Robust combinatorial optimization with knapsack uncertainty

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Abstract

We study in this paper min max robust combinatorial optimization problems for an uncertainty polytope that is defined by knapsack constraints, either in the space of the optimization variables or in an extended space. We provide exact and approximation algorithms that extend the iterative algorithms proposed by Bertismas and Sim (2003). We also study the limitation of the approach and point out $\mathcal{NP}$-hard situations. Then, we approximate axis-parallel ellipsoids with knapsack constraints and provide an approximation scheme for the corresponding robust problem. The approximation scheme is also adapted to handle the intersection of an axis-parallel ellipsoid and a box.

Keywords: robust optimization, combinatorial optimization, approximation algorithms, ellipsoidal uncertainty

1. Introduction

Robust optimization pioneered by [1] has become a key framework to address the uncertainty that arises in optimization problems. Stated simply, robust optimization characterizes the uncertainty over unknown parameters by providing a set that contains the possible values for the the uncertain parameters and considers the worst-case over the set. The popularity of robust optimization is largely due to its tractability for uncertainty handling, since linear robust op-

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timization problems are essentially as easy as their deterministic counterparts for many types of convex uncertainty sets [1], contrasting with the well-known difficulty of stochastic optimization approaches. In addition, robust optimization offers conservative approximation to stochastic programs with probabilistic constraints by choosing appropriate uncertainty sets [2, 3, 4].

The picture is more complex when it comes to robust combinatorial optimization problems. Let N denote a set of indices, with $|N| = n$, and $\mathcal{X} \subset \{0,1\}^n$ be the feasibility set of a combinatorial optimization problem, denoted CO. Given a bounded uncertainty set $\mathcal{U} \subset \mathbb{R}^n_+$, we consider in this paper the min max robust counterpart of CO, defined as

$$CO(\mathcal{U}) \qquad \min_{x\in\mathcal{X}} \max_{\xi\in\mathcal{U}} \xi^T x. \tag{1}$$

It is well known (e.g. [5, 6]) that a general uncertainty set $\mathcal{U}$ leads to a problem $CO(\mathcal{U})$ that is, more often than not, harder than the deterministic problem CO. This is the case, for instance, when $\mathcal{U}$ is an arbitrary ellipsoid [7] or a set of two arbitrary scenarios [6]. Robust combinatorial optimization witnessed a breakthrough with the introduction of budgeted uncertainty in [8], which keeps the tractability of the deterministic counterpart for a large class of combinatorial optimization problems. Specifically, Bertsimas and Sim considered uncertain cost functions characterized by the vector $c \in \mathbb{R}^n$ of nominal costs and the vector $d \in \mathbb{R}^n_+$ of deviations. Then, given a budget of uncertainty $\Gamma > 0$, they addressed

$$CO_{\boldsymbol{d}}(\mathcal{U}_\Gamma) \qquad \min_{x\in\mathcal{X}} \max_{\xi\in\mathcal{U}_\Gamma} \sum_{i\in N}(c_i + \xi_i d_i)x_i = \min_{x\in\mathcal{X}} \left(\sum_{i\in N} c_i x_i + \max_{\xi\in\mathcal{U}_\Gamma} \sum_{i\in N} \xi_i d_i x_i \right),$$

for the budgeted uncertainty set $\mathcal{U}_\Gamma := \left\{\xi : \sum_{i\in N} \xi_i \leq \Gamma, 0 \leq \xi_i \leq 1, i \in N\right\}$. Bertismas and Sim [8] proved two fundamental results:

Theorem 1 ([8])**.** *Problem $CO_{\boldsymbol{d}}(\mathcal{U}_\Gamma)$ can be solved by solving $n+1$ problems CO with modified costs.*

Theorem 2 ([8])**.** *If CO admits a polynomial-time $(1+\epsilon)$-approximation algorithm running in $O(f(n,\epsilon))$, then $CO_{\boldsymbol{d}}(\mathcal{U}_\Gamma)$ admits a polynomial-time $(1+\epsilon)$-approximation algorithm running in $O(nf(n,\epsilon))$.*

These positive complexity results have been extended up to some extent to optimization problem with integer variables (i.e. $\mathcal{X} \subseteq \mathbb{Z}^n$) and constraints uncertainty in [9, 10].

Another popular uncertainty model involves ellipsoids, and more particularly, axis-parallel ellipsoids, which we represent here trough the robust counterpart

$$CO_{\boldsymbol{d}}(\mathcal{U}_{ball}) \qquad \min_{x\in\mathcal{X}}\left(\sum_{i\in N} c_i x_i + \max_{\xi\in\mathcal{U}_{ball}} \sum_{i\in N} \xi_i d_i x_i\right),$$

where c now represents the center of the ellipsoid, d gives the length of its axes, and $\mathcal{U}_{ball}$ is a ball of radius Ω, $\mathcal{U}_{ball} := \{\xi : \|\xi\|_2 \leq \Omega\}$. Nikolova [11] proposes a counterpart of Theorem 2 for $CO_{\boldsymbol{d}}(\mathcal{U}_{ball})$ with a running time slightly worse than $O(\frac{1}{\epsilon^2} f(n,\epsilon))$. Her approach considers the problem as a two-objective optimization problem and approximates its pareto front. Other authors have addressed problem $CO_{\boldsymbol{d}}(\mathcal{U}_{ball})$, including Mokarami and Hashemi [12] who showed how the problem can be solved exactly by solving a pseudo-polynomial number of problems CO and [13, 14] who provide polynomial special cases. A drawback of $CO_{\boldsymbol{d}}(\mathcal{U}_{ball})$ from the practical viewpoint is that $\mathcal{U}_{ball}$ contains vectors with high individual values. For that reason, a popular variation considers instead the uncertainty set defined as the intersection of a ball and a box, formally defined as $\mathcal{U}_{ball}^{box} := \left\{\xi : \|\xi\|_2 \leq \Omega, -\underline{\xi} \leq \xi \leq \overline{\xi}\right\}$, for some $\underline{\xi}, \overline{\xi} \in \mathbb{R}^n_+$. While $\mathcal{U}_{ball}^{box}$ has been used in numerous papers dealing with robust optimization problems (e.g. [15, 16]), we are not aware of previous complexity results for $CO_{\boldsymbol{d}}(\mathcal{U}_{ball}^{box})$.

The main focus of this paper is to study robust combinatorial optimization problem for uncertainty polytopes defined by bounds restrictions and $s = |S|$ knapsack constraints, specifically

$$\mathcal{U}_{knap} := \left\{\xi \in \mathbb{R}^n : \sum_{i\in N} a_{ji}\xi_i \leq b_j, j \in S, 0 \leq \xi \leq \overline{\xi}\right\}, \tag{2}$$

where $a \in \mathbb{R}_+^{s\times n}$, $b \in \mathbb{R}^s_+$, and $\overline{\xi} \in \mathbb{R}^n_+$. Our definition $\mathcal{U}_{knap}$ is slightly more general than the multidimensional knapsack-contrained uncertainty set introduced in [17, 18] since we consider non-negative values for the constraint coefficients while [17, 18] assumes all of them equal to 1. The author of [17, 18] motivates the introduction of these complex polytopes in the context of multistage decision problems, where one wishes to correlate the value of uncertain parameters of a given period to those related to the precedent periods.

We relate next $\mathcal{U}_{knap}$ with the uncertainty polytopes that have been used in the literature for specific applications. Whenever $s = 1$, the resulting family of polytopes generalizes the uncertainty set $\left\{\xi \in \mathbb{R}^n : \sum_{i \in N} \xi_i \leq b, 0 \leq \xi \leq \overline{\xi}\right\}$ that has been used in scheduling [19]. For instance, our results imply that minimizing the sum of completion times is polynomial under less restrictive assumptions that those proposed in [19]. An application of $\mathcal{U}_{knap}$ with $s = 4$ arises for the vehicle routing problem with demand uncertainty for which the authors of [20, 21] partition the set of clients into four sets $N_1, \ldots, N_4$, according to the four geographic quadrants based on the coordinates in the benchmark problems, and limit the demand deviations for each of these four quadrants by b_j, formally, $\{\xi \in \mathbb{R}^n : \sum_{i \in N_j} \xi_i \leq b_j, j \in \{1, \ldots, 4\}, 0 \leq \xi \leq \overline{\xi}\}$. Hence, our result suggest solution algorithms based on solving a polynomial sequence of deterministic problems. Knapsack uncertainty sets with larger values of s have also been used in telecommunications network with demand uncertainty, under the name of Hose model (traced back to [22]). Given an undirected graph $G = (V, E)$ and a set of demands $D \subseteq V \times V$, the undirected Hose model is the polytope $\left\{\xi \in \mathbb{R}^{|D|} : \sum_{l:\{k,l\} \in D} \xi_{kl} \leq b_k, \text{ for each } k \in V \text{ s.t. } \exists \{k, l\} \in D\right\}$, where b_l is the bandwidth capacity of the terminal node $l \in V$. Therefore, when $|D|$ is small, our results could also be applied to robust network design with single path routing [23].

This paper is also dedicated to extensions of $\mathcal{U}_{knap}$ to (i) knapsack uncertainty polytopes described in an extended space rather than in the space of the optimization variables $\mathbb{R}^n$ and to (ii) decision-depend uncertainty. The first model can be used to describe a variant of $\mathcal{U}_\Gamma$ based on multiple intervals, which is a relevant approach when there is enough data to build a histogram of the values taken by the uncertain parameters [24]. We also show in the paper how knapsack uncertainty polytopes defined in extended spaces can be used to approximate axis-parallel ellipsoids and the intersection between an axis-parallel ellipsoid and a box, which have been used in the aforementioned applications. The second model describes uncertainty regions that are adjusted according to the values taken by the optimization variables [4, 25]. When the uncertainty is motivated by the probabilistic results from [3], it becomes natural to constraint the uncertainty with few knapsack constraints [4], which can be solved efficiently using the results of this paper.

Our contributions. In Section 2, we extend Theorems 1 and 2 to $\mathcal{U}_{knap}$, showing that similar positive complexity results hold when s is constant but that the robust problems are in general $\mathcal{NP}$-hard when s is part of the input. Section 3 considers uncertainty sets defined by s knapsack constraints in an extended space of dimension ℓ with $\ell > n$, yielding uncertainty set $\mathcal{U}_{ext}$. These models replace the cost function $g_i(\xi) = \xi_i d_i$ used in $CO_{\boldsymbol{d}}(\mathcal{U}_\Gamma)$ by a linear function $h : \mathbb{R}^\ell_+ \to \mathbb{R}^n_+$ that satisfies a technical assumption. We propose counterparts of Theorems 1 and 2 for the optimization problems built from $\mathcal{U}_{ext}$ and h and show that the problems are $\mathcal{NP}$-hard in general when the technical assumption is relaxed, even when s is constant. Section 4 considers problem $CO_{\boldsymbol{d}}(\mathcal{U}_{ball})$ and proposes an approximation scheme based on a piece-wise approximation of the quadratic function $\sum_{i=1}^n \xi_i^2$. Specifically, we propose a counterpart of Theorem 2 with running time $O(\frac{n^2}{\epsilon} f(n,\epsilon))$. The approach is finally extended in Section 5 to uncertainty set $\mathcal{U}_{ball}^{box}$, providing an approximation scheme for problem $CO_{\boldsymbol{d}}(\mathcal{U}_{ball}^{box})$.

2. Knapsack uncertainty

We provide in this section counterparts of Theorems 1 and 2 for $\mathcal{U}_{knap}$, which is defined by bounds restrictions and $s = |S|$ knapsack constraints. Specifically, we consider in this section the polytope Before stating our result, we introduce the following notation: let $A\theta = d$ be the matrix notation for the linear system of n equations given by $\sum_{j\in S} a_{ji}\theta_j = d_i, i \in N$, and define the larger system

$$A'\theta = d' \tag{3}$$

with $A' = \begin{pmatrix} A \\ Id \end{pmatrix}$ and $d' = \begin{pmatrix} d \\ \mathbf{0} \end{pmatrix}$, where Id is the $s \times s$ identity matrix and $\mathbf{0}$ is the vector of dimension s with all zeros. Further, consider the set $\Theta_{\mathbf{1}} \subseteq \mathbb{R}^s$ that is defined as follows: each element $\theta \in \Theta_{\mathbf{1}}$ is the unique solution of a subsystem of $A'\theta = d'$ formed by s linearly independent rows of A'.

Theorem 3. *Problem $CO_{\boldsymbol{d}}(\mathcal{U}_{knap})$ can be solved by solving $O(s^s n^s)$ linear systems with s variables and s equations, and $O(s^s n^s)$ nominal problems*

$$Z^* = \min_{\theta \in \Theta_{\mathbf{1}}} G^\theta,$$

where for each $\theta \in \Theta_{\mathbf{1}}$:

$$G^\theta = \sum_{j \in S} b_j \theta_j + \min_{x \in \mathcal{X}} \left\{ \sum_{i \in N} c_i x_i + \sum_{i \in N} x_i \overline{\xi}_i \max(0, d_i - \sum_{j \in S} a_{ji} \theta_j) \right\}.$$

Proof. Problem $CO_{\boldsymbol{d}}(\mathcal{U}_{knap})$ can be written as

$$\min_{x \in \mathcal{X}} \left(\sum_{i \in N} c_i x_i + \max \left\{ \sum_{i \in N} d_i \xi_i x_i : \sum_{i \in N} a_{ji} \xi_i \le b_j, j \in S, 0 \le \xi_i \le \overline{\xi}_i, i \in N \right\} \right). \quad (4)$$

Let us focus on the inner linear program in (4), and let y_i be the dual variables associated to the upper bounds and θ_j be the dual variable associated to the k-th inequality defining $\mathcal{U}_{knap}$. Dualizing the inner maximization we obtain

$$\min_{\theta, y \ge 0} \left\{ \sum_{j \in S} b_j \theta_j + \sum_{i \in N} \overline{\xi}_i y_i : \sum_{j \in S} a_{ji} \theta_j + y_i \ge d_i x_i, i \in N \right\} \quad (5)$$

$$= \min_{\theta \ge 0} \left\{ \sum_{j \in S} b_j \theta_j + \sum_{i \in N} \overline{\xi}_i \max(0, d_i x_i - \sum_{j \in S} a_{ji} \theta_j) \right\} \quad (6)$$

$$= \min_{\theta \ge 0} \left\{ \sum_{j \in S} b_j \theta_j + \sum_{i \in N} \overline{\xi}_i x_i \max(0, d_i - \sum_{j \in S} a_{ji} \theta_j) \right\}, \quad (7)$$

where y_i has been substituted by $\max(0, d_i x_i - \sum_{j \in S} a_{ji} \theta_j)$ in (6), and (7) holds because $x_i \in \{0, 1\}$ for each $i \in N$.

We consider next the piece-wise linear function

$$f_x(\theta) = \sum_{i \in N} c_i x_i + \sum_{j \in S} b_j \theta_j + \sum_{i \in N} x_i \overline{\xi}_i \max(0, d_i - \sum_{j \in S} a_{ji} \theta_j),$$

and denote $F(\theta) = \min_{x\in\mathcal{X}} f_x(\theta)$. Problem (4) can be rewritten as

$$\min_{x\in\mathcal{X}} \min_{\theta\geq 0} f_x(\theta) = \min_{\theta\geq 0} \min_{x\in\mathcal{X}} f_x(\theta) = \min_{\theta\geq 0} F(\theta).$$

For each $x \in \mathcal{X}$, the function f_x defined on $\mathbb{R}^s_+$ is piece-wise linear and $\min_{\theta\geq 0} f(\theta)$ is bounded because $\mathcal{U}_{knap}$ is bounded and non-empty. Hence, the minimum of $\min_{\theta\geq 0} f(\theta)$ is reached at some extreme point of the epigraph of f_x (recall that $\text{epi}(f_x) = \{(\theta, z) : \theta \geq 0, z \geq f_x(\theta)\}$). Let us denote $\text{ext}(\text{epi}(f_x))$ as Θ_x. Since the minimum of $f_x(\theta)$ is reached on Θ_x, we have that $\min\limits_{\theta\in\Theta_x} f_x(\theta) = \min\limits_{\theta\in \bigcup\limits_{y\in\mathcal{X}} \Theta_y} f_x(\theta)$ for each $x \in \mathcal{X}$. Therefore,

$$\min_{\theta\geq 0} F(\theta) = \min_{x\in\mathcal{X}} \min_{\theta\geq 0} f_x(\theta) = \min_{x\in\mathcal{X}} \min_{\theta\in\Theta_x} f_x(\theta) = \min_{\theta\in \bigcup\limits_{y\in\mathcal{X}} \Theta_y} \min_{x\in\mathcal{X}} f_x(\theta) = \min_{\theta\in \bigcup\limits_{y\in\mathcal{X}} \Theta_y} F(\theta),$$

and we are left to compute $\bigcup_{x\in\mathcal{X}} \Theta_x$.

We focus first on $\Theta_{\mathbf{1}}$ where $\mathbf{1} \in \mathbb{R}^s$ is the vector of all ones, formally defined as $\mathbf{1}_j = 1$ for each $j \in S$. Since, by definition, $\Theta_{\mathbf{1}} = \text{ext}(\text{epi}(f_{\mathbf{1}}))$, $\Theta_{\mathbf{1}}$ coincide with the set of vectors where $f_{\mathbf{1}}$ has no directional derivative. Therefore, one readily verifies that any vector θ in $\Theta_{\mathbf{1}}$ is the unique solution of a subsystem of the system $A'\theta = d'$, defined in (3), formed by s linearly independent rows of A', which can be solved in $O(s^3)$. Similarly, any vector in Θ_x is obtained by solving a subsystem of s independent linear constraints among

$$\begin{aligned} \theta_j &= 0, & j \in S \\ x_i \sum_{j\in S} a_{ji}\theta_j &= d_i x_i, & i \in N. \end{aligned}$$

We obtain that, for any $x \in \mathcal{X}$, $\Theta_x \subseteq \Theta_{\mathbf{1}}$, so that $\bigcup_{x\in\mathcal{X}} \Theta_x \subseteq \Theta_{\mathbf{1}}$. The result follows from the fact that computing $F(\theta)$ amounts to solve an instance of problem CO and $|\Theta_{\mathbf{1}}| = \sum\limits_{j=0}^{s} \binom{s}{j}\binom{n}{s-j} \leq O(s^s n^s)$. $\square$

Whenever CO is polynomially solvable and s is constant, Theorem 3 shows that the robust problem $CO_{\mathbf{d}}(\mathcal{U}_{knap})$ is polynomially solvable. The theorem also applies to polytopes more general than $\mathcal{U}_{knap}$. Recall that the down-monotone completion of a polytope $P \subseteq \mathbb{R}^n_+$ is given by

$$\text{dm}(P) = \{r \in \mathbb{R}^n_+ : \exists p \in P \text{ such that } r_i \leq p_i \text{ for each } i \in N\}.$$

We see that, for any polytope $P \subseteq \mathbb{R}^n_+$, $\mathrm{dm}(P)$ is a polytope that satisfies (2). The following simple result motivates the introduction of $\mathrm{dm}(P)$.

Lemma 1. *Let P and P' be polytopes included in $\mathbb{R}^n_+$. If $\mathrm{dm}(P) = \mathrm{dm}(P')$, then problems $CO_{\boldsymbol{d}}(P)$ and $CO_{\boldsymbol{d}}(P')$ have the same optimal solutions.*

Proof. The proof follows immediatly from the equality

$$\begin{aligned}
\min_{x\in\mathcal{X}}\left(\sum_{i\in N} c_i x_i + \max_{\xi\in P}\sum_{i\in N}\xi_i d_i x_i\right) &= \min_{x\in\mathcal{X}}\left(\sum_{i\in N} c_i x_i + \max_{\xi\in \mathrm{dm}(P)}\sum_{i\in N}\xi_i d_i x_i\right)\\
&= \min_{x\in\mathcal{X}}\left(\sum_{i\in N} c_i x_i + \max_{\xi\in P'}\sum_{i\in N}\xi_i d_i x_i\right)
\end{aligned}$$

□

Lemma 1 suggests that Theorem 3 can be an efficient way to solve $CO_{\boldsymbol{d}}(P)$ for any polytope P for which we can compute a description of the down-monotone completion that contains few knapsack constraints.

Whenever s is part of the input, the approach depicted in Theorem 3 has an exponential running-time, which is consistent with the hardness result below.

Corollary 1. *When s is part of the input, problem $CO_{\boldsymbol{d}}(\mathcal{U}_{knap})$ is at least as hard as solving $CO(\mathcal{U})$ where $\mathcal{U}$ contains 2 arbitrary vectors.*

Proof. Consider an optimization problem described by the feasibility set $\mathcal{X}$, and let us introduce its robust version $\min_{x\in\mathcal{X}}\max_{\eta\in\mathcal{U}^{2s}}\eta^T x$ for an uncertainty set $\mathcal{U}^{2s}$ that contains only two vectors, i.e. $\mathcal{U}^{2s} := \{\eta^1, \eta^2\}$. We characterize below an instance of $CO_{\boldsymbol{d}}(\mathcal{U}_{knap})$ that is equivalent to the above problem. We define $c_i = \min(\eta_i^1, \eta_i^2)$ and $d_i = \max(\eta_i^1 - c_i, \eta_i^2 - c_i)$ for each $i \in N$ and two 0-1 vectors ξ^1 and ξ^2 such that $\eta_i^1 = c_i + d_i\xi_i^1$ and $\eta_i^2 = c_i + d_i\xi_i^2$ for each $i \in N$.

We are left to define $\mathcal{U}_{knap}$ such that such that for any $x \in \mathcal{X}$

$$\max_{\eta\in\mathcal{U}^{2s}}\sum_{i\in N}\eta_i x_i = \sum_{i\in N} c_i x_i + \max_{\xi\in\mathcal{U}_{knap}}\sum_{i\in N}\xi_i d_i x_i. \tag{8}$$

If we were allowed to replace $\mathcal{U}_{knap}$ in the rhs of (8) by an arbitrary polytope $\mathcal{U}$, the equality could be enforced by defining $\mathcal{U}$ as the line segment joining ξ^1

and ξ^2, since in that case we would have

$$\sum_{i\in N} c_i x_i + \max_{\xi\in\mathcal{U}} \sum_{i\in N} \xi_i d_i x_i = \sum_{i\in N} c_i x_i + \max_{\xi\in\{\xi^1,\xi^2\}} \sum_{i\in N} \xi_i d_i x_i,$$

for any $x \in \mathcal{X}$, which is equal to $\max_{\eta\in\mathcal{U}^{2s}} \eta^T x$ by definition of ξ^1 and ξ^2.

Unfortunately, the above definition of $\mathcal{U}$ does not comply with the definition of $\mathcal{U}_{knap}$, provided in (2), so we instead construct a polytope $\mathcal{U}_{knap}$ that is the down-monotone completion of $\mathcal{U}$ and use Lemma 1. Specifically, we claim that the down-monotone completion of $\mathcal{U}$ can be defined as follows:

- For each $i \in N$ such that $\xi^1_i = \xi^2_i = 0$, we have $\overline{\xi}_i = 0$. The upper bounds $\overline{\xi}_i$ are set to 1 for the other indices.
- Let $N^1 \subseteq N$ be the set of indices such that $\xi^1_i = 1$ and $\xi^2_i = 0$, and define similarly N^2. We have the following knapsack constraints

$$\xi_i + \xi_j \leq 1 \text{ for each } (i,j) \in N^1 \times N^2. \tag{9}$$

To prove the claim, we must first verify that ξ^1 and ξ^2 belong to $\mathcal{U}_{knap}$, which is immediate from the definitions of the above knapsack constraints. To show that any extreme point of $\mathcal{U}_{knap}$ different from ξ^1 and ξ^2 is dominated by ξ^1 or ξ^2, we define the bipartite graph $G = (N, E)$ induced by the subsets N^1 and N^2, e.g. $N = N^1 \cup N^2$ and $E = N^1 \times N^2$. The knapsack constraints (9) are defined by the adjacency matrix of the graph, which is totally unimodular. Hence, all extreme points of $\mathcal{U}_{knap}$ are binary vectors ξ such that $\xi_i = 1$ for each $i \in N^*$ where either $N^* \subseteq N^1$ or $N^* \subseteq N^2$, proving the claim. $\square$

Corollary 1 implies that, when s is part of the input, $CO_{\boldsymbol{d}}(\mathcal{U}_{knap})$ is $\mathcal{NP}$-hard for the shortest path problem, the assignment problem, and the minimum spanning tree problem, since the robust versions of these problems are $\mathcal{NP}$-hard for two arbitrary scenarios.

Following the lines of Theorem 2 and assuming that s is constant, we can also obtain approximation algorithms for problems CO that are approximable. Let H be a polynomial time $(1+\epsilon)$-approximation algorithm for problem CO.

Algorithm 1: Approximation algorithm for $CO_{\mathbf{d}}(\mathcal{U}_{knap})$

for each $\theta \in \Theta_{\mathbf{1}}$ **find** an $(1+\epsilon)$-approximate solution x^θ for

$$\min_{x\in\mathcal{X}} \left\{ \sum_{i\in N} c_i x_i + \sum_{i\in N} \overline{\xi}_i x_i \max(0, d_i - \sum_{j\in S} a_{ji}\theta_j) \right\} ;$$

for each $\theta \in \Theta_{\mathbf{1}}$ **let** $Z^\theta = \sum_{i\in N} c_i x_i^\theta + \max_{\xi\in\mathcal{U}_{knap}} \sum_{i\in N} \xi_i d_i x_i^\theta$;

Let $\theta^\epsilon = \arg\min_{\theta\in\Theta_{\mathbf{1}}} Z^\theta$;

return: $x^\epsilon = x^{\theta^\epsilon}$ with cost equal to $Z^\epsilon = Z^{\theta^\epsilon}$

The approximation algorithm for $CO_{\mathbf{d}}(\mathcal{U}_{knap})$ is provided in Algorithm 1. Algorithm 1 clearly runs in polynomial time whenever the cardinality $\Theta_{\mathbf{1}}$ is bounded by a polynomial function of n, which is the case whenever s is constant.

Proposition 1. *Algorithm 1 returns an* $(1+\epsilon)$*-approximate solution to* $CO_{\mathbf{d}}(\mathcal{U}_{knap})$.

Proof. For each θ, we introduce G^θ such that

$$\min_{x\in\mathcal{X}} \left\{ \sum_{i\in N} c_i x_i + \sum_{i\in N} \overline{\xi}_i x_i \max(0, d_i - \sum_{j\in K} a_{ji}\theta_j) \right\}. \tag{10}$$

Let θ^* be the index such that $Z^* = G^{\theta^*}$ in Theorem 3 and x^{θ^*} be an $(1+\epsilon)$-approximate solution to problem (10). Then, we have

$$\begin{aligned}
Z^\epsilon &\le Z^{\theta^*} \\
&= \sum_{i\in N} c_i x_i^{\theta^*} + \max_{\xi\in\mathcal{U}_{knap}} \sum_{i\in N} \xi_i d_i x_i^{\theta^*} \\
&= \sum_{i\in N} c_i x_i^{\theta^*} + \min_{\theta\ge 0} \left\{ \sum_{j\in K} b_j\theta_j + \sum_{i\in N} \overline{\xi}_i x_i^{\theta^*} \max(0, d_i - \sum_{j\in K} a_{ji}\theta_j) \right\} && (11)\\
&\le \sum_{i\in N} c_i x_i^{\theta^*} + \sum_{j\in K} b_j\theta_j^* + \sum_{i\in N} \overline{\xi}_i x_i^{\theta^*} \max(0, d_i - \sum_{j\in K} a_{ji}\theta_j^*) && (12)\\
&\le (1+\epsilon)(G^{\theta^*} - \sum_{j\in K} b_j\theta_j^*) + \sum_{j\in K} b_j\theta_j^* && (13)\\
&\le (1+\epsilon)G^{\theta^*} && (14)\\
&= (1+\epsilon)Z^*, && (15)
\end{aligned}$$

where (11) follows from (7) and (13) follows from (10). □

We provide in the Appendix an application of the above results to variable uncertainty.

3. Extended knapsack uncertainty

The results from the previous section can be extended to polytopes described through certain types of extended formulations. Namely, let us consider the set of indices L, with $|L| = \ell$, and the linear mapping $h : \mathbb{R}_+^\ell \to \mathbb{R}_+^n$ characterized by the matrix (h_{il}) with non-negative coefficients. The robust problems studied in this section are then defined by replacing the product $d_i\xi_i$ present in $CO_{\boldsymbol{d}}(\mathcal{U}_{knap})$ by $h_i(\eta)$, obtaining the problem

$$CO_{\boldsymbol{h}}(\mathcal{U}_{ext}) \qquad \min_{x \in \mathcal{X}} \left(\sum_{i \in N} c_i x_i + \max_{\eta \in \mathcal{U}_{ext}} \sum_{i \in N} h_i(\eta) x_i \right)$$

defined for the extended uncertainty set

$$\mathcal{U}_{ext} := \left\{ \eta \in \mathbb{R}^\ell : \sum_{l \in L} a_{jl} \eta_l \le b_j, j \in S, 0 \le \eta \le \overline{\eta} \right\}.$$

Using a reduction from a robust scheduling problem studied in [26], we can show that problem $CO_{\boldsymbol{h}}(\mathcal{U}_{ext})$ is hard, even when s is constant.

Proposition 2. *Problem $CO_{\boldsymbol{h}}(\mathcal{U}_{ext})$ is $\mathcal{NP}$-hard in the strong sense even when $s = 1$.*

Proof. The result is obtained by considering the scheduling problem that minimizes the weighted sum of completion times, known to be solvable in polynomial time using Smith' rule. Its robust version for uncertainty set $\mathcal{U}_\Gamma$ has been proved strongly $\mathcal{NP}$-hard in [26]. The problem is defined as follows. Given a set of n jobs with weight w_j, mean processing time $\overline{p}_j$ and deviation $\hat{p}_j$ for each job j, and a budget of uncertainty Γ, the objective is to minimize the worst-case of the weighted sum of the completion times knowing the processing times vary in $[\overline{p}_j, \overline{p}_j + \hat{p}_j]$ and that at most Γ jobs reach simultaneously their upper values.

The problem can be cast in our framework using binary optimization variable $x_{ij} = 1$ if job i is scheduled prior to job j and letting the set $\mathcal{X}^{sched} \subset \{0,1\}^{n^2}$

contain all binary vectors x feasible for the problem, yielding

$$\min_{x \in \mathcal{X}^{sched}} \max \left\{ \sum_{i=1}^{n} \sum_{j=i}^{n} p_i w_j x_{ij} : \sum_{i=1}^{n} \frac{p_i - \overline{p}_i}{\hat{p}_i} \leq \Gamma, \overline{p}_i \leq p \leq \hat{p}_i, i \in N \right\}.$$

The above problem is a special case of $CO_{\boldsymbol{h}}(\mathcal{U}_{ext})$ obtained by defining $c_{ij} = \overline{p}_i w_j$ for $i \leq j$ and $c_{ij} = 0$ for $i > j$, $h_{ijl}(\eta) = \eta_i \hat{p}_i w_j$ for $i \leq j$ and $l = i$, and $h_{ijl}(\eta) = 0$ otherwise. Thus, considering an uncertainty set $\mathcal{U}_\Gamma$ of dimension n concludes the proof. $\square$

In view of the above hardness result, we focus below on a special type of functions h that satisfy the following technical assumption

$$\text{there exists a partition } L_1 \cup \cdots \cup L_N \text{ of } L \text{ such that } h_{il} > 0 \text{ iff } l \in L_i. \quad (16)$$

Assumption (16) is flexible enough to model the extension of the budgeted uncertainty set where multiple deviations are allowed, closely related to the histogram model studied in [24]. Namely, consider s budgets of uncertainty Γ_j and deviations $d_{ji} \in \mathbb{R}_+$. Then, $h_i(\eta) = \sum_{j \in S} d_{ji} \eta_{ji}$ and the uncertainty set is

$$\mathcal{U}^{M\Gamma} := \left\{ \eta \in \mathbb{R}^{|S| \times n} : \sum_{i \in N} \eta_{ji} \leq \Gamma_j, j \in S, 0 \leq \eta \leq 1 \right\}.$$

When s is constant, the following result shows that $CO_{\boldsymbol{h}}(\mathcal{U}^{M\Gamma})$ amounts to solve a polynomial number of problems CO.

Proposition 3. *Consider problem $CO_{\boldsymbol{h}}(\mathcal{U}_{ext})$ and suppose that* (16) *holds. The problem can be solved by solving $O(s^s \ell^s)$ linear systems with s variables and s equations and $O(s^s \ell^s)$ problems CO with modified costs.*

Proof. Problem $CO_{\boldsymbol{h}}(\mathcal{U}_{ext})$ can be written as

$$\min_{x \in \mathcal{X}} \left(\sum_{i \in N} c_i x_i + \max \left\{ \sum_{i \in N} \sum_{l \in L} h_{il} \eta_l x_i : \sum_{l \in L} a_{jl} \eta_l \leq b_j, j \in K, 0 \leq \eta \leq \overline{\eta}_l, l \in L \right\} \right).$$

Introducing the dual variables y_l and θ_j as in the proof of Theorem 3, the dual

Algorithm 2: Approximation algorithm for $CO_{\mathbf{h}}(\mathcal{U}_{ext})$

for each $\theta \in \Theta_{\mathbf{1}}^{ext}$ **find** an $(1+\epsilon)$-approximate solution x^θ for

$$\min_{x\in\mathcal{X}} \left\{ \sum_{i\in N} c_i x_i + \sum_{i\in N}\sum_{l\in L_i} \overline{\eta}_l x_i \max(0, h_{il} - \sum_{j\in S} a_{jl}\theta_j) \right\} ;$$

for each $\theta \in \Theta_{\mathbf{1}}^{ext}$ **let** $Z^\theta = \sum_{i\in N} c_i x_i^\theta + \max_{\eta\in\mathcal{U}_{ext}} \sum_{i\in N} h_i(\eta) x_i^\theta$;

Let $\theta^\epsilon = \arg\min_{\theta\in\Theta_{\mathbf{1}}^{ext}} Z^\theta$;

return: $x^\epsilon = x^{\theta^\epsilon}$ with cost equal to $Z^\epsilon = Z^{\theta^\epsilon}$

of the inner maximization problem reads

$$\min_{\theta,y\geq 0} \left\{ \sum_{j\in K} b_j\theta_j + \sum_{l\in L} \overline{\eta}_l y_l : \sum_{j\in K} a_{jl}\theta_j + y_l \geq \sum_{i\in N} h_{il}x_i, l\in L \right\} \tag{17}$$

$$= \min_{\theta,y\geq 0} \left\{ \sum_{j\in K} b_j\theta_j + \sum_{l\in L} \overline{\eta}_l y_l : \sum_{j\in K} a_{jl}\theta_j + y_l \geq h_{il}x_i, i\in N, l\in L_i \right\} \tag{18}$$

$$= \min_{\theta\geq 0} \left\{ \sum_{j\in K} b_j\theta_j + \sum_{i\in N}\sum_{l\in L_i} \overline{\eta}_l \max(0, h_{il}x_i - \sum_{j\in K} a_{jl}\theta_j) \right\} \tag{19}$$

$$= \min_{\theta\geq 0} \left\{ \sum_{j\in K} b_j\theta_j + \sum_{i\in N}\sum_{l\in L_i} \overline{\eta}_l x_i \max(0, h_{il} - \sum_{j\in K} a_{jl}\theta_j) \right\}, \tag{20}$$

where (19) holds because of property (16). The rest of the proof is identical to the proof of Theorem 3. □

Proposition 3 naturally leads to approximation algorithms for problems for which CO is approximable. Specifically, we introduce the set $\Theta_{\mathbf{1}}^{ext} \subseteq \mathbb{R}^s$ as in the previous section however considering here the linear system $h_{il} = \sum_{j\in S} a_{jl}\theta_j$, $i\in N, l\in L_i$, and $\theta_j = 0$, $j\in S$. The proof of correctness of Algorithm 2 is omitted as it is very similar to the proof of correctness of Algorithm 1.

4. Ellipsoidal uncertainty

We provide in the section a counterpart of Algorithm 1 for $CO_{\mathbf{d}}(\mathcal{U}_{ball})$. The first element of our approach follows an idea of [27] that approximates $\mathcal{U}_{ball}$

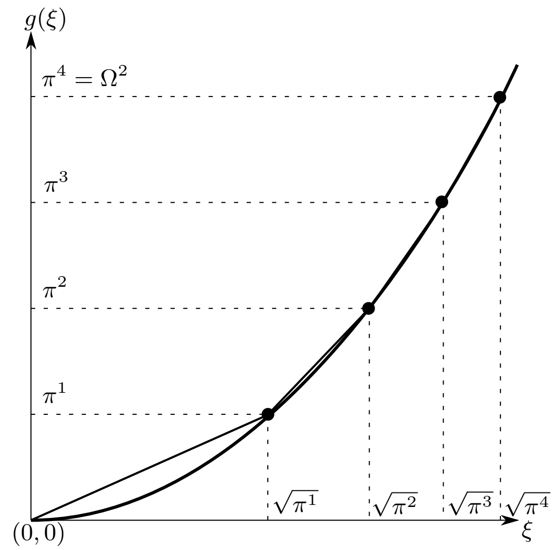

Figure 1: Piece-wise affine over approximation of the function ξ^2 with $m = 4$.

through a polytope $\mathcal{U}_m$ where the integer m parametrizes the precision of the approximation. Specifically, we approximate each function ξ_i^2 involved in the definition of $\mathcal{U}_{ball}$ by a piece-wise linear upper approximation $g(\xi_i)$ based on the equal division of the vertical axis $[0, \Omega^2]$ into m segments $[\pi^0 = 0, \pi^1 = \Omega^2/m], \dots, [\pi^{m-1}, \pi^m = \Omega^2]$ and their images on the horizontal axis $[0, \sqrt{\pi^1}], \dots, [\sqrt{\pi^{m-1}}, \sqrt{\pi^m}]$. Then, the graph of function g is defined as the union of the line segments joining $(\sqrt{\pi^k}, \pi^k)$ and $(\sqrt{\pi^{k+1}}, \pi^{k+1})$ for each $k \in \{0, \dots, m-1\}$, see Figure 1. The polytope $\mathcal{U}_m$ is then defined as

$$\mathcal{U}_m := \left\{ \xi \in \mathbb{R}^n : \sum_{i \in N} g(\xi_i) \leq \Omega^2 \right\}.$$

The first element of our approach shows that optimizing over $\mathcal{U}_m$ is equivalent to optimizing over an extended version of $\mathcal{U}_\Gamma$, defined as

$$\mathcal{U}'_m := \left\{ \eta \in \mathbb{R}^{n \times m} : \sum_{i \in N} \sum_{k \in M} \eta_i^k \leq m, 0 \leq \eta \leq 1 \right\},$$

where $M = \{1, \dots, m\}$.

Lemma 2. *Let $h : \mathbb{R}^{n \times m} \to \mathbb{R}^n$ be the linear mapping defined through $h_{ik} =$*

Algorithm 3: Approximation algorithm for $CO_{\boldsymbol{d}}(\mathcal{U}_{ball})$

Use Algorithm 2 with $\mathcal{U}_{ext} := \mathcal{U}'_m$ and h defined in Lemma 2 to obtain x^δ, an $(1+\delta)$-approximate solution to $CO_{\boldsymbol{h}}(\mathcal{U}'_m)$;

Compute the cost $Z^\delta = c^T x^\delta + \Omega\sqrt{\sum_{i\in N} d_i^2 x_i^\delta}$;

return: x^δ with cost equal to Z^δ

$d_i(\sqrt{\pi^k} - \sqrt{\pi^{k-1}})$ *for each* $i \in N$*. For any* x*, it holds that*

$$\max_{\xi\in\mathcal{U}_m} \sum_{i\in N} \xi_i d_i x_i = \max_{\eta\in\mathcal{U}'_m} \sum_{i\in N} h_i(\eta) x_i. \tag{21}$$

Proof. $\geq$: Let η be a maximizer of the rhs of (21). Since $\mathcal{U}'_m$ is integral, we can assume that η is binary. Further, we can assume that $\eta_i^k \geq \eta_i^{k+1}$ for each k and i because $h_{ik} > h_{ik+1}$. Let us then define the vector $\xi \in \mathcal{U}_m$ by $\xi_i = \sum_{k\in M}(\sqrt{\pi^k} - \sqrt{\pi^{k-1}})\eta_i^k$. One readily verifies that $\xi \in \mathcal{U}_m$ and, moreover, that $\xi_i d_i = h_i(\eta)$ for each $i \in N$.

$\leq$: Conversely, let $\xi \in \mathcal{U}_m$ and let $k(i)$ be the index $k \in M$ such that $\sqrt{\pi^{k-1}} \leq \xi_i < \sqrt{\pi^k}$ for each $i \in N$ (see Figure 1). Then, define $\eta_i^k = 1$ for each $1 \leq k \leq k(i) - 1$, $\eta_i^{k(i)} = \frac{\xi_i - \sqrt{\pi^{k(i)-1}}}{\sqrt{\pi^{k(i)}} - \sqrt{\pi^{k(i)-1}}}$, and $\eta_i^k = 0$ for $k \geq k(i) + 1$. We see that $\eta \in \mathcal{U}'_m$ and $\xi_i d_i = h_i(\eta)$ for each $i \in N$. $\square$

Using Algorithm 2, we obtain in polynomial time an approximate solution to $CO_{\boldsymbol{h}}(\mathcal{U}'_m)$, which is also an approximate solution to $CO_{\boldsymbol{d}}(\mathcal{U}_m)$ thanks to Lemma 2. The true cost of the solution is then computed for $\mathcal{U}_{ball}$ to obtain the desired approximate solution to $CO_{\boldsymbol{d}}(\mathcal{U}_{ball})$. The procedure is formally described in Algorithm 3 whose validity is stated below. The running time of Algorithm 3 is in $O(\frac{n^2}{\epsilon} f(n,\epsilon))$ where $O(f(n,\epsilon))$ is the running time required to compute an $(1+\epsilon)$-approximate solution to CO.

Theorem 4. *If* $m \geq \frac{n}{\delta}$ *and* $\epsilon = 4\delta$*, then Algorithm 3 returns a* $(1+\epsilon)$*-approximate solution to* $CO_{\boldsymbol{d}}(\mathcal{U}_{ball})$*.*

Let us introduce some notations before proving the theorem. First, we define $\mathcal{U}_{ball}(\alpha)$ as the ball of radius α centered at 0. Second, we define $F_{ball}(x) = c^T x+$

$\max_{\xi \in \mathcal{U}_{ball}} \sum_{i \in N} \xi_i d_i x_i = c^T x + \Omega \sqrt{\sum_{i \in N} d_i^2 x_i}$ and $F_m(x) = c^T x + \max_{\xi \in \mathcal{U}_m} \sum_{i \in N} \xi_i d_i x_i$.

To prove Theorem 4, we will bound the ratio $F_m(x)/F_{ball}(x)$ from above and from below. On the one hand, $g(\xi_i) \geq \xi_i^2$ for each $i \in N$ implies that $\mathcal{U}_m \subseteq \mathcal{U}_{ball}$ and we obtain immediately

$$\frac{F_m(x)}{F_{ball}(x)} \leq 1 \text{ for any } x \in \mathcal{X}. \tag{22}$$

On the other hand, proving that $\frac{F_m}{F_{ball}}$ is also bounded from below is more technical. We first show that $\mathcal{U}_{ball}(\rho(m)\Omega) \subseteq \mathcal{U}_m$ for a specific function $\rho(m)$.

Lemma 3. *If $m \geq n$, then $\mathcal{U}_{ball}(\Omega\sqrt{1 - n/m}) \subset \mathcal{U}_m$.*

Proof. It follows from the definition of g that

$$g(\xi_i) \leq \xi_i^2 + \frac{\Omega^2}{m}.$$

Consider then ξ be such that $\|\xi\|_2 \leq \Omega\sqrt{1 - n/m}$. Therefore,

$$\sum_{i \in N} g(\xi_i) \leq \sum_{i \in N} \xi_i^2 + n\frac{\Omega^2}{m} \leq \Omega^2,$$

proving that $\xi \in \mathcal{U}_m$. □

Using the above lemma, we can bound $\frac{F_m}{F_{ball}}$ from below.

Lemma 4. *Consider $\delta > 0$. If $m \geq \frac{n}{\delta}$, then $\frac{F_m(x)}{F_{ball}(x)} \geq 1 - \delta$ for any $x \in \mathcal{X}$.*

Proof. Lemma 3 implies that

$$F_m(x) \geq c^T x + \max_{\xi \in \mathcal{U}_{ball}(\Omega\sqrt{1-\frac{n}{m}})} \sum_{i \in N} \xi_i d_i x_i.$$

Hence,

$$\begin{aligned} F_m(x) &\geq c^T x + \Omega\sqrt{1 - \frac{n}{m}}\sqrt{\sum_{i \in N} d_i^2 x_i} \\ &\geq \sqrt{1 - \frac{n}{m}}\left(c^T x + \Omega\sqrt{\sum_{i \in N} d_i^2 x_i}\right) \\ &\geq \sqrt{1 - \frac{n}{m}} F_{ball}(x), \end{aligned}$$

and the results follows from taking $m = \lceil \frac{n}{\delta} \rceil$. □

Algorithm 4: Approximation algorithm for $\mathcal{U}_{\overline{ball}}CO_{\boldsymbol{d}}$

Use Algorithm 2 with $\mathcal{U}_{ext} := \mathcal{U}'_{\overline{m}}$ and h defined in Lemma 2 to obtain x^δ, an $(1+\delta)$-approximate solution to $\mathcal{U}'_{\overline{m}}CO_{\boldsymbol{h}}$;

Compute the cost $Z^\delta = c^T x^\delta + \max\limits_{\xi \in \mathcal{U}_{\overline{ball}}} \sum\limits_{i \in N} \xi_i d_i x_i$;

return: x^δ with cost equal to Z^δ

Proof. of Theorem 4. Let x^δ, x^m and x^{ball} denote the solution computed by Algorithm 3, and the optimal solutions of problems $\min\limits_{x \in \mathcal{X}} F_m(x)$ and $\min\limits_{x \in \mathcal{X}} F_{ball}(x)$, respectively. The following holds for $\delta > 0$ small enough:

$$\begin{aligned}
Z^\delta &= F_{ball}(x^\delta) \\
&\leq \frac{1}{1-\delta} F_m(x^\delta) && \text{(using Lemma 4)} \\
&\leq (1+2\delta) F_m(x^\delta) \\
&\leq (1+\delta)(1+2\delta) F_m(x^m) && \text{(by definition of } x^\delta \text{ and Lemma 2)} \\
&\leq (1+4\delta) F_m(x^m) \\
&\leq (1+4\delta) F_m(x^{ball}) && \text{(by definition of } x^m\text{)} \\
&\leq (1+4\delta) F_{ball}(x^{ball}) && \text{(follows from (22))} \\
&= (1+4\delta)\, \mathrm{opt}(CO_{\boldsymbol{d}}(\mathcal{U}_{ball})).
\end{aligned}$$

□

5. Ellipsoid with upper bounds

Rather than studying directly the problem $CO_{\boldsymbol{d}}(\mathcal{U}_{ball}^{box})$, we focus in this section on the problem $CO_{\boldsymbol{d}}(\mathcal{U}_{\overline{ball}})$ defined for an axis-parallel ellipsoid combined with upper bounds, namely

$$\mathcal{U}_{\overline{ball}} := \left\{ \xi \in \mathbb{R}^n : \sum_{i \in N} \|\xi\|_2 \leq \Omega, 0 \leq \xi \leq \overline{\xi} \right\}.$$

Since $\mathcal{U}_{\overline{ball}} \cap \mathbb{R}^n_+ = \mathcal{U}_{ball}^{box} \cap \mathbb{R}^n_+$ and $d \in \mathbb{R}^n_+$, problems $CO_{\boldsymbol{d}}(\mathcal{U}_{ball}^{box})$ and $CO_{\boldsymbol{d}}(\mathcal{U}_{\overline{ball}})$ have the same optimal solutions. Hence, the extension of Algorithm 3 to the problem $CO_{\boldsymbol{d}}(\mathcal{U}_{\overline{ball}})$, presented in the rest of this section, also applies to $CO_{\boldsymbol{d}}(\mathcal{U}_{ball}^{box})$.

The counterpart of $\mathcal{U}_m$ with $\mathcal{U}_{\overline{ball}}$ is given by

$$\mathcal{U}_{\overline{m}} := \left\{ \xi \in \mathbb{R}^n : \sum_{i \in N} g(\xi_i) \leq \Omega^2, 0 \leq \xi \leq \overline{\xi} \right\},$$

and one readily verifies that the counterpart of (21) is

$$\max_{\xi \in \mathcal{U}_m} \sum_{i \in N} \xi_i d_i x_i = \max_{\eta \in \mathcal{U}'_m} \sum_{i \in N} h_i(\eta) x_i, \tag{23}$$

where $\mathcal{U}'_{\overline{m}}$ is the following extended polytope with one knapsack constraint

$$\mathcal{U}'_{\overline{m}} := \left\{ \eta \in \mathbb{R}^{n \times m} : \sum_{i \in N} \sum_{k \in M} \eta_i^k \leq m, 0 \leq \eta \leq \overline{\eta} \right\},$$

$h : \mathbb{R}^{n \times m} \to \mathbb{R}^n$ is the linear mapping defined in Lemma 2, and $\overline{\eta} \in \mathbb{R}_+^{n \times m}$ is defined as follows. Let $k(i)$ be the index $k \in M$ such that $\sqrt{\pi^{k-1}} \leq \overline{\xi}_i < \sqrt{\pi^k}$ for each $i \in N$. We obtain $\overline{\eta}_i^k = 1$ for each $1 \leq k \leq k(i) - 1$, $\overline{\eta}_i^{k(i)} = \frac{\overline{\xi}_i - \sqrt{\pi^{k(i)-1}}}{\sqrt{\pi^{k(i)}} - \sqrt{\pi^{k(i)-1}}}$, and $\overline{\eta}_i^k = 0$ for $k \geq k(i) + 1$. Algorithm 3 is adapted in Algorithm 4 to handle the upper bounds, the proof of correctness of which is provided in Theorem 5 below.

Theorem 5. *If $m \geq \frac{n}{\delta}$ and $\epsilon = 4\delta$, then Algorithm 4 returns a $(1+\epsilon)$-approximate solution to $CO_{\boldsymbol{d}}(\mathcal{U}_{\overline{ball}})$.*

The proof of Theorem 5 follows closely the lines of the proof of Theorem 4 with one little difference explained below. Let us first extend the notations F_m and F_{ball} to $F_{\overline{m}}$ and $F_{\overline{ball}}$, respectively, and introduce

$$\mathcal{U}_{\overline{ball}}(\Omega, \overline{\xi}) := \left\{ \xi \in \mathbb{R}^n : \sum_{i \in N} \|\xi\|_2 \leq \Omega, 0 \leq \xi \leq \overline{\xi} \right\}.$$

Optimizing a linear function over the set $\mathcal{U}_{\overline{ball}}(\Omega, \overline{\xi})$ satisfies the useful property stated next.

Lemma 5. *Consider $\lambda > 0$ and $\Omega > 0$ and $\overline{\xi}, \mu \in \mathbb{R}_+^n$. It holds that*

$$\max_{\xi \in \mathcal{U}_{\overline{ball}}(\lambda\Omega, \lambda\overline{\xi})} \mu^T \xi = \lambda \max_{\xi \in \mathcal{U}_{\overline{ball}}(\Omega, \overline{\xi})} \mu^T \xi.$$

Proof. The results follows from the property

$$\mathcal{U}_{\overline{ball}}(\lambda\Omega, \lambda\bar{\xi}) = \lambda\mathcal{U}_{\overline{ball}}(\Omega, \bar{\xi})$$

by performing the change of variables $\phi = \lambda\xi$. $\square$

Proof of Theorem 5. Following the reasoning of the proof of Lemma 3, we see that that $m \geq n$ implies that $\mathcal{U}_{\overline{ball}}(\Omega\sqrt{1 - n/m}) \subset \mathcal{U}_{\overline{m}}$. Hence, we have

$$\begin{aligned} F_{\overline{m}}(x) &\geq c^T x + \max_{\xi \in \mathcal{U}_{\overline{ball}}(\Omega\sqrt{1-\frac{n}{m}}, \bar{\xi})} \sum_{i \in N} \xi_i d_i x_i && (24) \\ &\geq c^T x + \max_{\xi \in \mathcal{U}_{\overline{ball}}(\Omega\sqrt{1-\frac{n}{m}}, \bar{\xi}\sqrt{1-\frac{n}{m}})} \sum_{i \in N} \xi_i d_i x_i && (25) \\ &= c^T x + \sqrt{1 - \frac{n}{m}} \max_{\xi \in \mathcal{U}_{\overline{ball}}(\Omega, \bar{\xi})} \sum_{i \in N} \xi_i d_i x_i && (26) \\ &\geq \sqrt{1 - \frac{n}{m}} F_{\overline{ball}}(x), && (27) \end{aligned}$$

where (25) follows from

$$\mathcal{U}_{\overline{ball}}\left(\Omega\sqrt{1 - \tfrac{n}{m}}, \bar{\xi}\sqrt{1 - \tfrac{n}{m}}\right) \subseteq \mathcal{U}_{\overline{ball}}\left(\Omega\sqrt{1 - \tfrac{n}{m}}, \bar{\xi}\right),$$

and (26) follows from Lemma 5. Inequality (27) states the counterpart of Lemma 4 for $F_{\overline{m}}$. Moreover, one readily verifies that inequality $\frac{F_m(x)}{F_{ball}(x)} \leq 1$ also holds. The result is thus obtained by following the steps of the proof of Theorem 4. $\square$

6. Conclusion

We have investigated the complexity of min max robust combinatorial optimization under general uncertainty polytopes. We have shown that, if the down-monotone completion of the uncertainty polytope contains a constant number of linear inequalities, then the optimal solution to the robust problem can be obtained by solving a polynomial number of deterministic counterparts. We have extended these results to polytopes defined in extended spaces, in which case the complexity of the resulting robust problem also depend on the structure of the cost function. We have applied these results to problems where the uncertainty set is an axis-parallel ellipsoid or the intersection of the later with a box, obtaining approximation algorithms for robust whose deterministic counterparts are approximable.

From the practical viewpoint, our algorithms require to solve large numbers of deterministic problems. Hence, a future research direction could be dedicated to the efficient parallelization of this task, possibly exploiting the parallelization possibilities of dedicated algorithms for specific problems. A related question concerns the study of the stability of the optimal solutions under small changes in the objective functions. Another interesting question is whether it is possible to avoid solving the entire optimization problem at each iteration but instead focus on separation/pricing problems. For instance, consider a branch-and-cut-and-price algorithm for the vehicle routing problem that generates feasible routes in pricing problems. One can readily verify that the robust counterpart can be addressed by solving several pricing problems instead of solving several time the full problem.

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Appendix A. Decision-dependent uncertainty

We show below how the results from Section 2 apply to variable uncertainty introduced in [4, 28], and revived in [25] under the name "decision-dependent uncertainty". The framework considers robust problems where the uncertain parameters live in a point-to-set-mapping $\mathcal{U}(x) : \mathcal{X} \rightrightarrows \mathbb{R}^n$ instead of a fixed uncertainty set. We consider below a restricted type of variable uncertainty where only the rhs of the linear constraints characterizing the uncertainty point-to-set mapping depend affinely on the optimization variables, namely

$$\mathcal{U}_{knap}^{var}(x) := \left\{ \xi \in \mathbb{R}^n : \sum_{i \in N} a_{ji} \xi_i \leq b_j(x), j \in S, 0 \leq \xi \leq \overline{\xi} \right\},$$

where b_j is an affine function of x for each $j \in S$. Interestingly, Theorem 3 extends directly to $\mathcal{U}_{knap}^{var}(x)$.

Proposition 4. *Problem $\mathcal{U}_{knap}^{var}CO_{\mathbf{d}}$ can be solved by solving olving $O(s^s n^s)$ linear systems with s variables and s equations, and $O(s^s n^s)$ nominal problems CO with modified costs.*

Proof. The proof is almost identical to the proof of Theorem 3, with the difference that $b_j \theta_j$ is now replaced by $b_j(x)\theta_j$ in (17)–(20). □

One of the interests of variable uncertainty arises when allowing the rhs of $\mathcal{U}_\Gamma$, Γ, to depend on the optimization variables. Specifically, let us consider the variable budgeted uncertainty set, defined as

$$\mathcal{U}_\gamma(x) := \left\{ \xi \in \mathbb{R}^n : \sum_{i \in N} \xi_i \leq \gamma(x), 0 \leq \xi_i \leq 1, i \in N \right\}.$$

The author of [4, 28] has shown that if $\gamma(x)$ is constructed according to the probabilistic bounds proved in [3], then $\mathcal{U}_\gamma(x)$ yields the same probabilistic garantee as $\mathcal{U}_\Gamma$, albeit at a lower solution cost since $\mathcal{U}_\gamma(x) \subseteq \mathcal{U}_\Gamma$ for all x. For instance, a analytical choice for the function would be based on the weakest of the bounds proposed by [3], yielding $\alpha(x) = (-2\ln(\epsilon)\sum_i x_i)^{\frac{1}{2}}$. While the resulting point-to-set-mapping $\mathcal{U}_\alpha(x)$ cannot be used in Proposition 4 (because α is not an affine function of x), the point-to-set mapping can be approximated by a more conservative one defined by s tangent affine approximations of α, denoted $\gamma_1, \ldots, \gamma_s$,

$$\mathcal{U}_{\gamma\gamma}(x) := \left\{ \xi \in \mathbb{R}^n : \sum_{i \in N} \xi_i \leq \gamma_j(x), j \in S, 0 \leq \xi_i \leq 1, i \in N \right\}.$$

The point-to-set mapping is clearly a special case of $\mathcal{U}^{var}_{knap}(x)$, and can therefore be solved through Proposition 4. We refer to [4, 28, 25] for numerical experiments reporting the reduction in the *Price of robustness* offered by models $\mathcal{U}_\gamma(x)$ and $\mathcal{U}_{\gamma\gamma}(x)$ and the approximation of α through affine functions.