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How rotational vortices enhance transfers

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Inspired by recent observations of granular flow, we examine how rotational vortices contribute to heat or mass transfer enhancement in a fluid. We use a *tracer method* to simulate both diffusion and advection in systems of differing intrinsic diffusivities D_0 , vortex sizes R , vortex rotation frequencies f , and vortex lifetimes ℓ . The results reveal that these systems exhibit an effective diffusive behavior, characterized by an effective diffusivity D^{eff} . A striking finding is the existence of two regimes, dichotomised by the Péclet number $Pe = R^2 f / D_0$. When the Péclet number is less than one, there is no transfer enhancement, $D^{eff} = D_0$. For higher values, vortices produce some transfer enhancement with a corresponding power law $D^{eff}/D_0 \approx Pe^n$. The power n ranges from a lower bound of 0.5 for stationary vortices of lifetime infinity, to an upper bound of 1 for vortices of lifetimes shorter than half a rotation. This difference is attributed to two different internal mechanisms involving the coupling of diffusion and advection. These results could provide new insights on the transfer properties of fluid systems comprising rotational vortices, such as granular materials, suspensions, foams, and emulsions, as well as low Reynolds number stirred flows. © 2013 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4817671>]

I. INTRODUCTION

In a number of fluid systems, heat and mass transfers are intimately related to the nature of the flow field. In turbulent flows, stirred flows, and even dense granular flows, vortices appear to play a central role in enhancing transfers. Quantifying this advection-diffusion coupling is often a challenge.

In a recent study, we investigated the transfer of heat in dense granular flows.¹ This phenomenon is of considerable importance to both industrial and geophysical applications.² It was found that the heat transfer across a layer of grains is strongly enhanced when a shear deformation is applied. This enhancement was shown to arise from the development of granular vortices within the sheared layers.

The development of granular vortices is a fascinating property of dense granular flows. By analogy with turbulence, this phenomenon was coined *granulence*.^{3–5} There are, however, fundamental differences between turbulence and granulence. Turbulence results from a visco-inertial instability, and thus develops at high Reynolds numbers. By contrast, granulence results from a *steric* instability; grains must move across the laminar streamlines to avoid each other. This phenomena is related to the Reynolds dilatancy.⁶ Unlike visco-inertial instability, steric instability develops even at low strain rates, which is a significant practical advantage. Furthermore, granulence becomes more pronounced in the vicinity of the jamming transition where the granular flows exhibit transient islands of jammed grains with a typical size that increases and diverges as the flow decelerates and stops.^{7,8} The systematic characterization of granular vortices is still an emerging field of research. Nevertheless, two significant features have consistently been reported. First, granular vortices are ephemeral, with a typical lifetime much shorter than a shear deformation of 1.^{3–5} Second, their velocity field is approximately rotational, given that a jammed island of grains moves as a rigid

body. By contrast, the vortices that arise in turbulence are typically irrotational and the vortices can last for several revolutions.^{9,10}

Greater similarities may exist between granular vortices and the vortices that develop in viscous liquids that are mechanically stirred. Whilst their lifetimes tend to be controlled by the stirring protocol, these vortices are also rotational. Mixing viscous liquids such as melted polymers is an important industrial issue, and a number of studies have quantified the efficiency of various stirring protocols by focusing on the rate of homogenization of the mixture. Efficient mixing has been shown to occur with protocols producing chaotic pathlines and so-called *chaotic advection*.^{11–14} Stretching and folding of fluid elements were shown to play a crucial role, increasing the interface area between the two fluids and the concentration gradients, thereby intensifying the mixing process.^{15–20}

In the field of dense granular flows where grains tend to transiently rotate in rotational islands, the advection-diffusion coupling leads to a significant transfer enhancement.¹ However, the link between typical rotational vortex characteristics and the degree of transfer enhancement has not yet been clearly quantified. How do vortex characteristics such as their size, their rotation frequency, and lifetime influence heat and/or mass transfers? This question seems to be a fundamental step toward a comprehensive and unified understanding of transfer mechanisms underlying stirring protocols and fluid systems exhibiting granulence. These include dry granular materials,¹ suspensions,²¹ and other particulate fluids such as foams and emulsions.^{22,23}

In this paper, we tackle this question numerically by considering a model fluid system comprising a periodic array of rotational vortices, and we analyze the transfer properties obtained when varying the vortex size, rotation frequency, and lifetime. The study is based on a *passive Brownian tracer method*, which is used to solve for the advection-diffusion coupling and extract the effective transfer properties of the system. The tracer method and the characteristics of the simulated system will be presented in Sec. II. The results for stationary vortices and transient vortices will be successively introduced in Secs. III and IV, respectively. The mechanisms at the origin of the resulting transfer enhancement will be discussed in Sec. V.

II. SIMULATING ADVECTION-DIFFUSION WITH THE TRACER METHOD

We consider an idealised two-dimensional fluid system having an intrinsic diffusivity D_0 (m^2/s) and comprising a rotational vortex. This rotational vortex is circular, centred at $\vec{x}_c$, of radius R (m) and rotation frequency f (s^{-1}). Inside the vortex, the velocity field is given by $v_\theta(r) = 2\pi f r/R$ where r is the distance to the vortex centre, and $v_\theta(r)$ the angular velocity. Outside the vortex, for $r > R$, $v_\theta = 0$ (see Figure 1(a)). The radial velocity v_r is null both inside and outside the vortex.

In systems where both diffusion and advection occur simultaneously, the local evolution of the temperature or concentration field, here called c , satisfies the advection-diffusion equation. The idealized fluid we consider is incompressible and has a homogeneous and isotropic diffusivity D_0 . As is the case in dense granular flows, neither the diffusivity D_0 or the flow velocity field depend on the local value of c . Accordingly, the advection-diffusion equation reads

$$\dot{c} = D_0 \nabla^2 c - \vec{v} \cdot \nabla c. \quad (1)$$

$\vec{v}(t, \vec{x})$ is the local, possibly time dependent velocity field. Given a set of boundary and initial conditions for c , it is possible to simulate the local evolution of $c(t, \vec{x})$ according to the advection-diffusion equation (1). Methods based on a discretization of the gradient operator are available for this purpose.^{24–28}

We chose an alternative simulation method, coined the *tracer method* to investigate the effective transfer properties of these systems.^{21,29–32} Its principle is to inject a large number of passive Brownian tracers into the flow. Each tracer is advected by the local flow field but it is also assigned a random velocity characterised by a vector of constant designated norm v_0 and an orientation that changes randomly after each time step dt . The norm of the random velocity is set to achieve the desired intrinsic diffusivity, D_0 according to (in two-dimensions): $v_0 = 2\sqrt{D_0/dt}$. To produce an isotropic diffusivity, the orientation is randomly selected within the range $0-2\pi$ with a uniform likelihood.

FIG. 1. (a) Flow system for a stationary vortex. Blue arrows indicate the velocity of the tracers which are initially randomly distributed. (b) and (c) Mean square displacements versus time obtained for two systems with the same vortex ($R = 0.1$ m, $f = 1$ s $^{-1}$) but with different intrinsic diffusivity $D_0 = 10^{-8}$ m 2 /s (b) and $D_0 = 10^{-4}$ m 2 /s (c). The bottom (green) curve corresponds to the mean square displacements obtained in the absence of a vortex. Pictures represent the location of tracers tagged initially in blue and red in the bottom and top halves of the system.

The effective transfer properties of the system can simply be measured by tracking the tracers' trajectories and measuring their mean square displacement λ (m 2) which is defined at any time t as: $\lambda(t) = \frac{1}{N} \sum_{i=1}^N \|\vec{x}_i(t) - \vec{x}_i(0)\|^2$. The sum runs over all the tracers in the system, and $\vec{x}_i(t)$ refers to the position of tracer i . In a diffusive system, the mean square displacements increase linearly with time and the effective diffusivity D^{eff} (m 2 /s) is defined as $\lambda(t) = 4D^{eff}t$. In the absence of a vortex, $D^{eff} = D_0$. For such systems, the results from the method were validated against closed form analytic solutions. In the following, we investigate how the presence of a vortex changes the effective diffusivity of the system.

All simulations comprise 10 000 tracers initially randomly displaced in a square, two-dimensional domain of size $2R$, R being the radius of the vortex. The domain is periodic in both directions; tracers that leave the domain through one boundary, re-enter it through the opposite boundary. The tracer displacements are simulated over short time steps $dt = 10^{-5}$ s. We systematically checked that smaller time steps and/or more tracers produce the same results. The integration scheme comprises a first order Eulerian integration of the random motion, followed by a rigid body rotation of the tracers inside the vortex. This scheme is applied to each tracer at each time step.

III. TRANSFER ENHANCEMENT WITH A STATIONARY VORTEX

We first consider a rotational vortex with a fixed location, coined the *stationary vortex* (see Figure 1(a)). In this configuration, the adjustable parameters are the intrinsic diffusivity D_0 , the vortex rotation frequency f , and the vortex radius R which were varied in the ranges specified in Table I. For each run, the tracer motions were simulated over a period of time corresponding to 100 vortex rotations.

Figures 1(b) and 1(c) show the evolution over time of the mean square displacement for two systems with the same vortex ($R = 0.1$ m, $f = 1$ s $^{-1}$) and two different intrinsic diffusivities ($D_0 = 10^{-4}$ m 2 /s, $D_0 = 10^{-8}$ m 2 /s). The mean square displacements increase on average linearly

TABLE I. Range of parameters investigated for a stationary vortex (lifetime infinity).

D_0 (m 2 /s)	f (s $^{-1}$)	R (m)
$10^{-8} \rightarrow 10^{-4}$	$0.1 \rightarrow 10$	$10^{-3} \rightarrow 1$

FIG. 2. Effective diffusivity D^{eff} for a stationary vortex as a function of the Péclet number $Pe = R^2 f / D_0$. The data were obtained for different intrinsic diffusivities D_0 , vortex rotation frequencies f , and radii R (ranges reported in Table I).

in time with oscillations which correspond to the vortex rotation. The values of the mean square displacements taken after each rotation were used to determine the effective diffusivity.

The effective diffusivity measured using this method was obtained for different vortex sizes, rotation frequencies, and intrinsic diffusivities. The ratio D^{eff}/D_0 was used to quantify the transfer enhancement attributable to the presence of a vortex. This ratio corresponds to either the Nusselt number for heat transfers, or to the Sherwood number for mass transfers. The values obtained for this ratio range from 1 (no enhancement) to several orders of magnitude. When plotted as a function of the Péclet number $Pe = R^2 f / D_0$ (see Figure 2), the data collapse onto the same curve.

This scaling is observed for a wide range of intrinsic diffusivities, vortex sizes, and rotation frequencies. Figure 2 also reveals an interesting trend, exhibiting two main features. Namely, the Péclet number dichotomizes two regimes: there is no transfer enhancement for $Pe < 1$ ($D^{\text{eff}} = D_0$), and there is some (potentially considerable) transfer enhancement for $Pe > 1$. For $Pe > 1$, the effective diffusivity scales like: $\frac{D^{\text{eff}}}{D_0} \approx Pe^{\frac{1}{2}}$.

IV. TRANSFER ENHANCEMENT WITH A TRANSIENT VORTEX

Let us now consider the transfer properties of a system comprising a transient rotational vortex. After a designated lifetime ℓ , the vortex centre moves instantaneously to a new, random location akin to the transient vortices observed in granular flows.

A lifetime $\ell = 1/4$ means that a vortex completes a quarter of a rotation before its centre changes position. We ran simulations covering a large range of intrinsic diffusivities D_0 , vortex sizes R , rotation frequencies f , and different lifetimes between $1/8$ and 1 (see Table II).

Again the mean square displacement is found to increase linearly in time (see Figure 3). The effective diffusion coefficient can thus be measured directly from its slope. Figure 4(a) presents the effective diffusivity obtained for differing lifetimes. The effective diffusivity is almost constant for lifetimes shorter than $1/2$ a rotation, and tends to decrease for longer lifetimes. This decrease becomes steeper for high Péclet numbers. Figure 4(b) shows that the behavior of the ratio D^{eff}/D_0 is also governed by the Péclet number. For a given lifetime, the effective diffusivities of systems of differing intrinsic diffusivity, different vortex size and rotation frequency, collapse onto the same

TABLE II. Range of parameters investigated for a transient vortex.

D_0 (m ² /s)	f (s ⁻¹)	R (m)	ℓ
$10^{-8} \rightarrow 10^{-4}$	$1 \rightarrow 10$	$10^{-3} \rightarrow 1$	$1/8 \rightarrow 1$

FIG. 3. (Top) Velocity fields and (bottom) tracer positions for transient vortices at $t = 0$ s and $t = 50$ s. Mean square displacement of the tracers versus time.

curve

$$\frac{D^{eff}}{D_0} \approx Pe^n. \quad (2)$$

However, different lifetimes result in a different exponent n . This exponent is close to 1 for lifetimes shorter than half a rotation, and decreases toward 0.5 for lifetimes between half a rotation and a complete rotation, see Figure 4(c).

V. DISCUSSION

The above results show that the effective diffusivity of a system equals the intrinsic diffusivity D_0 for low Péclet numbers, and, for higher values, scales like $D^{eff}/D_0 \approx Pe^n$ with $n = 0.5$ for stationary vortices and $0.5 < n < 1$ for transient vortices. In this section, we seek to understand the origin of such dependencies.

Let us first consider the regime of low Péclet numbers, approximately $Pe < 1$, for which neither stationary or transient vortices have a significant effect on transfers. This result can be understood by comparing the time scale associated with vortex advection, f^{-1} , and the time scale associated with tracers diffusion across a vortex, R^2/D_0 . For $Pe = R^2 f/D_0 < 1$, the diffusion time is much shorter than the advection time. As a consequence, in this regime, tracers diffuse across a vortex before it

FIG. 4. (a) Effective diffusivity versus vortex lifetime obtained with transient rotational vortex systems of different intrinsic diffusivity D_0 , vortex size R , and rotation frequency f (ranges specified in Table II). (b) Same data versus Péclet number. The data scale like $D^{eff}/D_0 \approx Pe^n$. (c) Exponent n versus lifetime ℓ .

can significantly advect them, and the contribution of the vortex to the transfer is negligible. For $Pe > 1$, however, advection does significantly contribute to the transfer, as we will now discuss.

For a stationary vortex, let us consider the positions of the tracers at the end of each cycle of rotation of the vortex. The tracers that always remain outside or inside the vortex are not affected by the flow. Only the tracers that cross the vortex boundary during the rotation, coined *the crossers*, contribute toward enhancing the transfer since their displacement at the end of a cycle is much larger than if they only moved due to their intrinsic diffusion. The typical displacement of such a tracer during one cycle is R and its displacement occurs at a frequency f . Thus, the diffusion coefficient averaged over all tracers is

$$D^{eff} \propto R^2 f \mathcal{P}, \quad (3)$$

where $\mathcal{P}$ is the proportion of *crossers*. These crosser tracers are located within a distance $\delta \propto \sqrt{D_0/f}$ from the vortex boundary. Other tracers do not have enough time or do not diffuse fast enough to reach the vortex boundary within one rotation. Consequently, crossers are located within a ring of inner radius $R - \delta$ and outer radius $R + \delta$. The proportion of crossers thus scales like the ratio of the ring area to the total area of the system, $\mathcal{P} \propto R\delta/R^2 = \sqrt{D_0/f}/R = Pe^{-\frac{1}{2}}$. Introducing this proportion in (3) leads to the measured scaling for stationary vortices, $D^{eff}/D_0 \approx Pe^{\frac{1}{2}}$.

Note that a similar scaling with a power exponent of a half has been reported for non-rotational vortices.^{33,34} This suggests that the mechanism just described could also apply to these non-rotational vortices.

For transient vortices, the lifetime plays a central role in the power law $D^{eff}/D_0 \approx Pe^n$ (see Figures 4(b) and 4(c)). The exponent n is close to 1 for lifetimes shorter than half a rotation and drops to 0.5 for a lifetime of one rotation. The tracers advected by a vortex of lifetime ℓ typically move a distance $2R \sin(\pi\ell)$ during one lifetime. During each subsequent vortex lifetime, tracers experience the same kind of displacement, but with a new and random centre of rotation. The tracer motions during a series of lifetimes can thus be seen as a macroscopic random walk with a typical lengthscale $2R \sin(\pi\ell)$, and a frequency of step f/ℓ . This produces an effective diffusivity scaling like: $D^{eff} \propto R^2 f \sin^2(\pi\ell)/\ell$ which is consistent with the scaling $D^{eff}/D_0 \propto Pe$ obtained for short lifetimes.

For lifetimes longer than half a rotation, the typical step of the macroscopic random walk, $2R \sin(\pi\ell)$, decreases. It vanishes for a lifetime of $\ell = 1$ rotation, and this macroscopic random walk then has no effect on the effective diffusivity. In this case, the tracer motions during one rotation involve the same mechanism described for stationary vortices. These two systems thus have the same exponent $n \approx 0.5$.

Transient vortices with a short lifetime lead to a power law $n \approx 1$. The corresponding effective diffusivity can thus be expressed as $D^{eff} \approx R^2 f$. As a result, these vortices produce an effective diffusive behavior even if the intrinsic diffusivity is null, $D_0 = 0$. This is a consequence of the macroscopic random walk described above. This property is not shared with stationary vortices and transient vortices of lifetime one. With a power $n \approx 1/2$, their effective diffusivity can be expressed as $D^{eff} \approx R \sqrt{D_0 f}$. It is therefore null when D_0 is null. This is consistent with the fact that, with no intrinsic diffusivity, tracers advected by these vortices follow closed loop pathlines. After each rotation, they return to their initial positions, and the vortex alone does not lead to any transfer enhancement.

VI. CONCLUSION

By using a *Brownian tracer method*, we quantified how the transfer properties of a 2D fluid system are affected by the presence of a rotational vortex. Systematic simulations with differing intrinsic diffusivities D_0 , vortex sizes R , rotation frequencies f , and lifetimes ℓ revealed the following features: (i) These systems which involve both advection and diffusion exhibit a macroscopic diffusive behavior that can be characterized by an effective diffusion coefficient. (ii) The effective diffusivity is governed by the Péclet number $Pe = R^2 f/D_0$. (iii) The Péclet number dichotomizes two regimes. For $Pe < 1$, the vortex produces no significant transfer enhancement: $D^{eff} \approx D_0$. For $Pe > 1$,

the vortex does produce a significant transfer enhancement, and the ratio D^{eff}/D_0 increases as a power law of the Péclet number: $D^{\text{eff}}/D_0 \approx Pe^n$. The value of the power n ranges from a lower bound of 1/2 to an upper bound of 1 depending on the lifetime of the vortex.

The two extreme values of the power law exponent, i.e., $n = 1$ and $n = 0.5$, correspond to two distinct mechanisms that enhance transfers. The value $n \approx 1$ obtained for transient vortices with lifetimes shorter than half a rotation, results from a large scale random walk of the tracers induced by the transient vortices. By contrast, the value $n \approx 1/2$ obtained for stationary vortices and vortices with a lifetime of 1 rotation, involves a mechanism coupling the intrinsic diffusion and the vortex advection. In this case, only tracers located within a ring around the vortex boundary contribute to the transfer enhancement. The thickness of this ring is determined by the vortex rotation frequency, and the intrinsic diffusivity of the tracers.

This study, performed using a 2D model system, constitutes a first step toward a better understanding of the mechanisms underlying the transfer properties of fluid systems comprising rotational vortices, such as stirred flows and dense granular flows. Further investigations will be undertaken for more complex velocity fields such as irrotational vortices present in turbulent flows and actual velocity fields directly measured in granular systems. It is thought that the specific properties of the involved vortices, including their three-dimensional shape, their internal velocity field, and their spatial distribution may lead to a different advection diffusion coupling and different power law exponents n .

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