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Abstract In this work, we introduce a novel algorithm for the quasi-static nonlinear poroelasticity problem describing Darcian flow in a deformable saturated porous medium. The nonlinear elasticity operator is discretized using a Hybrid High-Order method while the heterogeneous diffusion part relies on a Symmetric Weighted Interior Penalty discontinuous Galerkin scheme. The method is valid in two and three space dimensions, delivers an inf-sup stable discretization on general meshes including polyhedral elements and nonmatching interfaces, allows arbitrary approximation orders, and has a reduced cost thanks to the possibility of statically condensing a large subset of the unknowns for linearized versions of the problem. Moreover, the proposed construction can handle rough variations of the permeability coefficient and vanishing specific storage coefficient. Numerical tests demonstrating the performance of the method are provided.

Key words: nonlinear poroelasticity, Hybrid High-Order, discontinuous Galerkin, general meshes

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1 Introduction

We consider in this work the nonlinear poroelasticity model obtained by generalizing the linear Biot's consolidation model of [1, 7] to nonlinear stress-strain constitutive laws. Our original motivation comes from applications in geosciences, where the support of polyhedral meshes and nonconforming interfaces is crucial.

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, denote a bounded connected polyhedral domain with Lipschitz boundary $\partial\Omega$ and outward normal $\mathbf{n}$. For a given finite time $t_\mathrm{F} > 0$, volumetric load $\mathbf{f}$, fluid source g, the considered nonlinear poroelasticity problem consists in finding a vector-valued displacement field $\mathbf{u}$ and a scalar-valued pore pressure field p solution of

$$-\nabla\cdot\sigma(\cdot,\nabla_\mathrm{s}\mathbf{u}) + \nabla p = \mathbf{f} \qquad \text{in } \Omega \times (0,t_\mathrm{F}), \tag{1a}$$
$$c_0\partial_t p + \partial_t \nabla\cdot\mathbf{u} - \nabla\cdot(\kappa(\cdot)\nabla p) = g \qquad \text{in } \Omega \times (0,t_\mathrm{F}), \tag{1b}$$

where ∇_s denotes the symmetric gradient, $c_0 \geq 0$ is the constrained specific storage coefficient, and $\kappa : \Omega \to (0,\overline{\kappa}]$ is the scalar-valued permeability field. Equations (1a) and (1b) express, respectively, the momentum equilibrium and the fluid mass balance. For the sake of simplicity, we assume that κ is piecewise constant on a partition P_Ω of Ω into bounded disjoint polyhedra and we consider the following homogeneous boundary conditions:

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \partial\Omega \times (0,t_\mathrm{F}), \tag{1c}$$
$$(\kappa(\cdot)\nabla p)\cdot\mathbf{n} = 0 \qquad \text{on } \partial\Omega \times (0,t_\mathrm{F}). \tag{1d}$$

The treatment of more general permeability fields and boundary conditions is possible up to minor modifications. Initial conditions are set prescribing $\mathbf{u}(\cdot,0) = \mathbf{u}^0$ and, if $c_0 > 0$, $p(\cdot,0) = p^0$. In the incompressible case $c_0 = 0$, we also need the following compatibility condition on g and zero-average constraint on p:

$$\int_\Omega g(\cdot,t) = 0 \quad \text{and} \quad \int_\Omega p(\cdot,t) = 0 \qquad \forall t \in (0,t_\mathrm{F}). \tag{1e}$$

We assume that the symmetric stress tensor $\sigma : \Omega \times \mathbb{R}^{d\times d}_\mathrm{sym} \to \mathbb{R}^{d\times d}_\mathrm{sym}$ is a Caratheodory function such that there exist real numbers $\overline{\sigma}, \underline{\sigma} \in (0,+\infty)$ and, for a.e. $\mathbf{x} \in \Omega$, and all $\tau, \eta \in \mathbb{R}^{d\times d}_\mathrm{sym}$, the following conditions hold:

$$\|\sigma(\mathbf{x},\tau) - \sigma(\mathbf{x},\mathbf{0})\|_{d\times d} \leq \overline{\sigma}\|\tau\|_{d\times d}, \qquad \text{(growth)} \tag{2a}$$
$$\sigma(\mathbf{x},\tau) : \tau \geq \underline{\sigma}\|\tau\|^2_{d\times d}, \qquad \text{(coercivity)} \tag{2b}$$
$$(\sigma(\mathbf{x},\tau) - \sigma(\mathbf{x},\eta)) : (\tau - \eta) \geq 0, \qquad \text{(monotonicity)} \tag{2c}$$

where $\tau : \eta := \sum_{i,j=1}^d \tau_{i,j}\eta_{i,j}$ and $\|\tau\|^2_{d\times d} = \tau : \tau$.

2 Mesh and notation

Denote by $\mathscr{H} \subset \mathbb{R}_*^+$ a countable set having 0 as unique accumulation point. We consider refined mesh sequences $(\mathscr{T}_h)_{h\in\mathscr{H}}$ where each $\mathscr{T}_h$ is a finite collection of disjoint open polyhedral elements T with boundary ∂T such that $\overline{\Omega} = \bigcup_{T\in\mathscr{T}_h} \overline{T}$ and $h = \max_{T\in\mathscr{T}_h} h_T$ with h_T diameter of T. We assume that mesh regularity holds in the sense of [4, Definition 1.38] and that, for all $h \in \mathscr{H}$, $\mathscr{T}_h$ is compatible with the partition P_Ω on which the permeability coefficient κ is piecewise constant, so that jumps of the permeability coefficient do not occur inside mesh elements.

Mesh faces are hyperplanar subsets of $\overline{\Omega}$ with positive $(d-1)$-dimensional Hausdorff measure and disjoint interiors. Interfaces are collected in the set $\mathscr{F}_h^{\mathrm{i}}$, boundary faces in $\mathscr{F}_h^{\mathrm{b}}$, and we assume that $\mathscr{F}_h := \mathscr{F}_h^{\mathrm{i}} \cup \mathscr{F}_h^{\mathrm{b}}$ is such that $\bigcup_{T\in\mathscr{T}_h} \partial T = \bigcup_{F\in\mathscr{F}_h} F$. For all $T \in \mathscr{T}_h$, $\mathscr{F}_T := \{F \in \mathscr{F}_h \mid F \subset \partial T\}$ denotes the set of faces contained in ∂T and, for all $F \in \mathscr{F}_T$, $\mathbf{n}_{TF}$ is the unit normal to F pointing out of T.

For $X \subset \overline{\Omega}$, we denote by $\|\cdot\|_X$ the norm in $L^2(X;\mathbb{R})$, $L^2(X;\mathbb{R}^d)$, and $L^2(X;\mathbb{R}^{d\times d})$. For $l \geq 0$, the space $\mathbb{P}^l(X;\mathbb{R})$ is spanned by the restriction to X of polynomials of total degree l. On regular mesh sequences, we have the following optimal approximation property for the L^2-projector $\pi_X^l : L^1(X;\mathbb{R}) \to \mathbb{P}^l(X;\mathbb{R})$: There exists $C_{\mathrm{ap}} > 0$ such that, for all $h \in \mathscr{H}$, all $T \in \mathscr{T}_h$, all $s \in \{1,\ldots,l+1\}$, and all $v \in H^s(T;\mathbb{R})$,

$$|v - \pi_T^l v|_{H^m(T;\mathbb{R})} \leq C_{\mathrm{ap}} h_T^{s-m} |v|_{H^s(T;\mathbb{R})} \qquad \forall m \in \{0,\ldots,s\}. \tag{3}$$

Other geometric and analytic results on regular meshes can be found in [4, Chapter 1] and [3]. In what follows, for an integer $l \geq 0$, we denote by $\mathbb{P}^l(\mathscr{T}_h;\mathbb{R})$, $\mathbb{P}^l(\mathscr{T}_h;\mathbb{R}^d)$, and $\mathbb{P}^l(\mathscr{T}_h;\mathbb{R}^{d\times d})$, respectively, the space of scalar-, vector-, and tensor-valued broken polynomials of total degree l on $\mathscr{T}_h$. The space of broken vector-valued polynomials of total degree l on the mesh skeleton is denoted by $\mathbb{P}^l(\mathscr{F}_h;\mathbb{R}^d)$.

3 Discretization

In this section we define the discrete counterparts of the elasticity and Darcy operators and of the hydro-mechanical coupling terms.

3.1 Nonlinear elasticity operator

The discretization of the nonlinear elasticity operator is based on the Hybrid High-Order method of [5]. Let a polynomial degree $k \geq 1$ be fixed. The degrees of freedom (DOFs) for the displacement are collected in the space $\underline{\mathbf{U}}_h^k := \mathbb{P}^k(\mathscr{T}_h;\mathbb{R}^d) \times \mathbb{P}^k(\mathscr{F}_h;\mathbb{R}^d)$. To account for the Dirichlet condition (1c) we define the subspace

$$\underline{\mathbf{U}}_{h,0}^k := \left\{ \underline{\mathbf{v}}_h := \left((\mathbf{v}_T)_{T\in\mathscr{T}_h}, (\mathbf{v}_F)_{F\in\mathscr{F}_h}\right) \in \underline{\mathbf{U}}_h^k \mid \mathbf{v}_F = \mathbf{0} \quad \forall F \in \mathscr{F}_h^{\mathrm{b}} \right\},$$

equipped with the discrete strain norm

$$\|\underline{\mathbf{v}}_h\|_{\varepsilon,h} := \left(\sum_{T\in\mathscr{T}_h} \|\underline{\mathbf{v}}_h\|_{\varepsilon,T}^2\right)^{1/2}, \quad \|\underline{\mathbf{v}}_h\|_{\varepsilon,T}^2 := \|\nabla_{\rm s}\mathbf{v}_T\|_T^2 + \sum_{F\in\mathscr{F}_T} \frac{\|\mathbf{v}_F - \mathbf{v}_T\|_F^2}{h_F}.$$

The DOFs corresponding to a function $\mathbf{v}\in H_0^1(\Omega;\mathbb{R}^d)$ are obtained by means of the reduction map $\underline{\mathbf{I}}_h^k : H_0^1(\Omega;\mathbb{R}^d) \to \underline{\mathbf{U}}_{h,0}^k$ such that $\underline{\mathbf{I}}_h^k\mathbf{v} := \big((\pi_T^k\mathbf{v})_{T\in\mathscr{T}_h}, (\pi_F^k\mathbf{v})_{F\in\mathscr{F}_h}\big)$. Using the H^1-stability of the L^2-projector and the trace inequality [4, Lemma 1.49], we infer the existence of $C_{\rm st}>0$ independent of h such that, for all $\mathbf{v}\in H_0^1(\Omega;\mathbb{R}^d)$,

$$\|\underline{\mathbf{I}}_h^k\mathbf{v}\|_{\varepsilon,h} \le C_{\rm st}\|\mathbf{v}\|_{H^1(\Omega;\mathbb{R}^d)}. \tag{4}$$

For all $T\in\mathscr{T}_h$, we denote by $\underline{\mathbf{U}}_T^k$ and $\underline{\mathbf{I}}_T^k$ the restrictions to T of $\underline{\mathbf{U}}_h^k$ and $\underline{\mathbf{I}}_h^k$, and we define the local symmetric gradient reconstruction $\mathsf{G}_{{\rm s},T}^k : \underline{\mathbf{U}}_T^k \to \mathbb{P}^k(T;\mathbb{R}^{d\times d}_{\rm sym})$ as the unique solution of the pure traction problem: For a given $\underline{\mathbf{v}}_T = \big(\mathbf{v}_T, (\mathbf{v}_F)_{F\in\mathscr{F}_T}\big) \in \underline{\mathbf{U}}_T^k$, find $\mathsf{G}_{{\rm s},T}^k\underline{\mathbf{v}}_T \in \mathbb{P}^k(T;\mathbb{R}^{d\times d}_{\rm sym})$ such that, for all $\tau\in\mathbb{P}^k(T;\mathbb{R}^{d\times d}_{\rm sym})$,

$$\int_T \mathsf{G}_{{\rm s},T}^k\underline{\mathbf{v}}_T : \tau = -\int_T \mathbf{v}_T\cdot(\nabla\cdot\tau) + \sum_{F\in\mathscr{F}_T}\int_F \mathbf{v}_F\cdot(\tau\mathbf{n}_{TF}). \tag{5}$$

The definition of $\mathsf{G}_{{\rm s},T}^k$ is justified by the following commuting property that, combined with (3), shows that $\mathsf{G}_{{\rm s},T}^k\underline{\mathbf{I}}_T^k$ has optimal approximation properties in $\mathbb{P}^k(T;\mathbb{R}^{d\times d}_{\rm sym})$.

Lemma 1. *For all $T\in\mathscr{T}_h$ and all $\mathbf{v}\in H^1(T;\mathbb{R}^d)$, $\mathsf{G}_{{\rm s},T}^k\underline{\mathbf{I}}_T^k\mathbf{v} = \pi_T^k(\nabla_{\rm s}\mathbf{v})$.*

Proof. Let $T\in\mathscr{T}_h$ and $\mathbf{v}\in H^1(T;\mathbb{R}^d)$. For all $\tau\in\mathbb{P}^k(T;\mathbb{R}^{d\times d}_{\rm sym})$, we have

$$\begin{aligned}\int_T \mathsf{G}_{{\rm s},T}^k\underline{\mathbf{I}}_T^k\mathbf{v} : \tau &= -\int_T \pi_T^k\mathbf{v}\cdot(\nabla\cdot\tau) + \sum_{F\in\mathscr{F}_T}\int_F \pi_F^k\mathbf{v}\cdot(\tau\mathbf{n}_{TF}) \\ &= -\int_T \mathbf{v}\cdot(\nabla\cdot\tau) + \sum_{F\in\mathscr{F}_T}\int_F \mathbf{v}\cdot(\tau\mathbf{n}_{TF}) = \int_T \pi_T^k(\nabla_{\rm s}\mathbf{v}) : \tau. \quad \square\end{aligned}$$

From $\mathsf{G}_{{\rm s},T}^k$ we define the local displacement reconstruction operator $\mathbf{r}_T^{k+1} : \underline{\mathbf{U}}_T^k \to \mathbb{P}^{k+1}(T;\mathbb{R}^d)$ such that, for all $\underline{\mathbf{v}}_T\in\underline{\mathbf{U}}_T^k$ and all $\mathbf{w}\in\mathbb{P}^{k+1}(T;\mathbb{R}^d)$, it holds

$$\int_T (\nabla_{\rm s}\mathbf{r}_T^{k+1}\underline{\mathbf{v}}_T - \mathsf{G}_{{\rm s},T}^k\underline{\mathbf{v}}_T) : \nabla_{\rm s}\mathbf{w} = 0$$

$$\int_T \mathbf{r}_T^{k+1}\underline{\mathbf{v}}_T = \int_T \mathbf{v}_T, \qquad \int_T \nabla_{\rm ss}\mathbf{r}_T^{k+1}\underline{\mathbf{v}}_T = \sum_{F\in\mathscr{F}_T}\int_F \tfrac{1}{2}\,(\mathbf{n}_{TF}\otimes\mathbf{v}_F - \mathbf{v}_F\otimes\mathbf{n}_{TF}),$$

where $\nabla_{\rm ss}$ denotes the skew-symmetric part of the gradient operator.

The discretization of the nonlinear elasticity operator is realized by the function $a_h : \underline{\mathbf{U}}_h^k\times\underline{\mathbf{U}}_h^k \to \mathbb{R}$ defined such that, for all $\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_h^k$,

$$a_h(\underline{\mathbf{w}}_h,\underline{\mathbf{v}}_h) := \sum_{T\in\mathscr{T}_h}\left(\int_T \sigma(\cdot,\mathsf{G}^k_{\mathrm{s},T}\underline{\mathbf{u}}_T):\mathsf{G}^k_{\mathrm{s},T}\underline{\mathbf{v}}_T + \sum_{F\in\mathscr{F}_T}\frac{\gamma}{h_F}\int_F \Delta^k_{TF}\underline{\mathbf{u}}_T\cdot\Delta^k_{TF}\underline{\mathbf{v}}_T\right), \quad (6)$$

where we penalize in a least-square sense the face-based residual $\Delta^k_{TF}:\underline{\mathbf{U}}^k_T\to\mathbb{P}^k(F;\mathbb{R}^d)$ such that, for all $T\in\mathscr{T}_h$, all $\underline{\mathbf{v}}_T\in\underline{\mathbf{U}}^k_T$, and all $F\in\mathscr{F}_T$,

$$\Delta^k_{TF}\underline{\mathbf{v}}_T := \pi^k_F(\mathbf{r}^{k+1}_T\underline{\mathbf{v}}_T-\mathbf{v}_F)-\pi^k_T(\mathbf{r}^{k+1}_T\underline{\mathbf{v}}_T-\mathbf{v}_T).$$

This definition ensures that Δ^k_{TF} vanishes whenever its argument is of the form $\underline{\mathbf{I}}^k_T\mathbf{w}$ with $\mathbf{w}\in\mathbb{P}^{k+1}(T;\mathbb{R}^d)$, a crucial property to obtain high-order error estimates (cf. [2, Theorem 12]). A possible choice for the scaling parameter $\gamma>0$ in (6) is $\gamma=\overline{\sigma}$. For all $\underline{\mathbf{v}}_h\in\underline{\mathbf{U}}^k_{h,0}$, it holds (the proof follows from [5, Lemma 4]):

$$C_{\mathrm{eq}}^{-1}\|\underline{\mathbf{v}}_h\|^2_{\varepsilon,h}\le\sum_{T\in\mathscr{T}_h}\left(\|\mathsf{G}^k_{\mathrm{s},T}\underline{\mathbf{v}}_T\|^2_T+\sum_{F\in\mathscr{F}_T}\frac{\gamma}{h_F}\|\Delta^k_{TF}\underline{\mathbf{v}}_T\|^2_F\right)\le C_{\mathrm{eq}}\|\underline{\mathbf{v}}_h\|^2_{\varepsilon,h},$$

where $C_{\mathrm{eq}}>0$ is independent of h. By (2b), this implies the coercivity of a_h.

3.2 Darcy operator

The discretization of the Darcy operator is based on the Symmetric Weighted Interior Penalty method of [6], cf. also [4, Sec. 4.5]. At each time step, the discrete pore pressure is sought in the broken polynomial space

$$P^k_h := \begin{cases}\mathbb{P}^k(\mathscr{T}_h;\mathbb{R}) & \text{if } c_0>0,\\ \mathbb{P}^k_0(\mathscr{T}_h;\mathbb{R}) := \left\{q_h\in\mathbb{P}^k(\mathscr{T}_h;\mathbb{R}) \mid \int_\Omega q_h=0\right\} & \text{if } c_0=0.\end{cases}$$

For all $F\in\mathscr{F}^{\mathrm{i}}_h$ and all $q_h\in\mathbb{P}^k(\mathscr{T}_h;\mathbb{R})$, we define the jump and average operators such that, denoting by q_T and κ_T the restrictions of q_h and κ to an element $T\in\mathscr{T}_h$,

$$[q_h]_F := q_{T_{F,1}}-q_{T_{F,2}}, \qquad \{q_h\}_F := \tfrac{\kappa_{T_{F,2}}}{\kappa_{T_{F,1}}+\kappa_{T_{F,2}}}q_{T_{F,1}}+\tfrac{\kappa_{T_{F,1}}}{\kappa_{T_{F,1}}+\kappa_{T_{F,2}}}q_{T_{F,2}},$$

where $T_{F,1},T_{F,2}\in\mathscr{T}_h$ are such that $F\subset T_{F,1}\cap T_{F,2}$. The bilinear form c_h on $P^k_h\times P^k_h$ is defined such that, for all $q_h,r_h\in P^k_h$,

$$\begin{aligned}c_h(r_h,q_h) :=& \int_\Omega\kappa\nabla_h r_h\cdot\nabla_h q_h+\sum_{F\in\mathscr{F}^{\mathrm{i}}_h}\tfrac{2\varsigma\kappa_{T_{F,1}}\kappa_{T_{F,2}}}{h_F(\kappa_{T_{F,1}}+\kappa_{T_{F,2}})}\int_F[r_h]_F[q_h]_F\\ &-\sum_{F\in\mathscr{F}^{\mathrm{i}}_h}\int_F\left([r_h]_F\{\kappa\nabla_h q_h\}_F+[q_h]_F\{\kappa\nabla_h r_h\}_F\right)\cdot\mathbf{n}_{T_{F,1}F},\end{aligned} \quad (7)$$

where ∇_h denotes the broken gradient and $\varsigma > 0$ is a user-defined penalty parameter chosen large enough to ensure the coercivity of c_h (cf. [4, Lemma 4.51]). In the numerical tests of Sec. 5, we took $\varsigma = (N_\partial + 0.1)k^2$, with N_∂ equal to the maximum number of faces between the elements in $\mathcal{T}_h$. The fact that the boundary terms only appear on internal faces in (7) reflects the Neumann boundary condition (1d).

3.3 Hydro-mechanical coupling

The hydro-mechanical coupling is realized by means of the bilinear form b_h on $\underline{\mathbf{U}}_{h,0}^k \times \mathbb{P}^k(\mathcal{T}_h;\mathbb{R})$ such that, for all $\underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_{h,0}^k$ and all $q_h \in \mathbb{P}^k(\mathcal{T}_h;\mathbb{R})$,

$$b_h(\underline{\mathbf{v}}_h, q_h) := -\sum_{T\in\mathcal{T}_h} \int_T \mathsf{G}_{\mathrm{s},T}^k \underline{\mathbf{v}}_T : q_{h|T} \mathsf{I}_d, \tag{8}$$

where $\mathsf{I}_d \in \mathbb{R}_{\mathrm{sym}}^{d\times d}$ is the identity matrix. A simple verification shows that there exists $C_{\mathrm{bd}} > 0$ independent of h such that $b_h(\underline{\mathbf{v}}_h, q_h) \le C_{\mathrm{bd}} \|\underline{\mathbf{v}}_h\|_{\varepsilon,h} \|q_h\|_\Omega$. Additionally, using definition (5) of $\mathsf{G}_{\mathrm{s},T}^k$, it can be proved that, for all $\underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_{h,0}^k$, $b_h(\underline{\mathbf{v}}_h, 1) = 0$. The following inf-sup condition expresses the stability of the coupling:

Proposition 1. *There is a real β independent of h such that, for all $q_h \in \mathbb{P}_0^k(\mathcal{T}_h;\mathbb{R})$,*

$$\|q_h\|_\Omega \le \beta \sup_{\underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_{h,0}^k \setminus \{\underline{\mathbf{0}}\}} \frac{b_h(\underline{\mathbf{v}}_h, q_h)}{\|\underline{\mathbf{v}}_h\|_{\varepsilon,h}}. \tag{9}$$

Proof. Let $q_h \in \mathbb{P}_0^k(\mathcal{T}_h;\mathbb{R})$. There is $\mathbf{v}_{q_h} \in H_0^1(\Omega;\mathbb{R}^d)$ such that $\nabla\cdot\mathbf{v}_{q_h} = q_h$ and $\|\mathbf{v}_{q_h}\|_{H^1(\Omega;\mathbb{R}^d)} \le C_{\mathrm{sj}} \|q_h\|_\Omega$, with $C_{\mathrm{sj}} > 0$ independent of h. Owing to (4) we get

$$\|\underline{\mathbf{I}}_h^k \mathbf{v}_{q_h}\|_{\varepsilon,h} \le C_{\mathrm{st}} \|\mathbf{v}_{q_h}\|_{H^1(\Omega;\mathbb{R}^d)} \le C_{\mathrm{st}} C_{\mathrm{sj}} \|q_h\|_\Omega.$$

Therefore, using the commuting property of Lemma 1, denoting by S the supremum in (9), and using the previous inequality, it is inferred that

$$\|q_h\|_\Omega^2 = \sum_{T\in\mathcal{T}_h} \int_T (\nabla_{\mathrm{s}} \mathbf{v}_{q_h} : q_h \mathsf{I}_d)_{|T} = -b_h(\underline{\mathbf{I}}_h^k \mathbf{v}_{q_h}, q_h) \le \mathsf{S} \|\underline{\mathbf{I}}_h^k \mathbf{v}_{q_h}\|_{\varepsilon,h} \le C_{\mathrm{st}} C_{\mathrm{sj}} \mathsf{S} \|q_h\|_\Omega. \square$$

If a HHO discretization were used also for the Darcy operator, only cell DOFs would be controlled by the inf-sup condition.

4 Formulation of the method

For the time discretization, we consider a uniform mesh of the time interval $(0, t_{\mathrm{F}})$ of step $\tau := t_{\mathrm{F}}/N$ with $N \in \mathbb{N}^*$, and introduce the discrete times $t^n := n\tau$ for all $0 \le$

$n \le N$. For any $\varphi \in C^l([0,t_F];V)$, we set $\varphi^n := \varphi(t^n) \in V$ and let, for all $1 \le n \le N$,

$$\delta_t \varphi^n := \frac{\varphi^n - \varphi^{n-1}}{\tau} \in V.$$

For all $1 \le n \le N$, the discrete solution $(\underline{\mathbf{u}}_h^n, p_h^n) \in \underline{\mathbf{U}}_{h,0}^k \times P_h^k$ at time t^n is such that, for all $(\underline{\mathbf{v}}_h, q_h) \in \underline{\mathbf{U}}_{h,0}^k \times \mathbb{P}^k(\mathscr{T}_h;\mathbb{R})$,

$$a_h(\underline{\mathbf{u}}_h^n, \underline{\mathbf{v}}_h) + b_h(\underline{\mathbf{v}}_h, p_h^n) = \sum_{T\in\mathscr{T}_h} \int_T \mathbf{f}^n \cdot \mathbf{v}_T, \tag{10a}$$

$$c_0 \int_\Omega (\delta_t p_h^n) q_h - b_h(\delta_t \underline{\mathbf{u}}_h^n, q_h) + c_h(p_h^n, q_h) = \int_\Omega g^n q_h. \tag{10b}$$

If $c_0 = 0$, we set the initial discrete displacement as $\underline{\mathbf{u}}_h^0 = \underline{\mathbf{I}}_h^k \mathbf{u}^0$. If $c_0 > 0$, the usual way to enforce the initial condition is to compute a displacement from the given initial pressure p^0. We let $p_h^0 := \pi_h^k p^0$ and set $\underline{\mathbf{u}}_h^0 \in \underline{\mathbf{U}}_{h,0}^k$ as the solution of

$$a_h(\underline{\mathbf{u}}_h^0, \underline{\mathbf{v}}_h) = \sum_{T\in\mathscr{T}_h} \int_T \mathbf{f}^0 \cdot \mathbf{v}_T - b_h(\underline{\mathbf{v}}_h, p_h^0) \qquad \forall \underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_{h,0}^k.$$

At each time step n the discrete nonlinear equations (10) are solved by the Newton's method using as initial guess the solution at step $n-1$. At each Newton's iteration the Jacobian matrix is computed analytically and in the linearized system the displacement element unknowns can be statically condensed (cf. [2, Section 5]).

5 Numerical results

We consider a regular exact solution in order to assess the convergence of the method for polynomial degree $k = 1$. Specifically, we solve problem (1) in the square domain $\Omega = (0,1)^2$ with $t_F = 1$ and physical parameters $c_0 = 0$ and $\kappa = 1$. As nonlinear constitutive law we take the Hencky–Mises relation given by

$$\sigma(\nabla_s \mathbf{u}) = (2e^{-\operatorname{dev}(\nabla_s \mathbf{u})} - 1)\operatorname{tr}(\nabla_s \mathbf{u})\mathsf{I}_d + (4 - 2e^{-\operatorname{dev}(\nabla_s \mathbf{u})})\nabla_s \mathbf{u},$$

where $\operatorname{tr}(\tau) := \tau : \mathsf{I}_d$ and $\operatorname{dev}(\tau) = \operatorname{tr}(\tau^2) - \frac{1}{d}\operatorname{tr}(\tau)^2$ are the trace and deviatoric operators. It can be checked that the previous stress-strain relation satisfies (2). The exact displacement $\mathbf{u}$ and exact pressure p are given by

$$\begin{aligned}
\mathbf{u}(\mathbf{x},t) &= t^2\big(\sin(\pi x_1)\sin(\pi x_2),\ \sin(\pi x_1)\sin(\pi x_2)\big),\\
p(\mathbf{x},t) &= -\pi^{-1} t(\sin(\pi x_1)\cos(\pi x_2) + \cos(\pi x_1)\sin(\pi x_2)).
\end{aligned}$$

The volumetric load $\mathbf{f}$, the source term g, and the boundary conditions are inferred from the exact solutions. The time step τ on the coarsest mesh is 0.2 and it decreases with the mesh size h according to $\tau_1/\tau_2 = 2h_1/h_2$. In Fig. 1 we display the convergence results obtained on two mesh families. The method exhibits second order convergence with respect to the mesh size h for both the energy norm of the displacement and the L^2-norm of the pressure at final time N. Further numerical tests, including higher-order approximation, will be considered in a future publication.

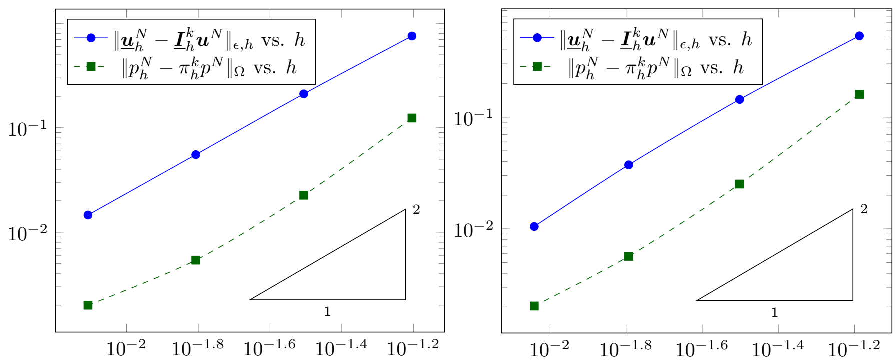

Fig. 1 Convergence tests on a Cartesian mesh family (left) and on a Voronoi mesh family (right).

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