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ARNAUD HEIBIG[1] *

[1] Institut Camille Jordan et Insa Lyon, Bât. Leonard de Vinci No. 401, 21 Avenue Jean Capelle, F-69621, Villeurbanne, France.

Abstract

We prove well posedness for ordinary differential equations with coefficients in Banach valued Besov spaces $B^s_{p,q}(]0,T[,E)$ with $\max\{-\frac{1}{2},\frac{1}{p}-1\} < s < 0$. In the linear case, a representation formula is given.

Keywords: Linear differential equations, differential equations, irregular coefficients, Poincaré inequality, well-posedness, Cauchy-Lipschitz theorem, Peano theorem, fractional Besov spaces, paraproduct.

1 Introduction.

It is well known that the Cauchy-Lipschitz theorem can be derived from the Picard fixed point theorem. A striking feature of the proof is the use of the time t as a contracting factor, whereas any other factor $\gamma(t)$ converging with t to zero would work as well. In fact, any problem of the type

$$M(t) = M_0 + \int_0^t \Big[\mathcal{H}\big(M\big)\Big](s)ds \tag{1.1}$$

can equally be solved under suitable assumptions on operator $\mathcal{H}$. Essentially, $\mathcal{H}$ must not derive more than once, and one may think at a $\delta < 1$ fractional derivative operator. For such operators, acting on wide scales of functional spaces and with always the same moderate loss of smoothness, solutions turns out to be C^∞ by a standard bootstrap argument. Consequently, the choices of a solving functional frame are overabundant, and the problem 1.1 is well posed in all of them.

This is not the case for an operator with irregular coefficients i.e wasting all the smoothness that is not required for its definition. The goal of this paper is to handle the case of such linear and nonlinear operators, and mainly to prove well-posedness. As a preliminary example (see section 6), we consider the model problem on $]0,T[$

$$\begin{cases} M' = \kappa M + \phi \\ M(0) = M_0 \end{cases} \tag{1.2}$$

and later on, some of its nonlinear extensions. In 1.2, κ, ϕ are assumed to belong to $B^s_{p,q}(]0,T[,E)$ with $s < 0$ and $M_0 \in E$ given. For the sake of simplicity, we restrict in this introductory part to $\phi = 0$ and $E = \mathbb{R}$.

*Corresponding author. E-mail: arnaud.heibig@insa-lyon.fr; Fax: +33 472438529

Recall that the elementary case $\kappa \in \mathrm{C}^0([0,\mathrm{T}])$ in 1.1 is well known (see [3], [17]); but, to our knowledge, the case of negative indexes s has not been discussed yet. Note also that the classical theory of evolutionary integral equations, essentially the Da Prato-Ianelli generation theorem (see [14]) such as described in [24], deals with operators of the type

$$\mathrm{u}(\mathrm{t}) = \mathrm{f}(\mathrm{t}) + \int_0^{\mathrm{t}} \mathrm{A}(\mathrm{t}-\tau)\mathrm{u}(\tau)d\tau$$

($\mathrm{A} \in \mathrm{L}^1_{\mathrm{loc}}([0,\mathrm{T}], \mathscr{B}(\mathrm{F},\mathrm{E}))$), which do not match equation 1.2. The nonlinear case, which can be found in classical textbooks such as [3], [10], [17], is even less favorable than the linear one. Unless working on suitable algebras, both cases are mostly restricted to smooth coefficients/lipschitz constitutive functions and turn out to be of some importance in many areas of mathematics, for instance when solving transport equations and related PDE's. See for example [16], [21], [6] and [5]. To be complete, let's mention that our primary motivation was the study of the Doi-Edwards configurational equation endowed with a transport operator with discontinuous coefficients, say:

$$\frac{\partial}{\partial t} + \Big(\kappa . u - \big(\kappa : u \otimes u\big)u\Big).\frac{\partial}{\partial u}$$

We shall deal with that problem in a forthcoming paper. See also [11] or [12].

In this introduction, we mainly focus on the functional frame and the organization of the article. We work within the frame of Besov spaces, but this a matter of convenience. We could probably consider Triebel-Lizorkin spaces, or restrict to the Sobolev-Slobodeckii scale, though this would not simplify the expository. However, notice that the scale H^{s} would not be large enough for our purposes, and that the same is true for the scale $\mathrm{B}^{\mathrm{s}}_{\mathrm{p,q}}$ when dealing with critical spaces.

Among others, one point that we have to clarify is the definition of the product $\kappa\mathrm{M}$. This can be done by means of elementary rules of paradifferential calculus. Assuming that $\kappa\mathrm{M}$ has exactly the same smoothness as $\kappa \in \mathrm{B}^{s}_{\mathrm{p,q}}(]0,\mathrm{T}[,\mathrm{E})$, equation $\mathrm{M}' = \kappa\mathrm{M}$ provides $\mathrm{M} \in \mathrm{B}^{s+1}_{\mathrm{p,q}}(]0,\mathrm{T}[,\mathrm{E})$. Looking at the remainder term, we obtain that the sum $\mathrm{s} + (\mathrm{s}+1) = 2\mathrm{s}+1$ of the regularity indexes must be positive. Hence, $\mathrm{s} \geq -1/2$. Setting $\mathrm{s} = -1/2 + \eta$, with $0 \leq \eta \leq 1/2$, we obtain the following conditions:

$$\kappa \in \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}} \text{ and } \mathrm{M} \in \mathrm{B}^{\frac{1}{2}+\eta}_{\mathrm{p,q}} \tag{1.3}$$

Finally, notice that condition $\mathrm{M}(0) = \mathrm{M}_0$ makes sense under condition:

$$\frac{1}{2} + \eta - \frac{1}{\mathrm{p}} > 0 \tag{1.4}$$

since in that case $\mathrm{B}^{\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[) \hookrightarrow \mathrm{C}^0(]0,\mathrm{T}[)$. Nevertheless, at this point, it is not clear wether very weak solutions could exist or not in a larger functional frame. In particular, one may expect that a suitable integral formulation would make useless the existence of a trace at $\mathrm{t} = 0$, i.e condition 1.4. This is not the case, and conditions 1.3, 1.4 are optimal. Consider the data $\phi = 0$ and $\kappa = \delta_{1/2} \in \mathrm{B}^0_{1,\infty}(]0,1[)$, where $\delta_{1/2}$ denotes the Dirac mass at 1/2. This corresponds to $\eta = 1/2$ and $\frac{1}{2} + \eta - \frac{1}{1} = 0$. With such data, system 1.2 admits formally the solution $\mathrm{M}(\mathrm{t}) = \mathrm{CH}(\mathrm{t}-1/2) + \mathrm{M}_0$, which does not make sense under the previous definition of the product $\kappa\mathrm{M}$. Note that the quantity $-1/2 + \eta - 1/\mathrm{p}$, which is strictly greater than -1 under condition 1.4, is the differential dimension of $\kappa \in \mathrm{B}^{-1/2+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[)$. And that for Dirac measures, this dimension is exactly equal to -1. See section 7 for a sharper analysis of some critical cases.

Since the Dirac measures are excluded from the κ-functional spaces $\mathrm{B}^{-1/2+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[,\mathrm{E})$, with $0 < \eta \leq 1/2$ and $1/2+\eta-1/\mathrm{p} > 0$, we obtain the uniform continuity - with respect to $]0,\mathrm{t}[$, $0 <$

$t < T$ - of the family of zero extension operator $E : B^{-1/2+\eta}_{p,q}(]0,t[,E) \to B^{-1/2+\eta}_{p,q}(\mathbb{R},E)$, which in turn provides uniformely bounded Poincaré's constants. Although the Poincaré's inequalities stimulated in the last decades an important amount of work, often in a more complicated context than ours, we gave up pulling this material from the literature (nevertheless, one may consult [8], [9], [18], [19], [20], [22], [23]). Essentially, we work with quotient norms, and if these norms are well suited for coretractation-retractation matters, they may not be uniformely equivalent to the usual inner norms (see [26] p. 208) when dealing with vanishing intervals $]0,t[$. As a consequence of this fact, remark that the norms of the embeddings $H^1(]0,t[) \hookrightarrow L^\infty(]0,t[)$ may or may not be uniformely bounded for $t \in]0,T[$ (see corollary 4.1 c) below). Finally, notice that t-independent estimates are so crucial for our purposes that we do not try to modify the linear framework in the nonlinear case. This stems from the fact that by taking the $\|.\|_{B^{-\frac{1}{2}+\eta}_{p,q}(]0,t[)}$ norm of both sides of equation $M' = \kappa M$, one obtains suitable inequalities on $\|M-M_0\|_{B^{\frac{1}{2}+\eta}_{p,q}(]0,t[)}$ provided that uniform Poincaré's inequalities hold true. However, it is likely that some extensions could be given in other functional settings. In the same spirit, extensions of the classical Osgood theory (see [4], p.124) could also be sought.

The nonlinear problem under consideration is the following

$$\begin{cases} M' = \mathcal{H}_{T,\alpha}(M) \\ M(0) = M_0 \end{cases} \tag{1.5}$$

with $\mathcal{H}_{T,\alpha} : D \subset B^{\frac{1}{2}+\alpha}_{p,q}(]0,T[,E) \to B^{-\frac{1}{2}+\eta}_{p,q}(]0,T[,E)$. Here, $0 < \alpha < \eta \leq 1/2$ and $1/2+\alpha-1/p > 0$. The difference $\eta - \alpha > 0$ will provide the contracting factor $t^{\eta-\alpha}$ in the Picard theorem, but a smaller factor could probably be obtain at the cost of a refined functional setting; see for instance [13] for logarithmic Besov spaces. The other requirements on the operator $\mathcal{H}_{T,\alpha}$ are the Lipschitz continuity and some localisation property. Under these assumptions, we prove well posedness for system 1.5. Relaxing the Lipschitz hypothesis, we also give a Peano's type existence theorem.

The paper is organized as follows. In the second section, we recall some notations and basic results, merely the definitions and some properties of the Besov spaces, and also the definition of the paraproduct and remainder. In a third part, we establish some preliminary "dyadic" lemmas, using the fact that the characteristic functions of an interval belongs to $B^{1/p}_{p,\infty}(\mathbb{R})$. The fourth part is devoted to the proof of uniform inequalities. The fifth part essentially deals with the definiton of the product κM via the Bony decomposition. The well posedness of equation 1.1 is established in the sixth part (theorem 6.1). It relies on a suitable L^p estimate combined with the Poincaré inequality. Section seven is devoted to the study of a few critical cases for κ, such as the derivative of a Cantor function. In a eighth part, we generalize some classical properties of the resolvent and establish the usual integral representation formula for solutions of 1.1. The last part deals with the nonlinear case. Here, the main issue is to define suitable localisation (in time) procedures. The main result of this section is theorem 9.2. As an example, we briefly discuss the case of the operator $\mathcal{H}_{T,\alpha} : D \subset B^{\frac{1}{2}+\alpha}_{p,q}(]0,T[,\mathcal{A}) \to B^{-\frac{1}{2}+\eta}_{p,q}(]0,T[,\mathcal{A})$ given by $\mathcal{H}_{T,\alpha}(M) = \sum_{j\in\mathbb{N}} \kappa_j M^j$ with $\kappa_j \in B^{-\frac{1}{2}+\eta}_{p,q}(]0,T[,\mathcal{A})$.

2 Notations and classical results.

- *Throughout this paper* E *and* F *denote two complex Banach spaces.* In the sequel, we consider Banach-valued distributions, and generalize, often without comments, scalar results to that context. The reader is refered to [1], [2], but we most often refer to the monographs of Triebel [26], [27], [28], or Bahouri-Chemin-Danchin [4], since the vector valued case follows by few additional arguments. Except for coretractation-retractation matters, which can be found in [1]. We avoid duality statements, since the use of brackets is enough for our purposes.

- For V and W two complex vector spaces, we denote by $\mathscr{L}(\mathrm{V},\mathrm{W})$ the space of linear applications from V to W. When $\mathrm{V}=\mathrm{W}$, we simply write $\mathscr{L}(\mathrm{V})$.
- For $\mathrm{x}\in\mathrm{V}$, $\mathrm{f}\in\mathscr{L}(\mathrm{V},\mathrm{W})$, we write f.x, fx or even xf instead of f(x) or $<\mathrm{f},\mathrm{x}>$.
- For $1\le \mathrm{r}\le\infty$, we denote by r' its conjugate exponent i.e $\mathrm{r}^{-1}+\mathrm{r}'^{-1}=1$. We denote by E^* the topological dual space of a Banach space E.
- For $0\le \mathrm{r}\le \mathrm{R}$, the shell $\{\xi\in\mathbb{R}^{\mathrm{n}}$ such that $\mathrm{r}\le|\xi|\le\mathrm{R}\}$ is denoted by $\mathcal{S}(\mathrm{r},\mathrm{R})$.
- Let $1\le \mathrm{p}\le\infty$ and $\alpha>0$. We will often use the notation

$$\omega(\eta,\mathrm{p})=\frac{1}{2}+\eta-\frac{1}{\mathrm{p}} \tag{2.1}$$

  We shall frequently impose condition $\omega(\eta,\mathrm{p})>0$ and $0<\eta\le 1/2$. It implies $\mathrm{p}>1$.

- The symbol $\hookrightarrow$ stands for classical continuous embeddings.
- The non-homogeneous Besov space $\mathrm{B}^{\mathrm{s}}_{\mathrm{p,q}}(\mathbb{R}^{\mathrm{n}},\mathrm{E})$ - or simply $\mathrm{B}^{\mathrm{s}}_{\mathrm{p,q}}(\mathbb{R}^{\mathrm{n}})$, or $\mathrm{B}^{\mathrm{s}}_{\mathrm{p,q}}$ - can be defined as the space of tempered distribution f such that (see[4]):

$$\|\mathrm{f}\|_{\mathrm{B}^{\mathrm{s}}_{\mathrm{p,q}}(\mathbb{R}^{\mathrm{n}},\mathrm{E})}:=\Big[\sum_{\mathrm{j}\ge-1}\big(2^{\mathrm{js}}\|\Delta_{\mathrm{j}}\mathrm{f}\|_{\mathrm{L}^{\mathrm{p}}(\mathbb{R}^{\mathrm{n}},\mathrm{E})}\big)^{\mathrm{q}}\Big]^{1/\mathrm{q}}<\infty \tag{2.2}$$

  with the usual modification in the case $\mathrm{q}=\infty$. In the above writings, the analytic functions $\Delta_{\mathrm{j}}\mathrm{f}$ are defined by the following standard dyadic procedure. Take $\mathfrak{X}\in\mathrm{C}^{\infty}(\mathbb{R}^{\mathrm{n}},[0,1])$ supported in the ball $\mathcal{S}(0,4/3)$. Set $\vartheta=\mathfrak{X}(\frac{\cdot}{2})-\mathfrak{X}$ and for $\mathrm{q}\in\mathbb{Z}$, $\xi\in\mathbb{R}^{\mathrm{n}}$, $\vartheta_{\mathrm{q}}(\xi)=\vartheta(2^{-\mathrm{q}}\xi)$. We can assume that $\mathrm{supp}\vartheta\subset\mathcal{S}(3/4,8/3)$. Finally, we write:

$$\Delta_{-1}\mathrm{f}=\big(\mathscr{F}^{-1}\mathfrak{X}\big)\star\mathrm{f}\ \text{and}\ \Delta_{\mathrm{q}}\mathrm{f}=\big(\mathscr{F}^{-1}\vartheta_{q}\big)\star\mathrm{f} \tag{2.3}$$

  ($q\in\mathbb{N}$) where $\star$ denotes the convolution with respect to t and $\mathscr{F}$ the Fourier transform. For $\mathrm{q}\le-2$ we set $\Delta_{\mathrm{q}}\mathrm{f}=0$, and for $\mathrm{q}\in\mathbb{Z}$, $\overset{o}{\Delta}_{\mathrm{q}}\mathrm{f}=\big(\mathscr{F}^{-1}\vartheta_{q}\big)\star\mathrm{f}$. Last, $\mathrm{S}_{\mathrm{p}}\mathrm{f}=\sum_{\mathrm{j}\le\mathrm{p}-1}\Delta_{\mathrm{j}}\mathrm{f}$.

- Let $\mathrm{n}\in\mathbb{N}^*$. For $\phi\in\mathscr{D}(\mathbb{R}^{\mathrm{n}},\mathbb{R})$ and $\mathrm{u}\in\mathscr{S}'(\mathbb{R}^{\mathrm{n}},\mathrm{E})$ we set $\phi(\mathrm{D})(\mathrm{u})=\mathscr{F}^{-1}(\phi\mathscr{F}\mathrm{u})$
- For $\mathrm{t}>0$, we denote by $\chi_{1/\mathrm{t}}\in\mathrm{B}^{\frac{1}{\mathrm{m}}}_{\mathrm{m},\infty}(\mathbb{R},\mathbb{R})$ $(1\le\mathrm{m}\le\infty)$ the characteristic function of $]0,\mathrm{t}[$. We set $\chi=\chi_1$. Similarly, $\chi_{\mathrm{J}}\in\mathrm{B}^{\frac{1}{\mathrm{m}}}_{\mathrm{m},\infty}(\mathbb{R},\mathbb{R})$ stands for the characteristic function of the interval J. Last, $\mathbf{1}_{]0,\mathrm{t}[}:]0,\mathrm{t}[\to\mathbb{R}$ is the unit function of $]0,\mathrm{t}[$.
- For $\mathrm{u}\in\mathscr{S}'(\mathbb{R}^{\mathrm{n}},\mathscr{L}(\mathrm{E},\mathrm{F}))$, $\mathrm{v}\in\mathscr{S}'(\mathbb{R}^{\mathrm{n}},\mathrm{E})$, the usual paraproduct (case $\mathrm{E}=\mathrm{F}=\mathbb{R}$) generalizes immediately as:

$$\Pi(\mathrm{u},\mathrm{v})=\sum_{\mathrm{p}\ge-1}\mathrm{S}_{\mathrm{p}-1}\mathrm{u}.\Delta_{\mathrm{p}}\mathrm{v}$$

  and for the remainder:

$$\mathfrak{R}(\mathrm{u},\mathrm{v})=\sum_{|\mathrm{p}-\mathrm{q}|\le1}\Delta_{\mathrm{q}}\mathrm{u}.(\Delta_{\mathrm{q}-1}+\Delta_{\mathrm{q}}+\Delta_{\mathrm{q}+1})\mathrm{v}$$

  so that formally, we get the Bony decomposition $\mathrm{u.v}=\Pi(\mathrm{u},\mathrm{v})+\Pi(\mathrm{u},\mathrm{v})+\mathfrak{R}(\mathrm{u},\mathrm{v})$. We shall use freely continuity results for the paradaproduct and remainder. See for instance [4] pp. 103-104 or [25] p.35.

- Let Ω be a (smooth) domain of $\mathbb{R}^n$. For any $A \subset \mathscr{D}'(\Omega)$, the restriction of a distribution $T \in \mathscr{D}'(\Omega)$ to a domain $\omega \subset \Omega$ is denoted by $T|_\omega$. The set $A|_\omega$ is the set of elements $T|_\omega$ with $T \in A$.

- Let $\Omega \subset \mathbb{R}^n$ be a smooth domain. The Besov space $B^s_{p,q}(\Omega, E)$ is defined as the restrictions of elements of $B^s_{p,q}(\mathbb{R}^n, E)$ to Ω. The space $B^s_{p,q}(\Omega, E)$ is endowed with the quotient norm:

$$\|u\|_{B^s_{p,q}(\Omega,E)} = \inf \|v\|_{B^s_{p,q}(\mathbb{R}^n,E)}$$

  the inf being taken on all the extensions $v \in B^s_{p,q}(\mathbb{R}^n, E)$ of u. This norm is well suited for extension-retractation operations, and we shall *never* use the symbol $\|.\|_{B^s_{p,q}(\Omega,E)}$ in any other sense. For $s - (1/p) > 0$, we shall also use the equivalent norm defined for $u \in B^s_{p,q}(\Omega, E)$ by:

$$\|u\|_{L^p(\Omega,E)} + \|u'\|_{B^{s-1}_{p,q}(\Omega,E)} \tag{2.4}$$

- Let Ω be a smooth domain of $\mathbb{R}^n$. For $1 < p, q < \infty$, $s \in \mathbb{R}$, the space of infinitely differentiable functions $C^\infty(\bar{\Omega}, E)$ is dense in $B^s_{p,q}(\Omega, E)$. See [26], p. 195.

- Let $\omega \subset \Omega$ be two domains of $\mathbb{R}^n$. For $u \in B^s_{p,q}(\Omega, E)$, we write $\|u\|_{B^s_{p,q}(\omega,E)} := \|u|_\omega\|_{B^s_{p,q}(\omega,E)}$.

- Let Ω be a smooth domain of $\mathbb{R}^n$. Then, for $1 < p, q < \infty$ and $\frac{1}{p} - 1 < s < \frac{1}{p}$ the set $\mathscr{D}(\Omega, E)$ is dense in $B^s_{p,q}(\Omega, E)$. Under the above hypothesis, we shall mostly use this result in the following form (see [26], pp. 210-211.):

  The extension by zero operator is continuous from $B^s_{p,q}(\Omega, E)$ to $B^s_{p,q}(\mathbb{R}^n, E)$ (2.5)

- The following classical proposition will be useful, when dealing with linear systems:

  Proposition 2.1. *Let* $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N}^*)$ *be a smooth domain, and let* $1 \le p, q \le \infty$, $0 \le \sigma < s$. *Then, for any* $\gamma > 0$, *there exists* $C_{\gamma,\Omega} > 0$ *such that, for any* $u \in B^s_{p,q}(\Omega, E)$*:*

$$\|u\|_{B^\sigma_{p,q}(\Omega,E)} \le \gamma \|u\|_{B^s_{p,q}(\Omega,E)} + C_{\gamma,\Omega}\|u\|_{L^p(\Omega,E)} \tag{2.6}$$

- In order to prove existence, uniqueness and the variation of constant formula, we have to define duality-like pairings. for vector valued distributions. The construction is very similar to the one given in [4], p.70 and p.101 for the duality bracket. Therefore we will be quite sketchy. We assume that E, F are two Banach spaces. We restrict to the case of an interval I, and stay within the range $0 < \eta \le 1/2$ and $\omega(\eta, p) > 0$. Set $s = \frac{1}{2} + \eta$. From the above conditions, we deduce that $\frac{1}{p'} - 1 < s < \frac{1}{p'}$ and $\frac{1}{p} - 1 < -s < \frac{1}{p}$. It entails that the extension by zero operator P_0 is continuous in both case:

  - $P_0 : B^s_{p',q'}(I, \mathscr{L}(E, F)) \to B^s_{p',q'}(\mathbb{R}, \mathscr{L}(E, F))$
  - $P_0 : B^{-s}_{p,q}(I, E) \to B^{-s}_{p,q}(\mathbb{R}, E)$

  where, as customory, we have denoted by the same letter the two operators. Hence, we define the pairing $< .,. >_{\eta,p,q,I}$: $B^s_{p',q'}(I, \mathscr{L}(E, F)) \times B^{-s}_{p,q}(I, E) \to F$ by:

$$< u, v >_{\eta,p,q,I} = \sum_{|k'-k| \le 1} \int_{\mathbb{R}} \Delta_k(P_0 u)(t).\Delta_{k'}(P_0 v)(t)dt \tag{2.7}$$

  Function $< .,. >_{\eta,p,q,I}$ extends continuously the pairing of $L^2(I, \mathscr{L}(E, F)) \times L^2(I, E) \to F$ given by $\int_I u(t)v(t)dt$. Notice that we are a little loose in our notations. In particular, we shall exchange the rules of the spaces E and $\mathscr{L}(E, F)$ without modifying the name of the bracket, and even exchange the places of u and v. Moreover, we shall often write $< .,. >_{\eta,p,q}$ in place of $< .,. >_{\eta,p,q,I}$.

3 The dyadic lemmas.

This section contains most of the proofs coming from Littlewood-Paley decomposition. They all rely on:

- the fact that the characteristic function χ_I of an interval belongs to $B^{\frac{1}{p}}_{p,\infty}$
- the use of the "differential dimension" $s - \frac{n}{p}$ (see [26]), mostly formulas 3.4, 3.5 below.

The first lemma will play a role when combined with the continuity of the zero-extension operator (see lemma 4.2 below), and will provide uniformely bounded Poincaré constants. It is more or less classical. In the case of a single, general, diffeomorphism, see [28]. See also [4] p.64, proposition 2.18. For future reference (formulas 3.5, 3.8), we give a standard proof using a dyadic decomposition.

We denote by $\psi_{x_0,\lambda} : \mathbb{R}^n \to \mathbb{R}^n$, the family of diffeomorphisms defined for $(x_0, \lambda) \in \mathbb{R}^n \times \mathbb{R}^*_+$ by $\psi_{x_0,\lambda}(y) = x_0 + \lambda y$. In the following, we mainly keep track of the relevant variables and omit some indexes in the writing of the constants.

Lemma 3.1. *Let $0 < \alpha < \beta < \infty$, $1 \le p, q \le \infty$, $s \in \mathbb{R}$. Let $n \in \mathbb{N}^*$ and Ω a domain of $\mathbb{R}^n$. Then, there exist $A_{\alpha,\beta} > 0$ and $C_{\alpha,\beta} > 0$ such that, for any $x_0 \in \mathbb{R}^n$ and $\alpha \le \lambda \le \beta$ and any $u \in B^s_{p,q}\big(\psi_{x_0,\lambda}(\Omega), E\big)$ we have:*

$$A_{\alpha,\beta}\|u\|_{B^s_{p,q}(\psi_{x_0,\lambda}(\Omega),E)} \le \|uo(\psi_{x_0,\lambda}|_\Omega)\|_{B^s_{p,q}(\Omega,E)} \le C_{\alpha,\beta}\|u\|_{B^s_{p,q}(\psi_{x_0,\lambda}(\Omega),E)} \tag{3.1}$$

Proof. We can assume that $x_0 = 0$ and $\alpha < 1 < \beta$.

1) We begin with the case $\Omega = \mathbb{R}^n$. Let $\alpha < \lambda < \beta$. We write u_λ in place of $uo\psi_{0,\lambda}$ and prove the right hand-side inequality of lemma 3.1.

1) a) *Estimates of the λ-blocks by the 1-blocks.* For $j \in \mathbb{Z}$, set:

$$w^\lambda = \mathfrak{X}\big(2^{\langle \log_2 \lambda\rangle - \log_2 \lambda} D\big)u_\lambda \tag{3.2}$$

$$z^\lambda_j = \vartheta(2^{\langle \log_2 \lambda\rangle - \log_2 \lambda - j} D)u_\lambda \tag{3.3}$$

where $\langle \log_2 \lambda\rangle$ denotes the integer part of $\log_2 \lambda$. Note that for any $\phi \in \mathscr{D}(\mathbb{R}^n, \mathbb{R})$ and $k \in \mathbb{Z}$, we have $\big[\phi(2^{\langle \log_2 \lambda\rangle - \log_2 \lambda - k} D)u_\lambda\big](x) = \big[\phi(2^{\langle \log_2 \lambda\rangle - k} D)u\big](\lambda x)$. Hence, the L^p norms of w^λ and z^λ_j satisfy:

$$\|w^\lambda\|_{L^p} = 2^{-n\langle \log_2 \lambda\rangle/p}\|\mathfrak{X}\big(2^{\langle \log_2 \lambda\rangle} D\big)u\|_{L^p} \tag{3.4}$$

$$2^{js}\|z^\lambda_j\|_{L^p} \le C_s \lambda^{s-\frac{n}{p}} 2^{(j-\langle \log_2 \lambda\rangle)s}\| \overset{o}{\Delta}_{j-\langle \log_2 \lambda\rangle}\, u\|_{L^p} \tag{3.5}$$

In order to estimate the right hand-side of formula 3.4, remark that for $K > 0$ large enough, we have $\mathrm{supp}\big(\mathfrak{X}\big) \cap \mathrm{supp}\theta\big(2^{-K}.\big) = \emptyset$. This implies that, for any $\lambda \in [\alpha, \beta]$, $\mathrm{supp}\mathfrak{X}(2^{\langle \log_2 \lambda\rangle}.) \cap \mathrm{supp}\theta\big(2^{-(K-\langle \log_2 \alpha\rangle+1)}.\big) = \emptyset$. Set $K_\alpha = K - \langle \log_2 \alpha\rangle + 1$. We deduce from the above, that for any $\lambda \in [\alpha, \beta]$

$$\begin{aligned}
\|\mathfrak{X}(2^{\langle \log_2 \lambda\rangle} D)u\|_{L^p} &= \|\mathfrak{X}(2^{\langle \log_2 \lambda\rangle} D) \sum_{k=-1}^{K_\alpha} \Delta_k u\|_{L^p} \\
&\le \|\mathscr{F}^{-1}\mathfrak{X}\|_{L^1} \sum_{k=-1}^{K_\alpha} \|\Delta_k u\|_{L^p} \\
&\le C_{\alpha,s}\|u\|_{B^s_{p,q}}
\end{aligned} \tag{3.6}$$

Arguing in the same way, we also have, for $-1 \leq \mathrm{j} \leq \langle \log_2 \beta \rangle + 2$

$$\| \overset{o}{\Delta}_{\mathrm{j}-\langle \log_2 \lambda \rangle} \mathrm{u} \|_{\mathrm{L^p}} \leq \mathrm{C}_{\alpha,\beta,\mathrm{s}} \| \mathrm{u} \|_{\mathrm{B^s_{p,q}}} \tag{3.7}$$

1)b) *High-frequencies.* Since $\mathrm{supp}\mathscr{F}(\vartheta) = \mathcal{S}(\frac{3}{4}, \frac{8}{3})$, we get that $\mathrm{supp}\mathscr{F}(\mathrm{z}^\lambda_j) \subset \mathcal{S}(\frac{3}{4}2^j, \frac{8}{3}2^{j+1})$. It follows from formula 3.3 that:

$$\overset{o}{\Delta}_j \mathrm{u}_\lambda = \sum_{|\mathrm{k}-\mathrm{j}| \leq 2} \overset{o}{\Delta}_{\mathrm{j}} \mathrm{z}^\lambda_k \tag{3.8}$$

Finally, inequality $\|\overset{o}{\Delta}_j \mathrm{f}\|_{\mathrm{L^p}} \leq \mathrm{C}\|\mathrm{f}\|_{\mathrm{L^p}}$ and 3.5, 3.8 provide:

$$\Big(\sum_{\mathrm{j} \geq \langle \log_2 \beta \rangle + 2} 2^{jqs} \|\Delta_{\mathrm{j}} u_\lambda\|^q_{L^p} \Big)^{1/q} \leq \mathrm{C}\lambda^{\mathrm{s}-\frac{n}{p}} \|\mathrm{u}\|_{\mathrm{B}^s_{\mathrm{p,q}}} \leq \mathrm{C}_{\alpha,\beta,\mathrm{s}} \|\mathrm{u}\|_{\mathrm{B^s_{p,q}}} \tag{3.9}$$

1)c) *Low-frequencies.* Let now $-1 \leq \mathrm{j} \leq \langle \log_2 \beta \rangle + 1$. Arguing as before, we get:

$$\Delta_{\mathrm{j}} \mathrm{u}_\lambda = \Delta_{\mathrm{j}} \Big[\mathrm{w}^\lambda + \sum_{\mathrm{k}=0}^{\langle \log_2 \beta \rangle + 3} \mathrm{z}^\lambda_k \Big] \tag{3.10}$$

hence, from inequalities $\|\overset{o}{\Delta}_{\mathrm{j}} \mathrm{f}\|_{\mathrm{L^p}} \leq \mathrm{C}\|\mathrm{f}\|_{\mathrm{L^p}}$, 3.4, 3.5, 3.6, 3.7 we get

$$\|\Delta_{\mathrm{j}} \mathrm{u}_\lambda\|_{\mathrm{L^p}} \leq \mathrm{C}_{\alpha,\beta,\mathrm{s}} \|\mathrm{u}\|_{B^{\mathrm{s}}_{\mathrm{p,q}}} \tag{3.11}$$

which finally provides

$$\sum_{\mathrm{j}=-1}^{\langle \log_2 \beta \rangle + 1} \big(2^{\mathrm{js}} \|\Delta_j u_\lambda\|_{L^p} \big)^q \leq \mathrm{C}_{\alpha,\beta,\mathrm{s}} \|\mathrm{u}\|_{B^{\mathrm{s}}_{\mathrm{p,q}}} \tag{3.12}$$

It follows from 3.9 and 3.12 that $\|\mathrm{u}_\lambda\|_{B^{\mathrm{s}}_{\mathrm{p,q}}} \leq \mathrm{C}_{\alpha,\beta,\mathrm{s}} \|\mathrm{u}\|_{B^{\mathrm{s}}_{\mathrm{p,q}}}$
2) We now adress the general case $\Omega \subset \mathbb{R}^{\mathrm{n}}$. Let $\mathrm{u} \in \mathrm{B}^{\mathrm{s}}_{\mathrm{p,q}}(\psi_{\mathrm{x_0},\lambda}(\Omega), \mathrm{E})$ and $\mathrm{v} \in \mathrm{B}^{\mathrm{s}}_{\mathrm{p,q}}(\mathbb{R}^{\mathrm{n}}, \mathrm{E})$ any extension of u. We deduce from the case $\Omega = \mathbb{R}^{\mathrm{n}}$ that:

$$\|\mathrm{u}o(\psi_{x_0,\lambda}|_\Omega)\|_{\mathrm{B^s_{p,q}}(\Omega,\mathrm{E})} \leq \|\mathrm{v}o\psi_{x_0,\lambda}\|_{\mathrm{B^s_{p,q}}(\mathbb{R}^{\mathrm{n}},\mathrm{E})} \leq \mathrm{C}_{\alpha,\beta,\mathrm{s}} \|\mathrm{v}\|_{\mathrm{B}^s_{\mathrm{p,q}}(\mathbb{R}^{\mathrm{n}},\mathrm{E})}$$

Taking the inf on all the extensions $\mathrm{v} \in \mathrm{B}^{\mathrm{s}}_{\mathrm{p,q}}(\mathbb{R}^{\mathrm{n}}, \mathrm{E})$ of u provides the result. ☐

The following result is a fractional integration theorem, replacing the "full" integration in use in the standard proof of Cauchy-Lipschitz theorem. Recall (see section 2) that for any $\mathrm{t} > 0$, $\chi_{1/t} : \mathbb{R} \to \mathbb{R}$ denotes the characteristic function of the interval $]0, \mathrm{t}[$ and set $\chi = \chi_1$.

Theorem 3.1. *a) Let* $\mathrm{t} > 0$, $1 \leq \mathrm{m} < \infty$ *and* $0 < \epsilon \leq 1/\mathrm{m}$. *Then*

$$\|\chi_{1/t}\|_{\mathrm{B}^{\frac{1}{\mathrm{m}}-\epsilon}_{\mathrm{m},\infty}(\mathbb{R})} \leq \mathrm{C}\|\chi\|_{\mathrm{B}^{\frac{1}{\mathrm{m}}}_{\mathrm{m},\infty}(\mathbb{R})} \mathrm{t}^\epsilon \tag{3.13}$$

b) Let $\mathrm{T} > 0$, $\mathrm{R} > 0$, $0 < \alpha < \eta < 1/2$, $1 \leq \mathrm{p,q} < \infty$ *with* $\omega(\alpha, \mathrm{p}) > 0$. *Let also* $\mathrm{u} \in \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0, \mathrm{T}[, \mathrm{E})$. *Then, for any* $\mathrm{t} \in]0, \mathrm{T}]$ *we have:* $\|\mathrm{u}\|_{\mathrm{B}^{-\frac{1}{2}+\alpha}_{\mathrm{p,q}}(]0,\mathrm{t}[,\mathrm{E})} \leq \mathrm{C}\|\mathrm{u}\|_{\mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{t}[,\mathrm{E})} \mathrm{t}^{\eta-\alpha}$.

Proof. a) We assume that $0 < \mathrm{t} < 1$, the case $\mathrm{t} > 1$ being simpler. We apply estimates 3.5 and 3.8 with $\lambda = 1/\mathrm{t}$, $\mathrm{s} = 1/\mathrm{m}$, $\mathrm{p} = \mathrm{m}$ and $\mathrm{n} = 1$. Let $\mathrm{j}_0 = \langle \log_2(1/\mathrm{t})\rangle > 0$. We obtain, for $\mathrm{j} > \mathrm{j}_0$:

$$2^{\frac{\mathrm{j}}{\mathrm{m}}}\|\Delta_{\mathrm{j}}\chi_{1/t}\|_{\mathrm{L}^{\mathrm{m}}} \leq \sum_{|\mathrm{k}-\mathrm{j}|\leq 2} \mathrm{C}2^{\frac{\mathrm{k}-\mathrm{j}_0}{\mathrm{m}}}\|\Delta_{\mathrm{k}-\mathrm{j}_0}\chi\|_{\mathrm{L}^{\mathrm{m}}} \tag{3.14}$$

Hence:

$$2^{\mathrm{j}(\frac{1}{\mathrm{m}}-\epsilon)}\|\Delta_{\mathrm{j}}\chi_{1/t}\|_{\mathrm{L}^{\mathrm{m}}} \leq \mathrm{C}\|\chi\|_{\mathrm{B}^{\frac{1}{\mathrm{m}}}_{\mathrm{m},\infty}} 2^{-\mathrm{j}\epsilon} \leq \mathrm{C}\|\chi\|_{\mathrm{B}^{\frac{1}{\mathrm{m}}}_{\mathrm{m},\infty}}\mathrm{t}^{\epsilon} \tag{3.15}$$

since $2^{-\mathrm{j}} \leq \mathrm{t}$. For $-1 \leq \mathrm{j} \leq \mathrm{j}_0$ with $\mathrm{h}_{\mathrm{j}} = \mathscr{F}^{-1}\vartheta$ for $\mathrm{j} \neq -1$ and $\mathrm{h}_{-1} = 2(\mathscr{F}^{-1}\mathfrak{X})(2.)$, we have

$$\begin{aligned}\|\Delta_j\chi_{1/t}\|_{\mathrm{L}^m} &= \|\int_0^t 2^j \mathrm{h}_{\mathrm{j}}[2^j(.-z)]dz\|_{\mathrm{L}^m}\\ &\leq \int_0^t 2^j\|\mathrm{h}_{\mathrm{j}}[2^j(.-z)]\|_{\mathrm{L}^m}dz \leq \int_0^t 2^j 2^{-j/m}\|\mathrm{h}_{\mathrm{j}}\|_{\mathrm{L}^m}dz \leq \|\mathrm{h}_{\mathrm{j}}\|_{\mathrm{L}^m}2^{j/m'}\mathrm{t}\end{aligned} \tag{3.16}$$

It follows that:

$$\begin{aligned}2^{\mathrm{j}(\frac{1}{\mathrm{m}}-\epsilon)}\|\Delta_{\mathrm{j}}\chi_{1/t}\|_{\mathrm{L}^{\mathrm{m}}} \leq \|\mathrm{h}_{\mathrm{j}}\|_{\mathrm{L}^{\mathrm{m}}}2^{\mathrm{j}(1-\epsilon)}\mathrm{t} &\leq \mathrm{C}\|\mathrm{h}_{\mathrm{j}}\|_{\mathrm{L}^{\mathrm{m}}}\mathrm{t}^{-(1-\epsilon)}\mathrm{t}\\ &\leq \mathrm{C}\|\mathrm{h}_{\mathrm{j}}\|_{\mathrm{L}^{\mathrm{m}}}\mathrm{t}^{\epsilon}\end{aligned} \tag{3.17}$$

Finally, a) of theorem 3.1 follows from 3.15 and 3.17.

b) In order to prove b), we first show that, for $\epsilon = \eta - \alpha$ with $0 < \alpha < \eta$, $\omega(\alpha, \mathrm{p}) > 0$, $0 < \mathrm{t} < \mathrm{T}$, and for any $\Theta \in \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(\mathbb{R}, \mathrm{E})$, the following inequality holds true:

$$\|\Theta\chi_{1/t}\|_{\mathrm{B}^{-\frac{1}{2}+\eta-\epsilon}_{\mathrm{p,q}}(\mathbb{R},\mathrm{E})} \leq \mathrm{C}\|\Theta\|_{\mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(\mathbb{R},\mathrm{E})}\mathrm{t}^{\epsilon} \tag{3.18}$$

In fact, taking in account $-\epsilon < 0$ and $-\frac{1}{2} + \eta - \frac{1}{\mathrm{p}} < 0$, we get:

$$\mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(\mathbb{R}, \mathrm{E}) \times \mathrm{B}^{\frac{1}{\mathrm{p}}-\epsilon}_{\mathrm{p,q}}(\mathbb{R}) \hookrightarrow \mathrm{B}^{-\frac{1}{2}+\eta-\frac{1}{\mathrm{p}}}_{\infty,\mathrm{q}}(\mathbb{R}, \mathrm{E}) \times \mathrm{B}^{\frac{1}{\mathrm{p}}-\epsilon}_{\mathrm{p},\infty}(\mathbb{R}) \xrightarrow{\Pi} \mathrm{B}^{-\frac{1}{2}+\eta-\epsilon}_{\mathrm{p,q}}(\mathbb{R}, \mathrm{E}) \tag{3.19}$$

$$\mathrm{B}^{\frac{1}{\mathrm{p}}-\epsilon}_{\mathrm{p,q}}(\mathbb{R}) \times \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(\mathbb{R}, \mathrm{E}) \hookrightarrow \mathrm{B}^{-\epsilon}_{\infty,\infty}(\mathbb{R}) \times \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(\mathbb{R}, \mathrm{E}) \xrightarrow{\Pi} \mathrm{B}^{-\frac{1}{2}+\eta-\epsilon}_{\mathrm{p,q}}(\mathbb{R}, \mathrm{E}) \tag{3.20}$$

Since $-\frac{1}{2} + \eta + \frac{1}{p}' - \epsilon = \frac{1}{2} + \eta - \frac{1}{p} - \epsilon = \omega(\alpha, \mathrm{p}) > 0$, we have, for the remainder:

$$\mathrm{B}^{\frac{1}{\mathrm{p}'}-\epsilon}_{\mathrm{p}',\mathrm{q}}(\mathbb{R}) \times \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(\mathbb{R}, \mathrm{E}) \xrightarrow{\mathfrak{R}} \mathrm{B}^{\omega(\alpha,\mathrm{p})}_{1,\mathrm{q}}(\mathbb{R}, \mathrm{E}) \hookrightarrow \mathrm{B}^{-\frac{1}{2}+\eta-\epsilon}_{\mathrm{p,q}}(\mathbb{R}, \mathrm{E}) \tag{3.21}$$

Noticing that $\chi_{1/t} \in \mathrm{B}^{\frac{1}{\mathrm{m}}-\epsilon}_{\mathrm{m,q}}(\mathbb{R})$ for $\mathrm{m} = \mathrm{p}$ and for $\mathrm{m} = \mathrm{p}'$, and using 3.19, 3.20, 3.21 and a), inequality 3.18 follows.

Now, for $\mathrm{u} \in \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0, \mathrm{T}[, \mathrm{E})$ and $0 < \mathrm{t} \leq \mathrm{T}$, we have, denoting by $\tilde{\mathrm{u}} \in \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(\mathbb{R}, \mathrm{E})$ any extension of $\mathrm{u}|_{]0,\mathrm{t}[} \in \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0, \mathrm{t}[, \mathrm{E})$ and invoking 3.18:

$$\|\mathrm{u}\|_{\mathrm{B}^{-\frac{1}{2}+\alpha}_{\mathrm{p,q}}(]0,\mathrm{t}[,\mathrm{E})} \leq \|\tilde{\mathrm{u}}\chi_{1/t}\|_{\mathrm{B}^{-\frac{1}{2}+\alpha}_{\mathrm{p,q}}(]0,\mathrm{t}[,\mathrm{E})} \leq \mathrm{C}\|\tilde{\mathrm{u}}\|_{\mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(\mathbb{R},\mathrm{E})}\mathrm{t}^{\eta-\alpha} \tag{3.22}$$

Taking the inf on all the extensions $\tilde{\mathrm{u}}$, we get b). □

The following lemma is a first version of Poincaré's inequalities.

Lemma 3.2. *Let* $\mathrm{T} > 0$, $1 \leq \mathrm{p}, \mathrm{q} \leq \infty$ *with* $0 < \eta \leq 1/2$ *and* $\omega(\eta, \mathrm{p}) > 0$.

a) There exists $\mathrm{C_T} > 0$ *such that, for any* $\mathcal{G} \in \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0, \mathrm{T}[, \mathrm{E})$ *and any* $0 < \mathrm{t} < \mathrm{T}$ *we have:*

$$\| < \mathcal{G}|_{]0,t[}, \mathbf{1}_{]0,t[} >_{\eta,\mathrm{p,q}} \|_{\mathrm{E}} \leq \mathrm{C_T} \|\mathcal{G}\|_{\mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{t}[,\mathrm{E})} \mathrm{t}^{\omega(\eta,p)} \tag{3.23}$$

b) There exists $\mathrm{C_T} > 0$ *such that, for any* $\mathrm{u} \in \mathrm{B}^{\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0, \mathrm{T}[, \mathrm{E})$ *and* $0 < \mathrm{t} < \mathrm{T}$, *we have:*

$$\|\mathrm{u} - \mathrm{u}(0)\|_{\mathrm{L}^\infty(]0,\mathrm{t}[,\mathrm{E})} \leq \mathrm{C_T} \|\mathrm{u}'\|_{\mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{t}[,\mathrm{E})} \mathrm{t}^{\omega(\eta,p)} \tag{3.24}$$

Proof. a) Set $\chi = \chi_1$. Let also $\mathcal{H} \in \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(\mathbb{R})$ be any extension of $\mathcal{G}|_{]0,\mathrm{t}[}$. In order to avoid cumbersome arguments, we assume that $0 < \mathrm{t} < inf\{1, \mathrm{T}\}$, the case $\mathrm{t} \geq 1$ being simpler. Invoking 3.5 and 3.8 with p' in place of p, $\lambda = 1/t$, $\mathrm{s} = 1/\mathrm{p}'$, $\mathrm{n} = 1$, $\mathrm{j} > \mathrm{j_0} := \langle \log_2(1/\mathrm{t}) \rangle > 0$, we get (see 3.14 and 3.15)

$$\|\Delta_{\mathrm{j}} \chi_{1/\mathrm{t}}\|_{\mathrm{L^{p'}}} \leq \mathrm{C} \|\chi\|_{\mathrm{B}^{\frac{1}{\mathrm{p}'}}_{\mathrm{p}',\infty}} 2^{-\mathrm{j}/\mathrm{p}'} \tag{3.25}$$

Next, $\mathcal{H} \in \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}$ implies that:

$$\|\Delta_{\mathrm{k}} \mathcal{H}\|_{\mathrm{L^p}} \leq \|\mathcal{H}\|_{\mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}} 2^{-(-\frac{1}{2}+\eta)\mathrm{k}} \tag{3.26}$$

Therefore, appealing to 3.25 and 3.26, we have:

$$\begin{aligned}
\sum_{|\mathrm{k}-\mathrm{j}|\leq 1, \mathrm{j}>\mathrm{j_0}} \|\Delta_{\mathrm{k}} \mathcal{H}\|_{\mathrm{L}^p} \|\Delta_{\mathrm{j}} \chi_{1/\mathrm{t}}\|_{\mathrm{L^{p'}}} &\leq \sum_{|\mathrm{k}-\mathrm{j}|\leq 1, \mathrm{j}>\mathrm{j_0}} \mathrm{C} \|\chi\|_{\mathrm{B}^{\frac{1}{\mathrm{p}'}}_{\mathrm{p}',\infty}} \|\mathcal{H}\|_{\mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}} 2^{-\mathrm{j}/\mathrm{p}'} 2^{(\frac{1}{2}-\eta)k} \\
&\leq \mathrm{C} \|\chi\|_{\mathrm{B}^{\frac{1}{\mathrm{p}'}}_{\mathrm{p}',\infty}} \|\mathcal{H}\|_{\mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}} \sum_{\mathrm{j}>\mathrm{j_0}} 2^{-\mathrm{j}\omega(\eta,\mathrm{p})} \\
&\leq \mathrm{C} \|\mathcal{H}\|_{\mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}} 2^{-\mathrm{j_0}\omega(\eta,\mathrm{p})} \leq \mathrm{C} \|\mathcal{H}\|_{\mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}} \mathrm{t}^{\omega(\eta,\mathrm{p})}
\end{aligned} \tag{3.27}$$

For the low frequency terms, appealing to 3.16 and 3.26 and , we have:

$$\begin{aligned}
\sum_{|\mathrm{k}-\mathrm{j}|\leq 1, -1\leq \mathrm{j}\leq \mathrm{j_0}} \|\Delta_{\mathrm{k}} \mathcal{H}\|_{\mathrm{L^p}} \|\Delta_{\mathrm{j}} \chi_{1/\mathrm{t}}\|_{\mathrm{L^{p'}}} &\leq \sum_{|\mathrm{k}-\mathrm{j}|\leq 1, -1\leq \mathrm{j}\leq \mathrm{j_0}} \mathrm{t} \|\mathrm{h_j}\|_{\mathrm{L^{p'}}} \|\mathcal{H}\|_{\mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}} 2^{\mathrm{j/p}} 2^{(\frac{1}{2}-\eta)\mathrm{k}} \\
&\leq \mathrm{C} \|\mathcal{H}\|_{\mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}} \sum_{\mathrm{j}=-1}^{\mathrm{j_0}} \mathrm{t} 2^{\mathrm{j}\left(1-\omega(\eta,\mathrm{p})\right)} \\
&\leq \mathrm{C} \|\mathcal{H}\|_{\mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}} \mathrm{t} 2^{\mathrm{j_0}(1-\omega(\eta,\mathrm{p}))} \leq \mathrm{C} \|\mathcal{H}\|_{\mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}} \mathrm{t}^{\omega(\eta,\mathrm{p})}
\end{aligned} \tag{3.28}$$

It follows from inequalities 3.27, and 3.28:

$$\begin{aligned}
\| < \mathcal{G}|_{]0,\mathrm{t}[}, \mathbf{1}_{]0,\mathrm{t}[} >_{\eta,\mathrm{p,q}} \|_{\mathrm{E}} &\leq \sum_{|\mathrm{k}-\mathrm{j}|\leq 1, \mathrm{j}\geq -1} \|\Delta_{\mathrm{k}} \mathcal{H}\|_{\mathrm{L^p}} \|\Delta_{\mathrm{j}} \chi_{1/t}\|_{\mathrm{L^{p'}}} \\
&\leq \mathrm{C_T} \|\mathcal{H}\|_{\mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(\mathbb{R})} \mathrm{t}^{\omega(\eta,\mathrm{p})}
\end{aligned} \tag{3.29}$$

Taking the inf on all the extensions $\mathcal{H} \in \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(\mathbb{R})$ of $\mathcal{G}|_{]0,\mathrm{t}[}$, a) is proved.

b) Follows from a) and lemma 3.3 below. □

The proof of lemma 3.3 depends on lemma 3.2 a). Nevertheless, we shall appeal to lemma 3.3 later, and therefore state it separately:

Lemma 3.3. *Let $0 \le \mathrm{t}_0 \le \mathrm{T}$, $0 < \gamma < 1/2$, $1 \le \mathrm{p}, \mathrm{q} < \infty$ with $\omega(\gamma, \mathrm{p}) > 0$. Let $\psi \in \mathrm{B}_{\mathrm{p},\mathrm{q}}^{-\frac{1}{2}+\gamma}(]0, \mathrm{t}_0[, \mathrm{E})$ and $\mathrm{M}_0 \in \mathrm{E}$. Then, problem: find $\mathrm{M} \in \mathrm{B}_{\mathrm{p},\mathrm{q}}^{\frac{1}{2}+\gamma}(]0, \mathrm{t}_0[, \mathrm{E})$ with:*

$$\begin{cases} \mathrm{M}' = \psi \\ \mathrm{M}(0) = \mathrm{M}_0 \end{cases} \tag{3.30}$$

admits exactly one solution, given by:

$$\mathrm{M}(\mathrm{t}) = \mathrm{M}_0 + < \psi|_{]0,\mathrm{t}[}, \mathbf{1}_{]0,t[} >_{\gamma,\mathrm{p},\mathrm{q},]0,\mathrm{t}[} \tag{3.31}$$

Proof. We only pove formula 3.31. For $\psi \in \mathrm{C}^\infty([0, \mathrm{t}_0], \mathrm{E})$, formula 3.31 reduces to the usual integral formula. In the general case $\psi \in \mathrm{B}_{\mathrm{p},\mathrm{q}}^{-\frac{1}{2}+\gamma}(]0, \mathrm{t}_0[, \mathrm{E})$, let $(\psi_\mathrm{n})_{\mathrm{n}\in\mathbb{N}}$ a sequence of $\mathrm{C}^\infty([0, \mathrm{t}_0], \mathrm{E})$ functions converging to ψ in $\mathrm{B}_{\mathrm{p},\mathrm{q}}^{-\frac{1}{2}+\gamma}(]0, \mathrm{t}_0[, \mathrm{E})$. Set $\mathrm{M}(\mathrm{t}) = \mathrm{M}_0 + < \psi|_{]0,\mathrm{t}[}, \mathbf{1}_{]0,t[} >_{\gamma,\mathrm{p},\mathrm{q}}$ and $\mathrm{M}_\mathrm{n}(\mathrm{t}) = \mathrm{M}_0 + < \psi_\mathrm{n}|_{]0,\mathrm{t}[}, \mathbf{1}_{]0,t[} >_{\gamma,\mathrm{p},\mathrm{q}}$. We deduce from the assumptions on ψ_n and the continuity of the bracket that $\mathrm{M}(\mathrm{t}) - \mathrm{M}_\mathrm{n}(\mathrm{t}) =< \psi - \psi_\mathrm{n}, \mathbf{1}_{]0,t[} >_{\gamma,\mathrm{p},\mathrm{q}} \to 0$ as $\mathrm{n} \to \infty$, everywhere in $0 < \mathrm{t} < \mathrm{t}_0$. Moreover, by 3.2 a)

$$\|\mathrm{M}_\mathrm{n}(\mathrm{t})\|_\mathrm{E} \le \|\mathrm{M}_0\|_\mathrm{E} + \mathrm{C}_\mathrm{T} \|\psi_\mathrm{n}\|_{\mathrm{B}_{\mathrm{p},\mathrm{q}}^{-\frac{1}{2}+\gamma}(]0,\mathrm{t}[,\mathrm{E})} \mathrm{t}^{\omega(\gamma,\mathrm{p})} \le \mathrm{C}_\mathrm{T} \tag{3.32}$$

with C_T independent of n since $(\|\psi_\mathrm{n}\|_{\mathrm{B}_{\mathrm{p},\mathrm{q}}^{-\frac{1}{2}+\gamma}(]0,\mathrm{t}_0[,\mathrm{E})})_{\mathrm{n}\in\mathbb{N}}$ is convergent. By Lebesgue theorem, it follows that

$$\mathrm{M}_\mathrm{n} \to \mathrm{M} \text{ in } \mathrm{L}^\mathrm{p}(]0, \mathrm{t}_0[, \mathrm{E}) \tag{3.33}$$

Since $\mathrm{M}'_\mathrm{n} = \psi_\mathrm{n}$, this implies that $\mathrm{M}' = \psi$ in $\mathscr{D}'(]0, \mathrm{T}[, \mathrm{E})$. Therefore, $\mathrm{M}'_\mathrm{n} \to \mathrm{M}'$ in $\mathrm{B}_{\mathrm{p},\mathrm{q}}^{-\frac{1}{2}+\gamma}(]0, \mathrm{t}_0[, \mathrm{E})$ due to the convergence $\psi_\mathrm{n} \to \psi$ in the same space. With 3.33, this entails that $\mathrm{M}_\mathrm{n} \to \mathrm{M}$ in $\mathrm{B}_{\mathrm{p},\mathrm{q}}^{\frac{1}{2}+\gamma}(]0, \mathrm{t}_0[, \mathrm{E})$, hence in the Holder-Zygmund space $\mathrm{B}_{\infty,\infty}^{\omega(\gamma,\mathrm{p})}(]0, \mathrm{t}_0[, \mathrm{E})$. From $\mathrm{M}_\mathrm{n}(0) = \mathrm{M}_0$, we finally deduce that $\mathrm{M}(0) = \mathrm{M}_0$, which completes the proof. ☐

4 Uniform inequalities.

In this section, we show that some classical inequalities hold true on certain function spaces with varying domains, but with uniformly bounded constants.

An application of lemma 3.1 provides uniform estimates in the case of a family of large enough domains:

Lemma 4.1. *Let $1 \le \mathrm{p}, \mathrm{q} < \infty$, $0 < \sigma < \mathrm{s} < 1$, $0 < \alpha < \beta < \infty$, and let $\Omega \subset \mathbb{R}^\mathrm{n}$ be a Lipschitz domain. Then:*
a) Let $\epsilon > 0$. Then, there exist $\mathrm{A}_{\epsilon,\alpha,\beta} > 0$ such that, for any $\lambda \in [\alpha, \beta]$, $\mathrm{x}_0 \in \mathbb{R}^n$ and $\mathrm{u} \in \mathrm{B}_{\mathrm{p},\mathrm{q}}^\mathrm{s}(\psi_{\mathrm{x}_0,\lambda}(\Omega), \mathrm{E})$, we have:

$$\|\mathrm{u}\|_{\mathrm{B}_{\mathrm{p},\mathrm{q}}^\sigma(\psi_{\mathrm{x}_0,\lambda}(\Omega),\mathrm{E})} \le \epsilon \|\mathrm{u}\|_{\mathrm{B}_{\mathrm{p},\mathrm{q}}^\mathrm{s}(\psi_{\mathrm{x}_0,\lambda}(\Omega),\mathrm{E})} + \mathrm{A}_{\epsilon,\alpha,\beta} \|\mathrm{u}\|_{\mathrm{L}^\mathrm{p}(\psi_{\mathrm{x}_0,\lambda}(\Omega),\mathrm{E})} \tag{4.1}$$

b) There exists $\mathrm{C}_{\alpha,\beta} > 0$ such that, for any $\lambda \in [\alpha, \beta]$, $\mathrm{x}_0 \in \mathbb{R}^\mathrm{n}$ and $\mathrm{u} \in \mathrm{B}_{\mathrm{p},\mathrm{q}}^\mathrm{s}(\psi_{\mathrm{x}_0,\lambda}(\Omega), \mathrm{E})$, we have:

$$\|\mathrm{u}\|_{\mathrm{B}_{\mathrm{p},\mathrm{q}}^\mathrm{s}(\psi_{\mathrm{x}_0,\lambda}(\Omega),\mathrm{E})} \le \mathrm{C}_{\alpha,\beta} \Big[\|\mathrm{u}\|_{\mathrm{L}^\mathrm{p}(\psi_{\mathrm{x}_0,\lambda}(\Omega),\mathrm{E})} + \|\mathrm{u}'\|_{\mathrm{B}_{\mathrm{p},\mathrm{q}}^{\mathrm{s}-1}(\psi_{\mathrm{x}_0,\lambda}(\Omega),\mathrm{E})} \Big] \tag{4.2}$$

Proof. Using proposition 2.1 in the fixed configuration Ω, and lemma 3.1 we get:

$$\begin{aligned} \|\mathrm{u}\|_{\mathrm{B}_{\mathrm{p},\mathrm{q}}^\sigma(\psi_{x_0,\lambda}(\Omega),\mathrm{E})} &= \|\mathrm{u} o \psi_{x_0,\lambda}\|_{\mathrm{B}_{p,q}^\sigma(\Omega,\mathrm{E})} \\ &\le \epsilon \|\mathrm{u} o \psi_{x_0,\lambda}\|_{\mathrm{B}_{\mathrm{p},\mathrm{q}}^\mathrm{s}(\Omega,\mathrm{E})} + \mathrm{A}_{\epsilon,\Omega} \|\mathrm{u} o \psi_{x_0,\lambda}\|_{\mathrm{L}^\mathrm{p}(\Omega,\mathrm{E})} \\ &\le \mathrm{C}_{\alpha,\beta} \epsilon \|\mathrm{u}\|_{\mathrm{B}_{\mathrm{p},\mathrm{q}}^\mathrm{s}(\psi_{x_0,\lambda}(\Omega),\mathrm{E})} + \mathrm{A}_{\epsilon,\Omega} \alpha^{-\mathrm{n}/\mathrm{p}} \|\mathrm{u}\|_{\mathrm{L}^\mathrm{p}(\psi_{x_0,\lambda}(\Omega),\mathrm{E})} \end{aligned} \tag{4.3}$$

which is a). Property b) follows in the same manner, using theorem 3.3.5, p. 202 in [26]. ☐

In order to handle the case of vanishing intervals for the Poincaré inequality, we first prove the uniform continuity of a family of zero-extension operators:

Lemma 4.2. *Let* $\mathrm{T} > 0$, $1 \le \mathrm{p}, \mathrm{q} \le \infty$ *and* $0 < \eta \le 1/2$ *with* $\omega(\eta, p) > 0$. *There exists* $\mathrm{C_T} > 0$ *such that, for any* $0 < \mathrm{t} < \mathrm{T}$ *and* $\mathrm{u} \in \mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]-1, \mathrm{t}[, \mathrm{E}))$ *with* $\mathrm{u}|_{]-1,0[} = 0$, *we have:*

$$\|\mathrm{u}\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]0,\mathrm{t}[,\mathrm{E})} \le \|\mathrm{u}\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]-1,\mathrm{t}[,\mathrm{E})} \le \mathrm{C_T}\|\mathrm{u}\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]0,\mathrm{t}[,\mathrm{E})} \tag{4.4}$$

Proof. The first inequality is obvious. We prove the second one. Let $\mathrm{z_t} \in \mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(\mathbb{R}, \mathrm{E})$ be any extension of $\mathrm{u}|_{]0,\mathrm{t}[}$. We first check that $\mathrm{u} = \big(\mathrm{z_t}\chi_{]0,2\mathrm{T}[}\big)|_{]-1,\mathrm{t}[}$. Indeed, $\mathrm{u}|_{]-1,0[} = \big(\mathrm{z_t}\chi_{]0,2\mathrm{T}[}\big)|_{]-1,0[}$ and $\mathrm{u}|_{]0,\mathrm{t}[} = \big(\mathrm{z_t}\chi_{]0,2\mathrm{T}[}\big)|_{]0,\mathrm{t}[[}$ are obvious. It follows that $\mathrm{supp}\big[\mathrm{u} - \big(\mathrm{z_t}\chi_{]0,2\mathrm{T}[}\big)|_{]-1,\mathrm{t}[}\big] \subset \{0\}$. Therefore, $\mathrm{u} - \big(\mathrm{z_t}\chi_{]0,2\mathrm{T}[}\big)|_{]-1,\mathrm{t}[} = \sum_{\mathrm{k}=0}^{\mathrm{r}} \mathrm{c_k}\delta_0^{(\mathrm{k})}$ where $\delta_0 \in \mathscr{D}'(]-1, \mathrm{t}[, \mathrm{E})$ is the Dirac measure at zero, $\mathrm{c_k} \in \mathrm{E}$ and $\mathrm{r} \in \mathbb{N}$. Since $\big[\mathrm{u} - \big(\mathrm{z_t}\chi_{]0,2\mathrm{T}[}\big)|_{]-1,\mathrm{t}[}\big] \in \mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]-1, \mathrm{t}[, \mathrm{E})$ and $\frac{1}{\mathrm{p}} - 1 < -\frac{1}{2} + \eta$ we obtain $\mathrm{u} - \big(\mathrm{z_t}\chi_{]0,2\mathrm{T}[}\big)|_{]-1,\mathrm{t}[} = 0$, as required.

In consequence:

$$\|\mathrm{u}\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]-1,\mathrm{t}[)} \le \|\mathrm{z_t}\chi_{]0,2\mathrm{T}[}\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(\mathbb{R},\mathrm{E})} \le \mathrm{C_T}\|\mathrm{z_t}\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(\mathbb{R},\mathrm{E})} \tag{4.5}$$

since $\chi_{]0,2\mathrm{T}[}$ is a multiplier for $\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(\mathbb{R}, \mathrm{E})$. Inequality 4.5 provides 4.4 by taking the inf on $\mathrm{z_t}$. □

We finally obtain uniform inequalities on intervals $]-1, \mathrm{t}[$ with $\mathrm{t} > 0$ - and even on $]0, \mathrm{t}[$ in the case of the Poincaré's inequality and the $\mathrm{L}^\infty([0, \mathrm{t}], \mathrm{E})$ inequality:

Corollary 4.1. *Let* $\mathrm{T} > 0$, $1 < \mathrm{p}, \mathrm{q} < \infty$.

a) Let $0 \le \sigma < \mathrm{s}$ *and* $\epsilon > 0$. *Then there exists* $\mathrm{A}_{\mathrm{T},\epsilon} > 0$ *such that, for any* $0 < \mathrm{t} < \mathrm{T}$ *and any* $\mathrm{u} \in \mathrm{B}_{\mathrm{p,q}}^{\mathrm{s}}(]-1, \mathrm{T}[, \mathrm{E})$, *we have:*

$$\|\mathrm{u}\|_{\mathrm{B}_{\mathrm{p,q}}^{\sigma}(]-1,\mathrm{t}[,\mathrm{E})} \le \epsilon\|\mathrm{u}\|_{\mathrm{B}_{\mathrm{p,q}}^{\mathrm{s}}(]-1,\mathrm{t}[,\mathrm{E})} + A_{\mathrm{T},\epsilon}\|\mathrm{u}\|_{\mathrm{L}^p(]-1,\mathrm{t}[,\mathrm{E})} \tag{4.6}$$

$$\|\mathrm{u}\|_{\mathrm{B}_{\mathrm{p,q}}^{\mathrm{s}}(]-1,\mathrm{t}[,\mathrm{E})} \le \mathrm{C_T}(\|\mathrm{u}\|_{\mathrm{L^p}(]-1,\mathrm{t}[,\mathrm{E})} + \|\mathrm{u}'\|_{\mathrm{B}_{\mathrm{p,q}}^{\mathrm{s}-1}(]-1,\mathrm{t}[,\mathrm{E})}) \tag{4.7}$$

b) Assume that $0 < \eta < 1/2$ *with* $\omega(\eta, p) > 0$. *Then there exists* $\mathrm{C_T} > 0$ *such that, for any* $0 < \mathrm{t} < \mathrm{T}$ *and any* $\mathrm{u} \in \mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}(]0, \mathrm{T}[, \mathrm{E})$, *we have:*

$$\|\mathrm{u} - \mathrm{u}(0)\|_{\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}(]0,\mathrm{t}[,\mathrm{E})} \le \mathrm{C_T}\|\mathrm{u}'\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]0,\mathrm{t}[,\mathrm{E})} \tag{4.8}$$

c) In the case $\mathrm{s} - \frac{1}{\mathrm{p}} > 0$, *there exists a universal constant* $\mathrm{C}_\infty > 0$ *such that, for any* $0 < \mathrm{t} < \mathrm{T}$ *and any* $\mathrm{u} \in \mathrm{B}_{\mathrm{p,q}}^{\mathrm{s}}(]0, \mathrm{t}[, \mathrm{E})$, *we have:*

$$\|\mathrm{u}\|_{\mathrm{L}^\infty([0,\mathrm{t}],\mathrm{E})} \le \mathrm{C}_\infty\|\mathrm{u}\|_{\mathrm{B}_{\mathrm{p,q}}^{s}(]0,\mathrm{t}[,\mathrm{E})} \tag{4.9}$$

Proof. a) For 4.6 and 4.7, apply lemma 4.1 and 4.2 with $\mathrm{n} = 1$, $\Omega =]0, 1[$, $\mathrm{x_0} = -1$, $\alpha = 1$, $\beta = \mathrm{T} + 1$, $\lambda = \mathrm{t} + 1$ $(0 < \mathrm{t} < \mathrm{T})$.

b) Let $\mathrm{u}^* :]-1, \mathrm{T}[\to \mathbb{R}$ be defined by $\mathrm{u}^*(\tau) = \mathrm{u}(\tau) - \mathrm{u}(0)$ for $0 < \tau < \mathrm{T}$, and zero otherwise. From $\frac{1}{p} - 1 < \frac{1}{2} + \eta - 1 < \frac{1}{p}$, we deduce that $\mathrm{u}^* \in \mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}(]-1, \mathrm{T}[, \mathrm{E})$ (see [26] p.208). Using 4.2 and arguing as in a), we get the $]-1, \mathrm{t}[$ inequality:

$$\|\mathrm{u}^*\|_{\mathrm{B}_{p,q}^{\frac{1}{2}+\eta}(]-1,t[,\mathrm{E})} \le \mathrm{C_T}[\|\mathrm{u}^*\|_{\mathrm{L^p}(]-1,t[,\mathrm{E})} + \|\mathrm{u}^{*\prime}\|_{\mathrm{B}_{p,q}^{-\frac{1}{2}+\eta}(]-1,t[,\mathrm{E})}] \tag{4.10}$$

Writing $\|u^*\|_{L^p(]-1,t[,E)} = \|u-u(0)\|_{L^p(]0,t[,E)}$, $\|u-u(0)\|_{B^{\frac{1}{2}+\eta}_{p,q}(]0,t[,E)} \leq \|u^*\|_{B^{\frac{1}{2}+\eta}_{p,q}(]-1,t[,E)}$ and using lemma 4.2, we obtain:

$$\|u-u(0)\|_{B^{\frac{1}{2}+\eta}_{p,q}(]0,t[,E)} \leq C_T[\|u-u(0)\|_{L^p(]0,t[,E)} + \|u'\|_{B^{-\frac{1}{2}+\eta}_{p,q}(]0,t[,E)}] \tag{4.11}$$

Now, due to 3.24, we obtain 4.8.

c) Since $s - \frac{1}{p} > 0$, inequality 4.9 follows from the embedding $B^s_{p,q}(\mathbb{R},E) \hookrightarrow L^\infty(\mathbb{R},E)$ and the definition of $\|u\|_{B^s_{p,q}(]0,t[,E)}$. ☐

As a consequence of Poincaré inequality, lemma 4.2 holds true for $B^{\frac{1}{2}+\eta}_{p,q}(]0,T[,E)$:

Lemma 4.3. *Let $T > 0$, $1 \leq p,q \leq \infty$ and $0 < \eta \leq 1/2$ with $\omega(\eta,p) > 0$. There exists $C_T > 0$ such that, for any $0 < t < T$ and $u \in B^{\frac{1}{2}+\eta}_{p,q}(]-1,t[,E)$ with $u|_{]-1,0[} = 0$, we have:*

$$\|u\|_{B^{\frac{1}{2}+\eta}_{p,q}(]0,t[,E)} \leq \|u\|_{B^{\frac{1}{2}+\eta}_{p,q}(]-1,t[,E)} \leq C_T\|u'\|_{B^{-\frac{1}{2}+\eta}_{p,q}(]0,t[,E)} \leq C^*_T\|u\|_{B^{\frac{1}{2}+\eta}_{p,q}(]0,t[,E)} \tag{4.12}$$

Proof. We easily derive form corollary 4.1 b) that, for any $0 < t < T$, $\|u\|_{B^{\frac{1}{2}+\eta}_{p,q}(]-1,t[,E)} \leq C_T\|u'\|_{B^{-\frac{1}{2}+\eta}_{p,q}(]-1,t[,E)}$. Hence, inequality $\|u\|_{B^{\frac{1}{2}+\eta}_{p,q}(]-1,t[,E)} \leq C_T\|u'\|_{B^{-\frac{1}{2}+\eta}_{p,q}(]0,t[,E)}$ follows from $u'|_{]-1,0[} = 0$ and lemma 4.2. In order to prove the last inequality, let $z \in B^{\frac{1}{2}+\eta}_{p,q}(\mathbb{R},E)$ be any extension of u. Then, $z' \in B^{-\frac{1}{2}+\eta}_{p,q}(\mathbb{R},E)$ is an extension of u', and by continuity of the derivation $\|z'\|_{B^{-\frac{1}{2}+\eta}_{p,q}(\mathbb{R},E)} \leq C\|z\|_{B^{\frac{1}{2}+\eta}_{p,q}(\mathbb{R},E)}$. It follows that $\|u'\|_{B^{-\frac{1}{2}+\eta}_{p,q}(]0,t[,E)} \leq C\|z\|_{B^{\frac{1}{2}+\eta}_{p,q}(\mathbb{R},E)}$, and taking the inf on the above extensions z, we obtain $\|u'\|_{B^{-\frac{1}{2}+\eta}_{p,q}(]0,t[,E)} \leq C\|u\|_{B^{\frac{1}{2}+\eta}_{p,q}(]0,t[,E)}$. ☐

5 Framework in the linear case

We now turn to define κM. We use the Bony decomposition.

Lemma 5.1. *Assume that $1 \leq p,q \leq \infty$ and $0 < \alpha \leq \eta \leq 1/2$ with $\omega(\alpha,p) > 0$. Then, for any open interval I, and any $\kappa \in B^{-\frac{1}{2}+\eta}_{p,q}(I,\mathscr{L}(F,E))$, $M \in B^{\frac{1}{2}+\alpha}_{p,q}(I,F)$, we have:*

$$\|\kappa M\|_{B^{-\frac{1}{2}+\eta}_{p,q}(I,E)} \leq C\|\kappa\|_{B^{-\frac{1}{2}+\eta}_{p,q}(I,\mathscr{L}(F,E))}\|M\|_{B^{\frac{1}{2}+\alpha}_{p,q}(I,F)} \tag{5.1}$$

The constant $C > 0$ depends on η, α, p, q but not on I.

Proof. 1) *Case* $I = \mathbb{R}$. In the sequel, we often omit $\mathbb{R}$ and/or E in the notations. We start with the remainder term and distinguish two cases. In the case $1 \leq p \leq 2 \leq p'$, we have (see [4], p.104):

$$B^{-\frac{1}{2}+\eta}_{p,q} \times B^{\frac{1}{2}+\alpha}_{p,q} \hookrightarrow B^{-\frac{1}{2}+\eta}_{p,q} \times B^{\frac{1}{2}+\alpha-\frac{1}{p}+1-\frac{1}{p}}_{p',\infty} \xrightarrow{\mathfrak{R}} B^{\alpha+\eta+1-\frac{2}{p}}_{1,\infty} \hookrightarrow B^{-\frac{1}{2}+\eta}_{p,q}$$

The arrow $\xrightarrow{\mathfrak{R}}$ follows from $\alpha+\eta+1-\frac{2}{p} = \omega(\alpha,p)+\omega(\eta,p) > 0$. The last injection follows from:

$$\alpha+\eta+1-\frac{2}{p}+\frac{1}{p}-1 = \alpha-\frac{1}{p}+\eta > -\frac{1}{2}+\eta$$

In the case $1 \leq p' \leq 2 \leq p$, we have:

$$B^{-\frac{1}{2}+\eta}_{p,q} \times B^{\frac{1}{2}+\alpha}_{p,q} \xrightarrow{\mathfrak{R}} B^{\eta+\alpha}_{p/2,\infty} \hookrightarrow B^{\eta+\alpha-\frac{1}{p}}_{p,\infty} \hookrightarrow B^{-\frac{1}{2}+\eta}_{p,q}$$

The last embedding is a consequence of $\eta+\alpha-\frac{1}{\mathrm{p}}-\Big(-\frac{1}{2}+\eta\Big)=\omega(\alpha,p)>0$. We now deal with the paraproducts. We have (see [4], p.103):

$$\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\alpha}\times\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}\hookrightarrow\mathrm{L}^\infty\times\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}\overset{\Pi}{\to}\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}$$

$$\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}\times\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\alpha}\hookrightarrow\mathrm{B}_{\infty,\mathrm{q}}^{-\frac{1}{2}+\eta-\frac{1}{\mathrm{p}}}\times\mathrm{B}_{\mathrm{p},\infty}^{\frac{1}{2}+\alpha}\overset{\Pi}{\to}\mathrm{B}_{\mathrm{p,q}}^{\alpha+\eta-\frac{1}{\mathrm{p}}}\hookrightarrow\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}$$

The second arrow $\overset{\Pi}{\to}$ follows from $-\frac{1}{2}+\eta-\frac{1}{\mathrm{p}}<0$, and the last injection is a consequence of $-\frac{1}{\mathrm{p}}+\alpha>-\frac{1}{2}$. Thus, the product is continuous from $\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(\mathbb{R})\times\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\alpha}(\mathbb{R})$ to $\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(\mathbb{R})$.

2) *Case of an interval* I.

2) a) Assume $\mathrm{p}\neq\infty$, $\mathrm{q}\neq\infty$. For $(\kappa,\mathrm{M})\in\mathrm{C}^\infty(\bar{\mathrm{I}},\mathscr{L}(\mathrm{E},\mathrm{F}))\times\mathrm{C}^\infty(\bar{\mathrm{I}},\mathrm{F})$ the product $\kappa\mathrm{M}$ is well defined, and for any extensions $\kappa_*\in\mathscr{D}(\mathbb{R},\mathscr{L}(\mathrm{E},\mathrm{F}))$ and $\mathrm{M}_*\in\mathscr{D}(\mathbb{R},\mathrm{F})$ of κ and M respectively, the first case provides $\|\kappa\mathrm{M}\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(\mathrm{I,E})}\leq\|\kappa_*\mathrm{M}_*\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(\mathbb{R},\mathrm{E})}\leq\mathrm{C}\|\kappa_*\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(\mathbb{R},\mathscr{L}(\mathrm{F,E}))}\|\mathrm{M}_*\|_{\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\alpha}(\mathbb{R},\mathrm{F})}$. Taking the inf on the set of the above $\mathscr{D}(\mathbb{R})$ extensions, and using a density argument, we get 5.1. In the general case $(\kappa,\mathrm{M})\in\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(\mathrm{I},\mathscr{L}(\mathrm{E},\mathrm{F}))\times\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\alpha}(\mathrm{I},\mathrm{F})$, the definition of the product $\kappa\mathrm{M}$ and inequality 5.1 follow by a uniform continuity/density argument.

2)b) Assume $\mathrm{p}\neq\infty$, $\mathrm{q}\neq\infty$.

We write $\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(\mathrm{I},\mathscr{L}(\mathrm{E},\mathrm{F}))\hookrightarrow\mathrm{B}_{\mathrm{p_1},1}^{-\frac{1}{2}+\eta_1}(\mathrm{I},\mathscr{L}(\mathrm{E},\mathrm{F}))$ and $\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\alpha}(\mathrm{I},\mathrm{F})\hookrightarrow\mathrm{B}_{\mathrm{p_1},1}^{\frac{1}{2}+\alpha_1}(\mathrm{I},\mathrm{F})$ for some $0<\eta_1<\eta\leq 1/2$, $0<\alpha_1<\inf\{\eta_1,\alpha\}\leq 1/2$, $1\leq\mathrm{p_1}<\infty$ with $\omega(\alpha_1,\mathrm{p_1})>0$. Hence, $\kappa\mathrm{M}$ is defined as an element of $\mathrm{B}_{\mathrm{p_1},1}^{-\frac{1}{2}+\eta_1}(\mathrm{I},\mathrm{F})$ by the case 2)a), and inequality 5.1 follows from 1). □

We shall also need an integration by part formula:

Lemma 5.2. *Let* $\mathrm{I}=]\mathrm{a},\mathrm{b}[$ *be a bounded interval. Assume that* $1<\mathrm{p},\mathrm{q}<\infty$, $0<\eta\leq 1/2$, $\omega(\eta,p)>0$, *and let* $\mathrm{u}\in\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}(\mathrm{I},\mathrm{E})$, $\mathrm{v}\in\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}(\mathrm{I},\mathscr{L}(\mathrm{E},\mathrm{F}))$. *Then:*
1) $\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}(\mathrm{I},\mathrm{E})\hookrightarrow\mathrm{B}_{\mathrm{p',q'}}^{\frac{1}{2}-\eta}(\mathrm{I},\mathrm{E})$.
2) $<\mathrm{u}',\mathrm{v}>_{\eta,\mathrm{p,q}}=-<\mathrm{v}',\mathrm{u}>_{\eta,\mathrm{p,q}}+<\mathrm{v},\mathrm{u}>_{\mathscr{L}(\mathrm{E,F}),\mathrm{E}}(\mathrm{b})-<\mathrm{v},\mathrm{u}>_{\mathscr{L}(\mathrm{E,F}),\mathrm{E}}(\mathrm{a})$

Proof. 1) Note that:

$$\omega(\eta,\mathrm{p})+\frac{1}{\mathrm{p}'}=\frac{1}{2}-\eta+2\omega(\eta,\mathrm{p})>\frac{1}{2}-\eta$$

Hence, $\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}(\mathrm{I},\mathrm{E})\hookrightarrow\mathrm{B}_{\mathrm{p',q}}^{\omega(\eta,\mathrm{p})+\frac{1}{\mathrm{p}'}}(\mathrm{I},\mathrm{E})\hookrightarrow\mathrm{B}_{\mathrm{p',q'}}^{\frac{1}{2}-\eta}(\mathrm{I},\mathrm{E})$.

2) Let V be a Banach space. In the sequel, $\mathrm{V}=\mathrm{E}$ or $\mathscr{L}(\mathrm{E},\mathrm{F})$. We deduce from $\omega(\eta,\mathrm{p})>0$ and the first part of the lemma that:

$$\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}(\mathrm{I},\mathrm{V})\hookrightarrow\mathrm{C}(\mathrm{I},\mathrm{V})\cap\mathrm{B}_{\mathrm{p',q'}}^{\frac{1}{2}-\eta}(\mathrm{I},\mathrm{V})\cap\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(\mathrm{I},\mathrm{V})\tag{5.2}$$

Let $\Delta(\mathrm{u},\mathrm{v}):=<\mathrm{u}',\mathrm{v}>_{\eta,\mathrm{p,q}}+<\mathrm{v}',\mathrm{u}>_{\eta,\mathrm{p,q}}-<\mathrm{v},\mathrm{u}>_{\mathscr{L}(\mathrm{E,F}),\mathrm{E}}(\mathrm{b})+<\mathrm{v},\mathrm{u}>_{\mathscr{L}(\mathrm{E,F}),\mathrm{E}}(\mathrm{a})$, which is well defined and continuous on $\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}(\mathrm{I},\mathrm{E})\times\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}(\mathrm{I},\mathscr{L}(\mathrm{E},\mathrm{F}))$ due to 5.2. Since $\Delta=0$ on $\mathrm{C}^\infty(\mathrm{I},\mathrm{E})\times\mathrm{C}^\infty(\mathrm{I},\mathscr{L}(\mathrm{E},\mathrm{F}))$ and $\mathrm{C}^\infty(\mathrm{I},V)$ is dense in $\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}(\mathrm{I},\mathrm{V})$, the lemma follows. □

6 Solutions of a linear system of equations with coefficients with negative power of derivability.

We now prove the existence of solutions for a system with coefficients with negative power of derivability. Derivatives are taken in the distributionnal sense. In the sequel, we omit in the writing the dependence with respect to the initial time t_0 and write $\mathrm{M}(\mathrm{t})$ in place of $\mathrm{M}(\mathrm{t}, \mathrm{t}_0)$. Hence, in the following, $\mathrm{M}' \in \mathscr{D}'(]0, \mathrm{T}[, \mathrm{E})$. We first define weak solutions of the following system:

$$\begin{cases} \mathrm{M}' = \kappa \mathrm{M} + \phi \\ \mathrm{M}(\mathrm{t}_0) = \mathrm{M}_0 \end{cases} \tag{6.1}$$

Definition 6.1. *Let* $\mathrm{I} =]\mathrm{a}, \mathrm{b}[$ *be an open, bounded interval,* $0 < \eta \leq 1/2$, $1 < \mathrm{p} \leq \infty$, $1 \leq \mathrm{q} \leq \infty$ *with* $\frac{1}{2} + \eta - \frac{1}{\mathrm{p}} > 0$. *Let* $\mathrm{t}_0 \in \mathrm{I}$ *be fixed, and assume that* $(\kappa, \phi, \mathrm{M}_0) \in \mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(\mathrm{I}, \mathscr{L}(\mathrm{E})) \times \mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(\mathrm{I}, \mathrm{E}) \times \mathrm{E}$.
1) We say that M *is a weak* $\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}(\mathrm{I}, \mathrm{E})$ *solution of system* 6.1 *if* $\mathrm{M}(\mathrm{t}_0) = \mathrm{M}_0$ *and, for any* $\theta \in \mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}(\mathrm{I}, \mathscr{L}(\mathrm{E}))$:

$$- < \theta', \mathrm{M} >_{\eta,\mathrm{p,q}} + \theta(\mathrm{b})\mathrm{M}(\mathrm{b}) - \theta(\mathrm{a})\mathrm{M}(\mathrm{a}) = < \kappa \mathrm{M} + \phi, \theta >_{\eta,\mathrm{p,q}} \tag{6.2}$$

2) A distributional solution M *of system* 6.1 *is defined as in 1), except that we take* $\theta \in \mathscr{D}(\mathrm{I}, \mathscr{L}(\mathrm{E}))$ *in place of* $\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}(\mathrm{I}, \mathscr{L}(\mathrm{E}))$

A weak solution is also a distributional solution, and, for $\mathrm{p} \neq \infty \neq \mathrm{q}$, the converse is true by lemma 5.2. Observe also that these definitions extend in an obvious way to non linear equations. We shall use such extensions without any comment.

We now prove global well-posedness for the above linear system.

Theorem 6.1. *Let* $\mathrm{I} =]0, \mathrm{T}[$ $(\mathrm{T} > 0)$, $0 < \eta \leq 1/2$, $1 < \mathrm{p} \leq \infty$, $1 \leq \mathrm{q} \leq \infty$ *with* $\frac{1}{2} + \eta - \frac{1}{p} > 0$. *Let* $(\kappa, \phi, \mathrm{M}_0) \in \mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]0, \mathrm{T}[, \mathscr{L}(\mathrm{E})) \times \mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]0, \mathrm{T}[, \mathrm{E}) \times \mathrm{E}$ *and set* $\mathrm{t}_0 = 0$. *Then:*

1) Problem 6.1 *admits exactly one distributional solution.*

2) Denote by $s_{\mathrm{t}_0}(\kappa, \phi, , \mathrm{M}_0)$ *the above solution. Then, function*

$$s_{\mathrm{t}_0} : \mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]0, \mathrm{T}[, \mathscr{L}(\mathrm{E})) \times \mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]0, \mathrm{T}[, \mathrm{E}) \times \mathrm{E} \to \mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}(]0, \mathrm{T}[, \mathrm{E})$$

is locally Lipschitz continuous.

Proof. a) Existence of a solution.

We begin with an a priori estimate in $\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}(]-1, \mathrm{T}[, \mathrm{E})$. We first restrict to $1 < p < \infty$. Assume that $(\tilde{\kappa}, \tilde{\phi}) \in \mathrm{C}^\infty([-1, \mathrm{T}], \mathscr{L}(\mathrm{E})) \times \mathrm{C}^\infty([-1, \mathrm{T}], \mathrm{E})$ with support in $[0, \mathrm{T}]$, and $\mathrm{M}_0 \in \mathrm{E}$, are given. Let $\tilde{\mathrm{M}} \in \mathrm{C}^\infty([-1, \mathrm{T}], \mathrm{E})$ be the unique solution of system:

$$\begin{cases} \tilde{\mathrm{M}}' = \tilde{\kappa} \tilde{\mathrm{M}} + \tilde{\phi} \\ \tilde{\mathrm{M}}(-1) = \mathrm{M}_0 \end{cases} \tag{6.3}$$

For another $(\tilde{\mu}, \tilde{\psi}) \in \mathrm{C}^\infty([-1, \mathrm{T}], \mathscr{L}(\mathrm{E})) \times \mathrm{C}^\infty([-1, \mathrm{T}], \mathrm{E})$ with support in $[0, \mathrm{T}]$ and $\mathrm{N}_0 \in \mathrm{E}$, we can define in a same manner a function $\tilde{\mathrm{N}} \in \mathrm{C}^\infty([-1, \mathrm{T}], \mathrm{E})$. Since we argue locally, we assume that all the norms of the data $\|\tilde{\kappa}\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]-1,\mathrm{T}[)}$, $\|\tilde{\phi}\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]-1,\mathrm{T}[)}$ ..., $\|\mathrm{N}_0\|_{\mathrm{E}}$ are bounded by some $\mathrm{R} > 0$. Set $\mathrm{Z} = \tilde{\mathrm{N}} - \tilde{\mathrm{M}}$ and $\Phi = (\tilde{\mu} - \tilde{\kappa})\tilde{\mathrm{N}} + (\tilde{\psi} - \tilde{\phi})$. Due to lemma 5.1:

$$\|\Phi\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]-1,\mathrm{t}[)} \leq \mathrm{C_T}\|\tilde{\mathrm{N}}\|_{\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}(]-1,\mathrm{t}[)}\|\tilde{\mu} - \tilde{\kappa}\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]-1,\mathrm{t}[)} + \|\tilde{\psi} - \tilde{\phi}\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]-1,\mathrm{t}[)} \tag{6.4}$$

Function Z satisfies equations $\mathrm{Z}' = \tilde{\kappa}\mathrm{Z} + \Phi$ and $\mathrm{Z}(-1) = \mathrm{N}_0 - \mathrm{M}_0$. Hence, for $0 \leq t \leq \mathrm{T}$:

$$\|\mathrm{Z}\|_{\mathrm{L}^p(]-1,\mathrm{t}[)} + \|\mathrm{Z}'\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]-1,\mathrm{t}[)}$$

$$\leq \|Z\|_{L^p(]-1,t[)} + C_T\|\tilde\kappa\|_{B_{p,q}^{-\frac12+\eta}(]-1,t[)}\|Z\|_{B_{p,q}^{\frac12+\alpha}(]-1,t[)} + \|\Phi\|_{B_{p,q}^{-\frac12+\eta}(]-1,t[)} \tag{6.5}$$

with $0<\alpha<\eta$, see lemma 5.1. Due to 6.5, 4.6 and 4.7, we have:

$$\begin{aligned}\gamma_T\|Z\|_{B_{p,q}^{\frac12+\eta}(]-1,t[)} &\leq \|Z\|_{L^p(]-1,t[)} + \|Z'\|_{B_{p,q}^{-\frac12+\eta}(]-1,t[)}\\ \leq &\|Z\|_{L^p(]-1,t[)} + C_T\|\tilde\kappa\|_{B_{p,q}^{-\frac12+\eta}(]-1,t[)}\big(\epsilon\|Z\|_{B_{p,q}^{\frac12+\eta}(]-1,t[)} + C_\epsilon\|Z\|_{L^p(]-1,t[)}\big) + \|\Phi\|_{B_{p,q}^{-\frac12+\eta}(]-1,t[)}\end{aligned}$$

with $\gamma_T>0$, $\epsilon=\gamma_T/(2C_TR)$ and $C_\epsilon>0$ choosen accordingly. Therefore, by definition of ϵ and R, we have:

$$\|Z\|_{B_{p,q}^{\frac12+\eta}(]-1,t[)} \leq C_{T,R}(\|Z\|_{L^p(]-1,t[)} + \|\Phi\|_{B_{p,q}^{-\frac12+\eta}(]-1,t[)}) \tag{6.6}$$

and, due to inequality 6.4 and 4.9 (recall that $\omega(\eta,p)>0$):

$$\|Z(t)\|_E \leq C_{T,R}\Big\{\|Z\|_{L^p(]-1,t[)} + \|\tilde N\|_{B_{p,q}^{\frac12+\eta}(]-1,t[)}\|\tilde\mu-\tilde\kappa\|_{B_{p,q}^{-\frac12+\eta}(]-1,t[)} + \|\tilde\psi-\tilde\phi\|_{B_{p,q}^{-\frac12+\eta}(]-1,t[)}\Big\} \tag{6.7}$$

Set $y(t)=\int_{-1}^t\|Z(\tau)\|_E^p d\tau$. Inequality 6.7 provides:

$$y'(t) \leq C_{T,R}y(t) + C_{T,R}\big(\|\tilde N\|_{B_{p,q}^{\frac12+\eta}(]-1,t[)}\|\tilde\mu-\tilde\kappa\|_{B_{p,q}^{-\frac12+\eta}(]-1,t[)} + \|\tilde\psi-\tilde\phi\|_{B_{p,q}^{-\frac12+\eta}(]-1,t[)}\big)^p \tag{6.8}$$

Inequality 6.8 holds true for $t\in[0,T]$. As $\tilde\kappa$, $\tilde\mu$, $\tilde\phi$ and $\tilde\psi$ are equal to zero on $[-1,0]$, we have $y(0)=\|N_0-M_0\|_E^p$. We deduce from Gronwall lemma that:

$$\begin{aligned}y(t) \leq \big[\|N_0-M_0\|_E^p + t(\|\tilde N\|_{B_{p,q}^{\frac12+\eta}(]-1,T[)}&\|\tilde\mu-\tilde\kappa\|_{B_{p,q}^{-\frac12+\eta}(]-1,T[)}\\ &+\|\tilde\psi-\tilde\phi\|_{B_{p,q}^{-\frac12+\eta}(]-1,T[)})^p\big]e^{C_{T,R}t}\end{aligned} \tag{6.9}$$

Finally, due to inequalities 6.4 and 6.6 and 6.9 we get:

$$\begin{aligned}\|\tilde N-\tilde M\|_{B_{p,q}^{\frac12+\eta}(]-1,T[)} \leq C_{T,R}\big[&\|N_0-M_0\|_E\\ &+\|\tilde N\|_{B_{p,q}^{\frac12+\eta}(]-1,T[)}\|\tilde\mu-\tilde\kappa\|_{B_{p,q}^{-\frac12+\eta}(]-1,T[)} + \|\tilde\psi-\tilde\phi\|_{B_{p,q}^{-\frac12+\eta}(]-1,T[)}\big]\end{aligned} \tag{6.10}$$

For $(\tilde\mu,\tilde\psi)=(0,0)$ and $N_0=0$, we obtain:

$$\|\tilde M\|_{B_{p,q}^{\frac12+\eta}(]-1,T[)} \leq A_{T,R} \tag{6.11}$$

We come back to the case of a general bounded set of data $(\tilde\mu,\tilde\psi,N_0)$. Using 6.11, with $\tilde N$ in place of $\tilde M$ in 6.10, we get:

$$\|\tilde N-\tilde M\|_{B_{p,q}^{\frac12+\eta}(]-1,T[)} \leq C_{T,R}\big[\|N_0-M_0\|_E + \|\tilde\mu-\tilde\kappa\|_{B_{p,q}^{-\frac12+\eta}(]-1,T[)} + \|\tilde\psi-\tilde\phi\|_{B_{p,q}^{-\frac12+\eta}(]-1,T[)}\big] \tag{6.12}$$

which is the required estimate. Assume now that $(\kappa,\mu)\in\mathscr{D}\big(]0,T[,\mathscr{L}(E)\big)^2$ and $(\phi,\psi)\in\mathscr{D}(]0,T[,E)^2$ with norms $\|\kappa\|_{B_{p,q}^{-\frac12+\eta}(]0,T[)}$, $\|\mu\|_{B_{p,q}^{-\frac12+\eta}(]0,T[)}$, $\|\phi\|_{B_{p,q}^{-\frac12+\eta}(]0,T[)}$, $\|\psi\|_{B_{p,q}^{-\frac12+\eta}(]0,T[)}$, bounded by $R>0$. These functions are restriction of functions $(\tilde\kappa,\tilde\mu)\in\mathscr{D}\big(]-1,T[,\mathscr{L}(E)\big)^2$ and $(\tilde\phi,\tilde\psi)\in\mathscr{D}(]-1,T[,E)^2$, to which we apply the above computations. Due to $\|\tilde N-\tilde M\|_{B_{p,q}^{\frac12+\eta}(]0,T[)} \leq \|\tilde N-\tilde M\|_{B_{p,q}^{\frac12+\eta}(]-1,T[)}$, estimate 6.12, lemma 4.2 (or simply the continuity of the zero-extension operator $B_{p,q}^{-\frac12+\eta}(]0,T[)\to B_{p,q}^{-\frac12+\eta}(]-1,T[)$) and definition of the tilde functions, we get:

$$\|\tilde N-\tilde M\|_{B_{p,q}^{\frac12+\eta}(]0,T[)} \leq C_{T,R}\big[\|N_0-M_0\|_E + \|\mu-\kappa\|_{B_{p,q}^{-\frac12+\eta}(]0,T[)} + \|\psi-\phi\|_{B_{p,q}^{-\frac12+\eta}(]0,T[)}\big] \tag{6.13}$$

Appealing to 6.3, we also have on $]0,\mathrm{T}[$:

$$\begin{cases} \tilde{\mathrm{M}}' = \kappa\tilde{\mathrm{M}} + \phi \\ \tilde{\mathrm{M}}(0) = \mathrm{M}_0 \end{cases} \tag{6.14}$$

and similarly for $\tilde{\mathrm{N}}$. Now, the existence part of the theorem, as well of the local Lipschitz continuity with respect to the data follow readily from the classical C^∞ theory, the density of $\mathscr{D}(]0,\mathrm{T}[,\mathrm{V})$ ($\mathrm{V} = \mathrm{E}$ or $\mathscr{L}(\mathrm{E})$) in $\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]0,\mathrm{T}[,\mathrm{V})$ for $\frac{1}{p} - 1 < -\frac{1}{2} + \eta < \frac{1}{p}$, 6.14 and 6.13.

b) Uniqueness of a solution.

We first restrict to $1 < p < \infty$ and $1 \leq q < \infty$. Let $\kappa \in \mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]0,\mathrm{T}[,\mathscr{L}(\mathrm{E}))$ be fixed. We prove by duality that a solution $\mathrm{M} \in \mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}(]0,\mathrm{T}[,\mathrm{E})$ of 6.1 with $\phi = 0$ and $\mathrm{M}_0 = 0$ is equal to zero. Let $\theta \in \mathscr{D}(]0,\mathrm{T}[,\mathrm{E}^*)$, and let $\mathrm{N} \in \mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}(]0,\mathrm{T}[,\mathrm{E}^*)$ be a solution of:

$$\begin{cases} \mathrm{N}' = -\kappa^*\mathrm{N} + \theta \\ \mathrm{N}(\mathrm{T}) = 0 \end{cases} \tag{6.15}$$

where, $\kappa^* \in \mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]0,\mathrm{T}[,\mathscr{L}(\mathrm{E}^*))$ is obtained from κ by adjonction. The existence of N follows from a). Using lemma 5.2, we get:

$$\begin{aligned} <\theta,\mathrm{M}>_{\eta,\mathrm{p,q}} &= <\mathrm{N}' + \kappa^*\mathrm{N},\mathrm{M}>_{\eta,\mathrm{p,q}} \\ &= <-\mathrm{M}',\mathrm{N}>_{\eta,\mathrm{p,q}} + <\kappa\mathrm{M},\mathrm{N}>_{\eta,\mathrm{p,q}} \\ &\qquad + <\mathrm{N}(\mathrm{T}),\mathrm{M}(\mathrm{T})>_{\mathrm{E}^*,\mathrm{E}} - <\mathrm{N}(0),\mathrm{M}(0)>_{\mathrm{E}^*,\mathrm{E}} \end{aligned} \tag{6.16}$$

Since $\mathrm{N}(\mathrm{T}) = \mathrm{M}(0) = 0$, we obtain:

$$<\theta,\mathrm{M}>_{\eta,\mathrm{p,q}} = <-\mathrm{M}' + \kappa,\mathrm{N}>_{\eta,\mathrm{p,q}} = 0 \tag{6.17}$$

Therefore, $\mathrm{M} = 0$, which proves the uniqueness.

c) Uniqueness and existence in the cases $p = \infty$ or $q = \infty$.

In the case $\mathrm{p} = \infty$ or $\mathrm{q} = \infty$, let $0 < \rho < 1$ and $1 < \gamma < \mathrm{p}$. Since $\omega(\eta,\mathrm{p}) > 0$, we can assume that $\omega(\rho\eta,\gamma) > 0$. Let also $0 < \rho < \rho' < 1$ with

$$\rho' > \rho + \frac{1}{\eta}(\frac{1}{\gamma} - \frac{1}{\mathrm{p}})$$

. It follows from these definitions that

$$\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]0,\mathrm{T}[,\mathrm{F}) \hookrightarrow \mathrm{B}_{\gamma,\gamma}^{-\frac{1}{2}+\rho'\eta}(]0,\mathrm{T}[,\mathrm{F}) \hookrightarrow \mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\rho\eta}(]0,\mathrm{T}[,\mathrm{F})$$

For such (ρ',γ), problem 6.1 admits a unique solution in $\mathrm{B}_{\gamma,\gamma}^{\frac{1}{2}+\rho'\eta}$, hence in the smaller space $\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}$. In order to prove the existence in $\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}$, let $\mathrm{M} \in \mathrm{B}_{\gamma,\gamma}^{\frac{1}{2}+\rho'\eta}$ be the solution of 6.1. We have $\mathrm{M}' = \kappa\mathrm{M} + \phi$, which belongs to $\mathrm{B}_{\gamma,\gamma}^{-\frac{1}{2}+\rho'\eta} \hookrightarrow \mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\rho\eta}$. In consequence, M belongs to $\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\rho\eta}$. Appealing once again to $\mathrm{M}' = \kappa\mathrm{M}+\phi$, $\kappa \in \mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}$ and lemma 5.1, we obtain $\mathrm{M} \in \mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}$, whence the existence. The proof of continuity with respect to the data is omitted. □

Appealing to standard argument, we also have:

Corollary 6.1. *Theorem* 6.1 *holds true for any* $\mathrm{t}_0 \in [0,\mathrm{T}]$.

7 Critical spaces.

We now turn to discuss a few critical cases that are not covered by theorem 6.1. For that purpose, we prove well-posedness when κ belongs to some set of smooth enough measures. We restrict to initial times $t_0 = 0$.

Let $T > 0$. Denote by $\mathscr{B}(0,T)$ the class of Borel sets in $[0,T]$ and let $\mathfrak{M}\big(\mathscr{B}(0,T),\mathscr{L}(E)\big)$ the space of all bounded countably additive (with respect to $\mathscr{B}(0,T)$) measures with value in $\mathscr{L}(E)$. For $\mu \in \mathfrak{M}\big(\mathscr{B}(0,T),\mathscr{L}(E)\big)$, we can define its variation $|\mu| \in \mathfrak{M}\big(\mathscr{B}(0,T),\mathbb{R}_+\big)$ by

$$|\mu|(A) = \sup\big(\Sigma\|\mu(A_i)\|_{\mathscr{L}(E)}\big) \tag{7.1}$$

the sup being being taken on all the denumerable partition of $A \in \mathscr{L}(E)$, i.e $A = \cup_{i\in\mathbb{N}} A_i$ with $A_i \in \mathscr{B}(0,T)$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. We say that $\mu \in \mathfrak{M}_\mathfrak{c}\big(\mathscr{B}(0,T),\mathscr{L}(E)\big)$ - or that μ is continuous - if the function

$$\begin{cases} [0,T[& \to \mathbb{R}_+ \\ \quad t & \mapsto |\mu|([0,t[) \end{cases} \tag{7.2}$$

is continuous. For such measures, and for $f \in C^0([0,T[,E)$, $0 \le t \le T$, the integral $\int_0^t f d\mu$ can be defined by elementary means (Riemann sums). Moreover, $\| \int_t^{t'} f d\mu\|_E \le \|f\|_{C^0([t,t'[,E)}|\mu|([t,t'])$ for any $0 \le t \le t' \le T$. With the above notations, we have the following simple

Theorem 7.1. *Let* $T > 0$, $M_0 \in E$ *and let* $\mu \in \mathfrak{M}_\mathfrak{c}\big(\mathscr{B}(0,T),\mathscr{L}(E)\big)$. *Let also* $\phi \in \mathfrak{M}_\mathfrak{c}\big(\mathscr{B}(0,T),E\big)$. *Then, the problem: find* $M \in C^0([0,T],E)$ *such that, for any* $0 \le t \le T$

$$M(t) = M_0 + \int_0^t M d\mu + \phi(]0,t[) \tag{7.3}$$

admits exactly one solution.

Proof. An application of Picard fixed point theorem in $C^0([0,T[,E)$. For any $0 \le \tau \le T$, let $\Lambda_\tau : C^0([0,\tau],E) \to C^0([0,\tau],E)$ be defined by $\Lambda_\tau(M)(t) = M_0 + \phi(]0,t[) + \int_0^t M d\mu$ for any $0 \le t \le \tau$.

a) Stability. For any $M \in C^0([0,\tau],E)$ and any $(t_1,t) \in [0,\tau]^2$ (for instance $t_1 \le t$) , we have: $\|\Lambda_\tau(M)(t) - \Lambda_\tau(M)(t_1)\|_E \le \|M\|_{C^0([0,\tau],E)}|\mu|([t_1,t]) + |\phi|([t_1,t]) \to 0$ as $t \to t_1$, due to the continuity of $|\mu|$ and $|\phi|$. Hence, $\Lambda_\tau(M) \in C^0([0,\tau],E)$

b) Contraction for $0 < \tau \le T$ small enough. For any $(M,N) \in C^0([0,\tau],E)^2$ and any $t \in [0,\tau]$, we have: $\|\Lambda_\tau(M)(t) - \Lambda_\tau(N)(t)\|_E \le \|M - N\|_{C^0([0,\tau],E}|\mu|([0,\tau])$. By continuity of μ, we can choose $\tau > 0$ such that $|\mu|([0,\tau]) \le 1/2$, i.e such that Λ_τ is 1/2-Lipschitz. It follows from a), b) an Picard fixed point theorem that the problem 7.3 is well posed for such small $\tau = \tau_* > 0$. The proof of the global existence is omitted. $\square$

We now apply theorem 7.1 to some critical cases excluded by theorem 6.1. Note the following diagram of critical κ-besov spaces $B_{p,q}^{-\frac{1}{2}+\eta}(]0,T[,\mathscr{L}(E))$ with $\omega(\eta,p) = 0$ or/and $\eta = 0$

$$\begin{array}{ccccc}
B^0_{1,1} \hookrightarrow ... & \hookrightarrow B^{\frac{1}{p}-1}_{p,1} \hookrightarrow ... & \hookrightarrow B^{-\frac{1}{2}}_{2,1} \hookleftarrow ... & \hookleftarrow B^{-\frac{1}{2}}_{r,1} \hookleftarrow ... & \hookleftarrow B^{-\frac{1}{2}}_{\infty,1} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
B^0_{1,q} \hookrightarrow ... & \hookrightarrow B^{\frac{1}{p}-1}_{p,q} \hookrightarrow ... & \hookrightarrow B^{-\frac{1}{2}}_{2,q} \hookleftarrow ... & \hookleftarrow B^{-\frac{1}{2}}_{r,q} \hookleftarrow ... & \hookleftarrow B^{-\frac{1}{2}}_{\infty,q} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
B^0_{1,\infty} \hookrightarrow ... & \hookrightarrow B^{\frac{1}{p}-1}_{p,\infty} \hookrightarrow ... & \hookrightarrow B^{-\frac{1}{2}}_{2,\infty} \hookleftarrow ... & \hookleftarrow B^{-\frac{1}{2}}_{r,\infty} \hookleftarrow ... & \hookleftarrow B^{-\frac{1}{2}}_{\infty,\infty}
\end{array}$$

$$ \tag{7.4} $$

where vertical arrows also denote continuous embeddings, and $p \le 2 \le r$. We shall only deal with the cases $B^0_{1,1}(]0,T[,\mathscr{L}(E))$ and $B^0_{1,\infty}(]0,T[,\mathscr{L}(E))$.

a) Case $\kappa \in B^0_{1,1}(]0,T[,\mathscr{L}(E))$. One can assume that $\phi = 0$, since it plays no significant role in the sequel. Our goal is to prove the existence of exactly one solution M of 6.1 in the space $B^1_{1,1}(]0,T[,E)$.

Notice that $B^0_{1,1}b(]0,T[,\mathscr{L}(E)) \hookrightarrow L^1(]0,T[,\mathscr{L}(E))$. Consequently, looking for $M \in C^0([0,T],E)$ solution of 7.3 is equivalent to looking for $M \in C^0([0,T],E)$ solution of 6.1 in the distributional sense. Hence, theorem 7.1 gives the existence of exactly one solution M to 6.1 in $C^0([0,T],E)$. This ensures uniqueness in the smaller space $B^1_{1,1}(]0,T[,E)$. In order to get the existence in the same space, we appeal to equation 6.1 and obtain that $M \in W^{1,1}(]0,T[)$. By Bernstein inequalities, $W^{1,1}(]0,T[) \hookrightarrow B^1_{1,\infty}(]0,T[) \cap L^\infty(]0,T[)$. It follows that κM is an element of $B^0_{1,1}(]0,T[,E)$. Indeed, using the paraproduct and remainder for spaces with domains equal to $\mathbb{R}$

$$
\begin{aligned}
&B^0_{1,1} \times W^{1,1} \hookrightarrow B^0_{1,1} \times B^1_{1,\infty} \hookrightarrow B^{-1}_{\infty,1} \times B^1_{1,\infty} \xrightarrow{\Pi} B^0_{1,1}\\
&W^{1,1} \times B^0_{1,1} \hookrightarrow L^\infty \times B^0_{1,1} \xrightarrow{\Pi} B^0_{1,1}\\
&W^{1,1} \times B^0_{1,1} \hookrightarrow B^1_{1,\infty} \times B^0_{1,1} \hookrightarrow B^0_{\infty,\infty} \times B^0_{1,1} \xrightarrow{\mathfrak{R}} B^0_{1,1}
\end{aligned}
\tag{7.5}
$$

Therefore, the product is continuous from $B^0_{1,1}(\mathbb{R}) \times W^{1,1}(\mathbb{R})$ to $B^0_{1,1}(\mathbb{R})$, and by usual arguments from $B^0_{1,1}(]0,T[) \times W^{1,1}(]0,T[)$ to $B^0_{1,1}(]0,T[)$. Finally, since $\kappa M \in B^0_{1,1}(]0,T[)$ equation 6.1 ensures that $M \in B^1_{1,1}(]0,T[)$. The problem 6.1 is well posed (existence and uniqueness) in $B^1_{1,1}(]0,T[,E)$.

b) Case $\kappa \in B^0_{1,\infty}(]0,T[,\mathbb{R})$. We assume that $\phi = 0$ and $T = 1$. We are looking for solutions M of 6.1 ibn $C^0([0,1],\mathbb{R})$ and give two examples.

- Let $\kappa = \delta_{1/2} \in B^0_{1,\infty}(]0,1[,\mathbb{R})$ (Dirac measure at 1/2).

As readily checked, necessarily $M = M_0$, which is not a solution. Hence, equations 6.1 are ill-posed within the above functional frame.

However, this may only be one aspect of the problem, and it is not clear that Besov spaces are well suited when dealing with critical cases. Actually, there exist irregular $\kappa \in B^0_{1,\infty}(]0,1[,\mathbb{R})$ for which existence of solutions of 6.1 in $C^0([0,1],\mathbb{R})$ can be proved.b As an example, we consider the case of $\kappa =$ derivative of the (ternary) Cantor function.

- Let $\kappa = \mu \in \mathfrak{M}_{\mathfrak{c}}\big(\mathscr{B}(0,1),\mathbb{R}\big)$ be the derivative of the Cantor function.

We only prove the existence of a solution to problem 6.1.

We first show that μ belongs to $B^0_{1,\infty}(]0,1[,\mathbb{R})$. Notice that μ can be written as the limit in $\mathscr{D}'(]0,1[,\mathbb{R})$ of functions $F_n|_{]0,1[}$ with

$$
F_n = (\frac{3}{2})^n \sum_{1\le k\le 2^n} \chi_{E_{n,k}}
\tag{7.6}
$$

Here, the sets $E_{n,k}$ are disjoint subintervals of $[0,1]$ with the same length $(\frac{1}{3})^n$ and $\chi_{E_{n,k}} : \mathbb{R} \to \mathbb{R}$ is the characteristic function of interval $E_{n,k}$. Appealing to inequality 3.13 with $m = 1$, $\epsilon = 1/m$ and $t = (1/3)^n$, we obtain

$$
\|F_n\|_{B^0_{1,\infty}(\mathbb{R})} \le (\frac{3}{2})^n \sum_{1\le k\le 2^n} \|\chi_{E_{n,k}}\|_{B^0_{1,\infty}(\mathbb{R})}
$$

$$\leq C(\frac{3}{2})^{n} \sum_{1\leq k\leq 2^{n}} \|\chi_{[0,1]}\|_{B^{1}_{1,\infty}(\mathbb{R})}(\frac{1}{3})^{n} \leq C \tag{7.7}$$

Notice that $B^0_{1,\infty}(\mathbb{R})$ can be identified with the dual space of $\overset{o}{B}{}^0_{\infty,1}(\mathbb{R})$, the completion of $\mathscr{S}(\mathbb{R})$ in $B^0_{\infty,1}(\mathbb{R})$ (see [26] p. 180). Extracting if necessary a subsequence, we infer from estimate 7.7 that $(F_n)_{n\in\mathbb{N}}$ converges weakly-* to some $F \in B^0_{1,\infty}(\mathbb{R})$. Consequently, $f_n = F_n|_{]0,1[} \to F|_{]0,1[}$ in $\mathscr{D}'(]0,1[,\mathbb{R})$. Therefore, $\mu = F|_{]0,1[} \in B^0_{1,\infty}(]0,1[)$. Moreover, $\mu \notin B^{-\frac{1}{2}+\eta}_{p,q}(]0,1[)$ with $\omega(\eta,p) > 0$ and $0 < \eta \leq 1/2$. This follows from the fact that, for such η, p, $B^{\frac{1}{2}+\eta}_{p,q}(]0,1[) \hookrightarrow B^{\omega(\eta,p)}_{\infty,\infty}(]0,1[)$, a Zygmund space, and that the Cantor function is not Holder continuous.

Observe now that $\mu \in \mathfrak{M}_{\mathfrak{c}}\big(\mathscr{B}(0,T),\mathscr{L}(E)\big)$. This implies the existence of exactly one solution M_* in $C^0([0,1],\mathbb{R})$ of equation 7.3 (see theorem 7.1). Invoking Fubini's theorem, we check that for any $\phi \in \mathscr{D}(]0,1[)$, we have

$$< \int_0^t M_* d\mu, \phi' >= - < M_*\mu, \phi >$$

Thus, $M_* \in C^0([0,1],\mathbb{R})$ is also a distributional solution of equation 6.1 for $\kappa = \mu$, the product $M_*\mu$ being directly defined in the sense of the measure theory. In contrast with the case $\kappa \in B^0_{1,1}(]0,T[,E)$, it seems that no smoothness on M_* can be recovered from equation 6.1 by the usual bootstrap procedure. For instance, starting from $M_* \in C^0([0,1],\mathbb{R}) \subset L^\infty([0,1]) \subset B^0_{\infty,\infty}(]0,1[)$ and $\kappa \in B^0_{1,\infty}(]0,1[)$, one obtains by usual means that $\kappa M_* \in B^{-1}_{\infty,\infty}(]0,1[)$ whereas $M_* \in B^0_{\infty,\infty}(]0,1[)$ is already known.

Finally, as easily checked by density, the usual representation formula $M_*(t) = M_0 \exp\big(\mu([0,t])\big)$ holds true.

8 Properties of the resolvent.

We now adress the case of $\mathcal{A}$-valued distributions, where $\mathcal{A}$ is a complex Banach algebra. We assume that $\mathcal{A}$ is endowed with a unit element $\mathbf{1}_{\mathcal{A}}$ and denote by $\mathcal{A}^\times$ the group of invertible elements of $\mathcal{A}$. As customary, we define the continuous morphism $\mathcal{A} \xrightarrow{r} \mathscr{L}(\mathcal{A})$ by $r(x).y = y.x$, $(x,y) \in \mathcal{A}^2$. A similar notation $\mathcal{A} \xrightarrow{l} \mathscr{L}(\mathcal{A})$ holds for the left multiplication. Last, recall that for $\kappa \in \mathscr{D}'(]0,T[,\mathcal{A}) = \mathscr{L}\big(\mathscr{D}(]0,T[),\mathcal{A}\big)$, we can define $r_*(\kappa) \in \mathscr{L}\big(\mathscr{D}(]0,T[),\mathscr{L}(\mathcal{A})\big) = \mathscr{D}'(]0,T[,\mathscr{L}(\mathcal{A}))$ by $r_*(\kappa) := ro\kappa$. For future reference, notice that if $\kappa \in B^{-\frac{1}{2}+\eta}_{p,q}(]0,T[,\mathcal{A})$, then $r_*(\kappa) \in B^{-\frac{1}{2}+\eta}_{p,q}(]0,T[,\mathscr{L}(\mathcal{A}))$. This follows from r Lipschitz. Denoting by R the resolvent associated with system 6.1,our first task is to prove that function $R(.,t_0)^{-1}$ is well defined and depends continuously on κ. The proof relies on theorem 6.1 and the fact that, for $\omega(\eta,p) > 0$, $B^{\frac{1}{2}+\eta}_{p,q}(]0,T[,\mathcal{A})$ is an algebra. In the following, for t_0 given, $R'(.,t_0)$ still denotes the distributional derivative of $R(.,t_0)$.

Proposition 8.1. *Let* $T > 0$, $0 < \eta \leq 1/2$, $1 < p \leq \gamma \leq \infty$, $1 \leq q \leq \infty$ *with* $\frac{1}{2} + \eta - \frac{1}{p} > 0$. *Let* $t_0 \in [0,T]$ *be fixed, and assume that* $\kappa \in B^{-\frac{1}{2}+\eta}_{p,q}(]0,T[,\mathcal{A})$. *Then:*
1) The problem:
find $R(.,t_0) \in B^{\frac{1}{2}+\eta}_{p,q}(]0,T[,\mathcal{A})$ *such that:*

$$\begin{cases} R'(.,t_0) = \kappa R(.,t_0) \\ R(t_0,t_0) = \mathbf{1}_{\mathcal{A}} \end{cases} \tag{8.1}$$

admits exactly one distributional solution. Moreover:
2) For any $(t,\tau,t_0) \in [0,T]^3$, $R(t,\tau)R(\tau,t_0) = R(t,t_0)$.

3) $\mathrm{R}(.,\mathrm{t}_0) \in \mathrm{B}^{\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[,\mathcal{A}^\times)$.
4) The mapping:

$$\begin{cases} \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[,\mathcal{A}) & \to \mathrm{B}^{\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[,\mathcal{A}^\times) \\ \kappa & \mapsto \mathrm{R}(.,\mathrm{t}_0)^{-1} \end{cases} \tag{8.2}$$

is well defined and continuous.
5) $\mathrm{R} \in \mathrm{B}^{\frac{1}{2}+\eta-\frac{1}{p}+\frac{1}{\gamma}}_{\gamma,q}(]0,\mathrm{T}[^2,\mathcal{A}^\times)$

Proof. 1) We know that $l_*(\kappa) \in \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[,\mathscr{L}(\mathcal{A}))$. Therefore, we can apply theorem 6.1 with $\mathrm{E} = \mathcal{A}$ and 1) follows.
2) For any $(\mathrm{t},\tau,\mathrm{t}_0) \in [0,\mathrm{T}]^3$, set $\mathrm{S}_{\tau,\mathrm{t}_0}(\mathrm{t}) = \mathrm{R}(\mathrm{t},\tau)\mathrm{R}(\tau,\mathrm{t}_0)$. Function $\mathrm{S}_{\tau,\mathrm{t}_0}$ satisfies 6.1 with $\phi = 0$ and $\mathrm{S}_{\tau,\mathrm{t}_0}(\tau) = \mathrm{R}(\tau,\mathrm{t}_0)$. By uniqueness, we have $\mathrm{S}_{\tau,\mathrm{t}_0}(\mathrm{t}) = \mathrm{R}(\mathrm{t},\mathrm{t}_0)$, which is 2).
3) Consequence of 1) and 2).
4) We first restrict to the case $1 \le \mathrm{p,q} < \infty$. Appealing to $r_*(\kappa) \in \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[,\mathscr{L}(\mathcal{A}))$ and to theorem 6.1, we can define $\mathrm{L}_{\mathrm{t}_0,\kappa} \in \mathrm{B}^{\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[,\mathcal{A})$, solution of problem 6.1 with $-r_*(\kappa)$ in place of κ, i.e $\mathrm{L}'_{\mathrm{t}_0,\kappa} = -\mathrm{L}_{\mathrm{t}_0,\kappa}\kappa$ and $\mathrm{L}_{\mathrm{t}_0,\kappa}(\mathrm{t}_0) = \mathbf{1}_{\mathcal{A}}$. Assume first that $\kappa \in \mathrm{C}^\infty([0,\mathrm{T}],\mathcal{A})$. We get $\mathrm{L}_{\mathrm{t}_0,\kappa} \in \mathrm{C}^\infty([0,\mathrm{T}],\mathcal{A})$ and $\mathrm{R}(.,\mathrm{t}_0) \in \mathrm{C}^\infty([0,\mathrm{T}],\mathcal{A})$. Due to definitions of $\mathrm{L}_{\mathrm{t}_0,\kappa}$ and $\mathrm{R}(.,\mathrm{t}_0)$, we have $\Big(\mathrm{L}_{\mathrm{t}_0,\kappa}\mathrm{R}(.,\mathrm{t}_0)\Big)' = \mathrm{L}'_{\mathrm{t}_0,\kappa}\mathrm{R}(.,\mathrm{t}_0) + \mathrm{L}_{\mathrm{t}_0,\kappa}\mathrm{R}'(.,\mathrm{t}_0) = 0$ and also $\mathrm{L}_{\mathrm{t}_0,\kappa}\mathrm{R}(.,\mathrm{t}_0)(0) = \mathbf{1}_{\mathcal{A}}$. Therefore:

$$\mathrm{L}_{\mathrm{t}_0,\kappa}\mathrm{R}(.,\mathrm{t}_0) = \mathbf{1}_{\mathcal{A}}. \tag{8.3}$$

Next, denote momentarily $\mathrm{R}(.,\mathrm{t}_0)$ by $\mathrm{R}_{\mathrm{t}_0,\kappa}$. Since $\omega(\eta,\mathrm{p}) > 0$, $\mathrm{B}^{\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[,\mathcal{A})$ is an algebra. Using theorem 6.1 2), we conclude that the application $\mathcal{W} : \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[,\mathcal{A}) \to \mathrm{B}^{\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[,\mathcal{A})$ defined by $\mathcal{W}(\kappa) = \mathrm{L}_{\mathrm{t}_0,\kappa}\mathrm{R}_{\mathrm{t}_0,\kappa}$ is continuous. Moreover, by 8.3, $\mathcal{W} = \mathbf{1}_{\mathcal{A}}$ on $\mathrm{C}^\infty([0,\mathrm{T}],\mathcal{A})$, a dense subset of $\mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[,\mathcal{A})$. Finally, $\mathcal{W} = \mathbf{1}_{\mathcal{A}}$ on the whole space $\mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[,\mathcal{A})$. It entails that $\mathrm{R}(.,\mathrm{t}_0)^{-1} = \mathrm{L}_{\mathrm{t}_0,\kappa} \in \mathrm{B}^{\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[,\mathcal{A})$, with continuous dependence with respect to $\kappa \in \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[,\mathcal{A})$. In the cases $\mathrm{p} = \infty$ or $\mathrm{q} = \infty$, we write as usual $\kappa \in \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}} \hookrightarrow \mathrm{B}^{-\frac{1}{2}+\rho\eta}_{\gamma,\gamma}$ with $1-\rho > 0$ small and $\gamma > 0$ large enough. The previous proof asserts that $\mathrm{R}(.,\mathrm{t}_0)^{-1} = \mathrm{L}_{\mathrm{t}_0,\kappa} \in \mathrm{B}^{-\frac{1}{2}+\rho\eta}_{\gamma,\gamma}$. But due to theorem 6.1, $\mathrm{L}_{\mathrm{t}_0,\kappa}$ also belongs to $\mathrm{B}^{\frac{1}{2}+\eta}_{\mathrm{p,q}}$, with local Lipschitz continuity with respect to $\kappa \in \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}$.
5) From 2) and 4), we have $\mathrm{R} = \mathrm{R}(.,0) \otimes \mathrm{R}(.,0)^{-1} \in \mathrm{B}^{\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[^2,\mathcal{A})$, which is 5) for $\gamma = p$. In the general case, we write $\kappa \in \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[,\mathcal{A}) \hookrightarrow \mathrm{B}^{\frac{1}{2}+\eta-\frac{1}{p}+\frac{1}{\gamma}}_{\gamma,q}(]0,\mathrm{T}[,\mathcal{A})$ and apply the previous result. ☐

Proposition 8.2 is the variation of constants formula in our functional setting. It essentially follows from the continuity of $\mathrm{R}(.,\mathrm{t}_0)^{-1}$ with respect to κ. In what follows, we restrict our statement to $\mathrm{t}_0 = 0$ and $1 \le \mathrm{p,q} < \infty$. Notation $\mathbb{I}_{[0,\mathrm{t}]}$ stands for the characteristic function of $[0,t] \subset [0,\mathrm{T}]$.

Proposition 8.2. *Let* $\mathrm{T} > 0$*,* $\mathrm{t}_0 = 0$*,* $0 < \eta \le 1/2$*,* $1 \le \mathrm{p,q} < \infty$ *with* $\frac{1}{2} + \eta - \frac{1}{p} > 0$*. Assume that* $\kappa \in \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[,\mathscr{L}(\mathrm{E}))$*,* $\phi \in \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[,\mathrm{E})$*,* $\mathrm{M}_0 \in \mathrm{E}$ *and let* $\mathrm{M} \in \mathrm{B}^{\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[,\mathrm{E})$ *be the solution of* 6.1*. Then, for any* $\mathrm{t} \in [0,\mathrm{T}]$*, we have:*

$$\mathrm{M}(\mathrm{t}) = \mathrm{R}(\mathrm{t},0).\mathrm{M}_0 + < \mathrm{R}(\mathrm{t},.), \mathbb{I}_{[0,\mathrm{t}]}\phi >_{\eta,\mathrm{p,q}} \tag{8.4}$$

Proof. Let $(\mathrm{t},\mathrm{t}_0) \in [0,\mathrm{T}]^2$. Set $G = \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[,\mathscr{L}(\mathrm{E})) \times \mathrm{B}^{-\frac{1}{2}+\eta}_{\mathrm{p,q}}(]0,\mathrm{T}[,\mathrm{E})$. Let Λ be the mapping:

$$\begin{cases} G & \to \mathrm{E} \\ (\kappa,\phi) & \mapsto \mathrm{M}_{\kappa,\phi}(\mathrm{t}) - \mathrm{R}_\kappa(\mathrm{t},0).\mathrm{M}_0 - < \mathrm{R}_\kappa(\mathrm{t},.), \mathbb{I}_{[0,\mathrm{t}]}\phi >_{\eta,\mathrm{p,q}} \end{cases} \tag{8.5}$$

where $\mathrm{M}_{\kappa,\phi}$ and R_κ denote respectively the solutions of 6.1 and 8.1. Function Λ is identically equal to zero on $\mathrm{C}^\infty([0,\mathrm{T}],\mathscr{L}(\mathrm{E}))\times\mathrm{C}^\infty([0,\mathrm{T}],\mathrm{E})$. It remains to prove that Λ is continuous. We just deal with the bracket in the definition of Λ, since the other terms are simpler. Writing $\mathrm{R}_\kappa(\mathrm{t},.)=\mathrm{R}_\kappa(\mathrm{t},0)\mathrm{R}_\kappa(0,.)$, we essentially have to prove that function $\Lambda_1 : G\to\mathrm{E}$ with $\Lambda_1(\kappa,\phi)=<\mathrm{R}_\kappa(0,.),\mathbb{I}_{[0,\mathrm{t}]}\phi>_{\eta,\mathrm{p},\mathrm{q}}$ is continuous. First, notice that:

$$\mathrm{B}_{\mathrm{p},\mathrm{q}}^{-\frac{1}{2}+\eta}(]0,\mathrm{T}[,\mathscr{L}(\mathrm{E}))\xrightarrow{R(0,.)}\mathrm{B}_{\mathrm{p},\mathrm{q}}^{\frac{1}{2}+\eta}(]0,\mathrm{T}[,\mathscr{L}(\mathrm{E}))\hookrightarrow\mathrm{B}_{\mathrm{p}',\mathrm{q}'}^{\frac{1}{2}-\eta}(]0,\mathrm{T}[,\mathscr{L}(\mathrm{E}))$$

In the first injection, $\mathrm{R}(0,.)$ is considered as a function of the variable κ. The second injection follows from lemma 5.2, 1). It remains to prove that function $\Lambda_2 : \mathrm{B}_{\mathrm{p},\mathrm{q}}^{-\frac{1}{2}+\eta}(]0,\mathrm{T}[,\mathrm{E})\to\mathrm{B}_{\mathrm{p},\mathrm{q}}^{-\frac{1}{2}+\eta}(]0,\mathrm{T}[,\mathrm{E})$ with $\Lambda_2(\phi)=\mathbb{I}_{[0,\mathrm{t}]}\phi$ is continuous. This follows from the fact that $\mathbb{I}_{[0,\mathrm{t}]}$ is a multiplier for $\mathrm{B}_{\mathrm{p},\mathrm{q}}^{-\frac{1}{2}+\eta}(]0,\mathrm{T}[,\mathrm{E})$. □

As a final remark, let's mention that all the classical results do not extend immediately to our functional frame. For instance, Floquet's theorem (case $\mathrm{E}=\mathbb{C}^n$) still holds true, whereas the perturbation/stability theory is unlikely to work, unless serious modifications.

9 The nonlinear case.

We now turn to generalize theorem 6.1 to the nonlinear case. The main issue is to define suitable restriction operations for a global operator, denoted below by $\mathcal{H}_{\mathrm{T},\alpha}$. Let $1<\mathrm{p},\mathrm{q}<\infty$, $0<\alpha<1/2$ with $\omega(\alpha,\mathrm{p})>0$. For any $\mathrm{t}\in\mathbb{R}_+$, $\rho>0$, $\mathrm{M}_0\in\mathrm{E}$, define $\mathscr{B}_{\mathrm{t},\alpha}(\mathrm{M}_0,\rho)$ as the open ball of $\mathrm{B}_{\mathrm{p},\mathrm{q}}^{\frac{1}{2}+\alpha}(]0,\mathrm{t}[,\mathrm{E})$ with center M_0 and radius ρ, and set :

$$\mathrm{B}_{\mathrm{t},\alpha}(\mathrm{M}_0\mathbf{1}_{]0,\mathrm{t}[},\rho)=\{\mathrm{M}\in\mathscr{B}_{\mathrm{t},\alpha}(\mathrm{M}_0\mathbf{1}_{]0,\mathrm{t}[},\rho)\text{ with }\mathrm{M}(0)=\mathrm{M}_0\}\tag{9.1}$$

Denote also by $\bar{\mathrm{B}}_{\mathrm{t},\alpha}(\mathrm{M}_0\mathbf{1}_{]0,\mathrm{t}[},\rho)$ its closure in $\mathrm{B}_{\mathrm{p},\mathrm{q}}^{\frac{1}{2}+\alpha}(]0,\mathrm{t}[,\mathrm{E})$ and a similar notation for $\mathscr{B}_{\mathrm{t},\alpha}(\mathrm{M}_0,\rho)$. Until the end of the paper, we abusively identify $\mathrm{M}_0\mathbf{1}_{]0,\mathrm{t}[}$ with M_0 and, for instance, often write $\mathrm{B}_{\mathrm{t},\alpha}(\mathrm{M}_0,\rho)$ in place of $\mathrm{B}_{\mathrm{t},\alpha}(\mathrm{M}_0\mathbf{1}_{]0,\mathrm{t}[},\rho)$. For future reference, we first prove

Lemma 9.1. *Let $1<\mathrm{p},\mathrm{q}<\infty$, $0<\alpha<1/2$ with $\omega(\alpha,\mathrm{p})>0$, $\rho>0$, $0\le\mathrm{t}_1\le\mathrm{t}_2$. Let $\mathrm{M}_0\in\mathrm{E}$. Then*
1) $\mathrm{B}_{\mathrm{t}_1,\alpha}(\mathrm{M}_0,\rho)=\mathrm{B}_{\mathrm{t}_2,\alpha}(\mathrm{M}_0,\rho)|_{]0,\mathrm{t}_1[}$.
2) $\mathscr{B}_{\mathrm{t}_1,\alpha}(\mathrm{M}_0,\rho)=\mathscr{B}_{\mathrm{t}_2,\alpha}(\mathrm{M}_0,\rho)|_{]0,\mathrm{t}_1[}$.

Proof. We only prove 1). Inclusion $\mathrm{B}_{\mathrm{t}_2,\alpha}(\mathrm{M}_0,\rho)|_{]0,\mathrm{t}_1[}\subset\mathrm{B}_{\mathrm{t}_1,\alpha}(\mathrm{M}_0,\rho)$ is clear. We prove the opposite inclusion. Let $\mathrm{M}\in\mathrm{B}_{\mathrm{t}_1,\alpha}(\mathrm{M}_0,\rho)$. Set $\epsilon=\dfrac{1}{2}(\rho-\|\mathrm{M}-\mathrm{M}_0\|_{\mathrm{B}_{\mathrm{p},\mathrm{q}}^{\frac{1}{2}+\alpha}(]0,\mathrm{t}_1[,\mathrm{E})})$. By definition of $\|.\|_{\mathrm{B}_{\mathrm{p},\mathrm{q}}^{\frac{1}{2}+\alpha}(]0,\mathrm{t}_1[,\mathrm{E})}$, there exists $\mathrm{M}_*\in\mathrm{B}_{\mathrm{p},\mathrm{q}}^{\frac{1}{2}+\alpha}(\mathbb{R},\mathrm{E})$ such that:

$$\mathrm{M}_*|_{]0,t_1[}=\mathrm{M}-\mathrm{M}_0\tag{9.2}$$

and also:

$$\|\mathrm{M}_*\|_{\mathrm{B}_{\mathrm{p},\mathrm{q}}^{\frac{1}{2}+\alpha}(]0,t_2[,\mathrm{E})}\le\|\mathrm{M}_*\|_{\mathrm{B}_{\mathrm{p},\mathrm{q}}^{\frac{1}{2}+\alpha}(\mathbb{R},\mathrm{E})}\le\|\mathrm{M}-\mathrm{M}_0\|_{\mathrm{B}_{\mathrm{p},\mathrm{q}}^{\frac{1}{2}+\alpha}(]0,t_1[,\mathrm{E})}+\epsilon<\rho\tag{9.3}$$

Set $\mathrm{M}_{**}=\mathrm{M}_*+\mathrm{M}_{0,*}$ with $\mathrm{M}_{0,*}\in\mathrm{B}_{\mathrm{p},\mathrm{q}}^{\frac{1}{2}+\alpha}(\mathbb{R},\mathrm{E})$ and $\mathrm{M}_{0,*}|_{]0,\mathrm{t}_2[}=\mathrm{M}_0$. By 9.2, $\mathrm{M}_{**}(0)=\mathrm{M}(0)=\mathrm{M}_0$. Hence, by 9.3, $\mathrm{M}_{**}|_{]0,\mathrm{t}_2[}\in\mathrm{B}_{\mathrm{t}_2,\alpha}(\mathrm{M}_0,\rho)$. With 9.2, this provides the lemma. □

Let now $R > 0$, $T > 0$, $0 < \alpha < \eta < 1/2$, $1 \leq p, q < \infty$ with $\omega(\alpha, p) > 0$, and $M_0 \in E$ be fixed. Let also $\mathfrak{L} \subset]0, T]$ with $T \in \mathfrak{L}$ and $0 \in \bar{\mathfrak{L}}$ (closure of $\mathfrak{L}$ in $\mathbb{R}$). *In the following,* V *denotes* $B_{T,\alpha}(M_0, R)$ *or* $\mathscr{B}_{T,\alpha}(M_0, R)$. Let finally $\mathcal{H}_{T,\alpha} : V \to B_{p,q}^{-\frac{1}{2}+\eta}(]0, T[, E)$. Consider the three following properties: for any $(M, N, t) \in V^2 \times \mathfrak{L}$, we have

- (L1) $\quad \|\mathcal{H}_{T,\alpha}(M) - \mathcal{H}_{T,\alpha}(N)\|_{B_{p,q}^{-\frac{1}{2}+\eta}(]0,T[,E)} \leq \mathfrak{C}_T \|M - N\|_{B_{p,q}^{\frac{1}{2}+\alpha}(]0,T[,E)}$
- (L1′) $\quad$ The operator $\mathcal{H}_{T,\alpha}$ is continuous on V.
- (L2) $\quad$ If $M|_{]0,t[} = N|_{]0,t[}$, then $\mathcal{H}_{T,\alpha}(M)|_{]0,t[} = \mathcal{H}_{T,\alpha}(N)|_{]0,t[}$

When condition L2 is satisfied, we can define for any $t \in \mathfrak{L}$ an operator:

$$\mathcal{H}_{t,\alpha} : V|_{]0,t[} \to B_{p,q}^{-\frac{1}{2}+\eta}(]0, t[, E)$$

by restriction. It means that, for any $M \in V|_{]0,t[}$ ($= B_{t,\alpha}(M_0, R)$ or $\mathscr{B}_{t,\alpha}(M_0, R)$, see lemma 9.1), we have:

$$\mathcal{H}_{t,\alpha}(M) = \mathcal{H}_{T,\alpha}(M_*)|_{]0,t[} \tag{9.4}$$

with $M_* \in V$ and $M_*|_{]0,t[} = M$.

Using L2, we now localize properties L1 and L1′. We need the following:

Lemma 9.2. *Let* $T > 0$, $0 < \alpha < 1/2$, $1 \leq p, q < \infty$ *with* $\omega(\alpha, p) > 0$.

1) There exists a bounded family $(P_t)_{0<t<T}$ *of extension operators* $P_t : B_{p,q}^{\frac{1}{2}+\alpha}(]-1, t[, E) \to B_{p,q}^{\frac{1}{2}+\alpha}(]-1, T[, E)$.

2) The family $(\phi_t)_{0<t<T}$ *of linear maps defined by*

$$\phi_t : \begin{cases} B_{p,q}^{\frac{1}{2}+\alpha}(]0, t[, E) & \to B_{p,q}^{\frac{1}{2}+\alpha}(]0, T[, E) \\ \qquad M & \mapsto M(0) \end{cases} \tag{9.5}$$

is bounded by $C^T := C_\infty \|\mathbf{1}_{]0,T[}\|_{B_{p,q}^{\frac{1}{2}+\alpha}(]0,t[,E)}$.

Proof. 1) Let $P : B_{p,q}^{\frac{1}{2}+\alpha}(]-1, 0[, E) \to B_{p,q}^{\frac{1}{2}+\alpha}(]-1, T[, E)$ be a continuous extension operator. With the notations of lemma 3.1, write $\mu = \psi_{t,t+1}$ and $\phi = \mu^{-1} = \psi_{-t(t+1)^{-1},(t+1)^{-1}}$. Note that $\mu|_{]-1,0[}(]-1, 0[) =]-1, t[$ and $\phi|_{]-1,T[}(]-1, T[) =]-1, \frac{T-t}{t+1}[\subset]-1, T[$. Therefore, for $u \in B_{p,q}^{\frac{1}{2}+\alpha}(]-1, t[, E)$, we can define $P_t(u) = P\big(u o \mu|_{]-1,0[}\big) o (\phi|_{]-1,T[})$. Due to $\phi(]-1, t[) =]-1, 0[$, we have $P_t(u)|_{]-1,t[} = u$. Now, applying lemma 3.1 and the continuity of the operator P , we get the boundedness of $(P_t)_{0<t<T}$, say:

$$\Lambda_{T,1} := \sup_{0<t<T}\Big(\|P_t\|_{B_{p,q}^{\frac{1}{2}+\alpha}(]-1,t[,E), B_{p,q}^{\frac{1}{2}+\alpha}(]-1,T[,E)}\Big) < \infty \tag{9.6}$$

2) Let $0 < t < T$ and $M \in B_{p,q}^{\frac{1}{2}+\alpha}(]0, t[, E)$. Then

$$\begin{aligned} \|M(0)\|_{B_{p,q}^{\frac{1}{2}+\alpha}(]0,T[,E)} &\leq \|M(0)\|_E \|\mathbf{1}_{]0,T[}\|_{B_{p,q}^{\frac{1}{2}+\alpha}(]0,T[,E)} \\ &\leq C_\infty \|\mathbf{1}_{]0,T[}\|_{B_{p,q}^{\frac{1}{2}+\alpha}(]0,t[,E)} \|M(0)\|_{B_{p,q}^{\frac{1}{2}+\alpha}(]0,t[,E)} \end{aligned} \tag{9.7}$$

due to inequality 4.9. This proves 2). ☐

In the sequel, the constant $\Lambda_{T,1} \geq 1$ is fixed, given by 9.6. It is independent of M_0 and R. For any $0 < t < T$, $0 < \alpha < 1/2$, $1 \leq p, q < \infty$ with $\omega(\alpha, p) > 0$ set $B^{\frac{1}{2}+\alpha}_{p,q,\mathbf{o}}(]0,t[,E) = \{f \in B^{\frac{1}{2}+\alpha}_{p,q}(]0,t[,E)$ with $f(0) = 0\}$ and define

$$E_t : B^{\frac{1}{2}+\alpha}_{p,q,\mathbf{o}}(]0,t[,E) \to B^{\frac{1}{2}+\alpha}_{p,q}(]-1,t[,E)$$

as the zero-extension operator. Set:

$$\Lambda_{T,2} := \sup_{0<t<T}\Big(\|E_t\|_{B^{\frac{1}{2}+\alpha}_{p,q,\mathbf{o}}(]0,t[,E),B^{\frac{1}{2}+\alpha}_{p,q}(]-1,t[,E)}\Big) \tag{9.8}$$

Clearly, $1 \leq \Lambda_{T,2} < \infty$ (see lemma 4.3). As usual, the dependence with respect to α in omited in the notations.

Lemma 9.3. *Let $T > 0$, $0 < \alpha < \eta < 1/2$, $1 \leq p, q < \infty$ with $\omega(\alpha, p) > 0$, $M_0 \in E$, $R > 0$.*

1) Let $\mathcal{H}_{T,\alpha} : B_{T,\alpha}(M_0, R) \to B^{-\frac{1}{2}+\eta}_{p,q}(]0,T[,E)$ satisfies properties (L1) and (L2). Then:

- (L1$_t$) *For any $t \in \mathfrak{L}$ and $(M,N) \in B_{t,\alpha}(M_0, R/\Lambda_{T,1}\Lambda_{T,2})^2$ we have*

$$\|\mathcal{H}_{t,\alpha}(M) - \mathcal{H}_{t,\alpha}(N)\|_{B^{-\frac{1}{2}+\eta}_{p,q}(]0,t[,E)} \leq \mathfrak{C}_T\Lambda_{T,1}\Lambda_{T,2}\|M - N\|_{B^{\frac{1}{2}+\alpha}_{p,q}(]0,t[,E)}$$

2) Let $\mathcal{H}_{T,\alpha} : \mathscr{B}_{T,\alpha}(M_0, R) \to B^{-\frac{1}{2}+\eta}_{p,q}(]0,T[,E)$ satisfies properties (L1′) and (L2). Then:

- (L1′$_t$) *For any $t \in \mathfrak{L}$, function $\mathcal{H}_{t,\alpha}$ is continuous on $\mathscr{B}_{t,\alpha}(M_0, R/\Lambda_{T,3})$.*

  Here, $\Lambda_{T,3} > 0$ is a constant depending only on T.

Proof. We only prove 2). Let $t \in \mathfrak{L}$ and $(M,N) \in \mathscr{B}_{t,\alpha}(M_0, R/\Lambda_{T,3})^2$. The constant $\Lambda_{T,3} \geq \Lambda_{T,1}\Lambda_{T,2}$ will be determined later. We have

$$\|\big(P_t o E_t\big)(M - M(0)\big)\|_{B^{\frac{1}{2}+\alpha}_{p,q}(]-1,T[,E)} \leq \Lambda_{T,1}\Lambda_{T,2}\|M - M(0)\|_{B^{\frac{1}{2}+\alpha}_{p,q}(]0,t[,E)} \tag{9.9}$$

Using lemma 9.2, this implies

$$\begin{aligned}
&\|M(0) + \big(P_t o E_t\big)(M - M(0)\big) - M_0\|_{B^{\frac{1}{2}+\alpha}_{p,q}(]0,t[,E)}\\
&\leq \|M(0) - M_0\|_{B^{\frac{1}{2}+\alpha}_{p,q}(]0,t[,E)} + \Lambda_{T,1}\Lambda_{T,2}\|M - M(0)\|_{B^{\frac{1}{2}+\alpha}_{p,q}(]0,t[,E)}\\
&\leq \big[C^T + \Lambda_{T,1}\Lambda_{T,2}(1 + C^T)\big]\,\|M - M_0\|_{B^{\frac{1}{2}+\alpha}_{p,q}(]0,t[,E)}
\end{aligned} \tag{9.10}$$

Hence, for $\Lambda_{T,3} := C^T + \Lambda_{T,1}\Lambda_{T,2}(1 + C^T)$, inequality 9.10 and $M \in \mathscr{B}_{t,\alpha}(M_0, R/\Lambda_{T,3})$ provides $M(0) + \big(P_t o E_t\big)(M - M(0)\big) \in \mathscr{B}_{t,\alpha}(M_0, R)$. The same holds true for N. As a consequence:

$$\begin{aligned}
&\|\mathcal{H}_{t,\alpha}(M) - \mathcal{H}_{t,\alpha}(N)\|_{B^{-\frac{1}{2}+\eta}_{p,q}(]0,t[,E)}\\
&\leq \|\mathcal{H}_{T,\alpha}\Big(M(0) + \big[(P_t o E_t)(M - M(0))\big]|_{]0,T[}\Big) -\\
&\mathcal{H}_{T,\alpha}\Big(N(0) + \big[(P_t o E_t)(N - N(0))\big]|_{]0,T[}\Big)\|_{B^{-\frac{1}{2}+\eta}_{p,q}(]0,T[,E)}
\end{aligned} \tag{9.11}$$

Recall that $\mathcal{H}_{T,\alpha}$, is continuous. Hence, due to 9.11, it's now enough to show that for any $t \in \mathcal{L}$, the function

$$\begin{cases} \mathscr{B}_{t,\alpha}(M_0, R/\Lambda_{T,3}) & \to B^{\frac{1}{2}+\alpha}_{p,q}(]0,T[,E)\\ \qquad M & \mapsto M(0) + \big[(P_t o E_t)(M - M(0))\big]|_{]0,T[} \end{cases} \tag{9.12}$$

is continuous. This follows from 9.6, 9.8 and lemma 9.2 2). □

From now on, we denote by $\mathfrak{D}_{\mathrm{T},\alpha}$ the constant $\mathrm{C_T}$ appearing in 4.12 with α in place of η, and by $\mathfrak{J}$ the constant C of theorem 3.1 b).

Theorem 9.1. *Let* $\mathrm{M}_0 \in \mathrm{E}$, $\mathrm{T} > 0$, $\mathrm{R} > 0$, $0 < \alpha < \eta < 1/2$, $1 \leq \mathrm{p}, \mathrm{q} < \infty$ *with* $\omega(\alpha, \mathrm{p}) > 0$.

I) Assume that $\mathcal{H}_{\mathrm{T},\alpha} : \mathrm{B}_{\mathrm{T},\alpha}(\mathrm{M}_0, \mathrm{R}) \to \mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]0, \mathrm{T}[, \mathrm{E})$ *satisfies conditions* (L1) *and* (L2). *Then, for any* $0 < \rho < \mathrm{R}/\Lambda_{\mathrm{T},1}\Lambda_{\mathrm{T},2}$ *there exists* $\mathrm{t}_0 \in \mathcal{L}$ *such that the problem: find* $\mathrm{M} \in \bar{\mathrm{B}}_{\mathrm{t}_0,\alpha}(\mathrm{M}_0, \rho)$ *with:*

$$\begin{cases} \mathrm{M}' = \mathcal{H}_{\mathrm{t}_0,\alpha}(\mathrm{M}) \\ \mathrm{M}(0) = \mathrm{M}_0 \end{cases} \tag{9.13}$$

admits exactly one distributional solution. This solution belongs to $\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}(]0, \mathrm{t}_0[, \mathrm{E})$. *Moreover, one can choose*

$$\mathrm{t}_0 \in \mathfrak{L} \cap \big]0, \mathit{inf}\Big(\mathrm{T}, \big(\frac{\mathfrak{D}_{\mathrm{T}}^{-1}\mathfrak{J}^{-1}\rho}{\mathfrak{C}_{\mathrm{T}}\Lambda_{\mathrm{T},1}\Lambda_{\mathrm{T},2}\rho + \|\mathcal{H}_{\mathrm{T},\alpha}(\mathrm{M}_0)\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]0,\mathrm{T}[,\mathrm{E})}}\big)^{\frac{1}{\eta-\alpha}}\Big)\big[\tag{9.14}$$

II) Assume that E *has finite dimension and assume that* $\mathcal{H}_{\mathrm{T},\alpha} : \mathscr{B}_{\mathrm{T},\alpha}(\mathrm{M}_0, \mathrm{R}) \to \mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]0, \mathrm{T}[, \mathrm{E})$ *satisfies conditions* (L1′) *and* (L2). *Then, there exists* $\rho > 0$ *and* $\mathrm{t}_1 \in \mathfrak{L}$ *such that, for any* $\mathrm{M}_1 \in \mathrm{E} \cap \bar{\mathscr{B}}_{\mathrm{t}_1,\alpha}(\mathrm{M}_0, \rho)$, *the problem: find* $\mathrm{M} \in \bar{\mathrm{B}}_{\mathrm{t}_1,\alpha}(\mathrm{M}_1, \rho)$ *with:*

$$\begin{cases} \mathrm{M}' = \mathcal{H}_{\mathrm{t}_1,\alpha}(\mathrm{M}) \\ \mathrm{M}(0) = \mathrm{M}_1 \end{cases} \tag{9.15}$$

admits at least one distributional solution. This solution belongs to $\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\eta}(]0, \mathrm{t}_1[, \mathrm{E})$.

Proof. An application of Picard and Schauder fixed point theorem.

I) a) Stability.

Let $0 < \rho < \mathrm{R}/\Lambda_{\mathrm{T},1}\Lambda_{\mathrm{T},2}$ and let $\mathrm{t}_0 \in \mathfrak{L}$ as in 9.14. Define:

$$\mathcal{S}_{\mathrm{t}_0} : \begin{cases} \bar{\mathrm{B}}_{\mathrm{t}_0,\alpha}(\mathrm{M}_0, \rho) & \to \mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\alpha}(]0, \mathrm{t}_0[, \mathrm{E}) \\ \qquad \tilde{\mathrm{M}} & \mapsto \mathrm{M} \end{cases} \tag{9.16}$$

where M is given by equation 3.31 with $\mathcal{H}_{\mathrm{t}_0,\alpha}(\tilde{\mathrm{M}})$ in place of ψ and $\alpha = \alpha$.
Let now $\tilde{\mathrm{M}} \in \bar{\mathrm{B}}_{\mathrm{t}_0,\alpha}(\mathrm{M}_0, \rho)$ and $\mathrm{M} = \mathcal{S}_{\mathrm{t}_0}(\tilde{\mathrm{M}})$. Appealing to lemmas 3.3, 4.3 and theorem 3.1 for $0 < \alpha < \eta$ we have:

$$\|\mathrm{M} - \mathrm{M}_0\|_{\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\alpha}(]0,\mathrm{t}_0[,\mathrm{E})} \leq \mathfrak{D}_{\mathrm{T},\alpha}\|\mathrm{M}'\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\alpha}(]0,\mathrm{t}_0[)} = \mathfrak{D}_{\mathrm{T},\alpha}\|\mathcal{H}_{\mathrm{t}_0,\alpha}(\tilde{\mathrm{M}})\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\alpha}(]0,\mathrm{t}_0[)} \tag{9.17}$$

$$\leq \mathfrak{D}_{\mathrm{T},\alpha}\mathfrak{J}{\mathrm{t}_0}^{\eta-\alpha}\big(\|\mathcal{H}_{\mathrm{t}_0,\alpha}(\tilde{\mathrm{M}}) - \mathcal{H}_{\mathrm{t}_0,\alpha}(\mathrm{M}_0)\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]0,\mathrm{t}_0[)} + \|\mathcal{H}_{\mathrm{t}_0,\alpha}(\mathrm{M}_0)\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]0,\mathrm{t}_0[)}\big) \tag{9.18}$$

Due to lemma 9.3, $\|\tilde{\mathrm{M}} - \mathrm{M}_0\|_{\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\alpha}(]0,\mathrm{t}_0[)} \leq \rho$ and 9.14, we get:

$$\begin{aligned} \|\mathrm{M} - \mathrm{M}_0\|_{\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\alpha}(]0,\mathrm{t}_0[,\mathrm{E})} &\leq \mathfrak{D}_{\mathrm{T}}\mathfrak{J}{\mathrm{t}_0}^{\eta-\alpha}(\mathfrak{C}_{\mathrm{T}}\Lambda_{\mathrm{T},1}\Lambda_{\mathrm{T},2}\|\tilde{\mathrm{M}} - \mathrm{M}_0\|_{\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\alpha}(]0,\mathrm{t}_0[)} + \|\mathcal{H}_{\mathrm{t}_0,\alpha}(\mathrm{M}_0)\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac{1}{2}+\eta}(]0,\mathrm{t}_0[)}) \\ &\leq \rho \end{aligned}$$

Hence, the stability is proved.

b) Proof that $\mathcal{S}_{\mathrm{t}_0}$ is a $\mathrm{B}_{\mathrm{p,q}}^{\frac{1}{2}+\alpha}(]0, \mathrm{t}_0[)$ contraction.

Let $\tilde{\mathrm{M}} \in \bar{\mathrm{B}}_{\mathrm{t}_0,\alpha}(\mathrm{M}_0,\rho)$, $\tilde{\mathrm{N}} \in \bar{\mathrm{B}}_{\mathrm{t}_0,\alpha}(\mathrm{M}_0,\rho)$ and set $\mathrm{M} = \mathcal{S}_{\mathrm{t}_0}(\tilde{\mathrm{M}})$ and $\mathrm{N} = \mathcal{S}_{\mathrm{t}_0}(\tilde{\mathrm{N}})$. Arguing as in a), we get:

$$\begin{aligned}\|\mathrm{M}-\mathrm{N}\|_{\mathrm{B}_{\mathrm{p,q}}^{\frac12+\alpha}(]0,\mathrm{t}_0[)} &\le \mathfrak{D}_{\mathrm{T}}\|(\mathrm{M}-\mathrm{N})'\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac12+\alpha}(]0,\mathrm{t}_0[)} \\ &\le \mathfrak{C}_{\mathrm{T}}\mathfrak{D}_{\mathrm{T}}\mathfrak{J}\Lambda_{\mathrm{T},1}\Lambda_{\mathrm{T},2}\mathrm{t}_0^{\eta-\alpha}\|\tilde{\mathrm{M}}-\tilde{\mathrm{N}}\|_{\mathrm{B}_{\mathrm{p,q}}^{\frac12+\eta}(]0,\mathrm{t}_0[)}\end{aligned} \tag{9.19}$$

with $\mathfrak{C}_{\mathrm{T}}\mathfrak{D}_{\mathrm{T}}\mathfrak{J}\Lambda_{\mathrm{T},1}\Lambda_{\mathrm{T},2}\mathrm{t}_0^{\eta-\alpha} < 1$ for t_0 as in I) a). Hence I) follows from Picard fixed point theorem and lemma 3.3.

II) a) Stability.

Let $0 < \rho < \mathrm{R}/2\Lambda_{\mathrm{T},3}$. Since $\mathcal{H}_{\mathrm{T},\alpha}$ is continuous at M_0, restricting if necessary ρ, we can impose that $\mathcal{H}_{\mathrm{T},\alpha}\big(\bar{\mathscr{B}}_{\mathrm{T},\alpha}(\mathrm{M}_0,2\rho)\big)$ is bounded in $\mathrm{B}_{\mathrm{p,q}}^{-\frac12+\eta}(]0,\mathrm{T}[,\mathrm{E})$ by $\mathrm{K} = \|\mathcal{H}_{\mathrm{T},\alpha}(\mathrm{M}_0)\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac12+\eta}(]0,\mathrm{T}[)} + 1$. Let $\mathrm{t}_1 \in \mathfrak{L}$ with

$$0 \le \mathrm{t}_1 \le (\rho/\mathrm{K}\mathfrak{D}_{\mathrm{T},\alpha}\mathfrak{J})^{\frac{1}{\eta-\alpha}} \tag{9.20}$$

Appealing to lemma 9.1, definition of $\mathcal{H}_{\mathrm{t}_1,\alpha}$ and the assumption on ρ, we obtain that $\mathcal{H}_{\mathrm{t}_1,\alpha}\big(\bar{\mathscr{B}}_{\mathrm{t}_1,\alpha}(\mathrm{M}_0,2\rho)\big)$ is bounded in $\mathrm{B}_{\mathrm{p,q}}^{-\frac12+\eta}(]0,\mathrm{t}_1[,\mathrm{E})$ by K. Fix $\mathrm{M}_1 \in \mathrm{E}\cap\bar{\mathscr{B}}_{\mathrm{t}_1,\alpha}(\mathrm{M}_0,\rho)$. For $\tilde{\mathrm{M}} \in \bar{\mathrm{B}}_{\mathrm{t}_1,\alpha}(\mathrm{M}_1,\rho) \subset \bar{\mathscr{B}}_{\mathrm{t}_1,\alpha}(\mathrm{M}_0,2\rho) \subset \mathscr{B}_{\mathrm{t}_1,\alpha}(\mathrm{M}_0,\mathrm{R}/\Lambda_{\mathrm{T},3})$, set $\mathrm{M} = \mathcal{S}_{\mathrm{t}_1}(\tilde{\mathrm{M}})$. Here, application $\mathcal{S}_{\mathrm{t}_1}$ is given by 9.16 with M_1 and t_1 in place of M_0 and t_0. Arguing as in I) a) (see 9.17), we have

$$\begin{aligned}\|\mathrm{M}-\mathrm{M}_1\|_{\mathrm{B}_{\mathrm{p,q}}^{\frac12+\alpha}(]0,\mathrm{t}_1[,\mathrm{E})} &\le \mathfrak{D}_{\mathrm{T},\alpha}\|\mathcal{H}_{\mathrm{t}_1,\alpha}(\tilde{\mathrm{M}})\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac12+\alpha}(]0,\mathrm{t}_1[,\mathrm{E})} \\ &\le \mathfrak{D}_{\mathrm{T},\alpha}\mathfrak{J}\|\mathcal{H}_{\mathrm{t}_1,\alpha}(\tilde{\mathrm{M}})\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac12+\eta}(]0,\mathrm{t}_1[,\mathrm{E})}\mathrm{t}_1^{\eta-\alpha} \le \mathrm{K}\mathfrak{D}_{\mathrm{T},\alpha}\mathfrak{J}\mathrm{t}_1^{\eta-\alpha}\end{aligned} \tag{9.21}$$

since $\|\mathcal{H}_{\mathrm{t}_1,\alpha}(\tilde{\mathrm{M}})\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac12+\alpha}(]0,\mathrm{t}_1[,\mathrm{E})} \le \mathrm{K}$ for $\tilde{\mathrm{M}} \in \bar{\mathscr{B}}_{\mathrm{t}_1,\alpha}(\mathrm{M}_0,2\rho)$. Hence, using 9.20, the stability $\mathcal{S}_{\mathrm{t}_1}(\bar{\mathrm{B}}_{\mathrm{t}_1,\alpha}(\mathrm{M}_1,\rho)) \subset \bar{\mathrm{B}}_{\mathrm{t}_1,\alpha}(\mathrm{M}_1,\rho)$ is proved.

Until the end of the proof, the notations and the definitions of II)a) hold.

b) Continuity.

For another $\tilde{\mathrm{N}} \in \bar{\mathrm{B}}_{\mathrm{t}_1,\alpha}(\mathrm{M}_1,\rho)$, set $\mathrm{N} = \mathcal{S}_{\mathrm{t}_1}(\tilde{\mathrm{N}})$. Arguing as in 9.17 and 9.18, we have

$$\|\mathrm{M}-\mathrm{N}\|_{\mathrm{B}_{\mathrm{p,q}}^{\frac12+\alpha}(]0,\mathrm{t}_1[,\mathrm{E})} \le \mathfrak{D}_{\mathrm{T},\alpha}\|\mathcal{H}_{\mathrm{t}_1,\alpha}(\tilde{\mathrm{M}}) - \mathcal{H}_{\mathrm{t}_1,\alpha}(\tilde{\mathrm{N}})\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac12+\alpha}(]0,\mathrm{t}_1[,\mathrm{E})} \tag{9.22}$$

Note that $\bar{\mathrm{B}}_{\mathrm{t}_1,\alpha}(\mathrm{M}_1,\rho) \subset \mathscr{B}_{\mathrm{t}_1,\alpha}(\mathrm{M}_0,\mathrm{R}/\Lambda_{\mathrm{T},3})$. Invoking lemma 9.3, we get that function $\mathcal{H}_{\mathrm{t}_1,\alpha}$ is continuous on $\bar{\mathrm{B}}_{\mathrm{t}_1,\alpha}(\mathrm{M}_1,\rho)$. Now, inequality 9.22 implies that $\mathcal{S}_{\mathrm{t}_1}$ is also continuous on $\bar{\mathrm{B}}_{\mathrm{t}_1,\alpha}(\mathrm{M}_1,\rho)$.

c) Compactness.

Let $0 < \alpha < \alpha' < \eta$. Arguing as in 9.17 and 9.18, we have

$$\begin{aligned}\|\mathrm{M}-\mathrm{M}_1\|_{\mathrm{B}_{\mathrm{p,q}}^{\frac12+\alpha'}(]0,\mathrm{t}_1[,\mathrm{E})} &\le \mathfrak{D}_{\mathrm{T},\alpha'}\|\mathrm{M}'\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac12+\alpha'}(]0,\mathrm{t}_1[,\mathrm{E})} = \mathfrak{D}_{\mathrm{T},\alpha'}\|\mathcal{H}_{\mathrm{t}_1,\alpha}(\tilde{\mathrm{M}})\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac12+\alpha'}(]0,\mathrm{t}_1[,\mathrm{E})} \\ &\le \mathfrak{D}_{\mathrm{T},\alpha'}\|\mathcal{H}_{\mathrm{t}_1,\alpha}(\tilde{\mathrm{M}})\|_{\mathrm{B}_{\mathrm{p,q}}^{-\frac12+\eta}(]0,\mathrm{t}_1[,\mathrm{E})} \le \mathrm{K}\mathfrak{D}_{\mathrm{T},\alpha'}\end{aligned} \tag{9.23}$$

Note now, the space E being finite dimensional, that the injection $\mathrm{B}_{\mathrm{p,q}}^{\frac12+\alpha'}(]0,\mathrm{t}_1[,\mathrm{E}) \hookrightarrow \mathrm{B}_{\mathrm{p,q}}^{\frac12+\alpha}(]0,\mathrm{t}_1[,\mathrm{E})$ is compact. Hence, due to 9.23, $\mathcal{S}_{\mathrm{t}_1}(\bar{\mathrm{B}}_{\mathrm{t}_1,\alpha}(\mathrm{M}_1,\rho))$ has compact closure in $\mathrm{B}_{\mathrm{p,q}}^{\frac12+\alpha}(]0,\mathrm{t}_1[,\mathrm{E})$.

d) Finally, since $\bar{\mathrm{B}}_{\mathrm{t}_1,\alpha}(\mathrm{M}_1,\rho)$ is a closed convex subset of $\mathrm{B}_{\mathrm{p,q}}^{\frac12+\alpha}(]0,\mathrm{t}_1[,\mathrm{E})$, the existence of a solution of 9.15 follows from II) a-c), Schauder fixed point theorem and lemma 3.3. $\square$

In order to apply theorem 9.1 to operators $\mathcal{H}_{T,\alpha} : U \subset B^{\frac{1}{2}+\alpha}_{p,q}(]0,T[,E) \to B^{-\frac{1}{2}+\eta}_{p,q}(]0,T[,E)$, it remains to identify the set

$$I(U) = \{M_0 \in E \text{ such that there exists } 0 < T_{M_0} \leq T \text{ with } M_0 \in U|_{]0,T_{M_0}[}\}$$

We also need some uniform estimate on the time T_{M_0}.

Proposition 9.1. *Let* $T > 0$, $1 \leq p, q < \infty$, $0 < \alpha < 1/2$ *with* $\omega(\alpha, p) > 0$, *and let* U *be an open subset of* $B^{\frac{1}{2}+\alpha}_{p,q}(]0,T[,E)$. *Then*

1) $I(U) = U(0)$, *where* $U(0)$ *denotes the set of initial values of elements of* U.

2) $U(0)$ *is an open subset of* E. *More precisely, for any* $M_0 \in I(U)$ *there exist* $\gamma > 0$, $R > 0$ *and* $T_0 > 0$ *such that for any* $M_1 \in E$ *with* $\|M_1 - M_0\|_E \leq \gamma$, *we have*

$$M_1 \mathbf{1}_{]0,T_0[} \in \mathscr{B}_{T_0,\alpha}(M_0 \mathbf{1}_{]0,T_0[}, R) \subset U|_{]0,T_0[} \tag{9.24}$$

Proof. The fact that $U(0)$ is an open subset of E is obvious, as well as the inclusion $I(U) \subset U(0)$, which is a direct consequence of the definition of $I(U)$. We prove the reverse inclusion -i.e that for any $M \in U$, $M(0) \in U|_{]0,T_0[}$ for some $0 < T_0 \leq T$- and 9.24 at the same time.

Let $M \in U$. For $R > 0$ small enough, we have $\mathscr{B}_{T,\alpha}(M, R) \subset U$. Set $\epsilon = R/\big[4(C_\infty + 2)\big]$ (see corollary 4.1 c)). Pick up $\psi_\epsilon \in \mathscr{B}_{T,\alpha}(M, R) \cap C^\infty([0,T], E)$ with $\|M - \psi_\epsilon\|_{B^{\frac{1}{2}+\alpha}_{p,q}(]0,T[,E)} \leq \epsilon$. Set $\phi_\epsilon = \psi_\epsilon - \psi_\epsilon(0) + M(0)$. By inequality 4.9, we have $\|M(0) - \psi_\epsilon(0)\|_E \leq C_\infty \|M - \psi_\epsilon\|_{B^{\frac{1}{2}+\alpha}_{p,q}(]0,T[,E)}$. Using definition of ϕ_ϵ and ψ_ϵ, this implies that $\|M - \phi_\epsilon\|_{B^{\frac{1}{2}+\alpha}_{p,q}(]0,T[,E)} \leq (C_\infty + 1)\epsilon$. Let now $\alpha < \eta < 1/2$ and $M_1 \in E$ with $\|M_1 - M(0)\|_E \leq \epsilon / \|\mathbf{1}_{]0,T[}\|_{B^{\frac{1}{2}+\alpha}_{p,q}(]0,T[,E)}$. Since $\phi_\epsilon(0) = M(0)$, appealing to theorem 3.1 b) and corollary 4.1 b) we get, for any $0 < t < T$

$$\begin{aligned} \|M - M_1\|_{B^{\frac{1}{2}+\alpha}_{p,q}(]0,t[,E)} &\leq \|M - \phi_\epsilon\|_{B^{\frac{1}{2}+\alpha}_{p,q}(]0,t[,E)} + \|\phi_\epsilon - \phi_\epsilon(0)\|_{B^{\frac{1}{2}+\alpha}_{p,q}(]0,t[,E)} + \\ + \|M(0) - M_1\|_{B^{\frac{1}{2}+\alpha}_{p,q}(]0,t[,E)} &\leq (C_\infty + 1)\epsilon + C_T \|\phi'_\epsilon\|_{B^{-\frac{1}{2}+\eta}_{p,q}(]0,t[,E)} t^{\eta-\alpha} + \epsilon \end{aligned} \tag{9.25}$$

Set

$$T_0 = \inf\left(\left(\frac{(R/2) - (C_\infty + 2)\epsilon}{C_T \|\phi'_\epsilon\|_{B^{-\frac{1}{2}+\eta}_{p,q}(]0,T[,E)}}\right)^{1/(\eta-\alpha)}, T\right) > 0$$

We infer from inequality 9.25 and lemma 9.1 b) that $M_1 \in \mathscr{B}_{T_0,\alpha}(M, R) = \mathscr{B}_{T,\alpha}(M, R)|_{]0,T_0[} \subset U|_{]0,T_0[}$, which proves the proposition. □

Let $T > 0$, $1 \leq p, q < \infty$, $0 < \alpha < \eta < 1/2$ with $\omega(\alpha, p) > 0$, and let U be an open subset of $B^{\frac{1}{2}+\alpha}_{p,q}(]0,T[,E)$. Consider an operator $\mathcal{H}_{T,\alpha} : U \to B^{-\frac{1}{2}+\eta}_{p,q}(]0,T[,E)$ satisfying (L2). We say that $\mathcal{H}_{T,\alpha}$ satisfies condition (L_o1) if, for any $M_0 \in I(U)$, there exists $R_{M_0} > 0$, $T_{M_0} \in \mathfrak{L}$ and $\mathfrak{C}_{T_{M_0}} > 0$ such that, for any $(M, N) \in \mathscr{B}_{T_{M_0},\alpha}(M_0, R_{M_0})^2$

- (L_o1) $\quad \|\mathcal{H}_{T_{M_0},\alpha}(M) - \mathcal{H}_{T_{M_0},\alpha}(N)\|_{B^{-\frac{1}{2}+\eta}_{p,q}(]0,T_{M_0}[,E)} \leq \mathfrak{C}_{T_{M_0}} \|M - N\|_{B^{\frac{1}{2}+\alpha}_{p,q}(]0,T_{M_0}[,E)}$

The following result is an immediate consequence of theorem 9.1 and proposition 9.1

Theorem 9.2. *Let* $T > 0$, $R > 0$, $0 < \alpha < \eta < 1/2$, $1 \leq p, q < \infty$ *with* $\omega(\alpha, p) > 0$, *and let* U *be an open subset of* $B^{\frac{1}{2}+\alpha}_{p,q}(]0,T[,E)$

I) Assume that $\mathcal{H}_{T,\alpha} : U \to B^{-\frac{1}{2}+\eta}_{p,q}(]0,T[,E)$ *satisfies conditions* (L2) *and* ($L_0$1). *Then, for any* $M_0 \in U(0)$, *there exists* $\rho > 0$ *and* $t_0 \in \mathfrak{L}$ *such that, for any* $M_1 \in E$ *with* $\|M_1 - M_0\|_E \leq \rho$ *the problem: find* $M \in \mathscr{B}_{t_0,\alpha}(M_0, \rho)$ *with:*

$$\begin{cases} M' = \mathcal{H}_{t_0,\alpha}(M) \\ M(0) = M_1 \end{cases} \tag{9.26}$$

admits exactly one distributional solution. This solution belongs to $\mathrm{B}_{p,q}^{\frac{1}{2}+\eta}(]0,t_0[,E)$.

II) Assume that E *is finite dimensional and assume that* $\mathcal{H}_{T,\alpha} : U \to \mathrm{B}_{p,q}^{-\frac{1}{2}+\eta}(]0,T[,E)$ *satisfies conditions* (L1′) *and* (L2). *Then, for any* $M_0 \in U(0)$, *there exists* $\rho > 0$ *and* $t_1 \in \mathfrak{L}$ *such that, for any* $M_1 \in E$ *with* $\|M_1 - M_0\|_E \leq \rho$ *the problem: find* $M \in \mathscr{B}_{t_1,\alpha}(M_0, \rho)$ *with:*

$$\begin{cases} M' = \mathcal{H}_{t_1,\alpha}(M) \\ M(0) = M_1 \end{cases} \tag{9.27}$$

admits at least one distributional solution. This solution belongs to $\mathrm{B}_{p,q}^{\frac{1}{2}+\eta}(]0,t_1[,E)$.

As an example, we apply theorem 9.2 to nonlinear operators with irregular coefficients. In the following proposition, we assume that $E = \mathcal{A}$ is an algebra such as in section 8. One may look for operators such as $\mathcal{H}_{T,\alpha}(M) = \int_{\mathbb{R}^n} \phi(s, M)\bar{\kappa}(s)d\mu(s)$ with $\bar{\kappa} : \mathbb{R}^n \to \mathrm{B}_{p,q}^{-\frac{1}{2}+\eta}(]0,T[,\mathcal{A})$, $\phi : \mathbb{R}^n \times \mathrm{B}_{p,q}^{\frac{1}{2}+\alpha}(]0,T[,\mathcal{A}) \to \mathrm{B}_{p,q}^{\frac{1}{2}+\alpha}(]0,T[,\mathcal{A})$ and μ a Radon measure on $\mathbb{R}^n$. For results on composition operators, see for instance [7], [4], [25]. In the following, we restrict to the simple case of a serie.

Proposition 9.2. *Let* $T > 0$, $R > 0$, $0 < \alpha < \eta < 1/2$, $1 < p, q < \infty$ *with* $\omega(\alpha, p) > 0$. *Let* $\kappa_j \in \mathrm{B}_{p,q}^{-\frac{1}{2}+\eta}(]0,T[,\mathcal{A})$ $(j \in \mathbb{N})$. *Assume that, for some* $R > 0$

$$\sum_{j\in\mathbb{N}} \|\kappa_j\|_{\mathrm{B}_{p,q}^{-\frac{1}{2}+\eta}(]0,T[,\mathcal{A})} R^j < \infty$$

Then, for $r > 0$ *small enough, operator* $\mathcal{H}_{T,\alpha} : \mathscr{B}_{T,\alpha}(0, r) \subset \mathrm{B}_{p,q}^{\frac{1}{2}+\alpha}(]0,T[,\mathcal{A}) \to \mathrm{B}_{p,q}^{-\frac{1}{2}+\eta}(]0,T[,\mathcal{A})$ *given by* $\mathcal{H}_{T,\alpha}(M) = \sum_{j\in\mathbb{N}} \kappa_j M^j$ *is well defined and satisfies properties* (L1) *and* (L2).

Proof. Denote by C_1 the constant in inequality 5.1. Note also, since $\omega(\alpha, p) > 0$, that there exists $C_2 > 1$ with $\|MN\|_{\mathrm{B}_{p,q}^{\frac{1}{2}+\alpha}(]0,T[,\mathcal{A})} \leq C_2 \|M\|_{\mathrm{B}_{p,q}^{\frac{1}{2}+\alpha}(]0,T[,\mathcal{A})} \|N\|_{\mathrm{B}_{p,q}^{\frac{1}{2}+\alpha}(]0,T[,\mathcal{A})}$ for any $(M, N)^2 \in \mathscr{B}_{T,\alpha}(0, R)^2$. Using lemma 5.1, we infer that the operator $\mathcal{H}_{T,\alpha}$ is well defined for $r = R/C_2 < R$. Moreover, for any $(M, N)^2 \in \mathscr{B}_{T,\alpha}(0, r)^2$

$$\begin{aligned} \|\mathcal{H}_{T,\alpha}(M) - \mathcal{H}_{T,\alpha}(N)\|_{\mathrm{B}_{p,q}^{-\frac{1}{2}+\eta}(]0,T[,\mathcal{A})} \\ \leq C_1 \Big(\sum_{j\in\mathbb{N}} \|\kappa_j\|_{\mathrm{B}_{p,q}^{-\frac{1}{2}+\eta}(]0,T[,\mathcal{A})(]0,T[,\mathcal{A})} j(C_2 r)^{j-1} \Big) \|M - N\|_{\mathrm{B}_{p,q}^{\frac{1}{2}+\alpha}(]0,T[,\mathcal{A})} \end{aligned} \tag{9.28}$$

Since $C_2 r = R$ and $\Big(\sum_{j\in\mathbb{N}} \|\kappa_j\|_{\mathrm{B}_{p,q}^{-\frac{1}{2}+\eta}(]0,T[,\mathcal{A})} jR^{j-1} < \infty \Big)$, the operator $\mathcal{H}_{T,\alpha}$ satisfies property (L1). Property (L2) is obvious. □

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