} \end{cases} \tag{124}$$

Let $\mathcal{V}=[0,1]\times[0,1]$, $m=\lambda_{[0,1]}\otimes\lambda_{[0,1]}$. For each probability distribution q on $\mathbb{Z}^k$, there exists a mapping $F_q:[0,1]\to\mathbb{Z}^k$ such that $F_q(V_1)$ has distribution q if V_1 is uniformly distributed on $[0,1]$. Then the transformation $\mathcal{T}$ in (107) is defined by (with $v=(v_1,v_2)$ and $\alpha=((q_x,\beta_x):\,x\in\mathbb{Z})$)

$$\mathcal{T}^{\alpha,x,v}\eta=\mathcal{T}_0^{x,F_{q_x}(v_1),b_x(F_{q_x}(v_1),.,.),v_2}\eta \tag{125}$$

The definition of j_2 in (26), applied to the generator (119), yields

$$j_2(\alpha,\eta)=\sum_{z\in\mathbb{Z}}zc_\alpha(0,z,\eta) \tag{126}$$

where

$$c_\alpha(x,y,\eta):=\sum_{i=1}^k c_\alpha^i(x,y,\eta)$$

Special cases.

1. The generalized misanthropes process is recovered for $k=1$, because then in (119) we have $c^1_\alpha(x,y,\eta)=q^1_x(y-x)b^1_x(y-x,\eta(x),\eta(y))$.

2. A *generalized disordered* k-step exclusion process is obtained if $K=1$ and $b^j_x(\underline{z},n,m)=\beta^j_x(\underline{z})n(1-m)$. In this process, if site x is the initial location of an attempted jump, and a particle is indeed present at x, a random path of length k with distribution q_x is picked, and the particle tries to find an empty location along this path. If it finds none, then it stays at x. Previous versions of the k-step exclusion process are recovered if one makes special choices for the distribution q_x:

2a. The usual k-step exclusion process with site disorder, whose generator was given by (5), corresponds to the case where q_x is the distribution of the first k steps of a random walk with kernel $p(.)$ absorbed at 0, and $\beta^j_x=\alpha(x)$.

2b. The exclusion process with overtaking, whose generator was given by (114), corresponds to the case where the random path is chosen as follows: first, one picks with equal probability a jumping direction (left or right); next, one moves in this direction by successive deterministic jumps of size 1.

3. For $K \geq 2$, the generalized k-step K-exclusion process ([9, Subsection 5.2]) corresponds to $b^j_x(\underline{z}, n, m) = \beta^j_x(\underline{z})\mathbf{1}_{\{n>0\}}\mathbf{1}_{\{m<K\}}$.

Returning to the general case, condition (118) is the relevant extension of the condition $\beta^j_x(\underline{z}) \leq \beta^{j-1}_x(\underline{z})$ in the exclusion process with overtaking. If $K \geq 2$, it means that *any* possible j-step jump has rate larger or equal than *any* $(j-1)$-step jump.

The monotonicity property (15) of the graphical construction, and thus the complete monotonicity of the process, is a consequence of the following lemma.

Lemma 6.1 *For every $(x, \underline{z}, u) \in \mathbb{Z} \times \mathbb{Z}^k \times [0, 1]$ and $b \in \mathcal{D}_0$, $\mathcal{T}_0^{x,\underline{z},b,u}$ is an increasing mapping from $\mathbf{X}$ to $\mathbf{X}$.*

Proof of lemma 6.1. Let $(\eta, \xi) \in \mathbf{X}^2$ with $\eta \leq \xi$. To prove that $\mathcal{T}_0^{x,\underline{z},b,u}\eta \leq \mathcal{T}_0^{x,\underline{z},b,u}\xi$, since η and ξ can only possibly change at sites x, $y := Y(x, \underline{z}, \eta)$ and $y' := Y(x, \underline{z}, \xi)$, it is sufficient to verify the inequality at these sites.

If $\xi(x) = 0$, then by (124), η and ξ are both unchanged by $\mathcal{T}_0^{x,\underline{z},b,u}$. If $\eta(x) = 0 < \xi(x)$, then $\mathcal{T}_0^{x,\underline{z},b,u}\xi(y') \geq \xi(y') \geq \eta(y') = \mathcal{T}_0^{x,\underline{z},b,u}\eta(y')$.

Now assume $\eta(x) > 0$. Then $\eta \leq \xi$ implies $N(x, \underline{z}, \eta) \leq N(x, \underline{z}, \xi)$. If $N(x, \underline{z}, \eta) = +\infty$, η and ξ are unchanged. If $N(x, \underline{z}, \eta) < N(x, \underline{z}, \xi) = +\infty$, then $\mathcal{T}_0^{x,\underline{z},b,u}\eta = \eta^{x,y}$, and $\xi(y) = K$. Thus, $\mathcal{T}_0^{x,\underline{z},b,u}\eta(x) = \eta(x) - 1 \leq \xi(x) = \mathcal{T}_0^{x,\underline{z},b,u}\xi(x)$, and $\mathcal{T}_0^{x,\underline{z},b,u}\xi(y) = \xi(y) = K \geq \mathcal{T}_0^{x,\underline{z},b,u}\eta(y)$.

In the sequel, we assume $N(x, \underline{z}, \eta)$ and $N(x, \underline{z}, \xi)$ both finite. Let $\beta := b^{N(x,\underline{z},\eta)}(\eta(x), \eta(y))$ and $\beta' := b^{N(x,\underline{z},\xi)}(\xi(x), \xi(y'))$.

1) Assume $N(x, \underline{z}, \eta) = N(x, \underline{z}, \xi) < +\infty$, then $y = y'$. If $u \geq \max(\beta, \beta')$, both η and ξ are unchanged. If $u < \min(\beta, \beta')$, $\mathcal{T}_0^{x,\underline{z},b,u}\eta = \eta^{x,y}$ and $\mathcal{T}_0^{x,\underline{z},b,u}\xi = \xi^{x,y}$, whence the conclusion. We are left to examine different cases where $\min(\beta, \beta') \leq u < \max(\beta, \beta')$.

a) If $\eta(x) = \xi(x)$, then $\beta' \leq \beta$, and $\beta' \leq u < \beta$ implies $\eta(y) < \xi(y)$. In this case, $\mathcal{T}_0^{x,\underline{z},b,u}\xi(x) = \xi(x) \geq \eta(x) > \mathcal{T}_0^{x,\underline{z},b,u}\eta(x)$ and $\mathcal{T}_0^{x,\underline{z},b,u}\xi(y) = \xi(y) = K \geq \mathcal{T}_0^{x,\underline{z},b,u}\eta(y)$.

b) If $\eta(x) < \xi(x)$, then $\mathcal{T}_0^{x,\underline{z},b,u}\xi(x) \geq \xi(x) - 1 \geq \eta(x) \geq \mathcal{T}_0^{x,\underline{z},b,u}\eta(x)$. If $\beta \leq u < \beta'$, then $\mathcal{T}_0^{x,\underline{z},b,u}\xi(y) = \xi(y) + 1 > \eta(y) = \mathcal{T}_0^{x,\underline{z},b,u}\eta(y)$. If $\beta' \leq u < \beta$, then $\eta(y) < \xi(y)$ and $\mathcal{T}_0^{x,\underline{z},b,u}\xi(y) = \xi(y) \geq \eta(y) + 1 = \mathcal{T}_0^{x,\underline{z},b,u}\eta(y)$.

2) Assume $N(x, \underline{z}, \eta) < N(x, \underline{z}, \xi) < +\infty$, hence $\beta \geq \beta'$ by (118), and $\eta(y) < \xi(y) = K$. If $u \geq \beta$, η and ξ are unchanged. If $u < \beta'$, then $\mathcal{T}_0^{x,\underline{z},b,u}\eta(y) = \eta(y) + 1 \leq \xi(y) = \mathcal{T}_0^{x,\underline{z},b,u}\xi(y)$, and $\mathcal{T}_0^{x,\underline{z},b,u}\xi(y') = \xi(y') + 1 \geq \mathcal{T}_0^{x,\underline{z},b,u}\eta(y') = \eta(y')$. If $\beta' \leq u < \beta$, then $\mathcal{T}_0^{x,\underline{z},\beta,u}\eta(x) = \eta(x) - 1 \leq \mathcal{T}_0^{x,\underline{z},b,u}\xi(x)$ and $\mathcal{T}_0^{x,\underline{z},b,u}\eta(y) = \eta(y) + 1 \leq \mathcal{T}_0^{x,\underline{z},b,u}\xi(y) = \xi(y) = K$. $\square$

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