Spectral spaces of countable abelian lattice-ordered groups

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Abstract. A compact topological space X is *spectral* if it is sober (i.e., every irreducible closed set is the closure of a unique singleton) and the compact open subsets of X form a basis of the topology of X, closed under finite intersections.

Theorem. *A topological space X is isomorphic to the spectrum of some countable Abelian ℓ-group with unit (resp., MV-algebra) iff X is spectral, has a countable basis of open sets, and for any points x and y in the closure of a singleton $\{z\}$, either x is in the closure of $\{y\}$ or y is in the closure of $\{x\}$.*

We establish this result by proving that a countable distributive lattice D with zero is isomorphic to the lattice of all principal ideals of an Abelian ℓ-group (we say that D is *ℓ-representable*) iff for all $a, b \in D$ there are $x, y \in D$ such that $a \vee b = a \vee y = b \vee x$ and $x \wedge y = 0$. On the other hand, we construct a non-ℓ-representable bounded distributive lattice, of cardinality $\aleph_1$, with an ℓ-representable countable $\mathscr{L}_{\infty,\omega}$-elementary sublattice. In particular, there is no characterization, of the class of all ℓ-representable distributive lattices, in arbitrary cardinality, by any class of $\mathscr{L}_{\infty,\omega}$ sentences.

1. Introduction

1.1. **Statement of the problem.** A *lattice-ordered group*, or *ℓ-group* for short, is a group G endowed with a translation-invariant lattice ordering. An *ℓ-ideal* of G is a convex, normal ℓ-subgroup I of G. We say that I is *prime* if $x \wedge y \in I$ implies that either $x \in I$ or $y \in I$, for all $x, y \in G$. We define the *ℓ-spectrum* of G as the set $\operatorname{Spec}_\ell G$ of all prime ℓ-ideals of G, endowed with the topology whose closed sets are exactly the sets $V_X \underset{\text{def}}{=} \{P \in \operatorname{Spec}_\ell G \mid X \subseteq P\}$, for $X \subseteq G$. Characterizing the spaces $\operatorname{Spec}_\ell G$, for *Abelian* ℓ-groups G, is a long-standing open problem. Although the work on that problem started more than twenty years ago, the first printed occurrence of its statement that we are aware of is Mundici [40, Problem 2], where it is stated in an equivalent form:

> *"Which topological spaces are homeomorphic to* $\operatorname{Spec}(A)$ *for some MV-algebra A?"*

To explain the connection, the concept of *MV-algebra* is tailored to describe, by a finite set of identities, the structure of intervals of the form $[0, u]$ in Abelian ℓ-groups, in terms of the operations $(x, y) \mapsto (x + y) \wedge u$ and $x \mapsto u - x$. There

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are natural concepts of ideal, then prime ideal, and thus also of spectrum (we will say *MV-spectrum*), in any MV-algebra, similar to those defined for Abelian ℓ-groups. In [39], Mundici constructs a category equivalence between the category of all Abelian ℓ-groups with unit, with unit-preserving ℓ-homomorphisms, and the variety (in the universal algebraic sense) of all MV-algebras (see also Marra and Mundici [35, 36]). Under that equivalence, the various concepts of ideal, prime ideal, and spectrum correspond, so the question about Abelian ℓ-groups with order-unit is equivalent to the one about MV-algebras.

1.2. **Completely normal spectral spaces.** The constructions, of the spectrum of an Abelian ℓ-group and the spectrum of an MV-algebra, are both particular cases of the following one. The spectrum of a distributive lattice D with zero can be defined in a similar way as the ℓ-spectrum of an Abelian ℓ-group. A lower subset I, in a distributive lattice D with zero, is an *ideal* if it is closed under finite joins (in particular, $0 \in I$). Further, we say that I is *prime* if $x \wedge y \in I$ implies that either $x \in I$ or $y \in I$, for all $x, y \in D$. This enables us to define the spectrum of D, and it is well known, since Stone [45], that spectra of bounded distributive lattices are exactly the so-called *spectral spaces* (cf. Definition 4.1). They are also the same as spectra of commutative, unital rings (cf. Hochster [27]). Moreover, it turns out that the ℓ-spectrum of an Abelian ℓ-group G is homeomorphic to the spectrum of the lattice $\mathrm{Id_c}\, G$ of all principal ℓ-ideals of G (see Section 4 for details).

Does every spectral space appear as the ℓ-spectrum of an Abelian ℓ-group? The answer has been known for a long time to be negative, and can be conveniently stated in terms of the *specialization order*. In any topological space X, let $x \leqslant y$ hold if y belongs to the closure of x, for all $x, y \in X$. The binary relation $\leqslant$ is a preorder on X, called the *specialization* (*pre*)*order on* X. It is antisymmetric iff X is T_0, which holds, in particular, if X is spectral.

A spectral space X is *completely normal*[1] if every principal filter of X is a chain, for every $p \in X$ (cf. Definition 4.3); that is, the specialization order is a *root system* (cf. Section 2.2). Not every spectral space is completely normal, but the ℓ-spectrum of any Abelian ℓ-group is completely normal. And then it turns out that completely normal spectral spaces also appear in the different context of *real spectra*: *The real spectrum of any commutative, unital ring is a completely normal spectral space* (cf. Coste and Roy [15], Dickmann [19]).

If we let go of the topology for a while, Cignoli and Torrens [14] characterized all posets (i.e., partially ordered sets) isomorphic to the specialization order on the MV-spectrum of some MV-algebra. An analogous result is proved in Dickmann, Gluschankof, and Lucas [20] about real spectra. It is noteworthy that both results represent the same class of root systems, called *spectral root systems*.

Is every completely normal spectral space an ℓ-spectrum? Delzell and Madden [17, Theorem 2] construct a completely normal spectral space, whose specialization order is a root system, which is not isomorphic to any MV-spectrum. However, that example is not *second countable* (i.e., its topology has no countable basis of open sets). The spectral space from Example 5.5, in the present paper, is similar, although not homeomorphic.

[1] In some references, a topological space X is completely normal if every subspace of X is normal. This definition is (strictly) stronger than the one used here, see Example 4.5.

How about the countable case? Trying to represent a second countable, completely normal spectral space X, a basic idea would be to express X as an inverse limit, for a suitable concept of morphism, of finite completely normal spectral spaces, then lift the arrows between individual building blocks, and then construct a direct limit (of Abelian ℓ-groups). Finite completely normal spectral spaces arise from finite root systems, and it has been known for a long time that those can be lifted by lexicographical powers of, say, the integers (see, for example, Bigard, Keimel, and Wolfenstein [8, Section 5.4]). However, such a plan would rely on the hope that our spectral space X be *profinite*. And Di Nola and Grigolia [18] found an example showing that this is not always the case; so we need to look elsewhere.

1.3. **Reduction to a problem about distributive lattices.** Any spectral space is determined, up to homeomorphism, by its (distributive) lattice of compact open subsets, and every distributive lattice with zero appears that way (Stone [45]; see Proposition 4.2). It follows that the problem, of characterizing ℓ-spectra of Abelian ℓ-groups, is equivalent to characterizing the distributive lattices with zero isomorphic to $\mathrm{Id_c}\, G$ for some Abelian ℓ-group G (cf. Lemma 4.7) — we shall call such lattices *ℓ-representable*. It follows that the question, of characterizing *second countable* ℓ-spectra, is equivalent to the following one:

Which countable distributive lattices are ℓ-representable?

Every ℓ-representable distributive lattice D satisfies a lattice-theoretical version of complete normality, which is equivalent to saying that for all $a, b \in D$ there are $x, y \in D$ such that $a \vee b = a \vee y = x \vee b$ and $x \wedge y = 0$ (Definition 4.3), and also to saying that the spectrum (of our distributive lattice) is completely normal (cf. Proposition 4.4). Completely normal lattices are studied in depth (under the name "relatively normal lattices") in Snodgrass and Tsinakis [43, 44]. Delzell and Madden's aforecited example shows that there are uncountable, non-ℓ-representable completely normal bounded distributive lattices. The most notable positive ℓ-representability result so far, namely Cignoli, Gluschankof, and Lucas [13, Theorem 3.3] (where the result is stated in terms of spectra), states that *Every completely normal distributive lattice D with zero, such that for all $a, b \in D$ there exists a smallest $x \in D$ with $a \leq b \vee x$, is ℓ-representable*; see also Iberkleid, Martínez, and McGovern [29, Theorem 3.1.1] (where the result is stated in lattice-theoretical terms).

Our main result is the following:

Theorem 11.1. *Every countable, completely normal distributive lattice with zero is ℓ-representable.*

As immediate corollaries of Theorem 11.1, we mention the following:

(1) *A second countable spectral space is an ℓ-spectrum iff it is completely normal* (Corollary 11.2);
(2) *For any countable ℓ-group G, there is a countable Abelian ℓ-group A such that the lattices of convex ℓ-subgroups of G and A are isomorphic* (Corollary 11.3);
(3) *Every second countable real spectrum is homeomorphic to some ℓ-spectrum* (Corollary 11.4).

1.4. **Overview of the paper.** In **Section 2**, we review the basic facts required in the proof of Theorem 11.1, mostly about distributive lattices, Heyting algebras, and topological vector spaces. In **Section 3**, we do the same for Abelian ℓ-groups, in particular recalling the classical description of free Abelian ℓ-groups and their ℓ-ideals, arising from the Baker-Beynon duality. In **Section 4** we present an overview of (generalized) spectral spaces. This enables us to reduce our problems, on ℓ-spectra, to problems on distributive lattices with zero (Lemma 4.7).

Section 5 contains a few non-ℓ-representability results, mainly

(1) *The class of all ℓ-representable distributive lattices is neither closed under infinite products* (Proposition 5.3), *nor under homomorphic images* (Example 5.6);
(2) *There is a non-ℓ-representable bounded distributive lattice, of cardinality $\aleph_1$, with an ℓ-representable countable $\mathscr{L}_{\infty,\omega}$-elementary sublattice* (Example 5.5).

As usual, $\mathscr{L}_{\infty,\omega}$ denotes the extension of first-order logic obtained by allowing infinite conjunctions and disjunctions (cf. Keisler and Knight [33], Bell [3]).

Section 6 introduces the crucial lattice-theoretical concepts of *consonance*, defined as a local version of complete normality, and of a *difference operation*. The latter concept, inspired by the dimension monoid construct of Wehrung [46], is designed to approximate the above-mentioned property, not valid in all completely normal distributive lattices, that for all a, b there is a smallest x such that $a \leq b \vee x$.

Section 7 introduces the main lattice-theoretical extension results required in our proof of Theorem 11.1, most notably the technical Lemma 7.3. Those extension results can be roughly described as follows. We are given a finite distributive lattice E, a bounded sublattice — in fact a Heyting subalgebra — D of E, a completely normal distributive lattice L with zero, and a 0-lattice homomorphism $f \colon D \to L$. We find convenient sufficient conditions for the existence of an extension of f to a homomorphism from E to L. In Lemma 7.3, those conditions are stated in terms of consonance of f and the join-irreducible elements of D.

Section 8 introduces the finite distributive lattices coming in replacement for the missing "completely normal finite building blocks" whose non-existence follows from Di Nola and Grigolia [18] (cf. Section 1.2). Our building blocks, typically denoted by $\mathrm{Op}(\mathcal{H})$, are generated by the open half-spaces associated with finitely many closed hyperplanes in a topological vector space $\mathbb{E}$. They are finite Heyting subalgebras of the lattice of all open subsets of $\mathbb{E}$. They are not, in general, completely normal.

The join-irreducible members of $\mathrm{Op}(\mathcal{H})$ are further investigated in **Section 9**. An important observation is that *Every join-irreducible member P of* $\mathrm{Op}(\mathcal{H})$ *is convex, and if P_* denotes the lower cover of P, then $P \setminus P_*$ is convex.* This enables us to reach, in Lemma 9.4, a specialization, to lattices of the form $\mathrm{Op}(\mathcal{H})$, of the homomorphism extension property established in Lemma 7.3. Further homomorphism extension lemmas, on lattices $\mathrm{Op}(\mathcal{H})$, are deduced in **Section 10**.

All those results are put together in **Section 11**, where we state our main theorem (Theorem 11.1) and a few corollaries. In **Section 12**, we discuss a few related results and open problems.

2. Basic concepts, notation, terminology

2.1. **General.** We set $[n] \underset{\text{def}}{=} \{1, 2, \dots, n\}$, for every natural number n. Also, $\omega = \{0, 1, 2, \dots\}$ is the first limit ordinal, and ω_1 is the first uncountable ordinal. A function f, with infinite domain X, is *almost constant* if there is a finite subset A of X such that $f{\restriction}_{X \setminus A}$ is constant; and then we denote that constant by $f(\infty)$.

Throughout the paper, "countable" means "at most countable".

In any commutative monoid, let $x \leq y$ hold if there is z such that $x+y=y$, and let $x \propto y$ hold if $x \leq ny$ for some positive integer n. We set $G^+ \underset{\mathrm{def}}{=} \{x \in G \mid 0 \leq x\}$, for every partially ordered group G (denoted additively).

2.2. **Posets and lattices.** A standard reference for lattice theory is Grätzer [25]. For any subsets X and Y in a poset P, we set

$$X \downarrow Y \underset{\mathrm{def}}{=} \{x \in X \mid (\exists y \in Y)(x \leq y)\}\ ,$$
$$X \uparrow Y \underset{\mathrm{def}}{=} \{x \in X \mid (\exists y \in Y)(x \geq y)\}\ ,$$

and we write $X \downarrow a$, $X \uparrow a$ instead of $X \downarrow \{a\}$, $X \uparrow \{a\}$, respectively, if there is no ambiguity on the ambient poset. We also write $\downarrow X$ and $\uparrow X$ instead of $P \downarrow X$ and $P \uparrow X$, respectively, if there is no ambiguity on P. We say that X is

— a *lower subset* (resp., *upper subset*) of P if $X = \downarrow X$ (resp., $X = \uparrow X$);

— a *coinitial* (resp., *cofinal*) subset of P if $P = \uparrow X$ (resp., $P = \downarrow X$).

For posets P and Q, a map $f\colon P \to Q$ is *isotone* (resp., *antitone*) if $x \leq y$ implies that $f(x) \leq f(y)$ (resp., $f(y) \leq f(x)$), for all $x, y \in P$.

A poset P is a *root system* if $P \uparrow p$ is a chain for every $p \in P$; that is, for all $p, x, y \in P$, if $p \leq x$ and $p \leq y$, then either $x \leq y$ or $y \leq x$.

For any maps f and g from a set X to a poset P, we set

$$[\![f \leq g]\!] \underset{\mathrm{def}}{=} \{x \in X \mid f(x) \leq g(x)\}\ ,$$

and we define similarly $[\![f \geq g]\!]$, $[\![f = g]\!]$, $[\![f < g]\!]$, $[\![f > g]\!]$, $[\![f \neq g]\!]$.

We will often denote by 0 (resp., 1) the smallest (resp., largest) element in a lattice L. For lattices K and L with zero (i.e., least element), a lattice homomorphism $f\colon K \to L$ is a 0-*lattice homomorphism* if $f(0_K) = 0_L$. We define similarly 1-lattice homomorphisms and $0,1$-lattice homomorphisms. We denote by $\mathrm{Ji}\,L$ (resp., $\mathrm{Mi}\,L$) the set of all join-irreducible (resp., meet-irreducible) elements in L.

Lemma 2.1 (folklore)**.** *Let D be a finite distributive lattice. Then every join-irreducible element p of D is* join-prime*, that is, it is nonzero and $p \leq x \vee y$ implies that $p \leq x$ or $p \leq y$, for all $x, y \in D$. Moreover, the subset $\{x \in D \mid p \nleq x\}$ has a largest element $p^\dagger$. The assignment $p \mapsto p^\dagger$ defines an order-isomorphism from* $\mathrm{Ji}\,D$ *onto* $\mathrm{Mi}\,D$.

We say that a (necessarily distributive) lattice D is a *generalized Heyting algebra* (cf. Johnstone [30]) if for all $a, b \in D$, there is a largest $x \in D$ such that $a \wedge x \leq b$, then denoted by $a \to b$ (or $a \to_D b$ if D needs to be specified) and called the *Heyting residue* of a by b. The operation $\to$ is then called the *Heyting implication* on D. Every generalized Heyting algebra has a largest element. A *Heyting algebra* is a generalized Heyting algebra with a least element. Of course, every finite distributive lattice is a Heyting algebra.

For a distributive lattice D, we set

$$a \ominus_D b \underset{\mathrm{def}}{=} \{x \in D \mid a \leq b \vee x\}\ , \quad \text{for all } a, b \in D\,. \tag{2.1}$$

The smallest element of $a \ominus_D b$, if it exists, is denoted by $a \smallsetminus_D b$. (In their paper [18], Di Nola and Grigolia call $a \smallsetminus_D b$ the *pseudo-difference of a and b*.) Hence, *D is a generalized dual Heyting algebra iff $a \smallsetminus_D b$ exists whenever $a, b \in D$*. We say that D *has countably based differences* if $a \ominus_D b$ has a countable coinitial subset, whenever $a, b \in D$.

A lattice L is *complete* if every subset of L has a join (equivalently, every subset of L has a meet). An element a of L is *compact* if for every $X \subseteq L$, if $a \leq \bigvee X$, then $a \leq \bigvee Y$ for some finite subset Y of X. A complete lattice L is *algebraic* if every element of L is a join of compact elements. A lattice is algebraic iff it is isomorphic to the ideal lattice of its $(\vee, 0)$-semilattice of compact elements.

2.3. **Topological spaces, vector spaces.** For a topological space X, we denote by $\mathrm{int}(Z)$ (resp., $\mathrm{cl}(Z)$) the interior (resp., the closure) of any subset Z of X. The lattice $\mathcal{O}(X)$ of all open subsets of X is a complete Heyting algebra (cf. Section 2.2), with Heyting implication given by

$$(U \to V) = \mathrm{int}\big((\complement U) \cup V\big)\,, \quad \text{for all } U, V \in \mathcal{O}(X)\,.$$

For any subset X in a real vector space $\mathbb{E}$, we denote by $\mathrm{conv}(X)$ the convex hull of X, and by $\mathrm{cone}(X)$ the convex cone generated by X; so $\mathrm{cone}(X) = \mathbb{R}^+ \cdot \mathrm{conv}(X)$.

The following is a well known consequence of the Hahn-Banach Theorem, usually known as Farkas' Lemma (see, for example, De Loera, Hemmecke, and Köppe [16, Section 1.2]). While this lemma is usually stated in the finite-dimensional case, the general case can easily be reduced to the finite-dimensional one by working in $\mathbb{E}/F$, where $F \underset{\mathrm{def}}{=} \ker(c) \cap \bigcap_{i=1}^{n} \ker(b_i)$.

Lemma 2.2. *Let $\mathbb{E}$ be a real vector space, let n be a nonnegative integer, and let $b_1, \dots, b_n$, c be linear functionals on $\mathbb{E}$. Then $\bigcap_{i=1}^{n} [\![b_i \geq 0]\!] \subseteq [\![c \geq 0]\!]$ iff $c \in \mathrm{cone}(\{b_1, \dots, b_n\})$.*

We also record two elementary lemmas, which will be useful in the sequel.

Lemma 2.3. *Let A and F be convex subsets in a real topological vector space $\mathbb{E}$, with F closed and $F \cap \mathrm{int}(A) \neq \varnothing$. Then $\mathrm{cl}(F \cap A) = F \cap \mathrm{cl}(A)$.*

Proof. Fix $u \in F \cap \mathrm{int}(A)$, and let $p \in F \cap \mathrm{cl}(A)$. Since F is convex, $(1-\lambda)p + \lambda u \in F$ for each $\lambda \in [0, 1]$. Moreover, since $u \in \mathrm{int}(A)$, $p \in \mathrm{cl}(A)$, and A is convex, $(1-\lambda)p + \lambda u$ belongs to A, thus to $F \cap A$, for each $\lambda \in (0, 1]$. Since $(1-\lambda)p + \lambda u$ converges to p, as λ goes to 0 and $\lambda > 0$, it follows that $p \in \mathrm{cl}(F \cap A)$. We have thus proved that $F \cap \mathrm{cl}(A) \subseteq \mathrm{cl}(F \cap A)$. The converse containment is trivial. □

Lemma 2.4. *Let F be the union of finitely many closed subspaces in a real topological vector space $\mathbb{E}$ and let Q be a convex subset of $\mathbb{E}$. Then either $Q \subseteq F$ or $Q \cap F$ is nowhere dense in Q.*

Proof. We first deal with the case where F is a closed subspace of $\mathbb{E}$. Suppose that $Q \cap F$ is not nowhere dense in Q. Since F is a closed subspace of $\mathbb{E}$, $Q \cap F$ is also relatively closed in Q, thus the relative interior U of $Q \cap F$ in Q is nonempty. Fix $u \in U$ and let $q \in Q$. Since Q is convex, $(1-\lambda)u + \lambda q \in Q$ for every $\lambda \in [0, 1]$. Since U is a relative neighborhood of u in Q, it follows that $(1-\lambda)u + \lambda q$ belongs to U, thus to F, for some $\lambda \in (0, 1]$. Since $\{u, (1-\lambda)u + \lambda q\} \subseteq F$ with $\lambda > 0$, it follows that $q \in F$, therefore completing the proof that $Q \subseteq F$.

Now in the general case, $F = \bigcup_{i=1}^{n} F_i$, where each F_i is a closed subspace of $\mathbb{E}$. If $Q \nsubseteq F$, then $Q \nsubseteq F_i$, thus, by the paragraph above, $Q \cap F_i$ is nowhere dense in Q, for all $i \in [n]$. Therefore, $Q \cap F = \bigcup_{i=1}^{n} (Q \cap F_i)$ is also nowhere dense in Q. □

3. Abelian lattice-ordered groups

In this section we will survey some well known facts about ℓ-groups, the functor $\mathrm{Id_c}$, from Abelian ℓ-groups to distributive lattices with zero, and the Baker-Beynon duality. A key point is that for a ℓ-homomorphism $f\colon A \to B$ of Abelian ℓ-groups, the map $\mathrm{Id_c}\, f\colon \mathrm{Id_c}\, A \to \mathrm{Id_c}\, B$ is a special kind of 0-lattice homomorphism, which we call a *closed map* (cf. Lemma 3.2).

Our basic reference on ℓ-groups will be Bigard, Keimel, and Wolfenstein [8].

In any ℓ-group G, we set $x^+ \underset{\mathrm{def}}{=} x \vee 0$, $x^- \underset{\mathrm{def}}{=} (-x) \vee 0$, and $|x| \underset{\mathrm{def}}{=} x \vee (-x)$. We denote by $\mathrm{Id}\, G$ the lattice of all ℓ-ideals of an Abelian ℓ-group G, and by $\mathrm{Id_c}\, G$ the lattice of all finitely generated ℓ-ideals of G. Denoting by $\langle a_1, \dots, a_n \rangle$ (or $\langle a_1, \dots, a_n \rangle_G$ in case G needs to be specified) the ℓ-ideal generated by $\{a_1, \dots, a_n\}$, $\mathrm{Id_c}\, G$ consists of all the ℓ-ideals of the form $\langle a_1, \dots, a_n \rangle$. Setting $a \underset{\mathrm{def}}{=} \sum_{i=1}^{n} |a_i|$, observe that $\langle a_1, \dots, a_n \rangle = \langle a \rangle = \{x \in G \mid |x| \propto a\}$. In particular, $\mathrm{Id_c}\, G = \{\langle a \rangle \mid a \in G^+\}$. Due to the elementary properties of ℓ-groups, the join and meet operation on $\mathrm{Id_c}\, G$ are given by

$$\langle x \rangle \vee \langle y \rangle = \langle x \vee y \rangle = \langle x + y \rangle\,, \quad \langle x \rangle \cap \langle y \rangle = \langle x \wedge y \rangle\,, \qquad \text{for all } x, y \in G^+\,. \tag{3.1}$$

In particular, $\mathrm{Id_c}\, G$ is a distributive lattice with zero.

The following lemma is well known, see, for example, Bigard, Keimel, and Wolfenstein [8, Section 2.3].

Lemma 3.1. *The following statements hold, for every Abelian ℓ-group G:*

(1) *For every ℓ-ideal I of G, the set $\boldsymbol{I} \underset{\mathrm{def}}{=} \{\langle x \rangle \mid x \in I \cap G^+\}$ is an ideal of $\mathrm{Id_c}\, G$. Furthermore, the assignment $\langle x \rangle_I \mapsto \langle x \rangle_G$ defines an isomorphism from $\mathrm{Id_c}\, I$ onto $\boldsymbol{I}$.*

(2) *For every ideal $\boldsymbol{I}$ of $\mathrm{Id_c}\, G$, the set $I \underset{\mathrm{def}}{=} \{x \in G \mid \langle x \rangle \in \boldsymbol{I}\}$ is an ℓ-ideal of G.*

(3) *The assignments $I \mapsto \boldsymbol{I}$ and $\boldsymbol{I} \mapsto I$, in* (1) *and* (2) *above, are mutually inverse.*

For an ℓ-homomorphism $f\colon G \to H$ between Abelian ℓ-groups, it follows easily from (3.1) that the map $\mathrm{Id_c}\, f\colon \mathrm{Id_c}\, G \to \mathrm{Id_c}\, H$, $\langle x \rangle \mapsto \langle f(x) \rangle$ is well defined, and is a 0-lattice homomorphism. This defines a functor $\mathrm{Id_c}$, from Abelian ℓ-groups with ℓ-homomorphisms, to distributive 0-lattices with 0-lattice homomorphisms. This functor preserves direct limits and finite direct products.

A map $\boldsymbol{f}$ *separates zero* if $\boldsymbol{f}^{-1}\{0\} = \{0\}$.

Lemma 3.2. *Let A and B be Abelian ℓ-groups and let $f\colon A \to B$ be an ℓ-homomorphism. Then the map $\boldsymbol{f} \underset{\mathrm{def}}{=} \mathrm{Id_c}\, f$ has the following properties:*

(1) *$\boldsymbol{f}$ is a 0-lattice homomorphism.*

(2) *Let $\boldsymbol{a}_0, \boldsymbol{a}_1 \in \mathrm{Id_c}\, A$ and let $\boldsymbol{b} \in \mathrm{Id_c}\, B$. If $\boldsymbol{f}(\boldsymbol{a}_0) \subseteq \boldsymbol{f}(\boldsymbol{a}_1) \vee \boldsymbol{b}$, then there exists $\boldsymbol{a} \in \mathrm{Id_c}\, A$ such that $\boldsymbol{a}_0 \subseteq \boldsymbol{a}_1 \vee \boldsymbol{a}$ and $\boldsymbol{f}(\boldsymbol{a}) \subseteq \boldsymbol{b}$. We say that the map $\boldsymbol{f}$ is* closed.

(3) *$\boldsymbol{f}$ separates zero iff it is one-to-one, iff f is one-to-one.*

The terminology *closed maps*, in Lemma 3.2(2) above, is borrowed from Iberkleid, Martínez, and McGovern [29].

Proof. Ad (1). This follows immediately from (3.1).

Ad (2). Let $a_0, a_1 \in A^+$ and $b \in B^+$ such that each $\boldsymbol{a}_i = \langle a_i \rangle_A$ and $\boldsymbol{b} = \langle b \rangle_B$. Then the assumption $\boldsymbol{f}(\boldsymbol{a}_0) \subseteq \boldsymbol{f}(\boldsymbol{a}_1) \vee \boldsymbol{b}$ means that there exists a positive integer n such that $f(a_0) \leq n(f(a_1) + b)$, which, since $b \geq 0$, is equivalent to $(f(a_0) - nf(a_1))^+ \leq nb$, that is, since f is an ℓ-homomorphism, $f\big((a_0 - na_1)^+\big) \leq nb$. Therefore, setting $\boldsymbol{a} \underset{\text{def}}{=} \langle (a_0 - na_1)^+ \rangle_A$, we get $\boldsymbol{a}_0 \subseteq \boldsymbol{a}_1 \vee \boldsymbol{a}$ and $\boldsymbol{f}(\boldsymbol{a}) \subseteq \boldsymbol{b}$.

Ad (3). It is clear that f is one-to-one iff f separates zero, iff $\boldsymbol{f}$ separates zero. Suppose that this condition holds, and let $\boldsymbol{a}_0, \boldsymbol{a}_1 \in \mathrm{Id_c}\, A$ such that $\boldsymbol{f}(\boldsymbol{a}_0) \subseteq \boldsymbol{f}(\boldsymbol{a}_1)$. By (2) above, there exists $\boldsymbol{a} \in \mathrm{Id_c}\, A$ such that $\boldsymbol{a}_0 \subseteq \boldsymbol{a}_1 \vee \boldsymbol{a}$ and $\boldsymbol{f}(\boldsymbol{a}) \subseteq \{0\}$. Since $\boldsymbol{f}$ separates zero, we get $\boldsymbol{a} = \{0\}$, whence $\boldsymbol{a}_0 \subseteq \boldsymbol{a}_1$. Hence, $\boldsymbol{f}$ is one-to-one. □

This motivates the following definition.

Definition 3.3. A distributive lattice with zero is *ℓ-representable* if it is isomorphic to $\mathrm{Id_c}\, G$ for some Abelian ℓ-group G. More generally, say that a diagram $\vec{D}$, of distributive lattices with zero and 0-lattice homomorphisms, is ℓ-representable, if it is isomorphic to $\mathrm{Id_c}\, \vec{G}$ for a diagram $\vec{G}$, of Abelian ℓ-groups and ℓ-homomorphisms.

Example 3.4. Using Lemma 3.2(2), it is easy to construct an example of a 0-sublattice $\boldsymbol{A}$ of a finite distributive lattice $\boldsymbol{B}$, both ℓ-representable, such that the inclusion map from $\boldsymbol{A}$ into $\boldsymbol{B}$ is not ℓ-representable: just define $\boldsymbol{B}$ as the square $\mathbf{2}^2$, and $\boldsymbol{A}$ as any of the two 3-element chains in $\boldsymbol{B}$.

Another example, this time using Lemma 3.2(3), of a non-ℓ-representable 0-lattice homomorphism between finite ℓ-representable distributive lattices, is the following: denote by $\mathbf{2}$ the two-element chain and by $\mathbf{3}$ the three-element chain. Then the unique zero-separating map $\boldsymbol{f} \colon \mathbf{3} \twoheadrightarrow \mathbf{2}$ is a surjective 0-lattice homomorphism. Since $\boldsymbol{f}$ is zero-separating but not one-to-one, it is not ℓ-representable.

Lemma 3.5. *Let G be an Abelian ℓ-group, let S be a distributive lattice with zero, and let $\varphi \colon \mathrm{Id_c}\, G \twoheadrightarrow S$ be a closed surjective $(\vee, 0)$-homomorphism. Then $I \underset{\text{def}}{=} \{x \in G \mid \varphi(\langle x \rangle) = 0\}$ is an ℓ-ideal of G, and there is a unique order-isomorphism $\psi \colon \mathrm{Id_c}(G/I) \to S$ such that $\psi(\langle x/I \rangle) = \varphi(\langle x \rangle)$ for every $x \in G^+$. In particular, S is a lattice and ψ is a lattice isomorphism.*

Proof. It is straightforward to verify that I is an ℓ-ideal of G and that there is a unique map $\psi \colon \mathrm{Id_c}(G/I) \to S$ such that $\psi(\langle x/I \rangle) = \varphi(\langle x \rangle)$ for every $x \in G^+$. Since φ is a surjective $(\vee, 0)$-homomorphism, so is ψ. It remains to verify that ψ is an order-embedding.

Let $x, y \in G^+$ such that $\psi(\langle x/I \rangle) \leq \psi(\langle y/I \rangle)$. This means that $\varphi(\langle x \rangle) \leq \varphi(\langle y \rangle)$, thus, since φ is a closed map, there exists $\boldsymbol{z} \in \mathrm{Id_c}\, G$ such that $\langle x \rangle \subseteq \langle y \rangle \vee \boldsymbol{z}$ and $\varphi(\boldsymbol{z}) = 0$. Writing $\boldsymbol{z} = \langle z \rangle$, for $z \in G^+$, this means that $z \in I$ and $x \leq ny + nz$ for some positive integer n. Therefore, $x/I \leq n(y/I)$, so $\langle x/I \rangle \subseteq \langle y/I \rangle$. □

It is well known that every free Abelian ℓ-group is a subdirect power of $\mathbb{Z}$, thus, *a fortiori*, of $\mathbb{R}$; this result originates in Henriksen and Isbell [26], Weinberg [49]. For a set I, denote by $\mathbb{R}^{(I)}$ the set of all families $x = (x_i \mid i \in I)$ of real numbers, such that the *support* of x, defined as $\mathrm{supp}(x) \underset{\text{def}}{=} \{i \in I \mid x_i \neq 0\}$, is finite.

Proposition 3.6. *Let I be a set and denote by $\mathrm{F}_\ell(I)$ the ℓ-subgroup of $\mathbb{R}^{\mathbb{R}^{(I)}}$ generated by the canonical projections $p_i \colon \mathbb{R}^{(I)} \twoheadrightarrow \mathbb{R}$, for $i \in I$. Then $\big(\mathrm{F}_\ell(I), (p_i \mid i \in I)\big)$ is the free Abelian ℓ-group on I.*

Note. Proposition 3.6 is usually stated with $\mathbb{Z}^I$, or $\mathbb{R}^I$, instead of $\mathbb{R}^{(I)}$. The result of Proposition 3.6 is not affected by that change.

The description of free Abelian ℓ-groups, given by Proposition 3.6, yields a convenient description of the ℓ-ideals of such ℓ-groups. This description is contained in the *Baker-Beynon duality*, see Baker [1], Beynon [6, 7]. The following consequence, of that duality, involves the description of $\mathrm{F}_\ell(I)$ given in Proposition 3.6.

Proposition 3.7. *Let I be a set, and denote by $\mathrm{Lat}(I)$ the sublattice, of the powerset lattice of $\mathbb{R}^{(I)}$, generated by all sets $[\![f > 0]\!]$, where f is a linear combination, with integer coefficients, of the projections $p_i \colon \mathbb{R}^{(I)} \twoheadrightarrow \mathbb{R}$. Then there is a unique isomorphism $\iota\colon \mathrm{Id_c}\,\mathrm{F}_\ell(I) \to \mathrm{Lat}(I)$ such that $\iota(\langle f\rangle) = [\![f \neq 0]\!]$ whenever $f \in \mathrm{F}_\ell(I)$.*

Note. The empty set $\varnothing = [\![0 > 0]\!]$ belongs to $\mathrm{Lat}(I)$, which is thus a 0-sublattice of the powerset lattice of $\mathbb{R}^{(I)}$. On the other hand, for every linear combination f of the projections, the subset $[\![f > 0]\!]$ omits the origin of $\mathbb{R}^{(I)}$. It follows that $\mathbb{R}^{(I)}$ does not belong to $\mathrm{Lat}(I)$, which is thus not a 1-sublattice of the powerset of $\mathbb{R}^{(I)}$.

4. Completely normal spaces, spectral spaces

In this section we shall survey some basic facts and examples about completely normal spaces and (generalized) spectral spaces. We will also reduce the MV-spectrum problem to a lattice-theoretical problem (cf. Lemma 4.7). Recall that $\mathcal{O}(X)$ denotes the lattice of all open subsets in a topological space X. Moreover, we shall denote by $\mathcal{K}(X)$ the set of all compact open subsets of X (i.e., the compact members of the lattice $\mathcal{O}(X)$, see Section 2.2).

Definition 4.1. A topological space X is

— *T_0* if for any distinct points $x, y \in X$, there is an open subset of X containing one element of $\{x, y\}$ and not the other;
— *sober* if every join-irreducible member, of the lattice of all closed subsets of X, is the closure of a unique singleton[2] ;
— *generalized spectral* if it is sober, $\mathcal{K}(X)$ is a basis for the topology of X, and the intersection of any two compact open subsets of X is compact;
— *spectral* if it is both generalized spectral and compact.

The *specialization order*[3] on a T_0 space X is defined by letting $x \leqslant y$ hold if y belongs to the closure of $\{x\}$, for all $x, y \in X$.

The statement, that $\mathcal{K}(X)$ is a basis for the topology of X, is equivalent to $\mathcal{O}(X)$ be an algebraic lattice (cf. Section 2.2 for terminology).

All generalized spectral spaces arise from distributive lattices with zero, as follows (see Johnstone [30, Section II.3] or Grätzer [25, Section II.5] for more details). The *spectrum* of a distributive lattice D with zero, denoted by $\mathrm{Spec}\,D$, is the set of all prime ideals of D, endowed with the topology whose open sets are exactly the subsets $\Omega_X \underset{\mathrm{def}}{=} \{P \in \mathrm{Spec}\,D \mid X \nsubseteq P\}$, for $X \subseteq D$. Denoting by $\langle X\rangle$ the ideal of D generated by X, it is obvious that $\Omega_X = \Omega_{\langle X\rangle}$. Moreover, it follows from Stone's representation Theorem, for distributive lattices, that every ideal I of D is the intersection of all prime ideals containing it; whence Ω_I characterizes I. The compact open subsets of $\mathrm{Spec}\,D$ are exactly the $\Omega_{\{a\}}$, for $a \in D$; and the assignment $a \mapsto \Omega_{\{a\}}$ defines an order-isomorphism of D onto $\mathcal{K}(\mathrm{Spec}\,D)$; whence

[2]In some references, the uniqueness of the singleton is not assumed. Our definition of sobriety ensures that every sober space is T_0.

[3]In some references, the specialization order is defined as the opposite of ours.

$D \cong \mathcal{K}(\operatorname{Spec} D)$. For prime ideals P and Q of D, $P \leqslant Q$ (for the specialization order) iff $P \subseteq Q$.

Proposition 4.2. *A topological space X is generalized spectral* (*resp., spectral*) *iff it is homeomorphic to the spectrum of a distributive lattice with zero* (*resp., a bounded distributive lattice*) *D. If this holds, then $D \cong \mathcal{K}(X)$.*

The *ℓ-spectrum*, $\operatorname{Spec}_\ell G$, of an Abelian ℓ-group G can be defined the same way as the spectrum of a distributive lattice with zero. Namely, $\operatorname{Spec}_\ell G$ consists of all prime ℓ-ideals of G, and its closed sets are the $V_I \underset{\mathrm{def}}{=} \{P \in \operatorname{Spec}_\ell G \mid I \subseteq P\}$, for ℓ-ideals I of G. Hence, by Lemma 3.1, the ℓ-spectrum of the ℓ-group G is homeomorphic to the spectrum of the distributive lattice $\operatorname{Id_c} G$. By Proposition 4.2, this reduces the description of ℓ-spectra of Abelian ℓ-groups to the one of ℓ-representable distributive lattices with zero.

Similarly, *MV-spectra* of MV-algebras are, *via* Mundici's equivalence [39], the same as ℓ-spectra of Abelian ℓ-groups with unit. For more detail we refer the reader to Belluce [4], Cignoli, Di Nola, and Lettieri [12], Cignoli and Torrens [14, Corollary 1.3]. If G is an Abelian ℓ-group with unit u, then the ℓ-spectrum of G is naturally isomorphic to the MV-spectrum of the MV-algebra $[0, u]$.

Definition 4.3.
(1) A distributive lattice D with zero is *completely normal* if for all $a, b \in D$, there exist $x, y \in D$ such that $a \leq b \vee x$, $b \leq a \vee y$, and $x \wedge y = 0$.
(2) A generalized spectral space X is *completely normal* if the specialization order on X is a root system.

The reader should be warned that the literature contains several non-equivalent definitions of (complete) normality, as well for lattices as for topological spaces:

— While our choice of terminology in Definition 4.3(1) is consistent with Johnstone [30], Cignoli [11] would rather call such lattices "dually completely normal"; and Snodgrass and Tsinakis [43, 44] would call them "relatively normal".

— Complete normality, of a topological space X, is sometimes defined by stating that every subspace of X — or, equivalently, every open subspace of X — is normal (cf. Munkres [41, Exercise IV.32.6]). Equivalently, the lattice $\mathcal{O}(X)$, of all open subsets of X, is completely normal in the sense of Definition 4.3(1). By the following Proposition 4.4, this implies complete normality as stated in Definition 4.3(2). However, as witnessed by the following Example 4.5, the converse implication does not hold, even for spectral spaces; that is, the two concepts of complete normality are not equivalent.

The following "local-global principle" is a restatement of Monteiro [38, Théorème V.3.1], see also Cignoli [11, Proposition 1.9]. We include a proof for convenience.

Proposition 4.4. *A generalized spectral space X is completely normal, in the sense of Definition* 4.3(1)*, iff its lattice $\mathcal{K}(X)$, of all compact open subsets, is completely normal.*

Proof. By Proposition 4.2, it suffices to prove that a distributive lattice D with zero is completely normal iff $\operatorname{Spec} D$ is a completely normal topological space (in the sense of Definition 4.3). Suppose first that D is completely normal and let P,

Q, R be prime ideals of D such that P is contained in both Q and R, and Q and R are incomparable. Pick $a \in Q \setminus R$ and $b \in R \setminus Q$. By assumption, there are $x, y \in D$ such that $a \leq b \vee x$, $b \leq a \vee y$, and $x \wedge y = 0$. Necessarily, $x \notin R$ and $y \notin Q$, thus $x, y \notin P$, in contradiction with $x \wedge y = 0$ together with the primeness of P.

Suppose, conversely, that $\operatorname{Spec} D$ is completely normal and let $a, b \in D$. We must prove that 0 belongs to the filter $C \underset{\text{def}}{=} \uparrow\{x \wedge y \mid (x, y) \in (a \ominus_D b) \times (b \ominus_D a)\}$ (cf. (2.1) for notation). Suppose otherwise. By a well known theorem of Stone (cf. Grätzer [25, Theorem 115]), there is a prime ideal P of D such that $C \cap P = \varnothing$. We claim that $\uparrow a \cap (P \vee \downarrow b) = \varnothing$ (where $P \vee \downarrow b$ denotes the join of P and $\downarrow b$ in the ideal lattice of D). Otherwise, there is $x \in P$ such that $a \leq b \vee x$; that is, $x \in a \ominus_D b$, thus $x \in C$, and thus $x \notin P$, a contradiction. This proves our claim. By the above-mentioned Stone's Theorem, there is a prime ideal Q such that $P \vee \downarrow b \subseteq Q$ and $Q \cap \uparrow a = \varnothing$; that is, $P \subseteq Q$, $b \in Q$, and $a \notin Q$. Likewise, there is a prime ideal R such that $P \subseteq R$, $a \in R$, and $b \notin R$. In particular, Q and R are incomparable, and they both contain P, which contradicts our assumption. □

Example 4.5. *A second countable spectral space X satisfying the following conditions:*

(1) *The lattice $\mathcal{K}(X)$ is a self-dual Heyting algebra.*
(2) *The lattice $\mathcal{K}(X)$ is completely normal.*
(3) *The lattice $\mathcal{O}(X)$ is not completely normal.*

Proof. Consider the finite chain $\mathbf{3} \underset{\text{def}}{=} \{0, 1, 2\}$. The set D, of all eventually constant sequences of elements of $\mathbf{3}$, is a countable, bounded sublattice of $\mathbf{3}^\omega$. Since $\mathbf{3}$ is a completely normal, self-dual Heyting algebra, so is D.

We claim that the ideal lattice $\operatorname{Id} D$ of D is not completely normal. Write the elements of D in the form $(1 \cdot X) \sqcup (2 \cdot Y)$, for disjoint subsets X and Y of ω, each of them either finite or cofinite. Recalling the usual convention $n = \{0, \dots, n-1\}$, we set

$$a_n \underset{\text{def}}{=} 2 \cdot n \text{ and } b_n \underset{\text{def}}{=} 1 \cdot (\omega \setminus n)\,, \quad \text{for all } n \in \omega\,.$$

Moreover, denote by $\boldsymbol{a}$ the ideal of D generated by $\{a_n \mid n \in \omega\}$. Suppose that there are ideals $\boldsymbol{u}$, $\boldsymbol{v}$ of D such that $\downarrow b_0 \subseteq \boldsymbol{a} \vee \boldsymbol{v}$, $\boldsymbol{a} \subseteq \downarrow b_0 \vee \boldsymbol{u}$, and $\boldsymbol{u} \cap \boldsymbol{v} = \{0\}$. The first condition implies that $b_m \in \boldsymbol{v}$ for some $m \in \omega$. The second condition implies that $a_n \in \boldsymbol{u}$ for all n. In particular, $a_{m+1} \in \boldsymbol{u}$, thus the element $1 \cdot \{m\} = a_{m+1} \wedge b_m$ belongs to $\boldsymbol{u} \cap \boldsymbol{v}$, a contradiction.

By Proposition 4.2, the spectrum X of D is a spectral space with $D \cong \mathcal{K}(X)$; hence the lattice $\mathcal{K}(X)$ is completely normal. On the other hand, $\mathcal{O}(X)$ is isomorphic to the ideal lattice of $\mathcal{K}(X)$, thus to the ideal lattice of D. By the above, this lattice is not completely normal. □

As already mentioned in Section 1.2, it is well known that the ℓ-spectrum of any Abelian ℓ-group is completely normal (see, for example, Bigard, Keimel, and Wolfenstein [8, Proposition 10.1.11]). It is worthwhile to pinpoint the lattice-theoretical content of that result.

Lemma 4.6. *Let G be an Abelian ℓ-group. Then $\operatorname{Id}_{\mathrm{c}} G$ is completely normal.*

Proof. Let $\boldsymbol{a}, \boldsymbol{b} \in \operatorname{Id}_{\mathrm{c}} G$. There are $a, b \in G^+$ such that $\boldsymbol{a} = \langle a \rangle$ and $\boldsymbol{b} = \langle b \rangle$. Set $\boldsymbol{x} \underset{\text{def}}{=} \langle a - (a \wedge b) \rangle$ and $\boldsymbol{y} \underset{\text{def}}{=} \langle b - (a \wedge b) \rangle$. Then $\boldsymbol{a} \subseteq \boldsymbol{b} \vee \boldsymbol{x}$, $\boldsymbol{b} \subseteq \boldsymbol{a} \vee \boldsymbol{y}$, and $\boldsymbol{x} \cap \boldsymbol{y} = \{0\}$. □

Lemma 4.7. *A topological space X is homeomorphic to the ℓ-spectrum of an Abelian ℓ-group iff it is generalized spectral and the lattice $\mathcal{K}(X)$ is ℓ-representable.*

Proof. For every Abelian ℓ-group G, the ℓ-spectrum X of G is isomorphic to the spectrum of the distributive lattice $\mathrm{Id_c}\, G$, thus it is generalized spectral (cf. Proposition 4.2). Moreover, the lattice $\mathcal{K}(X) \cong \mathrm{Id_c}\, G$ is ℓ-representable.

Let, conversely, X be a generalized spectral space and let G be an Abelian ℓ-group such that $\mathcal{K}(X) \cong \mathrm{Id_c}\, G$. Then $\mathcal{O}(X)$ is isomorphic to the ideal lattice of $\mathcal{K}(X)$, which is isomorphic to the ideal lattice of $\mathrm{Id_c}\, G$, which is isomorphic to the ideal lattice of G (cf. Lemma 3.1), which is isomorphic to $\mathcal{O}(\mathrm{Spec}_\ell\, G)$. Since X and $\mathrm{Spec}_\ell\, G$ are both sober spaces, it follows (cf. Johnstone [30, Section II.1]) that $X \cong \mathrm{Spec}_\ell\, G$. □

5. Non-ℓ-representability results

In this section we shall show that the class of ℓ-representable distributive lattices is neither first-order, nor closed under infinite products (resp., homomorphic images). Recall that lattices with countably based differences are introduced in Section 2.2. The following result is a restatement, in terms of lattices of principal ℓ-ideals, of Cignoli, Gluschankof, and Lucas [13, Theorem 2.2]; see also Iberkleid, Martínez, and McGovern [29, Proposition 4.1.2]. We include a proof for convenience.

Lemma 5.1. *Let G be a Abelian ℓ-group. Then the lattice $\mathrm{Id_c}\, G$ has countably based differences.*

Proof. Let $\boldsymbol{a}, \boldsymbol{b} \in \mathrm{Id_c}\, G$, and pick $a, b \in G^+$ such that $\boldsymbol{a} = \langle a \rangle$ and $\boldsymbol{b} = \langle b \rangle$. Set $\boldsymbol{c}_n \underset{\mathrm{def}}{=} \langle (a - nb)^+ \rangle$, for every nonnegative integer n. From $a \leq nb + (a - nb)^+$ it follows that $\boldsymbol{a} \leq \boldsymbol{b} \vee \boldsymbol{c}_n$. Let $\boldsymbol{x} \in \mathrm{Id_c}\, G$ such that $\boldsymbol{a} \leq \boldsymbol{b} \vee \boldsymbol{x}$. Let $x \in G^+$ such that $\boldsymbol{x} = \langle x \rangle$. There exists a nonnegative integer n such that $a \leq nb + nx$. Since $x \in G^+$, it follows that $(a - nb)^+ \leq nx$; whence $\boldsymbol{c}_n \leq \boldsymbol{x}$. □

Example 5.2. *A countable Abelian ℓ-group G, with order-unit, such that $\mathrm{Id_c}\, G$ is not a dual Heyting algebra.*

Proof. Let G consist of all maps $x \colon \omega \to \mathbb{Z}$ such that there are (necessarily unique) $\alpha, \beta \in \mathbb{Z}$ such that $x(n) = \alpha n + \beta$ for all large enough n. Then G, ordered componentwise, is an ℓ-subgroup of $\mathbb{Z}^\omega$. The constant function a, with value 1, and the identity function b on ω both belong to G^+, $a + b$ is an order-unit of G, and there is no least $\boldsymbol{x} \in \mathrm{Id_c}\, G$ such that $\langle b \rangle \subseteq \langle a \rangle \vee \boldsymbol{x}$. □

It is easy to see that the class of all ℓ-representable distributive lattices is closed under finite direct products. We shall now show that this observation does not extend to infinite products.

Proposition 5.3. *Let D be a distributive lattice with zero. If D is not a generalized dual Heyting algebra, then D^ω is not ℓ-representable.*

Proof. Denote by $\varepsilon \colon D \hookrightarrow D^\omega$ the diagonal embedding and suppose that D^ω is ℓ-representable. Since D is isomorphic to an ideal of D^ω, it is also ℓ-representable, thus, by Lemma 5.1, D has countably based differences. On the other hand, since D is not a generalized dual Heyting algebra, there are $a, b \in D$ such that $a \ominus_D b$

(cf. (2.1)) has no least element. Hence, the set $a \ominus_D b$ has a strictly descending coinitial sequence $(c_n \mid n \in \omega)$.

Now by Lemma 5.1, D^ω has countably based differences. In particular, the set $\varepsilon(a) \ominus_{D^\omega} \varepsilon(b)$ has a countable descending coinitial sequence $(\boldsymbol{e}_n \mid n \in \omega)$. For all $n, k \in \omega$, $a \leq b \vee \boldsymbol{e}_n(k)$, thus there exists $f(n,k) \in \omega$ such that $c_{f(n,k)} \leq \boldsymbol{e}_n(k)$. Set $\boldsymbol{x} \underset{\text{def}}{=} \bigl(c_{f(n,n)+1} \mid n \in \omega\bigr)$. Since $\varepsilon(a) \leq \varepsilon(b) \vee \boldsymbol{x}$, there exists $n \in \omega$ such that $\boldsymbol{e}_n \leq \boldsymbol{x}$. It follows that $c_{f(n,n)} \leq \boldsymbol{e}_n(n) \leq \boldsymbol{x}(n) = c_{f(n,n)+1}$, a contradiction. $\square$

By taking $D \underset{\text{def}}{=} \operatorname{Id_c} G$, for the ℓ-group of Example 5.2, we obtain immediately the following.

Corollary 5.4. *The class of all ℓ-representable bounded distributive lattices is not closed under infinite products.*

Our next example involves the infinitary logic $\mathscr{L}_{\infty,\omega}$, for which we refer the reader to Keisler and Knight [33] (see also Bell [3]), of which we will adopt the terminology, in particular about back-and-forth families. We say that a submodel M, of a model N, is an *$\mathscr{L}_{\infty,\omega}$-elementary submodel* of N, if for every $\mathscr{L}_{\infty,\omega}$ sentence φ, with (finitely many, by definition of a sentence) parameters from M, M satisfies φ iff N does. Our example will show that there is no class of $\mathscr{L}_{\infty,\omega}$ sentences whose class of models is the one of all ℓ-representable bounded distributive lattices.

Example 5.5. *A non-ℓ-representable bounded distributive lattice $\boldsymbol{D}_{\omega_1}$, of cardinality $\aleph_1$, with a countable ℓ-representable $\mathscr{L}_{\infty,\omega}$-elementary sublattice.*

Proof. Consider the finite chain $\mathbf{3} \underset{\text{def}}{=} \{0,1,2\}$. For any sets I and J with $I \subseteq J$, we denote by $[I]^{<\omega}$ the set of all finite subsets of I, and we set

$$\begin{aligned}
\boldsymbol{B}_J &\underset{\text{def}}{=} \{X \subseteq J \mid \text{either } X \text{ or } J \setminus X \text{ is finite}\}\,,\\
\boldsymbol{D}_{I,J} &\underset{\text{def}}{=} \bigl\{(X,k) \in \boldsymbol{B}_J \times \mathbf{3} \mid (k=0 \Rightarrow X \in [I]^{<\omega}) \text{ and } (k \neq 0 \Rightarrow J \setminus X \in [I]^{<\omega})\bigr\}\,,\\
\boldsymbol{D}_J &\underset{\text{def}}{=} \boldsymbol{D}_{J,J}\,.
\end{aligned}$$

(Observe, in particular, that if J is finite, then $\boldsymbol{D}_J = \boldsymbol{B}_J \times \mathbf{3}$.) We endow $\boldsymbol{D}_J$ and $\boldsymbol{D}_{I,J}$ with their componentwise ordering (i.e., $(X,k) \leq (Y,l)$ if $X \subseteq Y$ and $k \leq l$). They are obviously bounded distributive lattices. Further, we set

$$\varepsilon_{I,J}(X,k) \underset{\text{def}}{=} \begin{cases} (X,k)\,, & \text{if } k=0\,,\\ (X \cup (J \setminus I), k)\,, & \text{if } k \neq 0\,, \end{cases} \quad \text{for any } (X,k) \in \boldsymbol{D}_I\,.$$

For any sets I and J and any bijection $f\colon I \to J$, the map $\overline{f}\colon \boldsymbol{D}_I \to \boldsymbol{D}_J$, $(X,k) \mapsto (f[X],k)$ is a lattice isomorphism. The following claim states some elementary properties of the maps $\varepsilon_{I,J}$ and $\overline{f}$; its proof is straightforward and we omit it.

Claim 1.

(1) *For any sets $I \subseteq J$, $\boldsymbol{D}_{I,J}$ is a bounded sublattice of $\boldsymbol{D}_J$, and $\varepsilon_{I,J}$ defines an isomorphism from $\boldsymbol{D}_I$ onto $\boldsymbol{D}_{I,J}$.*
(2) *The maps $\varepsilon_{I,J}$ form a direct system: that is, $\varepsilon_{I,I} = \mathrm{id}_{\boldsymbol{D}_I}$ and $\varepsilon_{I,K} = \varepsilon_{J,K} \circ \varepsilon_{I,J}$ whenever $I \subseteq J \subseteq K$.*
(3) *For any set J, the set $\boldsymbol{D}_J$ is the ascending union of all subsets $\boldsymbol{D}_{I,J}$, for $I \in [J]^{<\omega}$.*

(4) *Let I', I'', J', J'' be sets with $I' \subseteq I''$ and $J' \subseteq J''$, let $g\colon I'' \to J''$ be a bijection with $g[I'] = J'$, and let f be the domain-range restriction of g from I' onto J'. Then $\overline{g} \circ \varepsilon_{I',I''} = \varepsilon_{J',J''} \circ \overline{f}$.*

For any set K, we denote by $\mathcal{L}_K$ the first-order language obtained by adding to the language $(\vee, \wedge, 0, 1)$, of bounded lattices, a collection of constant symbols indexed by $\boldsymbol{D}_K$. Then for every set I containing K, the lattice $\boldsymbol{D}_I$ is naturally equipped with a structure of model for $\mathscr{L}_K$, by interpreting every $\boldsymbol{a} \in \boldsymbol{D}_K$ by $\varepsilon_{K,I}(\boldsymbol{a})$.

For infinite sets I and J, a finite subset K of $I \cap J$, and finite sequences $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ of elements of $\boldsymbol{D}_I$ and $(\boldsymbol{y}_1, \dots, \boldsymbol{y}_n)$ of elements of $\boldsymbol{D}_J$, let the statement $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) \simeq_K (\boldsymbol{y}_1, \dots, \boldsymbol{y}_n)$ hold if there are $I' \in [I]^{<\omega}$ and $J' \in [J]^{<\omega}$, both containing K, and a bijection $f\colon I' \to J'$ extending the identity of K, together with elements $\boldsymbol{x}'_1, \dots, \boldsymbol{x}'_n \in \boldsymbol{D}_{I'}$, such that each $\boldsymbol{x}_i = \varepsilon_{I',I}(\boldsymbol{x}'_i)$ and each $\boldsymbol{y}_i = \varepsilon_{J',J}(\overline{f}(\boldsymbol{x}'_i))$.

Claim 2. *The relation $\simeq_K$ is a back-and-forth family for $(\boldsymbol{D}_I, \boldsymbol{D}_J)$, with respect to the language $\mathcal{L}_K$.*

Proof of Claim. Trivially, $\varnothing \simeq_K \varnothing$. Further, if $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) \simeq_K (\boldsymbol{y}_1, \dots, \boldsymbol{y}_n)$ holds *via* I', J', and f as above, then $\varepsilon_{J',J} \circ \overline{f} \circ \varepsilon_{I',I}^{-1}$ is an isomorphism from $\boldsymbol{D}_{I',I}$ onto $\boldsymbol{D}_{J',J}$, sending each $\boldsymbol{x}_i$ to $\boldsymbol{y}_i$ and each $\varepsilon_{K,I}(\boldsymbol{z})$, where $\boldsymbol{z} \in \boldsymbol{D}_K$, to $\varepsilon_{K,J}(\boldsymbol{z})$; whence $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ and $(\boldsymbol{y}_1, \dots, \boldsymbol{y}_n)$ satisfy the same quantifier-free formulas of $\mathcal{L}_K$.

Now let $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) \simeq_K (\boldsymbol{y}_1, \dots, \boldsymbol{y}_n)$, *via* I', J', $f\colon I' \to J'$, and elements $\boldsymbol{x}'_i \in \boldsymbol{D}_{I'}$. Let $\boldsymbol{x} \in \boldsymbol{D}_I$. We need to find $\boldsymbol{y} \in \boldsymbol{D}_J$ such that $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n, \boldsymbol{x}) \simeq_K (\boldsymbol{y}_1, \dots, \boldsymbol{y}_n, \boldsymbol{y})$. There are a finite set I'', with $I' \subseteq I'' \subseteq I$, and $\boldsymbol{x}'' \in \boldsymbol{D}_{I''}$, such that $\boldsymbol{x} = \varepsilon_{I'',I}(\boldsymbol{x}'')$. We set $\boldsymbol{x}''_i = \varepsilon_{I',I''}(\boldsymbol{x}'_i)$ for each i. Since J is infinite, we can extend f to a bijection $g\colon I'' \to J''$, with $J'' \subseteq J$. Then each $\boldsymbol{x}_i = \varepsilon_{I'',I}(\boldsymbol{x}''_i)$ and (using Claim 1) $\boldsymbol{y}_i = \varepsilon_{J'',J}\bigl(\overline{g}(\boldsymbol{x}''_i)\bigr)$. Hence, setting $\boldsymbol{y} \underset{\mathrm{def}}{=} \varepsilon_{J'',J}\bigl(\overline{g}(\boldsymbol{x}'')\bigr)$, we get the relation

$$(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n, \boldsymbol{x}) \simeq_K (\boldsymbol{y}_1, \dots, \boldsymbol{y}_n, \boldsymbol{y})\,. \tag{5.1}$$

Symmetrically, for all $\boldsymbol{y} \in \boldsymbol{D}_J$, there exists $\boldsymbol{x} \in \boldsymbol{D}_I$ such that (5.1) holds. $\square$ Claim 2.

By Karp's Theorem (cf. Karp [31], Barwise [2, Theorem VII.5.3], Keisler and Knight [33, Theorem 1.2.1]), it follows that $\boldsymbol{D}_I$ and $\boldsymbol{D}_J$ satisfy the same $\mathscr{L}_{\infty,\omega}$-sentences of the language $\mathcal{L}_K$. By letting $I \underset{\mathrm{def}}{=} \omega$, $J \underset{\mathrm{def}}{=} \omega_1$ and by letting K range over all finite subsets of ω, we thus obtain the following claim.

Claim 3. *The lattice $\boldsymbol{D}_{\omega,\omega_1}$ is an $\mathscr{L}_{\infty,\omega}$-elementary sublattice of $\boldsymbol{D}_{\omega_1}$.*

Now we move to ℓ-representability.

Claim 4. *Let I be countably infinite. Then $\boldsymbol{D}_I$ is ℓ-representable.*

Proof of Claim. While Claim 4 trivially follows from our main theorem (viz., Theorem 11.1), it is easy to give an explicit representation, which we shall do now. Since $\boldsymbol{D}_I \cong \boldsymbol{D}_\omega$, it suffices to prove that $\boldsymbol{D}_\omega$ is ℓ-representable. Consider the Abelian ℓ-group G of Example 5.2. Let $x \in G^+$. By the definition of G, there are integers α and β such that $x(n) = \alpha n + \beta$ for all large enough n. We set

$$\rho(x) \underset{\mathrm{def}}{=} \begin{cases} (\mathrm{supp}(x), 0)\,, & \text{if } \alpha = \beta = 0\,,\\ (\mathrm{supp}(x), 1)\,, & \text{if } \alpha = 0 \text{ and } \beta > 0\,,\\ (\mathrm{supp}(x), 2)\,, & \text{if } \alpha > 0\,. \end{cases}$$

It is straightforward to verify that $x \propto y$ iff $\rho(x) \leq \rho(y)$, for all $x, y \in G^+$. Furthermore, the range of ρ is equal to $\boldsymbol{D}_\omega$, thus ρ induces an isomorphism from $\mathrm{Id_c}\, G$ onto $\boldsymbol{D}_\omega$, defined by the assignment $\langle x \rangle \mapsto \rho(x)$. □ Claim 4.

Claim 5. *The lattice $\boldsymbol{D}_{\omega_1}$ does not have countably based differences. In particular, it is not ℓ-representable.*

Proof of Claim. The elements $\boldsymbol{a} \underset{\text{def}}{=} (\omega_1, 1)$ and $\boldsymbol{b} \underset{\text{def}}{=} (\omega_1, 2)$ both belong to $\boldsymbol{D}_{\omega_1}$. Furthermore, the set $\boldsymbol{b} \ominus_{\boldsymbol{D}_{\omega_1}} \boldsymbol{a} = \{(X, 2) \mid X \subseteq \omega_1 \text{ cofinite}\}$ has no countable coinitial subset. The second part of our claim follows from Lemma 5.1. □ Claim 5.

This claim finishes the proof of Example 5.5. □

Note. Denote by Z the completely normal spectral space constructed by Delzell and Madden in [17, Theorem 2]. Although there is an obvious $0,1$-lattice embedding from $\boldsymbol{D}_{\omega_1}$ into $\mathcal{K}(Z)$, it is not hard to see that the two lattices are not isomorphic. Hence, Z is not homeomorphic to the spectrum of $\boldsymbol{D}_{\omega_1}$.

See also Problem 1 in Section 12.

Example 5.6. *An ℓ-representable bounded distributive lattice of cardinality $\aleph_1$, with a non-ℓ-representable lattice homomorphic image.*

Proof. The set $\boldsymbol{D}$, of all almost constant maps from ω_1 to $\mathbf{3}$, is a $0,1$-sublattice of $\mathbf{3}^{\omega_1}$. It is straightforward to verify that $\boldsymbol{D} \cong \mathrm{Id_c}\, H$, where H denotes the Abelian ℓ-group of all almost constant maps from ω_1 to the lexicographical product of $\mathbb{Z}$ by itself. Now consider the non-ℓ-representable lattice $\boldsymbol{D}_{\omega_1}$ of Example 5.5. The map $\rho \colon \boldsymbol{D} \to \boldsymbol{D}_{\omega_1}$, $\boldsymbol{x} \mapsto (\mathrm{supp}(\boldsymbol{x}), x(\infty))$ is a surjective lattice homomorphism. □

6. Consonance and difference operations

The proof of our main theorem (Theorem 11.1) will make an extensive use of the following concept of consonance, which is a local version of complete normality.

Definition 6.1. Let A and B be distributive lattices with zero.

- Two elements x and y of A are *consonant*, in notation $x \sim y$, or $x \sim_A y$ if A needs to be specified, if there are $u, v \in A$ such that $x \leq y \vee u$, $y \leq x \vee v$, and $u \wedge v = 0$; then following Iberkleid, Martínez, and McGovern [29], we say that (u, v) is a *splitting* of (x, y).
- A subset Z of A is *consonant in A* if $x \sim_A y$ for all $x, y \in Z$.
- A map $f \colon A \to B$ is *consonant* if $f[A]$ is consonant in B.

Observe that the definition of complete normality, as stated in Definition 4.3(1), says that a distributive lattice D with zero is completely normal iff any two elements of D are consonant.

Note. In the context of Definition 6.1, if (u, v) is a splitting of (x, y), then so is $(u \wedge x, v \wedge y)$. Moreover, for all $u \leq x$ and $v \leq y$, (u, v) is a splitting of (x, y) iff $x \vee y = x \vee v = u \vee y$ and $u \wedge v = 0$.

Lemma 6.2. *The following statements hold, for every distributive lattice D with zero and all $a, b, c \in D$:*

(1) *$a \sim b$ iff $b \sim a$.*
(2) *If either a and b are comparable or $a \wedge b = 0$, then $a \sim b$.*

(3) *If $a \sim c$ and $b \sim c$, then $a \wedge b \sim c$ and $a \vee b \sim c$.*

Proof. (1) is trivial.

Ad (2). If $a \leq b$, then $(a, 0)$ is a splitting of (a, b). If $a \wedge b = 0$, then (a, b) is a splitting of (a, b).

Ad (3). If (x, x') is a splitting of (a, c) and (y, y') is a splitting of (b, c), then $(x \vee y, x' \wedge y')$ is a splitting of $(a \vee b, c)$ and $(x \wedge y, x' \vee y')$ is a splitting of $(a \wedge b, c)$. □

Definition 6.3. Let L be a lattice and let S be a $(\vee, 0)$-semilattice. A map $L \times L \to B$, $(x, y) \mapsto x \smallsetminus y$ is an *S-valued difference operation* on L if the following statements hold:

(D0) $x \smallsetminus x = 0$, for all $x \in L$.
(D1) $x \smallsetminus z = (x \smallsetminus y) \vee (y \smallsetminus z)$, for all $x, y, z \in L$ such that $x \geq y \geq z$.
(D2) $x \smallsetminus y = (x \vee y) \smallsetminus y = x \smallsetminus (x \wedge y)$, for all $x, y \in L$.

We say, further, that the difference operation $\smallsetminus$ is *normal* if the following condition holds:

(D3) $(x \smallsetminus y) \wedge (y \smallsetminus x) = 0$, for all $x, y \in L$.

Although we will need the following lemma only in case L is distributive, we found it worth noticing that it holds in full generality.

Lemma 6.4. *Let L be a lattice, let S be a $(\vee, 0)$-semilattice, and let $\smallsetminus$ be an S-valued difference operation on L. Then $x \smallsetminus z \leq (x \smallsetminus y) \vee (y \smallsetminus z)$, for all $x, y, z \in S$* (triangle inequality). *Furthermore, the map $(x, y) \mapsto x \smallsetminus y$ is isotone in x and antitone in y.*

Proof. Set $\delta(x, y) \underset{\text{def}}{=} y \smallsetminus x$, for all $x \leq y$ in L. Then the map δ satisfies the axioms also denoted (D0)–(D2) in Wehrung [46], namely, in that order,

- $\delta(x, x) = 0$, for all $x \in L$;
- $\delta(x, z) = \delta(x, y) \vee \delta(y, z)$, for all $x \leq y \leq z$ in L;
- $\delta(x, x \vee y) = \delta(x \wedge y, y)$, for all $x, y \in L$.

As in [46], denote by $\Delta(x, y)$, for $x \leq y$ in L, the canonical generators of the *dimension monoid* $\operatorname{Dim} L$ of L. By the universal property defining $\operatorname{Dim} L$, there exists a unique monoid homomorphism $\mu\colon \operatorname{Dim} L \to S$ such that $\mu(\Delta(x, y)) = \delta(x, y)$ for all $(x, y) \in L^{[2]}$. Set $\Delta^+(x, y) \underset{\text{def}}{=} \Delta(x \wedge y, x)$, for all $x, y \in L$.

Denoting by $\leq$ the algebraic preordering of $\operatorname{Dim} L$ (i.e., $\alpha \leq \beta$ if there exists γ such that $\beta = \alpha + \gamma$), we established in [46, Proposition 1.9] that $\Delta^+(x, z) \leq \Delta^+(x, y) + \Delta^+(y, z)$, for all $x, y, z \in L$. By taking the image of that inequality under the monoid homomorphism μ, it follows that $\delta(x \wedge z, x) \leq \delta(x \wedge y, x) \vee \delta(y \wedge z, y)$, that is, $x \smallsetminus z \leq (x \smallsetminus y) \vee (y \smallsetminus z)$, thus completing the proof of the triangle inequality.

Now let $x_1, x_2, y \in L$ with $x_1 \leq x_2$. By using (D2) and (D0), we get

$$x_1 \smallsetminus x_2 = x_1 \smallsetminus (x_1 \wedge x_2) = x_1 \smallsetminus x_1 = 0\,,$$

thus, by the triangle inequality, $x_1 \smallsetminus y \leq (x_1 \smallsetminus x_2) \vee (x_2 \smallsetminus y) = x_2 \smallsetminus y$. The proof, that $y_1 \leq y_2$ implies $x \smallsetminus y_2 \leq x \smallsetminus y_1$, is similar. □

Lemma 6.5. *Let L be a finite lattice, let S be a $(\vee, 0)$-semilattice, and let $\smallsetminus$ be an S-valued difference operation on L. Then the following statement holds:*

$$a \smallsetminus b = \bigvee \left(p \smallsetminus p_* \mid p \in \operatorname{Ji} L\,,\ p \leq a\,,\ p \nleq b\right)\,, \quad \textit{for all } a, b \in L\,. \tag{6.1}$$

Proof. Since neither side of (6.1) is affected by changing the pair (a, b) to $(a, a \wedge b)$, we may assume that $a \geq b$, and then prove (6.1) by induction on a. The result is trivial for $a = b$ (use (D0)). Dealing with the induction step, suppose that $a > b$. Pick $a' \in L$ such that $b \leq a'$ and a' is a lower cover of a. The set $\{x \in L \mid x \leq a \text{ and } x \nleq a'\}$ has a minimal element p. Necessarily, p is join-irreducible and $p_* \leq a'$, so $[p_*, p]$ projects up to $[a', a]$. By (D2), $p \smallsetminus p_* = a \smallsetminus a'$. Moreover, by the induction hypothesis,

$$a' \smallsetminus b = \bigvee (q \smallsetminus q_* \mid q \in \operatorname{Ji} L\,,\ q \leq a'\,,\ q \nleq b)\ .$$

Using (D1), we get $a \smallsetminus b = (a \smallsetminus a') \vee (a' \smallsetminus b) \geq \bigvee (q \smallsetminus q_* \mid q \in \operatorname{Ji} L\,,\ q \leq a\,,\ q \nleq b)$. For the converse inequality, let $q \in \operatorname{Ji} L$ such that $q \leq a$ and $q \nleq b$. Observing that $q \wedge b \leq q_* < q$ and $b < q \vee b \leq a$, we obtain

$$\begin{aligned} q \smallsetminus q_* &\leq q \smallsetminus (q \wedge b) && \text{(use the second statement of Lemma 6.4)} \\ &= (q \vee b) \smallsetminus b && \text{(use (D2))} \\ &\leq a \smallsetminus b && \text{(use again the second statement of Lemma 6.4).} \end{aligned}$$

□

Lemma 6.6. *Let D be a distributive lattice, let S be a $(\vee, 0)$-semilattice, let $\smallsetminus$ be an S-valued difference operation on D. Then for all $a_1, b_1, a_2, b_2 \in D$, if $a_1 \leq b_1 \vee a_2$ and $a_1 \wedge b_2 \leq b_1$ (within D), then $a_1 \smallsetminus b_1 \leq a_2 \smallsetminus b_2$ (within S).*

Proof. Changing each b_i to $a_i \wedge b_i$ affects neither the premise nor the conclusion of Lemma 6.6 (use the distributivity of D together with (D2)). Hence, we may assume that $a_i \geq b_i$, for $i \in \{1, 2\}$. Set $u_1 \underset{\text{def}}{=} a_1 \wedge a_2$, $v_1 \underset{\text{def}}{=} b_1 \wedge a_2$, $u_2 \underset{\text{def}}{=} u_1 \vee b_2$, $v_2 \underset{\text{def}}{=} v_1 \vee b_2$. Then the interval $[b_1, a_1]$ projects down to $[v_1, u_1]$, which projects up to $[v_2, u_2]$, which is contained in $[b_2, a_2]$ (cf. Figure 6.1). Since, by (D2), the difference

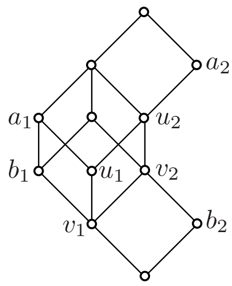

FIGURE 6.1. The sublattice of D generated by $\{a_1, b_1, a_2, b_2\}$

operation gives the same value to projective intervals, it follows that $a_1 \smallsetminus b_1 = u_1 \smallsetminus v_1 = u_2 \smallsetminus v_2$. Moreover, by (D1), $a_2 \smallsetminus b_2 = (a_2 \smallsetminus u_2) \vee (u_2 \smallsetminus v_2) \vee (v_2 \smallsetminus b_2) \geq u_2 \smallsetminus v_2$, so $a_2 \smallsetminus b_2 \geq a_1 \smallsetminus b_1$. □

The following lemma will be a crucial source of difference operations throughout the present paper. Although the results of the paper use only the finite case of the next two lemmas, we formulate those in the general case, with future applications in mind.

Lemma 6.7. *Let D be a sublattice in a generalized dual Heyting algebra S. Then the operation $\smallsetminus_S$ is an S-valued difference operation on D, and if D is consonant in S, then $\smallsetminus_S$ is a normal difference operation on D.*

Proof. It is a straightforward exercise to verify that $\smallsetminus_S$ is an S-valued difference operation on D. Now suppose that D is consonant in S and let $x, y \in D$. By assumption, $x \sim_S y$, thus there are $u, v \in S$ such that $x \leq y \vee u$, $y \leq x \vee v$, and $u \wedge v = 0$. Since $x \smallsetminus_S y \leq u$ and $y \smallsetminus_S x \leq v$, it follows that $(x \smallsetminus_S y) \wedge (y \smallsetminus_S x) = 0$. $\square$

Lemma 6.8. *The following statements hold, for every generalized dual Heyting algebra S and all elements $a_1, a_2, a, b_1, b_2, b \in S$:*

(1) $(a_1 \vee a_2) \smallsetminus_S b = (a_1 \smallsetminus_S b) \vee (a_2 \smallsetminus_S b)$.
(2) $a \smallsetminus_S (b_1 \wedge b_2) = (a \smallsetminus_S b_1) \vee (a \smallsetminus_S b_2)$.
(3) *If* $a_1 \sim_S a_2$, *then* $(a_1 \wedge a_2) \smallsetminus_S b = (a_1 \smallsetminus_S b) \wedge (a_2 \smallsetminus_S b)$.
(4) *If* $b_1 \sim_S b_2$, *then* $a \smallsetminus_S (b_1 \vee b_2) = (a \smallsetminus_S b_1) \wedge (a \smallsetminus_S b_2)$.

Proof. Ad (1). For every $s \in S$,

$$\begin{aligned}(a_1 \vee a_2) \smallsetminus_S b \leq s &\Leftrightarrow a_1 \vee a_2 \leq b \vee s\\ &\Leftrightarrow a_1 \leq b \vee s \text{ and } a_2 \leq b \vee s\\ &\Leftrightarrow a_1 \smallsetminus_S b \leq s \text{ and } a_2 \smallsetminus_S b \leq s\\ &\Leftrightarrow (a_1 \smallsetminus_S b) \vee (a_2 \smallsetminus_S b) \leq s\,.\end{aligned}$$

The desired conclusion follows.

Ad (2). For every $s \in S$,

$$\begin{aligned}a \smallsetminus_S (b_1 \wedge b_2) \leq s &\Leftrightarrow a \leq (b_1 \wedge b_2) \vee s\\ &\Leftrightarrow a \leq b_1 \vee s \text{ and } a \leq b_2 \vee s \qquad \text{(because } S \text{ is distributive)}\\ &\Leftrightarrow a \smallsetminus_S b_1 \leq s \text{ and } a \smallsetminus_S b_2 \leq s\\ &\Leftrightarrow (a \smallsetminus_S b_1) \vee (a \smallsetminus_S b_2) \leq s\,.\end{aligned}$$

The desired conclusion follows.

Ad (3). We first compute as follows:

$$\begin{aligned}a_1 \smallsetminus_S b &\leq \bigl(a_1 \smallsetminus_S (a_1 \wedge a_2)\bigr) \vee \bigl((a_1 \wedge a_2) \smallsetminus_S b\bigr) &&\text{(use Lemmas 6.4 and 6.7)}\\ &= (a_1 \smallsetminus_S a_2) \vee \bigl((a_1 \wedge a_2) \smallsetminus_S b\bigr) &&\text{(use (D2)).}\end{aligned}$$

Symmetrically, $a_2 \smallsetminus_S b \leq (a_2 \smallsetminus_S a_1) \vee \bigl((a_1 \wedge a_2) \smallsetminus_S b\bigr)$. By meeting the two inequalities, we obtain, by using the distributivity of S, the following inequality:

$$(a_1 \smallsetminus_S b) \wedge (a_2 \smallsetminus_S b) \leq \bigl((a_1 \smallsetminus_S a_2) \wedge (a_2 \smallsetminus_S a_1)\bigr) \vee \bigl((a_1 \wedge a_2) \smallsetminus_S b\bigr)\,.$$

Now our assumption $a_1 \sim_S a_2$ means that $(a_1 \smallsetminus_S a_2) \wedge (a_2 \smallsetminus_S a_1) = 0$, so we obtain the following inequality:

$$(a_1 \smallsetminus_S b) \wedge (a_2 \smallsetminus_S b) \leq (a_1 \wedge a_2) \smallsetminus_S b\,.$$

The converse inequality is trivial.

Ad (4). We first compute as follows:

$$\begin{aligned}a \smallsetminus_S b_1 &\leq \bigl(a \smallsetminus_S (b_1 \vee b_2)\bigr) \vee \bigl((b_1 \vee b_2) \smallsetminus_S b_1\bigr) &&\text{(use Lemmas 6.4 and 6.7)}\\ &= \bigl(a \smallsetminus_S (b_1 \vee b_2)\bigr) \vee (b_2 \smallsetminus_S b_1) &&\text{(use (D2)).}\end{aligned}$$

Symmetrically, $a \smallsetminus_S b_2 \leq \bigl(a \smallsetminus_S (b_1 \vee b_2)\bigr) \vee (b_1 \smallsetminus_S b_2)$. By meeting the two inequalities, we obtain, by using the distributivity of S, the following inequality:

$$(a \smallsetminus_S b_1) \wedge (a \smallsetminus_S b_2) \leq \bigl(a \smallsetminus_S (b_1 \vee b_2)\bigr) \vee \bigl((b_1 \smallsetminus_S b_2) \wedge (b_2 \smallsetminus_S b_1)\bigr)\,.$$

Now our assumption $b_1 \sim_S b_2$ means that $(b_1 \smallsetminus_S b_2) \wedge (b_2 \smallsetminus_S b_1) = 0$, so we obtain the following inequality:

$$(a \smallsetminus_S b_1) \wedge (a \smallsetminus_S b_2) \leq a \smallsetminus_S (b_1 \vee b_2)\,.$$

The converse inequality is trivial. $\square$

Lemma 6.9. *Let S be a generalized dual Heyting algebra and let $a_1, a_2, b_1, b_2 \in S$. If $a_1 \sim_S a_2$ and $a_1 \wedge a_2 \leq b_1 \wedge b_2$, then $(a_1 \smallsetminus_S b_1) \wedge (a_2 \smallsetminus_S b_2) = 0$.*

Proof. Set $b \underset{\mathrm{def}}{=} b_1 \wedge b_2$. We compute as follows:

$$\begin{aligned}
(a_1 \smallsetminus_S b_1) \wedge (a_2 \smallsetminus_S b_2) &\leq (a_1 \smallsetminus_S b) \wedge (a_2 \smallsetminus_S b) && \text{(because each } b_i \geq b\text{)}\\
&= (a_1 \wedge a_2) \smallsetminus_S b && \text{(use Lemma 6.8)}\\
&= 0 && \text{(by assumption)}\,.
\end{aligned}$$ $\square$

Although the results of the present paper use only the case where E and S are both finite, we will formulate our next lemma in the general case.

Lemma 6.10. *Let D and L be distributive lattices with zero, let E and S be generalized dual Heyting algebras, and let $g\colon E \to L$ be a 0-lattice homomorphism. We assume that D is a consonant 0-sublattice of E and that $g[D]$ is a consonant subset of S. Let G be a subset of D, generating D as a lattice. If $g(x \smallsetminus_E y) \leq g(x) \smallsetminus_S g(y)$ for all $x, y \in G$, then $g(x \smallsetminus_E y) \leq g(x) \smallsetminus_S g(y)$ for all $x, y \in D$.*

The situation in Lemma 6.10 is partly illustrated in Figure 6.2.

$$\begin{tikzcd}
D \arrow[r, hook, "\text{consonant}"] \arrow[d, "g \restriction D"', "\text{consonant}"] & E \arrow[d, "g"] \\
S \arrow[r, hook] & L
\end{tikzcd}$$

FIGURE 6.2. Illustrating Lemma 6.10

Proof. Let $x \in G$. We claim that the set $D_x \underset{\mathrm{def}}{=} \{y \in D \mid g(x \smallsetminus_E y) \leq g(x) \smallsetminus_S g(y)\}$ is equal to D. Indeed, it follows from our assumptions that $G \subseteq D_x$. For all $y_1, y_2 \in D_x$,

$$\begin{aligned}
g\big(x \smallsetminus_E (y_1 \vee y_2)\big) &= g\big((x \smallsetminus_E y_1) \wedge (x \smallsetminus_E y_2)\big)\\
&\qquad \text{(because } D \text{ is consonant in } E \text{ and by Lemma 6.8)}\\
&= g(x \smallsetminus_E y_1) \wedge g(x \smallsetminus_E y_2)\\
&\qquad \text{(because } g \text{ is a meet-homomorphism)}\\
&\leq \big(g(x) \smallsetminus_S g(y_1)\big) \wedge \big(g(x) \smallsetminus_S g(y_2)\big) \qquad \text{(because } y_1, y_2 \in D_x\text{)}\\
&= g(x) \smallsetminus_S \big(g(y_1) \vee g(y_2)\big)\\
&\qquad \text{(because } g[D] \text{ is consonant in } S \text{ and by Lemma 6.8)}\\
&= g(x) \smallsetminus_S g(y_1 \vee y_2) \quad \text{(because } g \text{ is a join-homomorphism)}\,,
\end{aligned}$$

that is, $y_1 \vee y_2 \in D_x$. The proof that $y_1 \wedge y_2 \in D_x$ is similar, although easier since it does not require any consonance assumption. Hence, D_x is a sublattice of D. Since it contains G, it contains D; whence $D_x = D$.

This holds for all $x \in G$, which means that for all $y \in D$, the set $D'_y \underset{\mathrm{def}}{=} \{x \in D \mid g(x \smallsetminus_E y) \leq g(x) \smallsetminus_S g(y)\}$ contains G. Moreover, by an argument similar to the one used in the paragraph above, D'_y is a sublattice of D. Hence, $D'_y = D$. This holds for all $y \in D$; the desired conclusion follows. $\square$

7. Homomorphism extension on distributive lattices

The key idea, of our proof of Theorem 11.1, is the possibility of extending certain lattice homomorphisms $f\colon D \to L$, where D and L are distributive 0-lattices with D finite and L completely normal, to finite, or countable, distributive extensions of D. The following example shows that this is hopeless in general.

Example 7.1. Let m be a positive integer, and denote by F_m the Abelian ℓ-group defined by generators a, b, c, and relations $a \geq b \geq c \geq 0$ together with $(a - mb) \vee (b - mc) \leq 0$. Then $0 < \langle c\rangle < \langle b\rangle < \langle a\rangle$ within $\mathrm{Id_c}\, F_m$, and the 4-element chain $\mathbf{4} \underset{\mathrm{def}}{=} \{0, \langle c\rangle, \langle b\rangle, \langle a\rangle\}$ is ℓ-representable (by the third lexicographical power of the integers). We claim that $\mathbf{4}$ is not a lattice-theoretical retract of $\mathrm{Id_c}\, F_m$. Indeed, suppose that $\rho\colon \mathrm{Id_c}\, F_m \twoheadrightarrow \mathbf{4}$ is such a retraction, set $\boldsymbol{x} = \rho\bigl((a - mb)^+\bigr)$, and $\boldsymbol{y} = \rho\bigl((b - mc)^+\bigr)$. Then $\langle a\rangle = \langle b\rangle \vee \boldsymbol{x}$ within $\mathbf{4}$, thus $\boldsymbol{x} = \langle a\rangle$. Likewise, $\langle b\rangle = \langle c\rangle \vee \boldsymbol{y}$, thus $\boldsymbol{y} = \langle b\rangle$. On the other hand, since ρ is a $(\wedge, 0)$-homomorphism, $\boldsymbol{x} \wedge \boldsymbol{y} = 0$, a contradiction.

The problem raised by Example 7.1 makes our road to Theorem 11.1 more convoluted than one could have expected. The present section contains the required lattice-theoretical extension results.

We will denote by J_2 the second entry of *Jaskowsky's sequence*, defined as the lattice of all lower subsets of the three-element set $\{\mathsf{a}, \mathsf{b}, 1\}$ with $\mathsf{a} < 1$ and $\mathsf{b} < 1$. Hence, J_2 is the distributive lattice obtained from the square, with atoms a and b, by adjoining a new top element 1. It is represented in Figure 7.1.

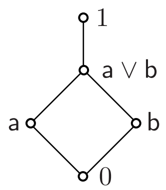

Figure 7.1. The lattice J_2

For any bounded distributive lattice D, the free distributive product $D * \mathsf{J}_2$, that is, the coproduct of D and J_2 in the category of bounded distributive lattices with 0, 1-lattice homomorphisms, can be identified with the sublattice of D^3 consisting of all triples $(x, y, z) \in D^3$ such that $z \leq x$ and $z \leq y$. This follows for example from Grätzer [24, Theorem 12.5]. For $x, y, z \in D$, the element $(x \wedge \mathsf{a}) \vee (y \wedge \mathsf{b}) \vee z$ of $D * \mathsf{J}_2$ can be identified with the triple $(x \vee z, y \vee z, z)$. We will need the following form of that observation, which no longer requires any boundedness assumption on D and which can be verified by a direct calculation.

Lemma 7.2. *Let D and E be distributive lattices such that E has a zero, and let $a, b \in E$ such that $a \wedge b = 0$. Then for every lattice homomorphism $f\colon D \to E$, the* *map* $g\colon D * \mathsf{J}_2 \to E$ *defined by*

$$g(x,y,z) \underset{\mathrm{def}}{=} \big(f(x) \wedge a\big) \vee \big(f(y) \wedge b\big) \vee f(z)\,, \qquad \text{for all } (x,y,z) \in D * \mathsf{J}_2\,,$$

is a lattice homomorphism. Moreover, $g(x,x,x) = f(x)$ *for every* $x \in D$.

The harder extension results in our paper rely upon the following crucial technical lemma.

Lemma 7.3. *Let E be a finite distributive lattice, let D be a* $0,1$-*sublattice of E, and let $a, b \in E$ such that the following conditions hold:*

(1) *E is generated, as a sublattice, by $D \cup \{a, b\}$.*
(2) *D is a Heyting subalgebra of E.*
(3) $a \wedge b = 0$.
(4) *For every $p \in \mathrm{Ji}\,D$, $p \leq p_* \vee a \vee b$ implies that either $p \leq p_* \vee a$ or $p \leq p_* \vee b$.*
(5) *Let $p, q \in \mathrm{Ji}\,D$. If $p \leq p_* \vee a$ and $q \leq q_* \vee b$, then p and q are incomparable.*

Let L be a generalized dual Heyting algebra and let $f\colon D \to L$ be a consonant 0-*lattice homomorphism. For every $t \in E$, we set*

$$f_*(t) \underset{\mathrm{def}}{=} \bigvee \left(f(p) \smallsetminus_L f(p_*) \mid p \in \mathrm{Ji}\,D \text{ and } p \leq p_* \vee t\right)\,,$$

$$f^*(t) \underset{\mathrm{def}}{=} \bigwedge \left(f(x) \mid x \in D \text{ and } t \leq x\right)\,,$$

*Call a pair (α, β) of elements of L f-*admissible *if there exists a (necessarily unique) lattice homomorphism $g\colon E \to L$, extending f, such that $g(a) = \alpha$ and $g(b) = \beta$. Then the following statements hold:*

(i) *$f_*(t) \leq f^*(t)$, for every $t \in E$.*
(ii) $f_*(a) \wedge f_*(b) = 0$.
(iii) *The f-admissible pairs are exactly the pairs (α, β) satisfying the inequalities $f_*(a) \leq \alpha \leq f^*(a)$, $f_*(b) \leq \beta \leq f^*(b)$, and $\alpha \wedge \beta = 0$.*

Note. Although the proof of our main result, Theorem 11.1, will require only the consideration of $(\alpha, \beta) = (f_*(a), f_*(b))$, we keep the more general formulation, due to possible relevance to further work. The proof of Lemma 7.3 is mostly unaffected by that slight increase in generality.

Proof. The uniqueness statement on g follows immediately from Assumption (1), so we need to deal only with existence.

Since f is consonant, the assignment $(x, y) \mapsto f(x) \smallsetminus_L f(y)$ defines an L-valued normal difference operation on D (use Lemma 6.7). By Lemma 6.5, it follows that

$$f(x) \smallsetminus_L f(y) = \bigvee \left(f(p) \smallsetminus_L f(p_*) \mid p \in \mathrm{Ji}\,D\,,\ p \leq x\,,\ p \nleq y\right)\,, \quad \text{for all } x, y \in D\,. \tag{7.1}$$

The remainder of our proof consists mainly of a series of claims.

Claim 1. *The following equation holds, for every $t \in E$:*

$$f_*(t) = \bigvee \left(f(x) \smallsetminus_L f(y) \mid x, y \in D \text{ and } x \leq y \vee t\right)\,. \tag{7.2}$$

Consequently, $x \leq y \vee t$ implies that $f(x) \leq f(y) \vee f_*(t)$, for all $x, y \in D$.

Proof of Claim. Let $x, y \in D$ such that $x \leq y \vee t$, and let $p \in \mathrm{Ji}\,D$ such that $p \leq x$ and $p \nleq y$. The latter relation means that $y \leq p^\dagger$. Hence, from $x \leq y \vee t$ it follows that $p \leq p^\dagger \vee t$, or, equivalently, $p \leq p_* \vee t$. Therefore, by the definition of $f_*(t)$, we get $f(p) \smallsetminus_L f(p_*) \leq f_*(t)$. Joining those inequalities over all possible values of p and invoking (7.1), we get $f(x) \smallsetminus_L f(y) \leq f_*(t)$. This proves that the right hand side of (7.2) is less than or equal to $f_*(t)$. The converse inequality being trivial, (7.2) follows. □ Claim 1.

Claim 2. *Let $x, y \in D$ such that $x \leq y \vee a \vee b$. Then $f(x) \leq f(y) \vee f_*(a) \vee f_*(b)$.*

Proof of Claim. We must prove that $f(x) \smallsetminus_L f(y) \leq f_*(a) \vee f_*(b)$. By (7.1), it suffices to prove that $f(p) \smallsetminus_L f(p_*) \leq f_*(a) \vee f_*(b)$, for every $p \in \operatorname{Ji} D$ such that $p \leq x$ and $p \nleq y$. The latter relation means that $y \leq p^\dagger$, thus, for any such p, the inequality $p \leq p^\dagger \vee a \vee b$, or, equivalently, $p \leq p_* \vee a \vee b$, holds. By Assumption (4), this implies that either $p \leq p_* \vee a$ or $p \leq p_* \vee b$. By the definition of $f_*(a)$ and $f_*(b)$, this implies that either $f(p) \smallsetminus_L f(p_*) \leq f_*(a)$ or $f(p) \smallsetminus_L f(p_*) \leq f_*(b)$. In both cases, $f(p) \smallsetminus_L f(p_*) \leq f_*(a) \vee f_*(b)$. □ Claim 2.

Claim 3. *$f_*(t) \leq f^*(t)$, for every $t \in E$.*

Proof of Claim. We must prove that for all $p \in \operatorname{Ji} D$ and all $x \in D$ such that $p \leq p_* \vee t$ and $t \leq x$, the inequality $f(p) \smallsetminus_L f(p_*) \leq f(x)$ holds. Since obviously, $p \leq p_* \vee x$, it follows, since p is join-prime in D, that $p \leq x$, so we obtain the inequalities $f(p) \smallsetminus_L f(p_*) \leq f(p) \leq f(x)$. □ Claim 3.

Claim 4. *$f_*(a) \wedge f_*(b) = 0$.*

Proof of Claim. It suffices to prove that for all $p, q \in \operatorname{Ji} D$ with $p \leq p_* \vee a$ and $q \leq q_* \vee b$, the relation $(f(p) \smallsetminus_L f(p_*)) \wedge (f(q) \smallsetminus_L f(q_*)) = 0$ holds. By Lemma 6.9, it suffices to prove that $f(p) \wedge f(q) \leq f(p_*) \wedge f(q_*)$. Since f is a meet-homomorphism, it suffices to prove that $p \wedge q \leq p_* \wedge q_*$. However, it follows from (5) that p and q are incomparable, so this is obvious. □ Claim 4.

Now it is clear that every f-admissible pair (α, β) satisfies $f_*(a) \leq \alpha \leq f^*(a)$, $f_*(b) \leq \beta \leq f^*(b)$, and $\alpha \wedge \beta = 0$. It thus remains to prove that conversely, every such pair (α, β) is f-admissible.

Claim 5. *There exists a unique map $g \colon E \to L$ such that*

$$g\big((x \wedge a) \vee (y \wedge b) \vee z\big) = (f(x) \wedge \alpha) \vee (f(y) \wedge \beta) \vee f(z)\,, \quad \text{for all } x, y, z \in D\,. \tag{7.3}$$

Moreover, $g(a) = \alpha$, $g(b) = \beta$, and g is a join-homomorphism extending f.

Proof of Claim. By Assumptions (1) and (3), every element t of E has the form $(x \wedge a) \vee (y \wedge b) \vee z$, where $x, y, z \in D$. This implies the uniqueness statement on g, and says that all we need to do is to verify that the right hand side of (7.3) depends only on t; the map g thus defined, *via* (7.3), would then automatically be a join-homomorphism extending f, satisfying, by virtue of the relations $f(0) = 0$, $\alpha \leq f(1)$, and $\beta \leq f(1)$, the equations $g(a) = \alpha$ and $g(b) = \beta$. Hence, we only need to verify that the following implications hold, for every $u \in D$:

$$u \leq (x \wedge a) \vee (y \wedge b) \vee z \Rightarrow f(u) \leq (f(x) \wedge \alpha) \vee (f(y) \wedge \beta) \vee f(z)\,, \tag{7.4}$$

$$u \wedge a \leq (x \wedge a) \vee (y \wedge b) \vee z \Rightarrow f(u) \wedge \alpha \leq (f(x) \wedge \alpha) \vee (f(y) \wedge \beta) \vee f(z)\,, \tag{7.5}$$

$$u \wedge b \leq (x \wedge a) \vee (y \wedge b) \vee z \Rightarrow f(u) \wedge \beta \leq (f(x) \wedge \alpha) \vee (f(y) \wedge \beta) \vee f(z)\,, \tag{7.6}$$

for all $u, x, y, z \in D$. Since E is distributive, the premise of (7.4) is equivalent to the conjunction of the following inequalities:

$$\begin{aligned} &u \leq x \vee y \vee z\,; \\ &u \leq x \vee b \vee z\,; \\ &u \leq a \vee y \vee z\,; \\ &u \leq a \vee b \vee z\,. \end{aligned}$$

Since f is a join-homomorphism and by Claims 1 and 2, together with the inequalities $f_*(a) \leq \alpha$ and $f_*(b) \leq \beta$, those inequalities imply the following inequalities:

$$\begin{aligned} &f(u) \leq f(x) \vee f(y) \vee f(z)\,; \\ &f(u) \leq f(x) \vee \beta \vee f(z)\,; \\ &f(u) \leq \alpha \vee f(y) \vee f(z)\,; \\ &f(u) \leq \alpha \vee \beta \vee f(z)\,. \end{aligned}$$

Since L is distributive, this implies, by reversing the argument above, the inequality $f(u) \leq (f(x) \wedge \alpha) \vee (f(y) \wedge \beta) \vee f(z)$, thus completing the proof of (7.4).

Further, since E is distributive and since $a \wedge b = 0$, the premise of (7.5) is equivalent to the inequality $u \wedge a \leq (x \wedge a) \vee z$, thus to the inequality $u \wedge a \leq x \vee z$, which can be written $a \leq (u \rightarrow_E (x \vee z))$. By Assumption (2), this is equivalent to $a \leq v$, where we set $v \underset{\mathrm{def}}{=} (u \rightarrow_D (x \vee z))$. Since $\alpha \leq f^*(a)$, this implies that $\alpha \leq f(v)$. Hence, $f(u) \wedge \alpha \leq f(u) \wedge f(v) = f(u \wedge v) \leq f(x \vee z) = f(x) \vee f(z)$. Since L is distributive, this implies in turn that

$$f(u) \wedge \alpha \leq (f(x) \wedge \alpha) \vee (f(y) \wedge \beta) \vee f(z)$$

thus completing the proof of (7.5). The proof of (7.6) is symmetric. $\square$ Claim 5.

In order to conclude the proof of Lemma 7.3, it is sufficient to prove that g is a meet-homomorphism. By Assumption (3) and since $\alpha \wedge \beta = 0$, respectively, it follows from Lemma 7.2 that there are unique lattice homomorphisms $d\colon D * \mathsf{J}_2 \to E$ and $\delta\colon D * \mathsf{J}_2 \to L$ such that

$$d(x, y, z) = (x \wedge a) \vee (y \wedge b) \vee z \text{ and } \delta(x, y, z) = \big(f(x) \wedge a\big) \vee \big(f(y) \wedge b\big) \vee f(z)$$

for all $(x, y, z) \in D * \mathsf{J}_2$. Then Claim 5 implies that $\delta = g \circ d$. Moreover, it follows from Assumption (1) that d is surjective. Now any two elements of E have the form $d(t_1)$ and $d(t_2)$, where $t_1, t_2 \in D * \mathsf{J}_2$, and

$$g(d(t_1)) \wedge g(d(t_2)) = \delta(t_1) \wedge \delta(t_2) = \delta(t_1 \wedge t_2) = g(d(t_1 \wedge t_2)) \leq g(d(t_1) \wedge d(t_2))\,.$$

The converse inequality $g(d(t_1) \wedge d(t_2)) \leq g(d(t_1)) \wedge g(d(t_2))$ is trivial. $\square$

8. Lattices of convex open polyhedral cones

Throughout this section we shall fix a topological vector space $\mathbb{E}$, not necessarily finite-dimensional, over the reals[4]. Moreover, in all subsequent sections, for any closed hyperplane H of $\mathbb{E}$, we shall denote by H^+ and H^- the open half-spaces with boundary H, with associated closed half-spaces $\overline{H}^+ \underset{\mathrm{def}}{=} \mathrm{cl}(H^+)$ and $\overline{H}^- \underset{\mathrm{def}}{=} \mathrm{cl}(H^-)$.

[4]Many results of the paper, on real vector spaces, extend to any arbitrary totally ordered field in place of the reals. A notable exception is Lemma 10.4 (and thus all subsequent results), which requires the field be Archimedean.

Notation 8.1. For a set $\mathcal{H}$ of closed hyperplanes of $\mathbb{E}$, we will set

$$\Sigma_{\mathcal{H}} \underset{\text{def}}{=} \left\{H^+ \mid H \in \mathcal{H}\right\} \cup \left\{H^- \mid H \in \mathcal{H}\right\} ,$$
$$\overline{\Sigma}_{\mathcal{H}} \underset{\text{def}}{=} \left\{\overline{H}^+ \mid H \in \mathcal{H}\right\} \cup \left\{\overline{H}^- \mid H \in \mathcal{H}\right\} .$$

Furthermore, we will denote by $\operatorname{Bool}(\mathcal{H})$ the Boolean algebra of subsets of $\mathbb{E}$ generated by $\Sigma_{\mathcal{H}}$ (equivalently, by $\overline{\Sigma}_{\mathcal{H}}$), and by $\operatorname{Clos}(\mathcal{H})$ (resp., $\operatorname{Op}(\mathcal{H})$) the lattice of all closed (resp., open) members of $\operatorname{Bool}(\mathcal{H})$.

In particular, for every $H \in \mathcal{H}$, the hyperplane $H = \mathbb{E} \setminus (H^+ \cup H^-)$ belongs to $\operatorname{Bool}(\mathcal{H})$. Moreover, the bounds of $\operatorname{Bool}(\mathcal{H})$ are $\varnothing$ and $\mathbb{E}$.

Trivially, $\operatorname{Clos}(\mathcal{H})$ and $\operatorname{Op}(\mathcal{H})$ are both $0,1$-sublattices of $\operatorname{Bool}(\mathcal{H})$, which is a $0,1$-sublattice of the powerset lattice of $\mathbb{E}$. On one extreme end, $\operatorname{Bool}(\varnothing) = \operatorname{Clos}(\varnothing) = \operatorname{Op}(\varnothing) = \{\varnothing, \mathbb{E}\}$. In general,

$$\operatorname{Bool}(\mathcal{H}) = \bigcup\left(\operatorname{Bool}(\mathcal{H}') \mid \mathcal{H}' \subseteq \mathcal{H} \text{ finite}\right) ,$$
$$\operatorname{Op}(\mathcal{H}) = \bigcup\left(\operatorname{Op}(\mathcal{H}') \mid \mathcal{H}' \subseteq \mathcal{H} \text{ finite}\right) ,$$
$$\operatorname{Clos}(\mathcal{H}) = \bigcup\left(\operatorname{Clos}(\mathcal{H}') \mid \mathcal{H}' \subseteq \mathcal{H} \text{ finite}\right) .$$

For the remainder of this section we shall fix a *nonempty* set $\mathcal{H}$ of closed hyperplanes of $\mathbb{E}$ through the origin.

Lemma 8.2. *For every $X \in \operatorname{Bool}(\mathcal{H})$, the subsets $\operatorname{cl}(X)$ and $\operatorname{int}(X)$ both belong to $\operatorname{Bool}(\mathcal{H})$. Moreover, $\operatorname{Op}(\mathcal{H})$ is generated, as a lattice, by $\Sigma_{\mathcal{H}} \cup \{\mathbb{E}\}$, and it is Heyting subalgebra of the Heyting algebra $\mathcal{O}(\mathbb{E})$ of all open subsets of $\mathbb{E}$.*

Proof. For the duration of the proof, we shall denote by $\operatorname{Clos}'(\mathcal{H})$ (resp., $\operatorname{Op}'(\mathcal{H})$) the sublattice of $\operatorname{Bool}(\mathcal{H})$ generated by $\overline{\Sigma}_{\mathcal{H}} \cup \{\varnothing\}$ (resp., $\Sigma_{\mathcal{H}} \cup \{\mathbb{E}\}$).

We first prove that the closure of any member of $\operatorname{Bool}(\mathcal{H})$ belongs to $\operatorname{Clos}'(\mathcal{H})$. Writing the elements of $\operatorname{Bool}(\mathcal{H})$ in disjunctive normal form, we see that every element of $\operatorname{Bool}(\mathcal{H})$ is a finite union of finite intersections of open half-spaces and closed half-spaces with boundaries in $\mathcal{H}$. Since $H^\sigma = \overline{H}^\sigma \setminus H$, for each $H \in \mathcal{H}$ and each $\sigma \in \{+,-\}$, it follows that every element of $\operatorname{Bool}(\mathcal{H})$ is a finite union of sets of the form $Q \setminus F$, where Q is a finite intersection of closed half-spaces with boundaries in $\mathcal{H}$ and F is a finite union of members of $\mathcal{H}$. Since the closure operator commutes with finite unions, the first statement of Lemma 8.2 thus reduces to verifying that $\operatorname{cl}(Q \setminus F)$ belongs to $\operatorname{Clos}'(\mathcal{H})$, for any Q and F as above. Now this follows from Lemma 2.4: if $Q \subseteq F$ then $\operatorname{cl}(Q \setminus F) = \varnothing$, and if $Q \nsubseteq F$, then $Q \cap F$ is nowhere dense in Q, thus $\operatorname{cl}(Q \setminus F) = \operatorname{cl}(Q) = Q$. The statement about the closure follows; in particular, $\operatorname{Clos}'(\mathcal{H}) = \operatorname{Clos}(\mathcal{H})$. By taking complements, the statement about the interior follows; in particular, $\operatorname{Op}'(\mathcal{H}) = \operatorname{Op}(\mathcal{H})$.

In particular, for all $X, Y \in \operatorname{Op}(\mathcal{H})$, the Heyting residue $X \to Y$, evaluated within the lattice $\mathcal{O}(\mathbb{E})$ of all open subsets of $\mathbb{E}$, is equal to $\operatorname{int}\big((\complement X) \cup Y\big)$, thus, as $(\complement X) \cup Y$ belongs to $\operatorname{Bool}(\mathcal{H})$ and by the paragraph above, it belongs to $\operatorname{Op}(\mathcal{H})$. □

In particular, the members of $\operatorname{Op}(\mathcal{H})$ are *open polyhedral cones*, that is, Boolean combinations of open half-spaces of $\mathbb{E}$. Lemma 8.2 also says that the topology on $\mathbb{E}$ could be, in principle, omitted from the study of $\operatorname{Bool}(\mathcal{H})$ and $\operatorname{Op}(\mathcal{H})$.

Corollary 8.3. *The subsets $\operatorname{Bool}(\mathcal{H})$ and $\operatorname{Op}(\mathcal{H})$ are independent of the structure of topological vector space on $\mathbb{E}$ that makes all members of $\mathcal{H}$ closed.*

Corollary 8.4. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be sets of closed hyperplanes of $\mathbb{E}$. If $\mathcal{H}_1 \subseteq \mathcal{H}_2$, then* $\mathrm{Op}(\mathcal{H}_1)$ *is a Heyting subalgebra of* $\mathrm{Op}(\mathcal{H}_2)$*.*

Define a *basic open member of* $\mathrm{Op}(\mathcal{H})$ as a nonempty finite intersection of open half-spaces with boundaries in $\mathcal{H}$. In particular, the empty intersection yields the basic open set $\mathbb{E}$.

Corollary 8.5. *Every join-irreducible element of* $\mathrm{Op}(\mathcal{H})$ *is basic open. In particular, it is convex.*

Proof. It follows from Lemma 8.2 that every element of $\mathrm{Op}(\mathcal{H})$ is a finite union of basic open sets. The desired conclusion follows immediately. □

It is easy to find examples showing that the converse of Corollary 8.5 does not hold: *A basic open member of* $\mathrm{Op}(\mathcal{H})$ *may not be join-irreducible.* On the other hand, it can be proved that the basic open members of $\mathrm{Op}(\mathcal{H})$ are exactly its convex members. Since this fact will not be needed in the paper, we omit its proof.

Corollary 8.6. *Let H be a closed hyperplane of $\mathbb{E}$, with associated open half spaces H^+ and H^-. Then the members of* $\mathrm{Op}(\mathcal{H} \cup \{H\})$ *are exactly the sets of the form*

$$X \cup (Y^+ \cap H^+) \cup (Y^- \cap H^-)\,, \tag{8.1}$$

where X, Y^+, and Y^- are basic open in $\mathrm{Op}(\mathcal{H})$*. Moreover, one can take $X \subseteq Y^+ \cap Y^-$ in* (8.1)*.*

Proof. For every basic open set X in $\mathrm{Op}(\mathcal{H} \cup \{H\})$, there is a basic open set Y in $\mathrm{Op}(\mathcal{H})$ such that either $X = Y$ or $X = Y \cap H^+$ or $X = Y \cap H^-$. By Lemma 8.2, every element of $\mathrm{Op}(\mathcal{H} \cup \{H\})$ is a finite union of basic open sets, thus it has the form (8.1). Moreover, changing Y^σ to $X \cup Y^\sigma$, for $\sigma \in \{+, -\}$, does not affect the right hand side of (8.1). Hence, one can take $X \subseteq Y^+ \cap Y^-$. □

Lemma 8.7. *The top element of* $\mathrm{Op}(\mathcal{H})$*, namely $\mathbb{E}$, is join-irreducible in* $\mathrm{Op}(\mathcal{H})$*. Consequently, the subset* $\mathrm{Op}^-(\mathcal{H}) \underset{\mathrm{def}}{=} \mathrm{Op}(\mathcal{H}) \setminus \{\mathbb{E}\}$ *is a* 0*-sublattice of* $\mathrm{Op}(\mathcal{H})$*. It is generated, as a sublattice, by $\Sigma_{\mathcal{H}}$.*

Proof. Any basic open member of $\mathrm{Op}(\mathcal{H})$, distinct from $\mathbb{E}$, omits the origin. Hence, any member of $\mathrm{Op}(\mathcal{H})$, distinct from $\mathbb{E}$, omits the origin, and so the union of any two such sets is distinct from $\mathbb{E}$. This proves that $\mathbb{E}$ is join-irreducible in $\mathrm{Op}(\mathcal{H})$. The verifications of the other statements of Lemma 8.7 are straightforward. □

Remark 8.8. Let $\mathcal{H}$ be finite. Then the unit of $\mathrm{Op}^-(\mathcal{H})$ is equal to $\mathbb{E} \setminus \bigcap \mathcal{H}$, which is distinct from the unit of $\mathrm{Op}(\mathcal{H})$, which is equal to $\mathbb{E}$. In particular, $\mathrm{Op}^-(\mathcal{H})$ is not a Heyting subalgebra of $\mathrm{Op}(\mathcal{H})$.

9. Join-irreducible members of $\mathrm{Op}(\mathcal{H})$

Throughout this section we shall fix a real topological vector space $\mathbb{E}$ and a nonempty *finite* set $\mathcal{H}$ of closed hyperplanes of $\mathbb{E}$ through the origin. After having conveniently described the join-irreducible members of $\mathrm{Op}(\mathcal{H})$ (cf. Lemma 9.2), we will specialize Lemma 7.3 to lattices of the form $\mathrm{Op}(\mathcal{H})$.

Notation 9.1. For every $U \in \mathrm{Op}(\mathcal{H})$, we set $\mathcal{H}_U \underset{\mathrm{def}}{=} \{H \in \mathcal{H} \mid H \cap U \neq \varnothing\}$. The intersection ∇U of all members of $\mathcal{H}_U$ is a closed subspace of $\mathbb{E}$.

The following lemma characterizes the join-irreducible elements in $\mathrm{Op}(\mathcal{H})$ in terms of the operator ∇ defined above. Recall (cf. Lemma 2.1) that $P^\dagger$ denotes the largest element of $\mathrm{Op}(\mathcal{H})$ not containing P, and that the assignment $P \mapsto P^\dagger$ defines an order-isomorphism from $\mathrm{Ji}\,\mathrm{Op}(\mathcal{H})$ onto $\mathrm{Mi}\,\mathrm{Op}(\mathcal{H})$.

Lemma 9.2. *A nonempty, convex member P of $\mathrm{Op}(\mathcal{H})$ is join-irreducible, within the lattice $\mathrm{Op}(\mathcal{H})$, iff $P \cap \nabla P$ is nonempty. Moreover, in that case, the lower cover P_* of P, in $\mathrm{Op}(\mathcal{H})$, is equal to $P \setminus \nabla P$, and $P^\dagger = \mathbb{E} \setminus (\mathrm{cl}(P) \cap \nabla P)$.*

Proof. Suppose first that P is join-irreducible. Suppose, by way of contradiction, that $P \cap \nabla P = \varnothing$, that is, $P \subseteq \bigcup_{H \in \mathcal{H}_P} (\mathbb{E} \setminus H)$. Since P is join-irreducible in the distributive lattice $\mathrm{Op}(\mathcal{H})$, it is join-prime in that lattice (cf. Lemma 2.1). Since each $\mathbb{E} \setminus H$ belongs to $\mathrm{Op}(\mathcal{H})$, there exists $H \in \mathcal{H}_P$ such that $P \subseteq \mathbb{E} \setminus H$; in contradiction with $H \in \mathcal{H}_P$.

Suppose, conversely, that $P \cap \nabla P \neq \varnothing$. The subset $P \setminus \nabla P$ belongs to $\mathrm{Op}(\mathcal{H})$ and it is a proper subset of P, thus we only need to prove that every proper subset X of P, belonging to $\mathrm{Op}(\mathcal{H})$, is contained in $P \setminus \nabla P$. It suffices to consider the case where X is join-irreducible in $\mathrm{Op}(\mathcal{H})$. By Corollary 8.5, X is basic open, so there are a subset $\mathcal{X}$ of $\mathcal{H}$ and a family $(\varepsilon_H \mid H \in \mathcal{X})$ of elements of $\{+, -\}$ such that $X = \bigcap_{H \in \mathcal{X}} H^{\varepsilon_H}$. Since $P \not\subseteq X$, there exists $H \in \mathcal{X}$ such that $P \not\subseteq H^{\varepsilon_H}$. Hence,

$$P \cap \overline{H}^{-\varepsilon_H} \neq \varnothing\,. \tag{9.1}$$

If $P \subseteq H^{-\varepsilon_H}$, then $X \subseteq H^{-\varepsilon_H}$, thus, since $X \subseteq H^{\varepsilon_H}$, we get $X = \varnothing$, a contradiction since X is join-irreducible. Hence, $P \not\subseteq H^{-\varepsilon_H}$, so

$$P \cap \overline{H}^{\varepsilon_H} \neq \varnothing\,. \tag{9.2}$$

By (9.1) and (9.2), and since P is convex, it follows that $P \cap H \neq \varnothing$, that is, $H \in \mathcal{H}_P$. Hence, $\nabla P \subseteq H$. Since $X \cap H = \varnothing$, it follows that $X \cap \nabla P = \varnothing$, thus completing the proof of the join-irreducibility of P.

Finally, it follows from Lemma 8.2 that the set $U \underset{\mathrm{def}}{=} \mathrm{int}\,\complement(P \cap \nabla P)$ belongs to $\mathrm{Op}(\mathcal{H})$. Moreover, $U = \complement\,\mathrm{cl}(P \cap \nabla P)$. Since $P \cap \nabla P \neq \varnothing$ and by Lemma 2.3, we get $U = \complement(\mathrm{cl}(P) \cap \nabla P)$. For every $V \in \mathrm{Op}(\mathcal{H})$, $P \not\subseteq V$ iff $P \cap V \subsetneqq P$, iff $P \cap V \subseteq P_*$, iff $P \cap V \cap \nabla P = \varnothing$, iff $V \subseteq \complement(P \cap \nabla P)$. Since V is open, this is equivalent to $V \subseteq U$. Therefore, $U = P^\dagger$. □

Note. The *atoms* of $\mathrm{Op}(\mathcal{H})$, called the *regions* of the hyperplane arrangement $\mathcal{H}$, have been studied extensively, starting with Björner, Edelman, and Ziegler [9]. For an atom P of $\mathrm{Op}(\mathcal{H})$, we get $\mathcal{H}_P = \varnothing$, thus $\nabla P = \mathbb{E}$.

Proposition 9.3. *Let P and Q be join-irreducible elements in $\mathrm{Op}(\mathcal{H})$. If $P \subsetneqq Q$, then $\nabla Q \subsetneqq \nabla P$.*

Proof. By definition, $\mathcal{H}_P \subseteq \mathcal{H}_Q$, thus $\nabla Q \subseteq \nabla P$. Since $P \subsetneqq Q$ and by Lemma 9.2, P is contained in $Q_* = Q \setminus \nabla Q$, thus $P \cap \nabla Q = \varnothing$. Since $P \cap \nabla P \neq \varnothing$, it follows that $\nabla P \neq \nabla Q$. □

Lemma 9.4. *Let H be a closed hyperplane of $\mathbb{E}$, let L be a generalized dual Heyting algebra, and let $f\colon \mathrm{Op}(\mathcal{H}) \to L$ be a consonant 0-lattice homomorphism. Then f extends to a unique lattice homomorphism $g\colon \mathrm{Op}(\mathcal{H} \cup \{H\}) \to L$ such that $g(H^+) = f_*(H^+)$ and $g(H^-) = f_*(H^-)$.*

We refer to Lemma 7.3 for the notations $f_*(H^+)$ and $f_*(H^-)$.

Proof. It suffices to verify that Conditions (1)–(5) of Lemma 7.3 are satisfied, where we set $D \underset{\text{def}}{=} \operatorname{Op}(\mathcal{H})$, $E \underset{\text{def}}{=} \operatorname{Op}(\mathcal{H} \cup \{H\})$, $a \underset{\text{def}}{=} H^+$, and $b \underset{\text{def}}{=} H^-$. Conditions (1) (use Corollary 8.6) and (3) are obvious. By Corollary 8.4, $\operatorname{Op}(\mathcal{H})$ is a Heyting subalgebra of $\operatorname{Op}(\mathcal{H} \cup \{H\})$; Condition (2) follows.

Let P be a join-irreducible element of $\operatorname{Op}(\mathcal{H})$ such that $P \subseteq P_* \cup H^+ \cup H^-$. By Lemma 9.2, this means that $P \cap \nabla P \subseteq H^+ \cup H^-$. Since $P \cap \nabla P$ is convex, this implies that $P \cap \nabla P$ is contained either in H^+ or in H^-, thus that P is contained either in $P_* \cup H^+$ or in $P_* \cup H^-$. Condition (4) follows.

For Condition (5), let $P, Q \in \operatorname{Ji}\operatorname{Op}(\mathcal{H})$ such that $P \subseteq P_* \cup H^+$ and $Q \subseteq Q_* \cup H^-$. Suppose that $P \subseteq Q$. This means that $P \cap \nabla P \subseteq H^+$, $Q \cap \nabla Q \subseteq H^-$, and $P^\dagger \subseteq Q^\dagger$, thus, by Lemma 9.2, $\operatorname{cl}(Q) \cap \nabla Q \subseteq \operatorname{cl}(P) \cap \nabla P$. Hence, $Q \cap \nabla Q \subseteq \overline{H}^+$, and hence $Q \cap \nabla Q = \varnothing$, a contradiction. If $Q \subseteq P$, we get a similar contradiction. □

10. Extending homomorphisms on open polyhedral cones

Throughout this section we shall fix a set[5] I, and consider the free real vector space $\mathbb{R}^{(I)}$ on I (see Section 3 for the notation). We endow $\mathbb{R}^{(I)}$ with its canonical inner product, defined by

$$(x|y) \underset{\text{def}}{=} \sum_{i \in I} x_i y_i\,, \quad \text{for all } x, y \in \mathbb{R}^{(I)}\,.$$

This enables us to identify every element $x \in \mathbb{R}^{(I)}$ with the linear functional $(x|\_)$. We endow $\mathbb{R}^{(I)}$ with its *weak topology*, making all those linear functionals continuous. A hyperplane H of $\mathbb{R}^{(I)}$ is closed iff it is the the kernel of some element of $\mathbb{R}^{(I)} \setminus \{0\}$. Since x is determined up to a scalar multiple, the support of x depends on H only, so we will denote it by $\operatorname{supp}(H)$. An element $x \in \mathbb{R}^{(I)}$ is *integral* if all its entries are integers, that is, $x \in \mathbb{Z}^{(I)}$, and we say that a hyperplane H of $\mathbb{R}^{(I)}$ is *integral* if it is the kernel of some element of $\mathbb{Z}^{(I)} \setminus \{0\}$. For a set $\mathcal{H}$ of closed hyperplanes of $\mathbb{R}^{(I)}$, we shall set $\operatorname{supp}(\mathcal{H}) \underset{\text{def}}{=} \bigcup(\operatorname{supp}(H) \mid H \in \mathcal{H})$. For any $i \in I$, we shall denote by δ_i the vector whose ith coordinate is 1 and all other coordinates are 0, and we shall denote by Δ_i the kernel of δ_i, that is, $\Delta_i \underset{\text{def}}{=} \bigl\{x \in \mathbb{R}^{(I)} \mid x_i = 0\bigr\}$. We shall also set

$$\Delta_i^+ \underset{\text{def}}{=} \Bigl\{x \in \mathbb{R}^{(I)} \mid x_i > 0\Bigr\}\,, \qquad \Delta_i^- \underset{\text{def}}{=} \Bigl\{x \in \mathbb{R}^{(I)} \mid x_i < 0\Bigr\}\,.$$

For $x \in \mathbb{R}^{(I)}$ and $S \subseteq I$, we shall denote by $x{\restriction}_S$ the restriction of x to S, extended by zero on $I \setminus S$.

Lemma 10.1. *Let $\mathcal{H}$ be a set of closed hyperplanes of $\mathbb{R}^{(I)}$, with support S, and let $Z \in \operatorname{Bool}(\mathcal{H})$. Then $x \in Z$ iff $x{\restriction}_S \in Z$, for all $x \in \mathbb{R}^{(I)}$.*

Proof. For each $H \in \mathcal{H}$, pick a vector $p_H \in \mathbb{R}^{(I)}$ with kernel H, and set $H^+ \underset{\text{def}}{=} [\![p_H > 0]\!]$, $H^- \underset{\text{def}}{=} [\![p_H < 0]\!]$. Then for every $x \in \mathbb{R}^{(I)}$, $x \in H^+$ iff $(p_H|x) > 0$, iff $(p_H|\,x{\restriction}_S) > 0$, iff $x{\restriction}_S \in H^+$. The proof for H^- is similar. Since the H^+ and H^- generate $\operatorname{Bool}(\mathcal{H})$ as a Boolean algebra, the general result follows easily. □

Our next two lemmas both involve the construction $D{*}\mathsf{J}_2$ introduced in Section 7.

[5] Although the case $I = \omega$ will be sufficient in the paper, the more general case might get relevant to further work.

Lemma 10.2. *Let $\mathcal{H}$ be a set of closed hyperplanes of $\mathbb{R}^{(I)}$ and let $i \in I \setminus \operatorname{supp}(\mathcal{H})$. We denote by $\varphi\colon \mathrm{Op}(\mathcal{H}) \hookrightarrow \mathrm{Op}(\mathcal{H}) * \mathsf{J}_2$ and $\psi\colon \mathrm{Op}(\mathcal{H}) \hookrightarrow \mathrm{Op}(\mathcal{H} \cup \{\Delta_i\})$ the diagonal embedding and the inclusion map, respectively, and we set $\varepsilon(X,Y,Z) \underset{\mathrm{def}}{=} (X \cap \Delta_i^+) \cup (Y \cap \Delta_i^-) \cup Z$, for all $(X,Y,Z) \in \mathrm{Op}(\mathcal{H}) * \mathsf{J}_2$. Then ε is an isomorphism, and $\psi = \varepsilon \circ \varphi$.*

We illustrate Lemma 10.2 on Figure 10.1.

$$\begin{tikzcd}
 & \mathrm{Op}(\mathcal{H}) \arrow[dl, hook', "\varphi"'] \arrow[dr, hook, "\psi"] & \\
\mathrm{Op}(\mathcal{H}) * \mathsf{J}_2 \arrow[rr, "\varepsilon"'] & & \mathrm{Op}(\mathcal{H} \cup \{\Delta_n\})
\end{tikzcd}$$

FIGURE 10.1. Illustrating Lemma 10.2

Proof. It is straightforward to verify that φ and ψ are both $0,1$-lattice homomorphisms, that ε is a $0,1$-join-homomorphism, and that $\psi = \varepsilon \circ \varphi$. Moreover, it follows from Corollary 8.6 that ε is surjective.

Set $S \underset{\mathrm{def}}{=} \operatorname{supp}(\mathcal{H})$. In order to prove that ε is one-to-one, it is sufficient to prove that every triple $(X,Y,Z) \in \mathrm{Op}(\mathcal{H}) * \mathsf{J}_2$ is determined by the set $T \underset{\mathrm{def}}{=} \varepsilon(X,Y,Z)$. Let $t \in \mathbb{R}^{(I)}$. Then $t{\restriction}_S \in \Delta_i$, thus $t{\restriction}_S \in T$ iff $t{\restriction}_S \in Z$, iff $t \in Z$ (cf. Lemma 10.1); hence T determines Z. Likewise, $t{\restriction}_S + \delta_i \in \Delta_i^+$, thus $t{\restriction}_S + \delta_i \in T$ iff $t{\restriction}_S + \delta_i$ belongs to $X \cup Z = X$, iff (using again Lemma 10.1) $t{\restriction}_S \in X$, iff $t \in X$. Symmetrically, $t{\restriction}_S - \delta_i \in T$ iff $t \in Y$. Therefore, T determines both X and Y. $\square$

Lemma 10.3. *Let $\mathcal{H}$ be a set of closed hyperplanes and let $i \in I \setminus \operatorname{supp}(\mathcal{H})$. Let L be a bounded distributive lattice, and let $a, b \in L$ such that $a \wedge b = 0$. Then every $0,1$-lattice homomorphism $f\colon \mathrm{Op}(\mathcal{H}) \to L$ extends to a unique $0,1$-lattice homomorphism $g\colon \mathrm{Op}(\mathcal{H} \cup \{\Delta_i\}) \to L$ such that $a = g(\Delta_i^+)$ and $b = g(\Delta_i^-)$.*

Proof. By Lemma 10.2, Lemma 10.3 can be reformulated, by replacing $\mathrm{Op}(\mathcal{H} \cup \{\Delta_i\})$ by $\mathrm{Op}(\mathcal{H}) * \mathsf{J}_2$, the inclusion map $\psi\colon \mathrm{Op}(\mathcal{H}) \hookrightarrow \mathrm{Op}(\mathcal{H} \cup \{\Delta_i\})$ by the canonical embedding $\varphi\colon \mathrm{Op}(\mathcal{H}) \hookrightarrow \mathrm{Op}(\mathcal{H}) * \mathsf{J}_2$, Δ_i^+ by $\varepsilon^{-1}(\Delta_i^+) = (\mathbb{E}, \varnothing, \varnothing)$, and Δ_i^- by $\varepsilon^{-1}(\Delta_i^-) = (\varnothing, \mathbb{E}, \varnothing)$.

By Lemma 7.2, the map $g\colon \mathrm{Op}(\mathcal{H}) * \mathsf{J}_2 \to L$ defined by

$$g(X,Y,Z) \underset{\mathrm{def}}{=} (f(X) \wedge a) \vee (f(Y) \wedge b) \vee f(Z)\,, \quad \text{for all } (X,Y,Z) \in \mathrm{Op}(\mathcal{H}) * \mathsf{J}_2\,,$$

is a lattice homomorphism, and $g(X,X,X) = f(X)$ for every $X \in \mathrm{Op}(\mathcal{H})$. The latter condition means that $f = g \circ \varphi$. Now $a = g(\mathbb{E}, \varnothing, \varnothing)$ and $b = g(\varnothing, \mathbb{E}, \varnothing)$. $\square$

Lemma 10.4. *Let $\mathcal{H}$ be a finite set of closed hyperplanes of $\mathbb{R}^{(I)}$, let $a, b \in \mathbb{R}^{(I)}$ with respective kernels A and B, both belonging to $\mathcal{H}$. We set*

$$A^+ \underset{\text{def}}{=} [\![a > 0]\!]\,, \quad A^- \underset{\text{def}}{=} [\![a < 0]\!]\,,$$
$$B^+ \underset{\text{def}}{=} [\![b > 0]\!]\,, \quad B^- \underset{\text{def}}{=} [\![b < 0]\!]\,,$$
$$C_m \underset{\text{def}}{=} \ker(a - mb)\,, \quad \mathcal{H}_m \underset{\text{def}}{=} \mathcal{H} \cup \{C_m\}\,,$$
$$C_m^+ \underset{\text{def}}{=} [\![a > mb]\!]\,, \quad C_m^- \underset{\text{def}}{=} [\![a < mb]\!]\,,$$

for any positive integer m. Then for all large enough m, the following statement holds: For every generalized dual Heyting algebra L, every consonant 0-lattice homomorphism $f\colon \mathrm{Op}(\mathcal{H}) \to L$ extends to a lattice homomorphism $g\colon \mathrm{Op}(\mathcal{H}_m) \to L$ such that $g(A^+ \smallsetminus_{\mathrm{Op}^-(\mathcal{H}_m)} B^+) = f(A^+) \smallsetminus_L f(B^+)$.

Note. The notation $A^+ \smallsetminus_{\mathrm{Op}^-(\mathcal{H}_m)} B^+$ might look a bit crowded, in particular due to the use of $\mathrm{Op}^-(\mathcal{H}_m)$ instead of $\mathrm{Op}(\mathcal{H}_m)$. In reality, that distinction is immaterial here, because $\mathrm{Op}^-(\mathcal{H}_m)$ is an ideal of $\mathrm{Op}(\mathcal{H}_m)$, thus $U \smallsetminus_{\mathrm{Op}^-(\mathcal{H}_m)} V = U \smallsetminus_{\mathrm{Op}(\mathcal{H}_m)} V$ for all $U, V \in \mathrm{Op}^-(\mathcal{H}_m)$.

Proof. We begin by stating exactly how large m should be.

Claim 1. *There exists a positive integer m_0 such that for all $m \geq m_0$ and all $X \in \mathrm{Op}(\mathcal{H})$, $C_m^- \subseteq X$ implies that $B^+ \subseteq X$.*

Proof of Claim. Every $P \in \mathrm{Ji}\,\mathrm{Op}(\mathcal{H})$ is basic open, thus both $\mathrm{cl}(P)$ and ∇P are intersections of closed half-spaces with boundaries in $\mathcal{H}$. Hence, there is a finite subset Φ_P of $\mathbb{R}^{(I)} \setminus \{0\}$ such that $\mathrm{cl}(P) \cap \nabla P = \bigcap_{x \in \Phi_P} [\![x \geq 0]\!]$ and $\ker(x) \in \mathcal{H}$ for every $x \in \Phi_P$. Denote by K_P the convex cone of $\mathbb{R}^{(I)}$ generated by Φ_P. We set

$$\mathcal{P} \underset{\text{def}}{=} \{P \in \mathrm{Ji}\,\mathrm{Op}(\mathcal{H}) \mid -b \notin K_P\}\,.$$

Since K_P is a finitely generated convex cone of $\mathbb{R}^{(I)}$, it has the form

$$\bigcap_{l=1}^{k} [\![p_l \geq 0]\!] \cap \bigcap_{i \in I \setminus J} \Delta_i\,,$$

for nonzero vectors $p_l \in \mathbb{R}^{(I)}$ and some finite $J \subseteq I$ (cf. De Loera, Hemmecke, and Köppe [16, Section 1.2]; the $\bigcap_{i \in I \setminus J} \Delta_i$ is put there in order to reduce the problem to the finite-dimensional case). Hence K_P is topologically closed, and hence there exists a positive integer m_0 such that

$$-b + (1/m)a \notin K_P\,, \text{ for all } P \in \mathcal{P} \text{ and all } m \geq m_0\,. \tag{10.1}$$

Now observe that for every $y \in \mathbb{R}^{(I)}$ and every $P \in \mathrm{Ji}\,\mathrm{Op}(\mathcal{H})$, $-y \in K_P$ iff $\bigcap_{x \in \Phi_P} [\![x \geq 0]\!] \subseteq [\![y \leq 0]\!]$ (use Lemma 2.2), iff $\mathrm{cl}(P) \cap \nabla P \subseteq [\![y \leq 0]\!]$ (by the definition of Φ_P), iff $[\![y > 0]\!] \subseteq P^\dagger$ (cf. Lemma 9.2). In particular, $-b \in K_P$ iff $B^+ \subseteq P^\dagger$, iff $P \nsubseteq B^+$ (because $B^+ \in \mathrm{Op}(\mathcal{H})$). Similarly, $-b + (1/m)a \in K_P$ iff $C_m^- \subseteq P^\dagger$ (we cannot infer that $P \nsubseteq C_m^-$, because usually $C_m^- \notin \mathrm{Op}(\mathcal{H})$). Hence, (10.1) means that $C_m^- \subseteq P^\dagger$ implies that $B^+ \subseteq P^\dagger$, whenever $m \geq m_0$ and $P \in \mathrm{Ji}\,\mathrm{Op}(\mathcal{H})$. Since every meet-irreducible element of $\mathrm{Op}(\mathcal{H})$ has the form $P^\dagger$ (cf. Lemma 2.1), and since every element of $\mathrm{Op}(\mathcal{H})$ is an intersection of meet-irreducible elements of $\mathrm{Op}(\mathcal{H})$, it follows that m_0 is as required. □ Claim 1.

We shall prove that every integer $m \geq m_0$ has the property stated in Lemma 10.4. Let L be a generalized dual Heyting algebra and let $f\colon \operatorname{Op}(\mathcal{H}) \to L$ be a consonant 0-lattice homomorphism. We consider the extension g of f, to a homomorphism from $\operatorname{Op}(\mathcal{H}_m)$ to L, given by Lemma 9.4, with $H := C_m$. In particular,

$$g(C_m^+) = \bigvee \left(f(P) \smallsetminus_L f(P_*) \mid P \in \operatorname{Ji}\operatorname{Op}(\mathcal{H})\,,\ P \subseteq P_* \cup C_m^+ \right)\,. \tag{10.2}$$

We claim that the following inequality holds:

$$f(A^+) \wedge g(C_m^+) \leq f(A^+) \smallsetminus_L f(B^+)\,. \tag{10.3}$$

Since L is distributive, this amounts to proving the following statement:

$$f(A^+) \wedge \big(f(P) \smallsetminus_L f(P_*)\big) \leq f(A^+) \smallsetminus_L f(B^+)\,,\quad \text{for every } P \in \operatorname{Ji}\operatorname{Op}(\mathcal{H}) \text{ such that } P \subseteq P_* \cup C_m^+\,. \tag{10.4}$$

Let $P \in \operatorname{Ji}\operatorname{Op}(\mathcal{H})$ such that $P \subseteq P_* \cup C_m^+$; that is, $P \cap \nabla P \subseteq C_m^+$. It follows that $\operatorname{cl}(P) \cap \nabla P = \operatorname{cl}(P \cap \nabla P) \subseteq \overline{C}_m^+$, that is, $C_m^- \subseteq P^\dagger$. It thus follows from the definition of m_0 (cf. Claim 1) that $B^+ \subseteq P^\dagger$, that is, $P \nsubseteq B^+$. Since $B^+ \in \operatorname{Op}(\mathcal{H})$, it follows that $P \cap B^+ \subseteq P_*$.

Now suppose that $P \subseteq A^+$. Since $P \cap B^+ \subseteq P_*$, the inequalities $P \subseteq P_* \cup A^+$ and $P \cap B^+ \subseteq P_*$ both hold, thus also $f(P) \leq f(P_*) \vee f(A^+)$ and $f(P) \wedge f(B^+) \leq f(P_*)$. Since $\smallsetminus_L$ is an L-valued difference operation on the range of f (cf. Lemma 6.7), it follows from Lemma 6.6 that $f(P) \smallsetminus_L f(P_*) \leq f(A^+) \smallsetminus_L f(B^+)$, which implies (10.4) right away.

It remains to handle the case where $P \nsubseteq A^+$. Due to the obvious containment $C_m^+ \subseteq A^+ \cup B^-$, we get $P \subseteq P_* \cup A^+ \cup B^-$, thus, since P is join-prime in $\operatorname{Op}(\mathcal{H})$, we get $P \subseteq B^-$, thus $f(P) \smallsetminus_L f(P_*) \leq f(B^-)$, and thus, by using the equation $f(B^+) \wedge f(B^-) = 0$ and the inequality $f(A^+) \leq f(B^+) \vee \big(f(A^+) \smallsetminus_L f(B^+)\big)$,

$$\begin{aligned}
f(A^+) \wedge \big(f(P) \smallsetminus_L f(P_*)\big) &\leq f(A^+) \wedge f(B^-)\\
&\leq \big(f(B^+) \wedge f(B^-)\big) \vee \Big(\big(f(A^+) \smallsetminus_L f(B^+)\big) \wedge f(B^-)\Big)\\
&= \big(f(A^+) \smallsetminus_L f(B^+)\big) \wedge f(B^-)\\
&\leq f(A^+) \smallsetminus_L f(B^+)\,,
\end{aligned}$$

thus completing the proof of (10.4) in the general case, and therefore of (10.3).

Now obviously, $A^+ \subseteq B^+ \cup (A^+ \cap C_m^+)$, thus $A^+ \smallsetminus_{\operatorname{Op}^-(\mathcal{H}_m)} B^+ \subseteq A^+ \cap C_m^+$, and thus

$$g(A^+ \smallsetminus_{\operatorname{Op}^-(\mathcal{H}_m)} B^+) \leq g(A^+ \cap C_m^+) = f(A^+) \wedge g(C_m^+) \leq f(A^+) \smallsetminus_L f(B^+)\,.$$

Since $f(A^+) \leq f(B^+) \vee g(A^+ \smallsetminus_{\operatorname{Op}^-(\mathcal{H}_m)} B^+)$, the converse inequality

$$f(A^+) \smallsetminus_L f(B^+) \leq g(A^+ \smallsetminus_{\operatorname{Op}^-(\mathcal{H}_m)} B^+)$$

holds, and therefore $f(A^+) \smallsetminus_L f(B^+) = g(A^+ \smallsetminus_{\operatorname{Op}^-(\mathcal{H}_m)} B^+)$. $\square$

Notation 10.5. We set

$$\mathcal{H}_\Phi \underset{\mathrm{def}}{=} \{\ker(x) \mid x \in \Phi\}\,,\quad \text{for every } \Phi \subseteq \mathbb{R}^{(I)} \setminus \{0\}\,.$$

Say that a subset Φ of $\mathbb{R}^{(I)}$ is *symmetric* if $-\Phi = \Phi$.

Lemma 10.6. *Let Φ be a symmetric subset of $\mathbb{R}^{(I)} \setminus \{0\}$ and set*

$$\Phi^+ \underset{\text{def}}{=} \Phi \cup \{x - y \mid x, y \in \Phi \text{ and } x \neq y\} .$$

Then Φ^+ is symmetric and $\mathrm{Op}^-(\mathcal{H}_\Phi)$ is consonant in $\mathrm{Op}^-(\mathcal{H}_{\Phi^+})$.

Proof. It is trivial that Φ^+ is symmetric. Since $\mathrm{Op}^-(\mathcal{H}_\Phi)$ is generated, as a lattice, by $\Sigma_{\mathcal{H}_\Phi}$ (cf. Lemma 8.7), it suffices, by Lemma 6.2, to prove that for any distinct $A, B \in \mathcal{H}_\Phi$, any two open half-spaces A^+ and B^+, with respective boundaries A and B, are consonant in $\mathrm{Op}^-(\mathcal{H}_{\Phi^+})$. Since Φ is symmetric, there are $a, b \in \Phi$, with respective kernels A and B, such that $A^+ = [\![a > 0]\!]$ and $B^+ = [\![b > 0]\!]$. The vectors $a - b$ and $b - a$ both belong to Φ^+, with $A^+ \subseteq B^+ \cup [\![a - b > 0]\!]$, $B^+ \subseteq A^+ \cup [\![b - a > 0]\!]$, and $[\![a - b > 0]\!] \cap [\![b - a > 0]\!] = \varnothing$. Hence, A^+ and B^+ are consonant in $\mathrm{Op}^-(\mathcal{H}_{\Phi^+})$. $\square$

Lemma 10.7. *Let $\mathcal{H}$ be a finite set of integral hyperplanes, let L be a completely normal distributive lattice with zero, let $f\colon \mathrm{Op}(\mathcal{H}) \to L$ be a 0-lattice homomorphism, let $U, V \in \mathrm{Op}^-(\mathcal{H})$, and let $\gamma \in L$ such that $f(U) \leq f(V) \vee \gamma$. Then there are a finite set $\widetilde{\mathcal{H}}$ of integral hyperplanes, containing $\mathcal{H}$, and a lattice homomorphism $g\colon \mathrm{Op}(\widetilde{\mathcal{H}}) \to L$ extending f, such that the following statements hold:*

(1) $\mathrm{Op}^-(\mathcal{H})$ *is consonant in* $\mathrm{Op}^-(\widetilde{\mathcal{H}})$.

(2) *There exists $W \in \mathrm{Op}^-(\widetilde{\mathcal{H}})$ such that $U \subseteq V \cup W$ and $g(W) \leq \gamma$.*

Proof. We may assume that $\mathcal{H}$ is nonempty. Fix an enumeration (A_0, B_0), $\dots$, (A_{n-1}, B_{n-1}) of all pairs of open half-spaces with boundary in $\mathcal{H}$. Moreover, fix a finite chain $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n$ of finite sublattices of L, such that S_0 contains $f[\mathrm{Op}(\mathcal{H})] \cup \{\gamma\}$ and S_i is consonant in S_{i+1} whenever $0 \leq i < n$. We construct inductively an ascending chain $\mathcal{H} = \mathcal{H}_0 \subseteq \mathcal{H}_1 \subseteq \cdots \subseteq \mathcal{H}_n$ of finite sets of integral hyperplanes, together with an ascending chain of lattice homomorphisms $f_l\colon \mathrm{Op}(\mathcal{H}_l) \to S_l$, for $0 \leq l \leq n$, such that $f_0 = f$ and

$$f_k\bigl(A_l \smallsetminus_{\mathrm{Op}^-(\mathcal{H}_k)} B_l\bigr) \leq f(A_l) \smallsetminus_{S_1} f(B_l)\,, \quad \text{whenever } 0 \leq l < k \leq n\,. \tag{10.5}$$

For $k = 0$ there is nothing to verify. Suppose having performed the construction up to level k, with $0 \leq k < n$. By applying Lemma 10.4, with $\mathcal{H}_k$ in place of $\mathcal{H}$, f_k in place of f, S_{k+1} (which is a finite distributive lattice, thus, *a fortiori*, a dual Heyting algebra) in place of L, and (A_k, B_k) in place of (A^+, B^+), we get a finite set $\mathcal{H}_{k+1}$ of integral hyperplanes, containing $\mathcal{H}_k$, together with a lattice homomorphism $f_{k+1}\colon \mathrm{Op}(\mathcal{H}_{k+1}) \to S_{k+1}$, extending f_k, such that

$$f_{k+1}\bigl(A_k \smallsetminus_{\mathrm{Op}^-(\mathcal{H}_{k+1})} B_k\bigr) = f(A_k) \smallsetminus_{S_{k+1}} f(B_k)\,.$$

Since S_{k+1} contains S_1, it follows that

$$f_{k+1}\bigl(A_k \smallsetminus_{\mathrm{Op}^-(\mathcal{H}_{k+1})} B_k\bigr) \leq f(A_k) \smallsetminus_{S_1} f(B_k)\,. \tag{10.6}$$

Since $\mathrm{Op}^-(\mathcal{H}_k)$ is a sublattice of $\mathrm{Op}^-(\mathcal{H}_{k+1})$ and since f_{k+1} extends f_k, it follows from the induction hypothesis (10.5) (with fixed k) that

$$f_{k+1}\bigl(A_l \smallsetminus_{\mathrm{Op}^-(\mathcal{H}_{k+1})} B_l\bigr) \leq f(A_l) \smallsetminus_{S_1} f(B_l)\,, \quad \text{whenever } 0 \leq l < k\,,$$

and hence, by (10.6),

$$f_{k+1}\bigl(A_l \smallsetminus_{\mathrm{Op}^-(\mathcal{H}_{k+1})} B_l\bigr) \leq f(A_l) \smallsetminus_{S_1} f(B_l)\,, \quad \text{whenever } 0 \leq l < k+1\,,$$

therefore completing the verification of the induction step.

At stage n, we obtain a finite set $\mathcal{H}_n$ of integral hyperplanes, containing $\mathcal{H}$, together with a homomorphism $f_n\colon \operatorname{Op}(\mathcal{H}_n) \to S_n$, extending f, such that

$$f_n(A_k \smallsetminus_{\operatorname{Op}^-(\mathcal{H}_n)} B_k) \leq f(A_k) \smallsetminus_{S_1} f(B_k)\,, \quad \text{whenever } 0 \leq k < n\,. \tag{10.7}$$

By Lemma 10.6, there is a finite set $\widetilde{\mathcal{H}}$ of integral hyperplanes, containing $\mathcal{H}_n$, such that $\operatorname{Op}(\mathcal{H}_n)$ (thus, *a fortiori*, $\operatorname{Op}(\mathcal{H})$) is consonant in $\operatorname{Op}(\widetilde{\mathcal{H}})$. Moreover, by Lemma 9.4, the homomorphism $f_n\colon \operatorname{Op}(\mathcal{H}_n) \to S_n$ extends to a homomorphism $g\colon \operatorname{Op}(\widetilde{\mathcal{H}}) \to \widetilde{S}$ for some finite sublattice $\widetilde{S}$ of L containing S_n. Hence, from (10.7) it follows that

$$g(A_k \smallsetminus_{\operatorname{Op}^-(\widetilde{\mathcal{H}})} B_k) \leq f(A_k) \smallsetminus_{S_1} f(B_k)\,, \quad \text{whenever } 0 \leq k < n\,. \tag{10.8}$$

Since the open half-spaces, with boundary in $\mathcal{H}$, generate $\operatorname{Op}^-(\mathcal{H})$ as a lattice

$$\begin{tikzcd}
\operatorname{Op}^-(\mathcal{H}) \arrow[r, hook, "\text{consonant}"] \arrow[d, "f\restriction_{\operatorname{Op}^-(\mathcal{H})} = g\restriction_{\operatorname{Op}^-(\mathcal{H})}"', "\text{consonant}"] & \operatorname{Op}^-(\widetilde{\mathcal{H}}) \arrow[d, "g\restriction_{\operatorname{Op}^-(\widetilde{\mathcal{H}})}"] \\
S_1 \arrow[r, hook] & \widetilde{S}
\end{tikzcd}$$

FIGURE 10.2. Illustrating the proof of Lemma 10.7

(cf. Lemma 8.7) and since every pair of such half-spaces has the form (A_k, B_k), it follows from Lemma 6.10, applied to (10.8) and the commutative square represented in Figure 10.2, that

$$g(X \smallsetminus_{\operatorname{Op}^-(\widetilde{\mathcal{H}})} Y) \leq f(X) \smallsetminus_{S_1} f(Y)\,, \quad \text{for all } X, Y \in \operatorname{Op}^-(\mathcal{H})\,.$$

In particular, $g(U \smallsetminus_{\operatorname{Op}^-(\widetilde{\mathcal{H}})} V) \leq f(U) \smallsetminus_{S_1} f(V) \leq \gamma$. Let $W \underset{\text{def}}{=} U \smallsetminus_{\operatorname{Op}^-(\widetilde{\mathcal{H}})} V$. □

Lemma 10.8. *Let $\mathcal{H}$ be a finite set of integral hyperplanes, let L be a completely normal distributive lattice with zero, let $f\colon \operatorname{Op}(\mathcal{H}) \to L$ be a 0-lattice homomorphism, let n be a positive integer, and let $\{(U_i, V_i, \gamma_i) \mid i \in [n]\}$ be a set of triples of elements of $\operatorname{Op}^-(\mathcal{H})^2 \times L$ such that $f(U_i) \leq f(V_i) \vee \gamma_i$ for each $i \in [n]$. Then there are a finite set $\widetilde{\mathcal{H}}$ of integral hyperplanes, containing $\mathcal{H}$, and a lattice homomorphism $g\colon \operatorname{Op}(\widetilde{\mathcal{H}}) \to L$ extending f, such that the following statements hold:*

(1) $\operatorname{Op}^-(\mathcal{H})$ *is consonant in* $\operatorname{Op}^-(\widetilde{\mathcal{H}})$.
(2) *For each $i \in [n]$, there exists $W_i \in \operatorname{Op}^-(\widetilde{\mathcal{H}})$ such that $U_i \subseteq V_i \cup W_i$ and $g(W_i) \leq \gamma_i$.*

Proof. Apply successively Lemma 10.7 to (U_1, V_1, γ_1) up to (U_n, V_n, γ_n). □

11. Representing countable completely normal lattices

This section is devoted to a proof of our main theorem (Theorem 11.1), together with a short discussion of some of its corollaries.

Theorem 11.1. *Every countable completely normal distributive lattice with zero is isomorphic to $\operatorname{Id_c} G$, for some Abelian ℓ-group G.*

Proof. Our goal is to represent a countable completely normal distributive lattice L with zero. The lattice $\overline{L}$, obtained from L by adding a new top element, is also completely normal, and L is an ideal of $\overline{L}$, so, by Lemma 3.1, any representation of $\overline{L}$ as $\mathrm{Id_c}\,\overline{G}$, for an Abelian ℓ-group $\overline{G}$, yields $L \cong \mathrm{Id_c}\,G$ for an ℓ-ideal G of $\overline{G}$. Hence, it suffices to consider the case where L is bounded.

We fix a generating subset $\{a_n \mid n \in \omega\}$ of L.

Denote by $\mathcal{H}_{\mathbb{Z}} = \{H_n \mid n \in \omega\}$ the set of all integral hyperplanes of $\mathbb{R}^{(\omega)}$. Moreover, let $\{(U_n, V_n, \gamma_n) \mid n \in \omega\}$ be an enumeration of all triples (U, V, γ), where $U, V \in \mathrm{Op}^-(\mathcal{H}_{\mathbb{Z}})$ and $\gamma \in L$.

We construct an ascending chain $(\mathcal{H}_n \mid n \in \omega)$ of nonempty finite subsets of $\mathcal{H}_{\mathbb{Z}}$, with union $\mathcal{H}_{\mathbb{Z}}$, together with an ascending sequence $(f_n \mid n \in \omega)$ of $0,1$-lattice homomorphisms $f_n \colon \mathrm{Op}(\mathcal{H}_n) \to L$, as follows.

Take $\mathcal{H}_0 \underset{\mathrm{def}}{=} \{\Delta_0\}$ (cf. Section 10); so $\mathrm{Op}(\mathcal{H}_0) = \{\varnothing, \Delta_0^+, \Delta_0^-, \Delta_0^+ \cup \Delta_0^-, \mathbb{R}^{(\omega)}\}$ is isomorphic to J_2 (cf. Section 7). Let $f_0 \colon \mathrm{Op}(\mathcal{H}_0) \to \{0, a_0, 1\}$ be the unique homomorphism such that $f_0(\Delta_0^+) = a_0$, $f_0(\Delta_0^-) = 0$, and $f_0\big(\mathbb{R}^{(\omega)}\big) = 1$.

Suppose $f_n \colon \mathrm{Op}(\mathcal{H}_n) \to L$ already constructed.

Let $n = 3m$ for some integer m, denote by k the first nonnegative integer outside $\mathrm{supp}(\mathcal{H}_n)$, and set $\mathcal{H}_{n+1} \underset{\mathrm{def}}{=} \mathcal{H}_n \cup \{\Delta_k\}$. By Lemma 10.3, there is a unique lattice homomorphism $f_{n+1} \colon \mathrm{Op}(\mathcal{H}_{n+1}) \to L$, extending f, such that $f_{n+1}(\Delta_k^+) = a_m$ and $f_{n+1}(\Delta_k^-) = 0$. This will take care of the surjectivity of the restriction, to $\mathrm{Op}^-(\mathcal{H}_{\mathbb{Z}})$, of the union of the f_n.

Let $n = 3m+1$ for some integer m, and set $\mathcal{H}_{n+1} \underset{\mathrm{def}}{=} \mathcal{H}_n \cup \{H_m\}$. Since L is completely normal and the range of f_n is finite, there is a finite sublattice S of L such that the range of f_n is consonant in S. By Lemma 9.4, f_n extends to a lattice homomorphism f_{n+1} from $\mathrm{Op}(\mathcal{H}_{n+1})$ to S, thus to L. This will take care of the union of all f_n be defined on $\mathrm{Op}(\mathcal{H}_{\mathbb{Z}})$.

Let, finally, $n = 3m+2$ for some integer m. By Lemma 10.8, there is a finite subset $\mathcal{H}_{n+1}$ of $\mathcal{H}_{\mathbb{Z}}$, containing $\mathcal{H}_n$, such that $\mathrm{Op}(\mathcal{H}_n)$ is consonant in $\mathrm{Op}(\mathcal{H}_{n+1})$, together with an extension $f_{n+1} \colon \mathrm{Op}(\mathcal{H}_{n+1}) \to L$, such that for every $k \leq n$, if $\{U_k, V_k\} \subseteq \mathrm{Op}^-(\mathcal{H}_n)$ and $f_n(U_k) \leq f_n(V_k) \vee \gamma_k$, then $f_{n+1}(U_k \smallsetminus_{\mathrm{Op}^-(\mathcal{H}_{n+1})} V_k) \leq \gamma_k$. This will take care of the union of the f_n be *closed* (as defined in the statement of Lemma 3.2) on $\mathrm{Op}^-(\mathcal{H}_{\mathbb{Z}})$.

The union f of all the f_n is a surjective lattice homomorphism from $\mathrm{Op}(\mathcal{H}_{\mathbb{Z}})$ onto L. Furthermore, the restriction f^- of f to $\mathrm{Op}^-(\mathcal{H}_{\mathbb{Z}})$ is a closed, surjective lattice homomorphism from $\mathrm{Op}^-(\mathcal{H}_{\mathbb{Z}})$ onto L. Now by the Baker-Beynon duality (cf. Proposition 3.7), $\mathrm{Id_c}\,\mathrm{F}_\ell(\omega)$ is isomorphic to $\mathrm{Lat}(\omega)$, which is identical to $\mathrm{Op}^-(\mathcal{H}_{\mathbb{Z}})$ (cf. Lemma 8.7). Hence, the map f^- induces a closed, surjective lattice homomorphism $g \colon \mathrm{Id_c}\,\mathrm{F}_\ell(\omega) \twoheadrightarrow L$. By Lemma 3.5, this map factors through an isomorphism from $\mathrm{Id_c}(\mathrm{F}_\ell(\omega)/I)$ onto L, for a suitable ℓ-ideal I of $\mathrm{F}_\ell(\omega)$. □

Recall that Delzell and Madden's results in [17] imply that Theorem 11.1 does not extend to the uncountable case.

Corollary 11.2. *A second countable generalized spectral space X is homeomorphic to the ℓ-spectrum of an Abelian ℓ-group iff it is completely normal.*

Proof. By Theorem 11.1 and Lemma 4.7, it remains to prove that if the topology of X has a countable basis, say $\mathcal{B}$, then $\mathcal{K}(X)$ is countable. Every $A \in \mathcal{K}(X)$ is the union of a subset $\mathcal{B}_A$ of $\mathcal{B}$, which, by the compactness of A, may be taken finite. Since the assignment $A \mapsto \mathcal{B}_A$ is one-to-one, it determines a one-to-one map from $\mathcal{K}(X)$ into the finite subsets of $\mathcal{B}$; whence $\mathcal{K}(X)$ is countable. $\square$

Recall that the complete normality of a spectral space X is equivalent to the specialization order on X be a root system (cf. Proposition 4.4).

It is well known that the lattice $\mathcal{C}(G)$, of all convex ℓ-subgroups of any ℓ-group (not necessarily Abelian) G, is the ideal lattice a completely normal distributive lattice with zero (see Iberkleid, Martínez, and McGovern [29, Section 1.2] for a short overview). Of course, in the Abelian case, $\mathcal{C}(G)$ is isomorphic to the ideal lattice of $\mathrm{Id_c}\, G$. A direct application of Theorem 11.1 yields the following.

Corollary 11.3. *For every countable ℓ-group G, there exists a countable Abelian ℓ-group A such that $\mathcal{C}(G) \cong \mathcal{C}(A)$.*

The results of Kenoyer [34] and McCleary [37] imply that Corollary 11.3 does not extend to the uncountable case. The question handled in both papers was credited, in the second paper, to Paul Conrad in a Workshop on ordered groups, Bowling Green, Ohio, 1985.

The real spectrum $\mathrm{Spec_r}\, R$, of any commutative unital ring R, is a completely normal spectral space (cf. Coste and Roy [15], Dickmann [19]). Moreover, it has a basis of open sets which is indexed by finite sequences of elements of R; in particular, if R is countable, then $\mathrm{Spec_r}\, R$ is second countable. A direct application of Corollary 11.2 yields the following.

Corollary 11.4. *For every countable, commutative, unital ring R, there exists a countable Abelian ℓ-group A with unit such that* $\mathrm{Spec_r}\, R$ *and* $\mathrm{Spec}_\ell\, A$ *are homeomorphic.*

12. Discussion

12.1. **Ideal lattices of dimension groups.** A partially ordered Abelian group G is a *dimension group* if G is directed, unperforated (i.e., $mx \geq 0$ implies that $x \geq 0$, whenever $x \in G$ and m is a positive integer), and G^+ satisfies the Riesz refinement property (cf. Goodearl [22]). The construction $\mathrm{Id_c}\, G$, for an Abelian ℓ-group G, extends naturally to arbitrary dimension groups, by replacing "ℓ-ideal" by "directed convex subgroup" (in short *ideal*). However, now $\mathrm{Id_c}\, G$ is only a $(\vee, 0)$-semilattice. This semilattice is always distributive (i.e., it satisfies the Riesz refinement property), but it may not be a lattice. In fact, *Every countable distributive* $(\vee, 0)$-*semilattice is isomorphic to* $\mathrm{Id_c}\, G$ *for some countable dimension group* G (this is implicit in Bratteli and Elliott [10], Hofmann and Thayer [28], Bergman [5], and explicitly stated as Goodearl and Wehrung [23, Theorem 5.2]); *moreover, the countable size is optimal* (Wehrung [47]).

In particular, it follows from Goodearl and Wehrung [23, Theorem 4.4] that for every distributive lattice L with zero, there exists a dimension group G such that $\mathrm{Id_c}\, G \cong L$ (without any restriction on the cardinality of L). One could then hope to be able to apply Theorem 1 of Elliott and Mundici [21], in order to infer that if L is completely normal, then G is lattice-ordered. However, this would already fail for the lattice $L \underset{\mathrm{def}}{=} \boldsymbol{D}_{\omega_1}$ of Example 5.5, simply because $\boldsymbol{D}_{\omega_1}$ is not ℓ-representable. The problem lies in the fact that one cannot read, on $\mathrm{Id_c}\, G$ (equivalently, on the spectrum of G) alone, that every prime quotient of G be totally ordered. This fact is illustrated by the following much easier example, also implied on page 181 in [21]:

let G be any non totally ordered simple dimension group (e.g., $G = \mathbb{Q} \times \mathbb{Q}$ with positive cone consisting of all (x, y) with either $x = y = 0$ or $x > 0$ and $y > 0$). Then $\mathrm{Id_c}\, G \cong \mathbf{2}$, yet G is not totally ordered.

12.2. **Lattices of ℓ-ideals in non-Abelian ℓ-groups.** It is proved in Růžička, Tůma, and Wehrung [42, Theorem 6.3] that *Every countable distributive* $(\vee, 0)$-*semilattice is isomorphic to* $\mathrm{Id_c}\, G$ *for some ℓ-group G; moreover, this result does not extend to semilattices of cardinality* $\aleph_2$. The gap at size $\aleph_1$ is not filled yet.

12.3. **Open problems.** While Example 5.5 gives an example of a non-ℓ-representable distributive lattice with an ℓ-representable $\mathscr{L}_{\infty,\omega}$-elementary sublattice, we do not know whether the opposite situation may occur:

Problem 1. Let D be an $\mathscr{L}_{\infty,\omega}$-elementary sublattice of a distributive lattice E. If E is ℓ-representable, is D also ℓ-representable?

While Theorem 11.1 implies a positive answer to Problem 1 in the countable case, the uncountable case remains open. Analogues of Problem 1, for other functors than $\mathrm{Id_c}$ on Abelian ℓ-groups, may lead to different situations. Consider, for example, the functor $\boldsymbol{L}$, which to every von Neumann regular ring R associates the lattice $\boldsymbol{L}(R)$ of all principal right ideals of R. Lattices of the form $\boldsymbol{L}(R)$ are said to be *coordinatizable*. In [48], the author constructs a countable, coordinatizable lattice with a non-coordinatizable elementary sublattice.

In that paper, it is also proved that the class of all coordinatizable lattices is not the class of models of any $\mathscr{L}_{\infty,\infty}$ sentence. This suggests the following problem.

Problem 2. Is the class of all ℓ-representable lattices the class of all models of an $\mathscr{L}_{\infty,\infty}$ sentence?

Recall from Example 5.5 that the class of all ℓ-representable lattices is not the class of all models of any class of $\mathscr{L}_{\infty,\omega}$ sentences.

The analogy between ℓ-spectra and real spectra (cf. Section 1.2), together with Corollary 11.2, suggests the following problem.

Problem 3. Is every second countable completely normal spectral space homeomorphic to the real spectrum of some commutative, unital ring?

The more general question, of characterizing real spectra of commutative, unital rings, is part of Problem 12 in Keimel's survey paper [32].

Problem 4. Is every retract, of an ℓ-representable lattice, also ℓ-representable?

Recall from Example 5.6 that the class of all ℓ-representable lattices is not closed under homomorphic images.

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