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Efficient computation of the multiplier based μ upper bound on large frequency intervals

Jérémy Lesprier, Clément Roos and Jean-Marc Biannic

Abstract—An algorithm for computing and validating the multiplier based μ upper bound of Fu and Barabanov (1997) on large frequency intervals is developed in this paper. It avoids the use of a frequency grid and removes the risk of missing a critical frequency. The μ -sensitivities are also introduced as a way to evaluate which uncertainties are worth considering when optimizing the multiplier. This allows to drastically reduce the computation time with only a slight loss of accuracy. The proposed technique is validated on three benchmarks and gives an interesting alternative to the classical (D,G)-scalings based μ upper bound when the latter is too conservative.

I. INTRODUCTION

μ -analysis (see [5]) is considered as a mature and efficient technique for checking robust stability and performance of a large class of systems in the presence of parametric uncertainties and neglected dynamics. Unfortunately, the exact computation of the structured singular value μ is known to be NP-hard in the general case (see [3]), so only lower and upper bounds are usually obtained. The latter gives a guaranteed but pessimistic robust stability margin, while the former allows to evaluate the conservatism of the upper bound. The methods for computing the lower bound are generally satisfactory (see e.g. [7] and [6]). Concerning the upper bound, one of the most widely used approaches is the (D,G)-scalings based formulation of [12]. It provides a satisfactory μ upper bound in many cases, but it sometimes yields very conservative results.

A generalization is proposed in [8], known as the multiplier based μ upper bound. Interestingly, it turns out to be less conservative than the upper bound of [12]. However, the multiplier method only allows to compute a bound at a given frequency. Obtaining a guaranteed robustness margin on the whole frequency interval $[0, +\infty[$ thus requires to solve the problem at each point of a frequency grid. This solution can be unreliable for two reasons. First, for some systems, and especially for flexible ones, it is impossible to guarantee that a critical μ peak is not missed between two points of the grid. Then, the computational complexity can become excessively large if a fine grid is used. To overcome these difficulties, a technique is proposed in this paper to determine all the frequency intervals on which a multiplier based μ upper bound computed at a single frequency remains valid.

Moreover, the computation time of the multiplier based μ upper bound increases exponentially with the number of

uncertain parameters. It is proposed in [8] to compute a looser but more computationally-friendly bound by considering only a limited number of uncertainties, so that the number of LMIs (Linear Matrix Inequalities) to be solved remains reasonable. However no way of efficiently choosing these uncertainties is suggested, so that the increase in the conservatism is the smallest. In this context, a method using the μ -sensitivities introduced in [2] is also proposed in this paper to address this difficulty.

The main contribution of this work is thus to propose a practical algorithm which determines a guaranteed multiplier based μ upper bound on a large frequency range, without missing any critical frequency. The use of the μ -sensitivities allows to drastically reduce the computation time, making this algorithm applicable to many realistic applications. The objective is to propose an interesting alternative to the classical (D,G)-scalings based μ upper bound when the latter is not accurate enough.

This paper is organized as follows. First, a brief review of the structured singular value and the (D,G)-scalings based μ upper bound is proposed in Section II. Then, the multiplier based μ upper bound is recalled in Section III and the μ -sensitivities are introduced to compute a slightly looser upper bound more quickly using a limited number of uncertainties. In Section IV a technique to validate the multipliers on large frequency intervals is presented. Section V finally shows the results obtained on several benchmarks.

II. PRELIMINARIES

A. Introduction to the structured singular value

Let us consider the standard interconnection of Fig. 1 where $M(s) \in \mathcal{RH}_\infty^{n \times n}$ is a stable linear time-invariant plant representing the nominal closed-loop system and Δ is a block-diagonal time-invariant operator gathering all the model uncertainties. This paper focuses on real parametric uncertainties, but this is not restrictive. Complex uncertainties such as neglected dynamics tend indeed to reduce conservatism without introducing a lot more optimization variables when computing μ upper bounds. The Δ matrix is thus of the form:

$$\Delta = \text{diag}(\delta_1 I_{k_1}, \dots, \delta_r I_{k_r}) \quad (1)$$

where the δ_i are real k_i -repeated scalars such that $\sum_{i=1}^r k_i = n$. Consider the set Δ which contains the $n \times n$ real matrices with the same structure as Δ in (1). Let then $\mathcal{B}_\Delta = \{\Delta \in \Delta : |\delta_i| \leq 1\}$ and $\bar{\mathcal{B}}_\Delta = \{\Delta \in \Delta : \delta_i = \pm 1\}$.

As is standard in robust stability analysis, the aim is to find the maximum value k_{max} such that the interconnection

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Fig. 1. Standard interconnection for robust stability analysis.

of Fig. 1 is stable for all $\Delta \in k_{max}\mathcal{B}_\Delta$. To do so the most relevant technique especially when dealing with high-order systems is μ -analysis, which relies on the structured singular value $\mu_\Delta(M)$ defined below.

Definition 1 (structured singular value). Let $M = M(j\omega)$ denote the frequency response of $M(s)$ at a given frequency ω . If no matrix $\Delta \in \Delta$ makes $I_n - \Delta M$ singular, then $\mu_\Delta(M) = 0$. Otherwise:

$$\mu_\Delta(M) = \frac{1}{\min \{k \in \mathbb{R}_+ : \exists \Delta \in k\mathcal{B}_\Delta, \det(I_n - \Delta M) = 0\}}$$

Theorem 1. The inverse of the maximum value of $\mu_\Delta(M(j\omega))$ over the frequency range $\mathbb{R}^+$ gives the robustness margin k_{max} :

$$k_{max} = \left[\sup_{\omega \in \mathbb{R}^+} \mu_\Delta(M(j\omega)) \right]^{-1}$$

Since the exact computation of μ_Δ is NP-hard in the general case, both lower and upper bounds are usually evaluated over the whole frequency range. This paper focuses on the upper bound.

B. (D,G)-scalings based μ upper bound

The most commonly used μ upper bound is recalled in Proposition 1, which is extracted from [12].

Proposition 1. Let β be a positive scalar and $M = M(j\omega)$ denote the frequency response of $M(s)$ at a given frequency ω . If there exist some scaling matrices $D \in \mathcal{D}_n$ and $G \in \mathcal{G}_n$ such that:

$$M^*DM + j(GM - M^*G) \leq \beta^2 D \quad (2)$$

where:

- $\mathcal{D}_n = \{D \in \mathbb{C}^{n \times n}, D = D^* > 0 : \forall \Delta \in \Delta, D\Delta = \Delta D\}$
- $\mathcal{G}_n = \{G \in \mathbb{C}^{n \times n}, G = G^* : \forall \Delta \in \Delta, G\Delta = \Delta G\}$

then $\mu_\Delta(M) \leq \beta$.

The minimization of β in (2) gives the so called (D,G)-scalings based μ upper bound, denoted $\bar{\mu}_\Delta(M)$ in the sequel.

The latter is only computed for one particular frequency ω in Proposition 1, but frequency sweeping techniques can be applied, such as the one presented in [10] and [1]. The frequency intervals are determined, for which $\bar{\mu}_\Delta(M)$, D and G still satisfy (2). Such a strategy is then repeated for a few other frequencies until the union of all intervals is equal to $\mathbb{R}^+$. A guaranteed μ upper bound over the whole frequency range is finally obtained. However, this bound is

sometimes conservative in the sense that the gap with respect to the exact value of μ is quite large. A less conservative μ upper bound is thus exposed in the next section.

III. THE MULTIPLIER BASED μ UPPER BOUND

A. Review of the method

In this section, the multiplier based μ upper bound introduced in [8] is reviewed. The main results are based on the following lemma.

Lemma 1. A matrix family $\mathcal{A}$ is nonsingular if there exists another matrix C (multiplier) such that $\text{He } C^*A < 0$ for all $A \in \mathcal{A}$, with $\text{He } X$ the hermitian part of the matrix X .

Proof of Lemma 1. Assume $\exists A \in \mathcal{A}$ such that A is singular. Then $\exists v \neq 0, Av = 0$. We thus get that $v^*(CA + A^*C^*)v = 0$, which means that $\text{He } C^*A$ is not negative definite. $\square$

For the considered case where Δ contains only purely real scalar blocks, the following theorem simply derives from Lemma 1 and Definition 1.

Theorem 2. ([8]) Let α be a positive scalar and $M = M(j\omega)$ denote the frequency response of $M(s)$ at a given frequency ω . If there exists a multiplier $C \in \mathbb{C}^{n \times n}$ such that:

$$\text{He } C^* (I_n - \Delta M) < 0 \quad (3)$$

for all $\Delta \in \frac{1}{\alpha}\mathcal{B}_\Delta$, then $\mu_\Delta(M) \leq \alpha$.

Equation (3) is linear in δ_i , so $\mathcal{B}_\Delta$ can be equivalently replaced with $\bar{\mathcal{B}}_\Delta$. The aim is thus to find the infimum $\alpha > 0$ such that (3) holds for each of the 2^r matrices $\Delta \in (1/\alpha)\bar{\mathcal{B}}_\Delta$, which can be formulated as a GEVP (Generalized EigenValue minimization Problem):

$$\nu_\Delta(M) = \min_{\alpha \in \mathbb{R}^+, C \in \mathbb{C}^{n \times n}} \alpha \quad (4)$$

subject to $\text{He } C^* (\alpha I_n - \Delta M) < 0, \forall \Delta \in \bar{\mathcal{B}}_\Delta$

where $\nu_\Delta(M)$ is called the multiplier based μ upper bound.

Note that, as already proposed in [8], if M has the following special structure:

$$M = AB^* \quad (5)$$

with $A, B \in \mathbb{C}^{n \times q}$ and $q = \text{rank}(M) \leq n$, then (3) can be equivalently replaced with:

$$\text{He } \hat{C}^* (I_q - B^* \Delta A) < 0 \quad (6)$$

where $\hat{C} \in \mathbb{C}^{q \times q}$. Hence, if M does not have full rank, solving (6) is obviously advantageous since the size of the multiplier is $q \times q$, rather than $n \times n$ in (3).

The following theorem then shows that $\nu_\Delta(M)$ is less conservative than the (D,G)-scalings based upper bound $\bar{\mu}_\Delta(M)$.

Theorem 3. (adapted from [8]) For every structure Δ and matrix M , $\nu_\Delta(M) \leq \bar{\mu}_\Delta(M)$.

To prove this, the following Lemma, from [9], is introduced:

Lemma 2. Given $U, V \in \mathbb{C}^{n \times n}$ and $Q = Q^* \in \mathbb{C}^{n \times n}$, there exists $T \in \mathbb{C}^{n \times n}$ such that:

$$Q + U^*TV + V^*T^*U < 0 \quad (7)$$

if and only if both of the following conditions are satisfied:

$$U_\perp^*QU_\perp < 0, \quad U_\perp^*QV_\perp < 0 \quad (8)$$

where $U_\perp$ and $V_\perp$ are any matrices whose columns form bases of the null spaces of U and V , respectively.

The proof of Theorem 3, inspired from [8], is now recalled.

Proof of Theorem 3. Let $\Phi_\beta(D, G) = M^*DM + j(GM - M^*G) - \beta^2D$. Two steps are needed to show that $\Phi_\beta(D, G) < 0$ implies $\text{He } C^*(I_n - \Delta M) < 0$ for all $\Delta \in (1/\beta)\bar{B}_\Delta$, thus proving that $\nu_\Delta(M) \leq \bar{\mu}_\Delta(M)$.

Step 1) Let us show that $\Phi_\beta(D, G) < 0$ if and only if there exists some $C \in \mathbb{C}^{n \times n}$ such that:

$$W(C, D, G) = \begin{pmatrix} C + C^* + M^*DM & C^* + jM^*G \\ C - jGM & -\beta^2D \end{pmatrix} < 0 \quad (9)$$

To do this, first rewrite (9) as follows:

$$W(C, D, G) = \begin{pmatrix} M^*DM & jM^*G \\ -jGM & -\beta^2D \end{pmatrix} + \begin{pmatrix} I_n \\ I_n \end{pmatrix} C \begin{pmatrix} I_n & 0 \end{pmatrix} + \begin{pmatrix} I_n \\ 0 \end{pmatrix} C^* \begin{pmatrix} I_n & I_n \end{pmatrix} < 0 \quad (10)$$

Notice then that $\begin{pmatrix} 0 \\ I_n \end{pmatrix}$ and $\begin{pmatrix} -I_n \\ I_n \end{pmatrix}$ form bases of the null spaces of $\begin{pmatrix} I_n & 0 \end{pmatrix}$ and $\begin{pmatrix} I_n & I_n \end{pmatrix}$ respectively. Using Lemma 2, (10) holds if and only if $-\beta^2D < 0$ and:

$$M^*DM + j(GM - M^*G) - \beta^2D < 0 \quad (11)$$

The first condition is redundant and (11) is the same as $\Phi_\beta(D, G) < 0$.

Step 2) Define $Z = \begin{pmatrix} I_n \\ -\Delta M \end{pmatrix}$. It is straightforward that:

$$Z^*WZ = C^*(I_n - \Delta M) + (I_n - \Delta M)^*C + M^*D^{1/2}(I - \beta^2\Delta^2)D^{1/2}M \quad (12)$$

since $D\Delta = \Delta D$ and $G\Delta = \Delta G$. Hence, $\Phi_\beta(D, G) < 0 \Leftrightarrow W(C, D, G) < 0$ implies that $\text{He } C^*(I_n - \Delta M) < 0$ for all $\Delta \in (1/\beta)\bar{B}_\Delta$, which shows that $\nu_\Delta(M) \leq \bar{\mu}_\Delta(M)$. $\square$

B. Use of the μ -sensitivities to reduce the computation time

The GEVP (4) involves 2^r LMIs, which can be computationally very costly if the number of uncertainties is large. A solution is to solve (4) only for a limited number of uncertainties $r_1 < r$ and to use the (D,G)-scalings based formulation for the $r - r_1$ remaining ones. The obtained upper bound $\nu_\Delta^p(M)$ is thus between $\nu_\Delta(M)$ and $\bar{\mu}_\Delta(M)$, yielding a slightly looser but more computationally-friendly bound than $\nu_\Delta(M)$. This solution is already proposed in [8], but with no way of efficiently choosing the r_1 blocks of Δ so that the reduction of the conservatism is the highest.

The μ -sensitivities [2] can be used to address this issue, which is the first contribution of this paper. Indeed, they allow to determine how the value of $\bar{\mu}_\Delta(M)$ changes when a single δ_j varies, and thus to isolate the most "critical" uncertainties. They are defined below.

Definition 2 (μ -sensitivity). Let $M = M(j\omega)$ denote the frequency response of $M(s)$ at a given frequency ω . Consider that Δ is pre- and post-multiplied by a diagonal matrix $A(\epsilon_j) = \text{diag}(I_{k_1}, \dots, (1 - \epsilon_j)I_{k_j}, \dots, I_{k_r})$ so that the j th uncertainty is reduced by a small factor $(1 - \epsilon_j)^2$ while the others stay at their nominal values. The j th μ -sensitivity can be defined as:

$$S_j^\mu = \frac{\partial}{\partial \epsilon_j} \bar{\mu}_\Delta(M_{\epsilon_j}) = \lim_{\epsilon_j \rightarrow 0^+} \frac{|\bar{\mu}_\Delta(M) - \bar{\mu}_\Delta(M_{\epsilon_j})|}{\epsilon_j}$$

with $M_{\epsilon_j} = A(\epsilon_j)MA(\epsilon_j)$ (note that $M_0 = M$).

Using the above formulation to compute the μ -sensitivities is numerically sensitive and is therefore not recommended. The algorithm proposed in [4] is preferred in this work.

The μ -sensitivities can be used to select the most "sensitive" r_1 blocks of Δ . M and Δ are thus reordered, so that $\Delta = \text{diag}(\Delta_1, \Delta_2)$ with $\Delta_1 \in \mathbf{\Delta}_1 \subset \mathbb{R}^{p \times p}$ containing the selected r_1 blocks ($p = \sum_{i=1}^{r_1} k_i$) and Δ_2 the other $r - r_1$ blocks. Let $D \in \mathcal{D}_{n-p}$ and $G \in \mathcal{G}_{n-p}$ be the scaling matrices associated to Δ_2 . A rank decomposition $M = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \begin{pmatrix} B_1^* & B_2^* \end{pmatrix}$ is also performed to reduce the number of optimization variables, where $A_1, B_1 \in \mathbb{C}^{p \times q}$, $A_2, B_2 \in \mathbb{C}^{(n-p) \times q}$ and $q = \text{rank}(M)$. The following problem must now be solved:

$$\begin{aligned} \nu_\Delta^p(M) &= \min_{\alpha \in \mathbb{R}^+, \hat{C} \in \mathbb{C}^{q \times q}, G \in \mathcal{G}_{n-p}, D \in \mathcal{D}_{n-p}} \alpha \\ \text{subject to } &\begin{pmatrix} L_{11} & \hat{C}^*B_2^* + jA_2^*G \\ * & -\alpha^2D \end{pmatrix} < 0, \quad \forall \Delta_1 \in \frac{1}{\alpha}\bar{B}_{\Delta_1} \end{aligned}$$

where $L_{11} = 2\text{He } \hat{C}^*(I_q - B_1^*\Delta_1A_1) + A_2^*DA_2$ (13)

With this new formulation there are only 2^{r_1} LMIs to solve instead of 2^r , and, thanks to the μ -sensitivities, only the most relevant ones are considered. Note that the solution of problem (13) is also a solution of the initial problem (6).

Theorem 4. (adapted from [8]) Suppose that there exist $\alpha > 0$ and $\hat{C} \in \mathbb{C}^{q \times q}$ such that (13) is satisfied. Then $\text{He } \hat{C}^*(I_q - B^*\Delta A) < 0$ for all $\Delta \in (1/\alpha)\bar{B}_\Delta$.

Proof of Theorem 4. Let $\alpha > 0$, $\hat{C} \in \mathbb{C}^{q \times q}$, $D \in \mathcal{D}_{n-p}$ and $G \in \mathcal{G}_{n-p}$ such that:

$$L(\hat{C}, D, G) = \begin{pmatrix} L_{11} & \hat{C}^*B_2^* + jA_2^*G \\ * & -\alpha^2D \end{pmatrix} < 0, \quad \forall \Delta_1 \in \frac{1}{\alpha}\bar{B}_{\Delta_1}$$

Define $Y = \begin{pmatrix} I_q \\ -\Delta_2A_2 \end{pmatrix}$. Simple calculations lead to:

$$\begin{aligned} Y^*LY &= \hat{C}^*(I_q - A^*\Delta B)^* + (I_q - A^*\Delta B)\hat{C} \\ &\quad + A_2^*D_2^{1/2}(I - \alpha^2\Delta_2^2)D_2^{1/2}A_2 \end{aligned} \quad (14)$$

Thus $\text{He } \hat{C}^*(I_q - B^* \Delta A) < 0$ for all $\Delta \in (1/\alpha) \bar{\mathcal{B}}_\Delta$. $\square$

IV. VALIDATION OF THE MULTIPLIERS ON LARGE FREQUENCY INTERVALS

The previous section is dedicated to the computation of the multiplier based μ upper bound at a given frequency. This computation can be done by considering either the full Δ matrix as in (4) or only a selected part, using the μ -sensitivities as in (13), to reduce the number of LMIs. Nevertheless, computing a robustness margin over the whole frequency range $[0, +\infty[$ is not obvious. Indeed, using a fine frequency grid can involve a very large computation time without the proof that no critical frequency is missed. In this section an algorithm is developed, which is the second contribution of this paper. It determines for which frequency intervals the multiplier C (or $\hat{C}$) and the corresponding μ upper bound computed at one particular frequency remain valid. It is based on the following theorem.

Theorem 5. *Let (A_M, B_M, C_M, D_M) be a state-space representation of $M(s)$ such that A_M is Hurwitz. Assume that there exist $\alpha > 0$ and $C \in \mathbb{C}^{n \times n}$ such that, for a given frequency ω and a given $\Delta \in \frac{1}{\alpha} \bar{\mathcal{B}}_\Delta$:*

$$\text{He } C^*(I_n - \Delta M(j\omega)) < 0 \quad (15)$$

Let $X = 2\text{He } C^(I_n - \Delta D_M)$. If X is nonsingular, then $\text{He } C^*(I_n - \Delta M(j\tilde{\omega}))$ is singular if and only if $j\tilde{\omega}$ is an eigenvalue of the following Hamiltonian matrix:*

$$H = \begin{pmatrix} A_M & 0 \\ 0 & -A_M^* \end{pmatrix} + \begin{pmatrix} B_M & \\ C_M^* \Delta C & \end{pmatrix} X^{-1} (C^* \Delta C_M - B_M^*)$$

Theorem 5 can be used to compute the frequency intervals $\mathcal{I}_v$ on which Equation (15) is satisfied for the considered uncertainty matrix Δ and multiplier C . Let $j\tilde{\omega}_1, \dots, j\tilde{\omega}_N$ be the purely imaginary eigenvalues of H such that $0 < \tilde{\omega}_1 < \dots < \tilde{\omega}_N$. It is obvious that $[0 \ \tilde{\omega}_1] \cup [\tilde{\omega}_1 \ \tilde{\omega}_2] \cup \dots \cup [\tilde{\omega}_{N-1} \ \tilde{\omega}_N] \cup [\tilde{\omega}_N + \infty[= \mathbb{R}^+$. It suffices to pick a frequency ω_t inside each of these $N + 1$ intervals and to test whether $\text{He } C^*(I_n - \Delta M(j\omega_t)) < 0$. $\mathcal{I}_v$ is composed of all the intervals for which this condition is satisfied.

Remark 1. The nonsingularity condition on X in Theorem 5 is not restrictive in practice. Indeed, the feedthrough matrix D_M is generally a null matrix when realistic applications are considered. In this case, X is always nonsingular, since $X = 2\text{He } C^* < 0$. Moreover, several benchmarks have been studied, for which D_M is not zero. In all cases, the condition on X was satisfied.

The following lemma is introduced in order to prove Theorem 5.

Lemma 3. *Let $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ be a matrix with P_{11} and P_{22} invertible. The following equalities hold:*

$$\begin{aligned} \det(P) &= \det(P_{11}) \det(P_{22} - P_{21} P_{11}^{-1} P_{12}) \\ &= \det(P_{22}) \det(P_{11} - P_{12} P_{22}^{-1} P_{21}) \end{aligned} \quad (16)$$

Proof of Lemma 3. The following equality holds:

$$P \begin{pmatrix} I & -P_{11}^{-1} P_{12} \\ 0 & I \end{pmatrix} = \begin{pmatrix} P_{11} & 0 \\ P_{21} & P_{22} - P_{21} P_{11}^{-1} P_{12} \end{pmatrix} \quad (17)$$

Taking the determinant of (17) then leads to:

$$\begin{aligned} \det(P) &= \det \left(\begin{pmatrix} P_{11} & 0 \\ P_{21} & P_{22} - P_{21} P_{11}^{-1} P_{12} \end{pmatrix} \right) \\ &= \det(P_{11}) \det(P_{22} - P_{21} P_{11}^{-1} P_{12}) \end{aligned} \quad (18)$$

The second equality is obtained the same way by post-multiplying P by $\begin{pmatrix} I & 0 \\ -P_{22}^{-1} P_{21} & I \end{pmatrix}$. $\square$

Proof of Theorem 5. Suppose that there exist $\alpha > 0$ and $C \in \mathbb{C}^{n \times n}$ such that $F(j\omega) = 2\text{He } C^*(I_n - \Delta M(j\omega)) < 0$. The following equality holds:

$$F(j\omega) = \begin{pmatrix} (j\omega I - A_M)^{-1} B_M \\ I \end{pmatrix}^* \begin{pmatrix} 0 & -C_M^* \Delta C \\ -C^* \Delta C_M & X \end{pmatrix} \begin{pmatrix} (j\omega I - A_M)^{-1} B_M \\ I \end{pmatrix} \quad (19)$$

with $X = 2\text{He } C^*(I_n - \Delta D_M)$. Then define the matrix:

$$Q = \begin{pmatrix} j\tilde{\omega} I - A_M & 0 & -B_M \\ 0 & j\tilde{\omega} I + A_M^* & -C_M^* \Delta C \\ -C^* \Delta C_M & B_M^* & X \end{pmatrix} \quad (20)$$

Since A_M is Hurwitz, $j\tilde{\omega} I - A_M$ and $j\tilde{\omega} I + A_M^*$ are nonsingular. By assumption X is also nonsingular. Applying Lemma 3 finally leads to:

$$\begin{aligned} \det(Q) &= \det(j\tilde{\omega} I - H) \underbrace{\det(X)}_{\neq 0} \\ &= \underbrace{\det(j\tilde{\omega} I + A_M^*)}_{\neq 0} \det(F(j\tilde{\omega})) \underbrace{\det(j\tilde{\omega} I - A_M)}_{\neq 0} \end{aligned}$$

after a few matrix manipulations. Hence, $F(j\tilde{\omega})$ is singular if and only if $j\tilde{\omega}$ is an eigenvalue of H . $\square$

Theorem 5 is only valid for a multiplier C with full rank n . For the case where a multiplier $\hat{C} \in \mathbb{C}^{q \times q}$, $q < n$, is computed (e.g. using (6) or (13)), the following theorem is applied first to derive a valid full-rank multiplier $C \in \mathbb{C}^{n \times n}$ so that Theorem 5 can then be used.

Theorem 6. *Let $M = M(j\omega)$ denote the frequency response of $M(s)$ at a given frequency ω . Consider the reduced-rank structure $M = AB^*$, with $A, B \in \mathbb{C}^{n \times q}$ and $q = \text{rank}(M)$. Assume that there exist $\hat{C} \in \mathbb{C}^{q \times q}$ and $\alpha > 0$ such that:*

$$\text{He } \hat{C}^*(I_q - B^* \Delta A) < 0, \quad \forall \Delta \in \frac{1}{\alpha} \bar{\mathcal{B}}_\Delta \quad (21)$$

Let $C = B \hat{C} B^ - \epsilon E E^* \in \mathbb{C}^{n \times n}$, with $\epsilon > 0$ small enough and $E \in \mathbb{C}^{n \times (n-q)}$ such that $\begin{pmatrix} B & E \end{pmatrix}$ has full rank. Then:*

$$\text{He } C^*(I_n - \Delta M) < 0, \quad \forall \Delta \in \frac{1}{\alpha} \bar{\mathcal{B}}_\Delta \quad (22)$$

TABLE I
COMPARISON BETWEEN THE (D,G)-SCALINGS AND THE MULTIPLIER BASED APPROACHES

Benchmark	Description	States	Structure of Δ	$\underline{\mu}_\Delta$	(D,G)-scalings			Multiplier				
					$\bar{\mu}_\Delta$	Time (s)	η	r_1	ν_Δ^p	Time (s)	Iterations	η
1	Academic example	4	2×2	1.711	2.413	1	41.03%	2	1.732	42	27	1.23%
2	Aeronautical vehicle	8	4×1	1.620	2.067	6	27.59%	2	1.850	120	50	14.20%
								4	1.622	477	89	0.09%
3	Rigid aircraft	9	14×1	0.184	0.229	11	24.46%	3	0.222	100	4	20.38%
								5	0.208	133	4	12.88%
								9	0.194	1190	6	5.34%

Proof of Theorem 6. Let $C = B\hat{C}B^* - \epsilon EE^*$. The following equality holds:

$$\begin{pmatrix} B & E \end{pmatrix} \begin{pmatrix} \hat{C}^*(I_q - B^*\Delta A) & 0 \\ \epsilon E^*\Delta A & -\epsilon I \end{pmatrix} \begin{pmatrix} B^* \\ E^* \end{pmatrix} \quad (23)$$

Hence:

$$\begin{pmatrix} B & E \end{pmatrix} \underbrace{\begin{pmatrix} \text{He } \hat{C}^*(I_q - B^*\Delta A) & \frac{\epsilon}{2} A^*\Delta E \\ \frac{\epsilon}{2} E^*\Delta A & -\epsilon I \end{pmatrix}}_R \begin{pmatrix} B^* \\ E^* \end{pmatrix} \quad (24)$$

Applying the Schur complement then leads to:

$$R < 0 \Leftrightarrow \begin{cases} -\epsilon I < 0 \\ \text{He } \hat{C}^*(I_q - B^*\Delta A) + \frac{\epsilon}{4}(A^*\Delta E)(A^*\Delta E)^* < 0 \end{cases} \quad (25)$$

The first condition on the right part of (25) is redundant, and the second one is obviously satisfied for a small enough $\epsilon > 0$, yielding $R < 0$. And since by assumption $\begin{pmatrix} B & E \end{pmatrix}$ has full rank, it can be claimed that $\text{He } C^*(I_n - \Delta M) < 0$. $\square$

Remark 2. Numerically speaking, a full-rank multiplier C is thus quickly computed from $\hat{C}$ using the above theorem. Indeed it suffices to choose a small enough ϵ and a random complex matrix E such that $\begin{pmatrix} B & E \end{pmatrix}$ has full rank. Note that the latter assumption is not restrictive and that a suitable E can be easily obtained.

The following algorithm is now proposed to compute a guaranteed multiplier based μ upper bound on the whole frequency range.

Algorithm 1.

- 1) Let $\mathcal{I} = \mathbb{R}^+$ be the initial list of frequency intervals to be investigated. Define the number $r_1 \leq r$ of uncertainties to be considered when using the μ -sensitivities (see Section III-B). Set $\nu = 0$.
- 2) While $\mathcal{I} \neq \emptyset$, repeat:
 - a) Choose an interval $I \in \mathcal{I}$ and a frequency $\omega \in I$.
 - b) Solve problem (13) to compute the μ upper bound $\nu_\Delta^p(M(j\omega))$ and a multiplier $\hat{C} \in \mathbb{C}^{q \times q}$.
 - c) If $q < n$, use Theorem 6 to find the corresponding full-rank multiplier $C \in \mathbb{C}^{n \times n}$. Otherwise, set $C = \hat{C}$.

- d) Set $\nu = \max(\nu, \nu_\Delta^p(M(j\omega)))$. Apply Theorem 5 for each uncertainty matrix $\Delta_j \in (1/\nu)\bar{\mathcal{B}}_\Delta$ to compute the frequency intervals $\mathcal{I}_{\nu_j}$ on which the upper bound ν and the multiplier C are valid.
- e) Compute $\mathcal{I}_\nu = \bigcap_j \mathcal{I}_{\nu_j}$. Update $\mathcal{I} \leftarrow \mathcal{I} \setminus \mathcal{I}_\nu$.
- 3) $\nu_\Delta^p = \nu$ is a guaranteed μ upper bound on $\mathbb{R}^+$.

V. NUMERICAL RESULTS

To demonstrate the efficiency of the proposed method, three benchmarks issued from the Systems Modeling Analysis and Control (SMAC) Toolbox for Matlab [11] are considered in this paper. They are of interest since the (D,G)-scalings based approach of [12] does not give satisfactory μ upper bounds. To reduce the conservatism, Algorithm 1 is applied and the results are presented in Table I which contains:

- the description of each benchmark,
- the number of state variables,
- the structure of the uncertainty matrix Δ ($m \times p$ means that there are m real diagonal blocks of size $p \times p$),
- the best μ lower bound $\underline{\mu}_\Delta$ on $\mathbb{R}^+$ among all existing algorithms [6],
- the (D,G)-scalings based μ upper bound $\bar{\mu}_\Delta$ on $\mathbb{R}^+$ with the corresponding computation time and the gap η between $\underline{\mu}_\Delta$ and $\bar{\mu}_\Delta$,
- the multiplier based μ upper bound ν_Δ^p obtained using Algorithm 1, with the number of optimized blocks r_1 (see (13)), the computation time, the number of iterations needed to validate the bound on $\mathbb{R}^+$, and the gap η between $\underline{\mu}_\Delta$ and ν_Δ^p .

Globally the conservatism is drastically reduced with a reasonable computation time. Note that for benchmark 3 the use of the reduced-rank structure of M greatly decreases the computational burden. This property can be exploited for many realistic systems.

It is also interesting to notice that the number of iterations to validate the multiplier based μ upper bound on the whole frequency range is relatively small. On the contrary, using the standard method of [8] with a frequency grid usually requires at least 100 frequencies to get reasonably accurate results. This implies a much larger computation time in addition to an impossibility to ensure that no critical μ peak is missed. An illustration is given in Table II for the first benchmark using

both a fine and a rough frequency grid. In the first case, the computation time is clearly larger than the one from Table I. In the second case, the computation time remains close to the one obtained with Algorithm 1, but the μ upper bound is not valid since it is lower than the best known μ lower bound.

TABLE II
MULTIPLIER BASED μ UPPER BOUND USING A FREQUENCY GRID

Benchmark	Grid	ν_{Δ}^p	Time (s)
1	Fine (100 frequencies)	1.7255	115
	Rough (30 frequencies)	1.5715	38

Finally 50 runs of Algorithm 1 are performed on benchmarks 2 and 3 so that at each iteration, the r_1 uncertainty blocks are randomly chosen in step 2)b). The corresponding results are presented in Table III, which displays the minimum, the mean, and the maximum values of the gap η between $\underline{\mu}_{\Delta}$ and ν_{Δ}^p , and compares them with the gap obtained using the μ -sensitivities. This clearly shows the relevance of using the μ -sensitivities.

TABLE III
RESULTS SHOWING THE RELEVANCE OF USING THE μ -SENSITIVITIES

Benchmark	r_1	Gap η between $\underline{\mu}_{\Delta}$ and ν_{Δ}^p (%)	
		With the μ -sensitivities	Min/mean/max values for random choices of the r_1 uncertainties (50 runs)
2	2	14.20	20.55/20.84/20.93
3	5	12.88	10.35/18.27/22.89
	9	5.34	5.34/11.49/17.88

VI. CONCLUSION

An alternative to the classical (D,G)-scalings based approach giving a less conservative μ upper bound is developed in this paper. It extends the multiplier based method of [8]: the bounds and the multipliers can now be validated on large frequency intervals, whose union covers the whole frequency range. The resulting robustness margin is guaranteed and no critical frequency is missed, which is not true with a classical grid-based technique. Moreover, the rather low number of iterations and the ability of the proposed validation technique to exploit a possible low-rank structure of the system drastically reduce the computation time. Finally, introducing the μ -sensitivities in the proposed algorithm allows to focus on the most relevant uncertainties only. It significantly reduces the size of the LMI problems to be solved and the associated computation time with only a slight loss of accuracy.

The proposed method is successfully validated on three benchmarks. It may reach its limits if the number of uncertainties becomes much larger, due to the large number of LMIs and optimization variables. Nevertheless, it can be quite useful in many practical applications where the (D,G)-scalings based approach is not accurate enough.

It must finally be emphasized that only real parametric uncertainties are considered in this paper. Using the more general formulations of [8], it is very easy to extend the method to the mixed real/complex and the purely complex cases (involving neglected dynamics). But this is probably unnecessary since conservatism is usually not an issue for such applications. Indeed, the faster (D,G)-scalings based approach typically gives a sufficiently accurate μ upper bound, so that the present method is not needed.

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