Explicit generators of some pro-p groups via Bruhat-Tits theory

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Abstract

Given a semi-simple group over a local field of residual characteristic p, its topological group of rational points admits maximal pro-p subgroups. Those of quasi-split simply-connected semi-simple groups can be described in the combinatorial terms of a valued root groups datum, thanks to the Bruhat-Tits theory. In this context, it becomes possible to compute explicitly a minimal generating set of the (all conjugated) maximal pro-p subgroups thanks to parametrizations of a suitable maximal torus and of the corresponding root groups. We show that the minimal number of generators is then linear with respect to the rank of a suitable root system.

Contents

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1 Introduction

In this paper, a smooth connected affine group scheme of finite type over a field K will be called a K-group. Given a base field K and a K-group denoted by G, we get an abstract group called the group of rational points, denoted by $G(K)$. When K is a non-Archimedean local field, this group inherits a topology from the field. In particular, the topological group $G(K)$ is totally disconnected and locally compact. The maximal compact or pro-p subgroups of such a group $G(K)$, when they exist, provide a lot of examples of profinite groups. Thus, one can investigate maximal pro-p subgroups from the profinite group theory point of view.

1.1 Minimal number of generators

When H is a profinite group, we say that H is **topologically generated** by a subset X if H is equal to its smallest closed subgroup containing X;

such a set X is called a **generating set**. We investigate the minimal number of generators of a maximal pro-p subgroup of the group of rational points of an algebraic group over a local field.

Suppose that $K = \mathbb{F}_q((t))$ is a nonzero characteristic local field, where $q = p^m$ and G is a simple K-split simply-connected K-group of rank l. By a recent result of Capdeboscq and Rémy [CR14, 2.5], we know that any maximal pro-p subgroup of $G(K)$ admits a finite generating set X; moreover, the minimal number of elements of such an X is $m(l+1)$.

We would like to generalize this result to more general algebraic groups defined over any local field. If G is a smooth algebraic group scheme over a local field K of residual characteristic p, we know by [Loi16, 1.4.3] that $G(K)$ admits maximal pro-p subgroups (called pro-p Sylows) if, and only if, G is quasi-reductive (that means the split unipotent radical over K of G is trivial). When K is of characteristic 0, this corresponds to reductive groups because a unipotent group is always split over a perfect field. To provide explicit descriptions of a pro-p Sylow thanks to Bruhat-Tits theory, we restrict the study to the case of a simply-connected quasi-split semi-simple group G over a local field K.

Such a group G can be decomposed as a direct product of K-simple groups. Moreover, by [BT65, 6.21] for a simply connected group, we know that for any simply-connected K-simple group H, there exists a finite extension of local fields K'/K and an absolutely simple K'-group H' such that H is isomorphic to the Weil restriction $R_{K'/K}(H')$, that means H' seen as a K-group. Since $H(K) = H'(K')$ by definition of the Weil restriction, we will assume that the simply-connected semi-simple group G is absolutely simple.

In the Bruhat-Tits theory, given a reductive K-group G, we define a poly-simplicial complex $X(G,K)$ (a Euclidean affine building), called the Bruhat-Tits building of G over K together with a suitable action of $G(K)$ onto $X(G,K)$. There exists an unramified extension K'/K such that the K-group G quasi-splits over K'. There are two steps in the theory. The first part, corresponding to chapter 4 of [BT84], provides the building $X(G',K')$ of $G_{K'}$ by gluing together affine spaces, called apartments. The second part, corresponding to chapter 5 of [BT84], applies a Galois descent to the base field K, using fixed point theorems.

In the non quasi-split case, the geometry of the building does not faithfully reflect the structure of the group. There is an anisotropic kernel of the action of $G(K)$ on $X(G,K)$. As an example, when G is anisotropic over K, its Bruhat-Tits building is a point; the Bruhat-Tits theory completely fails to be explicit in combinatorial terms for anisotropic groups. Thus, the general case may require, moreover, arithmetical methods. Hence, to do explicit computations with a combinatorial method based on Lie theory, we have to assume that G contains a torus with enough characters over K. More precisely, we say that a reductive group G is **quasi-split** if it admits a Borel subgroup defined over K or, equivalently, if the centralizer of any maximal

K-split torus is a torus [BT84, 4.1.1].

Now, assume that K is any non-Archimedean local field of residual characteristic $p \neq 2$ and residue field $\kappa \simeq \mathbb{F}_q$ where $q = p^m$. Let G be an absolutely-simple simply-connected quasi-split K-group.

1.1.1 Theorem. *Denote by l the rank of the relative root system of G, and by n the rank of the absolute root system of G. Assume that $l \geq 2$. If G has a relative root system Φ of type G_2 or BC_l, assume that $p \neq 3$. Let P be a maximal pro-p subgroup of $G(K)$. Denote by $d(P)$ the minimal number of generators of P. Then, we have:*

$$d(P) = m(l+1) \text{ or } m(n+1)$$

depending on whether the minimal splitting field extension of short roots is ramified or not.

This theorem is formulated more precisely and proven in Corollary 5.2.2. According to [Ser94, 4.2], we know that $d(P)$ can also be computed via cohomological methods: $d(P) = \dim_{\mathbb{Z}/p\mathbb{Z}} H^1(P, \mathbb{Z}/p\mathbb{Z}) = \dim_{\mathbb{Z}/p\mathbb{Z}} \mathrm{Hom}(P, \mathbb{Z}/p\mathbb{Z})$.

From now on, we need to be more explicit. In the following, given a local field L, we denote by ω_L the discrete valuation on L, by $\mathcal{O}_L$ the ring of integers, by $\mathfrak{m}_L$ its maximal ideal, by ϖ_L a uniformizer, and by $\kappa_L = \mathcal{O}_L/\mathfrak{m}_L$ the residue field. Because we have to compare valuations of elements in L^*, we will normalize the discrete valuation $\omega_L : L^* \to \mathbb{Q}$ so that $\omega_L(L^*) = \mathbb{Z}$. When $l \in \mathbb{R}$, we denote by $\lfloor l \rfloor$ the largest integer less than or equal to l and by $\lceil l \rceil$ the smallest integer greater than or equal to l.

If it is clear in the context, we can omit the index L in these notations. When L/K is an extension, we denote by G_L the extension of scalars of G from K to L. When H is an algebraic L-group, we denote by $R_{L/K}(G)$ the K-group obtained by the Weil restriction functor $R_{L/K}$ defined in [DG70, I§1 6.6].

1.2 Pro-p Sylows and their Frattini subgroups

Let K be a non-Archimedean local field and G be a semi-simple K-group. We consider a maximal pro-p subgroup P of $G(K)$. When G is simply connected, we know by Bruhat-Tits theory [Loi16, 1.5.3], that there exists a model $\mathcal{G}$, that means a $\mathcal{O}_K$-group with generic fiber $\mathcal{G}_K = G$, such that we can identify P with the kernel of the natural surjective quotient morphism $\mathcal{G}(\mathcal{O}_K) \to \Big(\mathcal{G}_\kappa / \mathcal{R}_u(\mathcal{G}_\kappa)\Big)(\kappa)$. More precisely, $\mathcal{G}$ is a model of an Iwahori subgroup corresponding to the unique maximal facet (the alcove) fixed by P. The reductive group $\Big(\mathcal{G}_\kappa / \mathcal{R}_u(\mathcal{G}_\kappa)\Big)$ is, in fact, a κ-torus. Indeed, it is quasi-split since κ is a finite field and its root system over κ given by [BT84, 4.6.12 (i)] is empty. Thus, in other words, the pro-p Sylow P is the inverse image of a p-Sylow along the surjective homomorphism $\mathcal{G}(\mathcal{O}_K) \to \mathcal{G}(\kappa)$.

To compute the minimal number of generators, the theory of profinite groups provides a method consisting of computing the Frattini subgroup. The Frattini subgroup of a pro-p group P consists of non-generating elements and can be written as $\mathrm{Frat}(P) = \overline{[P,P]P^p}$, the smallest closed subgroup generated by p-powers and commutators of elements of P [DdSMS99, 1.13]. Once the elements of the group $\mathrm{Frat}(P)$ has been determined in order to obtain the structure of the quotient group $P/\mathrm{Frat}(P)$, it becomes immediate to provide a minimal topologically generating set X of P, arising from a finite generating set of $P/\mathrm{Frat}(P)$.

From this writing, we observe that the computation of the Frattini subgroup of P is mostly the computation of its derived subgroup. Despite P is close to be an Iwahori subgroup I of $G(K)$ (in fact, $I = \mathcal{N}_{G(K)}(P)$ is an Iwahori subgroup and P has finite index in I), we cannot use the results of [PR84, §6] because there are fewer semi-simple elements in P than in I. However, computations of Section 4 have some similarities with computations of Prasad and Raghunathan.

A question that is not considered here is to study the minimal presentations of a pro-p group. We say that P is finitely presented as pro-p group if there exists a closed normal subgroup R of the free pro-p group $\widehat{F_n}^p$ generated by n elements such that $P \simeq \widehat{F_n}^p/R$ and R is finitely generated as a pro-p group. Let $r(P)$ be the minimum of all the $d(R)$ among the R and $n \geq d(P)$. According to [Ser94, 4.3], P is finitely presented as pro-p group if, and only if P is finitely generated as a pro-p-group and $H^2(P,\mathbb{Z}/p\mathbb{Z})$ is finite. In this case, we get $r(P) = \dim_{\mathbb{Z}/p\mathbb{Z}} H^2(P,\mathbb{Z}/p\mathbb{Z})$ and, for any R, we have $d(R) = n - d(P) + r(P)$. Note that $r(P)$ does not depend on the choice of a generating set and we can choose simultaneously a minimal generating set and a minimal family of relations. More generally, Lubotzky has shown [Lub01, 2.5] that any finitely presented profinite group P can be presented by a minimal presentation as a profinite group. If we can show that $H^2(P,\mathbb{Z}/p\mathbb{Z})$ is finite, then, by [Wil99, 12.5.8], we get the Golod-Shafarevich inequality $r(P) \geq \frac{d(P)^2}{4}$. This has to be the case according to study of $\mathcal{O}_K$-standard groups of Lubotzky and Shalev [LS94, 5.2], at least for simply connected split groups in positive characteristic.

Here, the main result is a description of the Frattini subgroup of P, denoted by $\mathrm{Frat}(P)$, in terms of valued root groups datum. We assume that K is a non-Archimedean local field of residue characteristic p and that G is a semi-simple and simply-connected quasi-split K-group. To simplify the statements, we assume, moreover, that G is absolutely almost simple; this is equivalent to assuming that the absolute root system $\widetilde{\Phi}$ is irreducible. We know that it is possible to describe a maximal pro-p subgroup P of $G(K)$ in terms of the valued root groups datum [Loi16, 3.2.9]. A maximal polysimplex of the building $X(G,K)$, seen as poly-simplicial complex, is called an alcove. Any maximal pro-p subgroup of $G(K)$ fixes a unique alcove $\mathbf{c}$

that is a fundamental domain of the action of $G(K)$ on $X(G,K)$. It is then possible to describe the Frattini subgroup in terms of the valued root groups datum, as stated in the following theorem:

1.2.1 Theorem. *We assume that $p \neq 2$ and we denote by Φ the relative root system of G over the ground field K. If Φ is of type G_2 or BC_l, we assume that $p \geq 5$.*

Then the pro-p group P is topologically of finite type and, in particular, $\mathrm{Frat}(P) = P^p[P,P]$. Moreover, when Φ is not of type BC_1, we have $P^p \subset [P,P]$.

The Frattini subgroup $\mathrm{Frat}(P)$ can be written as the image of the map induced by multiplication from a direct product of some groups expressed in terms of the valuation of a root group datum.

When Φ is reduced (that means is not of type BC_l), then $\mathrm{Frat}(P)$ is the maximal pro-p subgroup of the pointwise stabilizer in $G(K)$ of the combinatorial ball centered at $\mathbf{c}$ of radius 1 (see Definition 3.1.11).

For a more precise version, see Theorems 5.1.1 and 5.1.2.

1.3 Structure of the paper

We assume that G is a simply-connected quasi-split absolutely simple K-group. We fix a Borel subgroup B of G defined over K. By [Bor91, 20.5, 20.6 (iii)], there exists a maximal K-split torus S in G such that its centralizer, denoted by $T = \mathcal{Z}_G(S)$, is a maximal K-torus of G contained in B. We fix a separable closure K_s of K; by [Bor91, 8.11], there exists a unique smallest Galois extension of K, denoted by $\widetilde{K}$, splitting T, hence also splitting G by [Bor91, 18.7]. We call **the relative root system**, denoted by Φ, the root system of G relatively to S. We call **the absolute root system**, denoted by $\widetilde{\Phi}$, the root system of $G_{\widetilde{K}}$ relatively to $T_{\widetilde{K}}$. In particular, the choice of B determines an order Φ^+ of the root system Φ and a basis Δ.

In Section 2.1.2, we recall the definition of a $\mathrm{Gal}(K_s/K)$-action on $\widetilde{\Phi}$ which preserves the Dynkin diagram structure of $\mathrm{Dyn}(\widetilde{\Delta})$ and on its basis $\widetilde{\Delta}$ corresponding to the Borel subgroup B. According to [BT84, 4.1.16], when G is absolutely simple (hence $\mathrm{Dyn}(\widetilde{\Delta})$ is connected), the group $\mathrm{Aut}\big(\mathrm{Dyn}(\widetilde{\Delta})\big)$ is a finite group of order $d \leq 6$. As a consequence, the degree of each splitting field extension is small and does not interact a lot with Lie theory. One can note that a major part of proofs in this paper is taken by the non-reduced BC_l cases and the trialitarian D_4 cases.

From this action and thanks to a rank 1 consideration, we define, according to [BT84, §4.2], a coherent system of parametrizations of root groups in Section 2.1.3 together with a filtration of the root groups in Section 2.1.4. This provides us with a valued root groups datum $\Big(T(K), \big(U_a(K), \varphi_a\big)_{a\in\Phi}\Big)$ built from $(G,S,K,\widetilde{K})$. This filtration corresponds to a prescribed affinisation of the spherical root system Φ. From this, we compute, in Sections 2.2

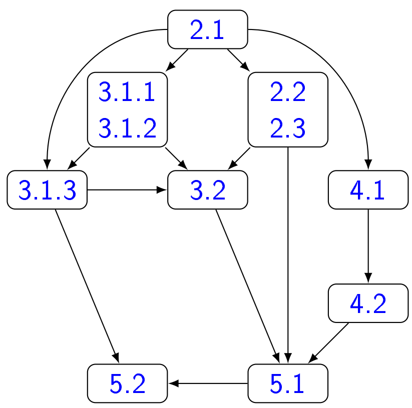
and 2.3, various commutation relations between unipotent and semi-simple elements in rank 1. This will be useful to describe, in Section 3.2, the action of P onto a combinatorial ball centered at $\mathbf{c}$ of radius 1. This will also be useful in Section 5.1 to generate semi-simple elements of $\mathrm{Frat}(P)$.

We denote by $\mathbb{A} = A(G, S, K)$ the "standard" apartment and we choose an alcove $\mathbf{c}_{\mathrm{af}} \subset \mathbb{A}$, to be a fundamental domain of the action of $G(K)$ on $X(G, K)$. Those objects will be described in Section 3.1.1 and 3.1.2 respectively thanks to the sets of values, defined in Section 2.1.5, which measure where the gaps between two terms of the filtration are and, in the non-reduced case, what kind of gaps we must deal with. From this, we deduce, in Section 3.1.3, the geometrical description of the combinatorial ball centered at $\mathbf{c}$ of radius 1. Consequently, the geometric situation provides, in Section 3.2, an upper bound for $\mathrm{Frat}(P)$, that means a group Q containing $\mathrm{Frat}(P)$.

Thus, we seek a generating set of Q contained in $\mathrm{Frat}(P)$. As the Frattini subgroup can be expressed as $\mathrm{Frat}(P) = \overline{P^p[P, P]}$, we seek such a generating set by commuting elements of P. In Section 4.1, we invert the commutation relations provided by [BT84, A] in the quasi-split case from which we deduce, in Section 4.2, a list of unipotent elements contained in $[P, P]$.

From these unipotent elements and from semi-simple elements obtained by the rank 1 case, we obtain, in Section 5.1, a generating set and a description of the Frattini subgroup as a product of groups. In Section 3.1.3, we go a bit further than Bruhat-Tits in the study of quotient subgroups of filtered root groups. From this, we can compute the finite quotient $P/\mathrm{Frat}(P)$ and deduce, in Section 5.2, a minimal generating set of P. The minimal number of elements of such a family is stated in Corollary 5.2.2.

We summarize this in the following graph:



2 Rank 1 subgroups inside a valued root group datum

We keep notations of Section 1.3. In particular, we always denote by K a field and by G a quasi-split absolutely simple K-group. From Section 2.1.4, we will assume that K is a non-Archimedean local field, and we will assume that G is simply-connected. In order to compute the Frattini subgroup of a maximal pro-p subgroup of $G(K)$, we adopt the point of view of valued root groups datum. In Section 2.1, we recall how we define a valuation on root groups, and how these groups can be parametrized. Thanks to these parametrizations, given in Section 2.1.3, we compute explicitly, in Sections 2.2 and 2.3, the various possible commutators, and the p-powers of elements in a rank 1 subgroup corresponding to a given root. The rank 1 case is not only useful to define filtrations of root groups, but also useful to compute elements in the Frattini subgroup corresponding to elements of the maximal torus T. There are exactly two root systems of rank 1, up to isomorphism, whose types are named A_1 and BC_1, corresponding to groups SL_2 (Section 2.2) and $\mathrm{SU}(h) \subset \mathrm{SL}_3$ (Section 2.3) respectively.

We denote by $T(K)_b$ the maximal bounded subgroup of $T(K)$, defined in [BT84, 4.4.1]. We denote by $T(K)_b^+$ the (unique) maximal pro-p subgroup of $T(K)_b$.

2.1 Valued root groups datum

We want to describe precisely the derived group of a maximal pro-p subgroup. We do it in combinatorial terms, thanks to a filtration of root groups. Because we have to deal with non-reduced root systems, we recall the following definitions:

2.1.1 Definition. Let Φ be a root system. A root $a \in \Phi$ is said to be **multipliable** if $2a \in \Phi$; otherwise, it is said to be **non-multipliable**. A root $a \in \Phi$ is said to be **divisible** if $\frac{1}{2}a \in \Phi$; otherwise, it is said to be **non-divisible**.

The set of non-divisible roots, denoted by Φ_{nd}, is a root system; the set of non-multipliable roots, denoted by Φ_{nm}, is a root system.

2.1.1 Root groups datum

For each root $a \in \Phi$, there is a unique unipotent subgroup U_a of G whose Lie algebra is the weight subspace with respect to a. In order to define an action of $G(K)$ on a spherical building with suitable properties, it suffices to have suitable relations of the various root groups $U_a(K)$. These required relations are the axioms given in the definition [BT72, 6.1.1] of a root groups datum.

Now, given a connected reductive group G over a field K, with a relative root system Φ, we provide a root groups datum of $G(K)$. Let $a \in \Phi$. By [Bor91, 14.5 and 21.9], there exists a unique closed K-subgroup of G, denoted by U_a, which is connected, unipotent, normalized by T and whose Lie algebra is $\mathfrak{g}_a$ if the root a is non-multipliable and $\mathfrak{g}_a + \mathfrak{g}_{2a}$ if the root a is multipliable, where $\mathfrak{g}_a$ denotes the root subspace of the Lie algebra $\mathfrak{g}$ of G with respect to a, as defined in [Bor91, 8.17]. This group U_a is called the **root group** of G associated to a. By [BT84, 4.1.19], there exists cosets M_a such that $\Big(T(K), \big(U_a(K), M_a\big)_{a\in\Phi}\Big)$ is a generating root groups datum of $G(K)$ of type Φ.

2.1.2 The Galois action on the absolute root system and splitting extension fields of root groups

As before, for simplicity, G is a quasi-split absolutely simple K-group. As in Section 1.3, we denote by $\widetilde{K}$ the minimal splitting field of G over K (uniquely defined in a given separable closure K_s of K).

We consider the canonical action of the absolute Galois group $\Sigma = \mathrm{Gal}(K_s/K)$ on the abstract group $G(K_s)$. Since G is quasi-split, we can choose a maximal K-split torus S and we get a maximal torus $T = \mathcal{Z}_G(S)$ of G defined over K. Thus, we define an action of Σ on $X^*(T_{K_s})$ by:

$$\forall \sigma \in \Sigma,\ \forall \chi \in X^*(T_{K_s}),\ \sigma \cdot \chi = t \mapsto \sigma\Big(\chi\big(\sigma^{-1}(t)\big)\Big)$$

2.1.2 Notation (The Galois action on the absolute root system)**.** This is a summary of [BT65, §6] for a quasi-split absolutely simple group G. Denote by $\widetilde{\Delta}$ the set of absolute simple roots corresponding to the Borel subgroup B and by $\mathrm{Dyn}(\widetilde{\Delta})$ its associated Dynkin diagram. The above action induces an action of the Galois group $\Sigma = \mathrm{Gal}(\widetilde{K}/K)$ on $\mathrm{Dyn}(\widetilde{\Delta})$ which preserves the diagram structure. This action can be extended, by linearity, to an action of Σ on $\widetilde{V}^* = X^*(T_{\widetilde{K}}) \otimes_{\mathbb{Z}} \mathbb{R}$, and on $\widetilde{\Phi}$. The restriction morphism $j = \iota^* : X^*(T) \to X^*(S)$, where $\iota : S \subset T$ is the inclusion morphism, can be extended to an endomorphism of the Euclidean space $\rho : \widetilde{V}^* \to \widetilde{V}^*$. This morphism ρ is the orthogonal projection onto the subspace V^* of fixed points by the action of Σ on $\widetilde{V}^*$. From the inclusion of $\widetilde{\Phi}$ in the Euclidean space $\widetilde{V}^*$ providing a geometric realization of absolute roots, we deduce a geometric realization of $\Phi = \rho(\widetilde{\Phi})$ in V^*. The orbits of the action of Σ on $\widetilde{\Phi}$ are the fibers of the map $\rho : \widetilde{\Phi} \to \Phi$.

2.1.3 Notation (A separable field extension)**.** According to [BT84, 4.1.2], by definition of $\widetilde{K}$ as minimal splitting extension, the action of $\Sigma = \mathrm{Gal}(\widetilde{K}/K)$ on $\mathrm{Dyn}(\widetilde{\Delta})$ is faithful since G is semi-simple. Because we assumed that G is absolutely simple, its absolute root system $\widetilde{\Phi}$ is irreducible. Denote by d the degree of the extension $\widetilde{K}/K$. Because of the classification of root systems, the index d is an element of $\{1, 2, 3, 6\}$ [BT84, 4.1.16].

If $d = 1$, we let $L' = \widetilde{K} = K$.

If $d = 2$, we let $L' = \widetilde{K}$; we fix $\tau \in \mathrm{Gal}(\widetilde{K}/K)$ to be the non-trivial element.

If $d \geq 3$, we let L' be a separable sub-extension of $\widetilde{K}$ (possibly non-Galois) of degree 3 over K; we fix $\tau \in \mathrm{Gal}(\widetilde{K}/K)$ to be an element of order 3.

Thus, we denote $d' = [L' : K] \in \{1, 2, 3\}$. In practice, $d' = \min(d, 3)$. Note that if $d \geqslant 3$, then $\widetilde{\Phi}$ is of type D_4 and Φ is of type G_2. Thus, if there is a multipliable root (i.e. Φ is of type BC_n), then we are in the case $d = 2$.

2.1.4 Definition. Let $\alpha \in \widetilde{\Phi}$ be an absolute root. Denote by Σ_α be the stabilizer of α for the canonical Galois action. The **field of definition** of the root α is the subfield of $\widetilde{K}$ fixed by Σ_α, denoted by $L_\alpha = \widetilde{K}^{\Sigma_\alpha}$.

Let $a = \alpha|_S$. The **splitting field extension class** of a is the isomorphism class of the field extension L_α/K, denoted by L_a/K.

Proof that this definition makes sense. We know, by [BT65, §6], that the set $\{\alpha \in \widetilde{\Phi},\ a = \alpha|_S\}$ is a non-empty orbit of the canonical Galois action on $\widetilde{\Phi}$. Hence, by abuse of notation, we denote $a = \{\alpha \in \widetilde{\Phi},\ a = \alpha|_S\}$. Thus, given any relative root $a \in \Phi$, the field extension class L_α/K does not depend of the choice of $\alpha \in a$. □

2.1.5 Remark. If $a \in \Phi$ is a multipliable root, then there exists $\alpha, \alpha' \in a$ such that $\alpha + \alpha' \in \widetilde{\Phi}$ [BT84, 4.1.4 Cas II]. Because a is an orbit, we can write $\alpha' = \sigma(\alpha)$ where $\sigma \in \Sigma$ is of order 2. As a consequence, the extension of fields $L_\alpha/L_{\alpha+\alpha'}$ is quadratic. By abuse of notation, we denote this extension class by L_a/L_{2a}; the ramification of this extension will be considered later.

2.1.3 Parametrization of root groups

In order to value the root groups (we do it in Section 2.1.4) thanks to the valuation of the local field, we have to define a parametrization of each root group. Moreover, these valuations have to be compatible. That is why we furthermore have to get relations between the parametrizations.

Let $(\widetilde{x}_\alpha)_{\alpha\in\widetilde{\Phi}}$ be a Chevalley-Steinberg system of $G_{\widetilde{K}}$. This is a parametrization of the absolute root groups $\widetilde{x}_\alpha : \mathbb{G}_a \to U_\alpha$ over $\widetilde{K}$ satisfying some compatibility relations, that will be exploited to get commutation relations in Section 4.1. We recall the precise definition and that such a system exists in Section 4.1. Such a Chevalley-Steinberg system determines parametrisations x_a of relative root groups U_a over K.

Let $a \in \Phi$ be a relative root. To compute commutators between elements of opposite root groups, or between elements of a torus and of a root group, it is sufficient to compute inside the semi-simple K-group $G^a = \langle U_{-a}, U_a \rangle$ generated by the two opposite root groups U_{-a} and U_a. Note that if G is simply connected, then so is G^a. Let $\pi : \widetilde{G}^a \to G^a$ be the universal covering of the quasi-split semi-simple K-subgroup of relative rank 1 generated by U_a and U_{-a}. The group $\widetilde{G}^a$ splits over L_a (this explains the terminology of "splitting field" of a root). A parametrization of the simply-connected group $\widetilde{G}^a$ is given by [BT84, 4.1.1 to 4.1.9]. We now recall notations and the matrix realization that we will use later.

The non-multipliable case: Let $a \in \Phi$ be a relative root such that $2a \notin \Phi$ and choose $\alpha \in a$. By [BT84, 4.1.4], the rank-1 group $\widetilde{G}^a$ is isomorphic to $R_{L_\alpha/K}(\mathrm{SL}_{2,L_\alpha})$. Inside the classical group SL_{2,L_α}, a maximal L_α-split torus of SL_{2,L_α} can be parametrized by the following homomorphism:

$$\begin{array}{cccc} z: & \mathbb{G}_{m,L_\alpha} & \to & \mathrm{SL}_{2,L_\alpha} \\ & t & \mapsto & \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \end{array}$$

The corresponding root groups can be parametrized by the following homomorphisms:

$$\begin{array}{cccc} y_-: & \mathbb{G}_{a,L_\alpha} & \to & \mathrm{SL}_{2,L_\alpha} \\ & v & \mapsto & \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{cccc} y_+: & \mathbb{G}_{a,L_\alpha} & \to & \mathrm{SL}_{2,L_\alpha} \\ & u & \mapsto & \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \end{array}$$

According to [BT84, 4.1.5], there exists a unique L_α-group isomorphism, denoted by $\xi_\alpha : \mathrm{SL}_{2,L_\alpha} \to \widetilde{G}^\alpha$, satisfying $\widetilde{x}_{\pm\alpha} = \pi \circ \xi_\alpha \circ y_\pm$ where $\widetilde{G}^\alpha$ is the simple factor of $\widetilde{G}^a_{\widetilde{K}}$ of index α.

Thus, we define a K-homomorphism $x_a = \pi \circ R_{L_\alpha/K}(\xi_\alpha \circ y_+)$ which is a K-group isomorphism between $R_{L_\alpha/K}(\mathbb{G}_{a,L_\alpha})$ and U_a. We also define the following K-group homomorphism:

$$\widetilde{a} = \pi \circ R_{L_\alpha/K}(\xi_\alpha \circ z) : R_{L_\alpha/K}(\mathbb{G}_{m,L_\alpha}) \to T^a$$

where $T^a = T \cap \langle U_{-a}, U_a \rangle$.

The multipliable case: Let $a \in \Phi$ be a relative root such that $2a \in \Phi$. Let $\alpha \in a$ be an absolute root from which a arises, and let $\tau \in \Sigma$ be as in Notation 2.1.3 so that $\alpha + \tau(\alpha)$ is again an absolute root. To simplify notations, we let (up to compatible isomorphisms in Σ) $L = L_a = L_\alpha$ and $L_2 = L_{2a} = L_{\alpha+\tau(\alpha)}$ whenever there is no possible confusion on the considered root a. By [BT84, 4.1.4], the K-group $\widetilde{G}^a$ is isomorphic to $R_{L_2/K}(\mathrm{SU}(h))$, where h denotes the hermitian form on $L \times L \times L$ given by the formula:

$$h : (x_{-1}, x_0, x_1) \mapsto \sum_{i=-1}^{1} x_i {}^{\tau}x_{-i}$$

The group $\widetilde{G}^a_{L_2}$ can be written as $\widetilde{G}^a_{L_2} = \prod_{\sigma \in \mathrm{Gal}(L_2/K)} \widetilde{G}^{\sigma(\alpha),\sigma(\tau(\alpha))}$ where each $\widetilde{G}^{\sigma(\alpha),\sigma(\tau(\alpha))}$ denotes a simple factor isomorphic to $\mathrm{SU}(h)$, so that $\mathrm{SU}(h)_L \simeq \mathrm{SL}_{3,L}$.

We define a connected unipotent L_2-group scheme by providing the L_2-subvariety $H_0(L, L_2) = \{(u, v) \in L \times L,\ u{}^\tau u = v + {}^\tau v\}$ of $L \times L$ with the following group law:

$$(u, v), (u', v') \mapsto (u + u', v + v' + u{}^\tau u')$$

Then, we let $H(L, L_2) = R_{L_2/K}(H_0(L, L_2))$. For the rational points, we get $H(L, L_2)(K) = \{(u, v) \in L \times L,\ u{}^\tau u = v + {}^\tau v\}$.

We parametrize a maximal torus of $\mathrm{SU}(h)$ by the isomorphism

$$\begin{array}{cccc} z: & R_{L/L_2}(\mathbb{G}_{m,L}) & \to & \mathrm{SU}(h) \\ & t & \mapsto & \begin{pmatrix} t & 0 & 0 \\ 0 & t^{-1}{}^\tau t & 0 \\ 0 & 0 & {}^\tau t^{-1} \end{pmatrix} \end{array}$$

We parametrize the corresponding root groups of $\mathrm{SU}(h)$ by the homomorphisms:

$$\begin{array}{cccc} y_-: & H_0(L, L_2) & \to & \mathrm{SU}(h) \\ & (u, v) & \mapsto & \begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ -v & -{}^\tau u & 1 \end{pmatrix} \end{array}$$

and

$$\begin{array}{cccc} y_+: & H_0(L, L_2) & \to & \mathrm{SU}(h) \\ & (u, v) & \mapsto & \begin{pmatrix} 1 & -{}^\tau u & -v \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

By [BT84, 4.1.9], there exists a unique L_2-group isomorphism, denoted by $\xi_\alpha : \mathrm{SU}(h) \to \widetilde{G}^{\alpha,\tau(\alpha)}$, satisfying $\widetilde{x}_{\pm\alpha} = \pi \circ \xi_\alpha \circ y_\pm$. From this, we define a K-homomorphism $x_a = \pi \circ R_{L_2/K}(\xi_\alpha \circ y_+)$ which is a K-group isomorphism between the K-group $H(L, L_2)$ and the root group U_a. We also define the following K-group homomorphism:

$$\widetilde{a} = \pi \circ R_{L_2/K}(\xi_\alpha \circ z) : R_{L/K}(\mathbb{G}_{m,L}) \to T^a$$

where $T^a = T \cap \langle U_{-a}, U_a \rangle$.

2.1.6 Notation. For any multipliable root $a \in \Phi$, in [BT84, 4.2.20] are furthermore defined the following notations:

- $L^0 = \{y \in L,\ y + {}^\tau y = 0\}$, this is an L_2-vector space of dimension 1;
- $L^1 = \{y \in L,\ y + {}^\tau y = 1\}$, this is an L^0-affine space.

Indeed, if K is not of characteristic 2, then $L^0 = \ker(\tau + \mathrm{id})$ is of dimension 1 because $L_2 = \ker(\tau - \mathrm{id})$ is of dimension 1 and ± 1 are the eigenvalues of $\tau \in \mathrm{Gal}(L/L_2)$. If K is of characteristic 2, then $L^0 = \ker(\tau + \mathrm{id}) = L_2$. Moreover, in both cases, since the quadratic extension L/L_2 is separable, the trace map $L \to L_2$ is surjective and, therefore, the affine space L^1 is non-empty.

2.1.7 Remark (Interest of such notations). For any $\lambda \in L^0$ so that $\lambda \neq 0$, we have an isomorphism of abelian groups given by the relation

$$\begin{array}{ccc} L_2 & \to & L^0 \\ y & \mapsto & \lambda y \end{array}$$

so that $x_a(0, \lambda y) = x_{2a}(y)$. This constitutes an additional uncertainty when we want to perform computations in $G(K)$. Because of valuation considerations, we will have to choose a λ whose valuation is minimal. To avoid confusion, it is better to work with the isomorphism of abelian groups

$$\begin{array}{ccc} L^0 & \to & U_{2a}(K) \\ y & \mapsto & x_a(0, y) \end{array}$$

in order to realize this group as a subgroup of $U_a(K)$.

The affine space L^1 has an interest in the context of a valued field. In particular, since L^1 is non-empty, we can write $L = L_2\lambda \oplus L^0$ with a suitable $\lambda \in L^1$.

2.1.4 Valuation of a root groups datum

If L/K is a finite extension of local fields, the valuation ω over $K^\times$ can be extended uniquely to a valuation over $L^\times$, still denoted by ω because of its uniqueness.

For each root group, we now use its parametrization to define an exhaustion by subgroups. In order to define an action of $G(K)$ on an affine building with suitable properties, it suffices to have suitable relations between the terms of filtration of root groups. More precisely, given a quasi-split reductive group G over a non-Archimedean local field K, with a relative root system Φ, we provide a valued root groups datum of $G(K)$ (see Definition [BT72, 6.2.1]). We define a valuation $(\varphi_a)_{a \in \Phi}$ of the rational points $U_a(K)$ of each root group by:

- $\varphi_a(x_a(y)) = \omega(y)$ if a is a non-multipliable and non-divisible root, and if $y \in L_a$;
- $\varphi_a(x_a(y, y')) = \frac{1}{2}\omega(y')$ if a is a multipliable root and if $(y, y') \in H(L_a, L_{2a})$;
- $\varphi_{2a}(x_a(0, y')) = \omega(y')$ if a is a multipliable root and if $y' \in L_a^0$.

By [BT84, §4.2], the family $\left(T, \left(U_a(K), M_a, \varphi_a\right)_{a\in\Phi}\right)$ is a valued root groups datum. In the following, we denote by $U_{a,l} = \varphi_a^{-1}([l,+\infty])$ for any root $a \in \Phi$ and any value $l \in \mathbb{R}$.

2.1.5 Set of values

We let $\Gamma_L = \omega(L^\times)$.

Because we considered a discrete valuation ω, the terms of filtration indexed by $\mathbb{R}$ can, in fact, be indexed by discrete subsets. These subsets will be used in Section 3.1, to provide an "affinisation" of the spherical root system.

Let $a \in \Phi$ be a root. We define the following sets of values:

- $\Gamma_a = \varphi_a(U_a(K) \setminus \{\mathbf{1}\})$;
- $\Gamma'_a = \{\varphi_a(u),\ u \in U_a(K) \setminus \{\mathbf{1}\} \text{ and } \varphi_a(u) = \sup \varphi_a(uU_{2a}(K))\}$;

where U_{2a} denotes the trivial group for $2a \notin \Phi$. Furthermore, for any value $l \in \mathbb{R}$, we denote $l^+ = \min\{l' \in \Gamma_a,\ l' > l\}$. This is the lowest value, greater than l, for which we will detect a change in the valued root groups $(U_{a,l'})_{l'>l}$. In order to characterize Γ'_a, we complete the notations of 2.1.6 introducing the following $L^1_{a,\max} = \{z \in L^1_a,\ \omega(z) = \sup\{\omega(y),\ y \in L^1_a\}\}$ for a multipliable root a. It is the subset of L^1_a whose elements reach the maximum of the valuation.

2.1.8 Remark. Be careful that the value l^+ also depends on a.

The sense of Γ'_a will be provided by Lemma 3.1.14, as the set of values parametrizing the affine roots.

2.1.9 Lemma. *If a is a non-multipliable non-divisible root, then we have $\Gamma_a = \Gamma'_a = \Gamma_{L_a}$.*

Proof. This is obvious by the isomorphism between $U_a(K)$ and L_a. □

Now, we assume that $a \in \Phi$ is a multipliable root.

Let p be the residue characteristic of K. Even if the sets of values can be computed for any p, we assume here that $p \neq 2$. This assumption provides a short cut in the computation of sets of values (mostly because $\frac{1}{2} \in L^1_{a,\max}$ in this case), and will be necessary later for more algebraic reasons.

Since ω is a discrete valuation and since for any $y \in L^1_a$, we have $\omega(y) \leq 0$, it is clear that $L^1_{a,\max}$ is non-empty. Moreover, when $p \neq 2$, we have $\frac{1}{2} \in L^1_{a,\max}$. Hence, by [BT84, 4.2.21 (4)], we know that $\Gamma_a = \frac{1}{2}\Gamma_{L_a}$ and that $\Gamma'_a = \Gamma_{L_a}$.

By [BT84, 4.3.4], we know that:

- when the quadratic extension L_a/L_{2a} is unramified, we have the equalities $\Gamma_{2a} = \Gamma'_{2a} = \omega(L^0 \setminus \{0\}) = \Gamma_{L_a} = \Gamma_{L_{2a}}$;

- when the quadratic extension L_a/L_{2a} is ramified, we have the equalities $\Gamma_{2a} = \Gamma'_{2a} = \omega(L^0 \setminus \{0\}) = \omega(\varpi_{L_a}) + \Gamma_{L_{2a}}$.

2.1.10 Lemma (Summary)**.** *Let $a \in \Phi$ be a multipliable root. If we normalize the valuation ω so that $\Gamma_{L_a} = \mathbb{Z}$, then we get:*

L_a/L_{2a}	*unramified*	*ramified*
Γ_{L_a}	$\mathbb{Z}$	$\mathbb{Z}$
$\Gamma_{L_{2a}}$	$\mathbb{Z}$	$2\mathbb{Z}$
Γ_a	$\frac{1}{2}\mathbb{Z}$	$\frac{1}{2}\mathbb{Z}$
Γ_{2a}	$\mathbb{Z}$	$1+2\mathbb{Z}$
Γ'_a	$\mathbb{Z}$	$\mathbb{Z}$

2.1.11 Remark. The case of a divisible root has been treated. It is the case $2a$ of a multipliable root a.

2.1.12 Remark (The case of residue characteristic 2). When the residue characteristic is any prime number (and in particular if $p = 2$), it can be seen via further investigations, that the set $L^1_{a,\max}$ is non-empty and we let $\{\delta\} = \omega(L^1_{a,\max})$. We can compute the sets of values, depending on δ and on the ramification of L_a/L_{2a}. We get the following results:

- $\Gamma'_a = \frac{1}{2}\delta + \Gamma_{L_a}$;
- $\Gamma_a = \Gamma'_a \cup \frac{1}{2}\Gamma_{2a} = \frac{1}{2}\Gamma_{L_a}$;
- if L_a/L_{2a} is ramified, then $\Gamma'_a \cap \frac{1}{2}\Gamma_{2a} = \emptyset$ and $\Gamma_{2a} = \delta + \omega(\varpi_{L_a}) + \Gamma_{L_{2a}}$;
- if L_a/L_{2a} is unramified, then $\Gamma'_a \cap \frac{1}{2}\Gamma_{2a} \neq \emptyset$ and $\Gamma_{2a} = \Gamma_{L_{2a}} = \Gamma_{L_a}$.

  Because $\delta = 0$ when $p \neq 2$, this is, in fact, the generalisation to any residue characteristic.

2.2 The reduced case

Let $a \in \Phi$ be a non-multipliable root of Φ arising from an absolute root $\alpha \in \widetilde{\Phi}$. In this section, in order to simplify notation, we denote $L = L_\alpha = L_a$. The universal covering $\pi : R_{L/K}(SL_{2,L}) \to G^a$ is a central K-isogeny, which allows us to compute relations between the elements of U_a, U_{-a} and T by the parametrizations x_a, x_{-a} and $\widetilde{a}$ thanks to matrix realizations in SL_2.

We denote by $T^a = T \cap G^a$ the maximal torus of G^a and by $T^a(K)^+_b$ the image of the group homomorphism $\widetilde{a} : 1 + \mathfrak{m}_L \to T^a(K)$. If G^a is simply-connected, then the torus T^a is an induced torus and $T^a(K)^+_b = T(K)^+_b \cap T^a(K)$ is the maximal pro-p subgroup of $T^a(K)$, by [Loi16, 3.2.10].

2.2.1 Lemma (Commutation relation $[T, U_a]$ in the reduced case)**.**
(1) Let $t \in T(K)$. Then, for any $x \in L$, we have

$$\Big[x_a(x), t\Big] = x_a\Big(\big(1 - \alpha(t)\big)x\Big)$$

(2) Normalize the valuation ω by $\Gamma_a = \Gamma_L = \mathbb{Z}$. For any $l \in \Gamma_a$, we have:

$$U_{a,l+1} \supset \left[T(K)_b^+, U_{a,l}\right] \supset \left[T^a(K)_b^+, U_{a,l}\right] = \begin{cases} U_{a,l+1} & \text{if } p \neq 2 \\ U_{a,l+2} & \text{if } p = 2 \text{ and } L \neq \mathbb{Q}_2 \\ U_{a,l+3} & \text{if } L = \mathbb{Q}_2 = K \end{cases}$$

Proof. (1) By definitions, $tx_a(x)t^{-1} = x_a\big(\alpha(t)x\big)$. Hence $\big[x_a(x), t\big] = x_a\big(x\big)x_a\big(-\alpha(t)x\big) = x_a\Big(\big(1-\alpha(t)\big)x\Big)$.

(2) Since $\alpha(T(K)_b^+)$ is a pro-p-subgroup of $L^\times$, it is contained in $1 + \mathfrak{m}_L$. Thus, we deduce $\big[T^a(K)_b^+, U_{a,l}\big] \subset \big[T(K)_b^+, U_{a,l}\big] \subset U_{a,l+1}$ from (1).

We prove $U_{a,l+i} = \big[T^a(K)_b^+, U_{a,l}\big]$ where $i \in \{1,2,3\}$ is, as in statement, depending on L. Let $t \in T(K)_b^+$ and $u \in U_{a,l}$. Write $u = x_a(x)$ with $x \in L$ such that $\omega(x) \geq l$. Write $t = \widetilde{a}(1+z)$ with $z \in \mathfrak{m}_L$ so that $\alpha(t) = (1+z)^2$. Applying (1), we get $\big[x_a(x), t\big] = x_a\Big(-(2+z)zx\Big) \in U_a(K)$. Thus $\varphi_a\Big(\big[x_a(x), t\big]\Big) = \omega(x) + \omega(z) + \omega(2+z) \geq l+i$. Thus $[T^a(K)_b^+, U_{a,l}] \subset U_{a,l+i}$.

If $p \neq 2$, then $\varphi_a\Big(\big[x_a(x), t\big]\Big) = \omega(x) + \omega(z) \geq l+1$. Conversely, let $y \in L$ be such that $\omega(y) \geq l+1$. Let z be a uniformizer of $\mathcal{O}_L$ and $x = -(2z+z^2)^{-1}y$. Then $\omega(x) = \omega(y) - 1 \geq l$ and $x_a(y) = \big[x_a(x), \widetilde{a}(1+z)\big]$. This gives $\big[T^a(K)_b^+, U_{a,l}\big] = U_{a,l+1}$.

If $p = 2$ and $L \neq \mathbb{Q}_2$, then $\omega(2+z) + \omega(z) \geq 2$. Hence $\varphi_a\Big(\big[x_a(x), t\big]\Big) \geq l+2$. Conversely, let $y \in L$ be such that $\omega(y) \geq l+2$. If 2 is a uniformizer of $\mathcal{O}_L$, that means L is an unramified (non-trivial) extension of $\mathbb{Q}_2$, then there is a unit $u \in \mathcal{O}_L^\times$ such that $1+u$ is also a unit. We take $z = 2u$ and $x = -\big(4u(1+u)\big)^{-1}y$. Otherwise, if ϖ be a uniformizer of $\mathcal{O}_L$, then so is $2+\varpi$. We take $z = \varpi$ and $x = -\big(z(2+z)\big)^{-1}y$ Then, in both cases, we have $\omega(x) = \omega(y) - 2 \geq l$ and $x_a(y) = \big[x_a(x), \widetilde{a}(1+z)\big]$. This gives $\big[T^a(K)_b^+, U_{a,l}\big] = U_{a,l+2}$.

If $L = K = \mathbb{Q}_2$, then $\omega(2+z) + \omega(z) \geq 3$. Hence $\varphi_a\Big(\big[x_a(x), t\big]\Big) \geq l+3$. Conversely, let $y \in L$ be such that $\omega(y) \geq l+3$. Let $z = 2$ and $x = -(8)^{-1}y$. Then $\omega(x) = \omega(y) - 3 \geq l$ and $x_a(y) = \big[x_a(x), \widetilde{a}(1+2)\big]$. This gives $\big[T^a(K)_b^+, U_{a,l}\big] = U_{a,l+3}$. □

2.2.2 Lemma (Commutation relation $[U_{-a,l}, U_{a,l'}]$ in the reduced case)**.** *(1) For any $x, y \in L$ such that $1 + xy \in \mathcal{O}_L^\times$, we have:*

$$\Big[x_{-a}(y), x_a(x)\Big] = x_{-a}\left(\frac{xy^2}{1+xy}\right)\widetilde{a}\left(1+xy\right)x_a\left(\frac{-x^2y}{1+xy}\right)$$

(2) Let $l, l' \in \Gamma_a$ be such that $l + l' > 0$. The product $U_{-a,l}T^a(K)_b^+U_{a,l'}$ is a group.

(3) Let $l, l' \in \Gamma_a$ be such that $l + l' > 0$. Normalize ω by $\Gamma_a = \Gamma_L = \mathbb{Z}$, so that $l + l' \geq 1$. Then $[U_{-a,l}, U_{a,l'}] \subset U_{-a,l+1}T^a(K)_b^+U_{a,l'+1}$.

Proof. (1) If $x, y \in L$ satisfy $1 + xy \in \mathcal{O}_L^\times$, then in $\mathrm{SL}_2(L)$, we have:

$$\left[\begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ -\frac{xy^2}{1+xy} & 1 \end{pmatrix} \begin{pmatrix} 1+xy & 0 \\ 0 & \frac{1}{1+xy} \end{pmatrix} \begin{pmatrix} 1 & \frac{-x^2y}{1+xy} \\ 0 & 1 \end{pmatrix}$$

Applying π to this equality, we get the desired equality.

(3) For any $x, y \in L$ such that $\omega(x) \geq l'$ and $\omega(y) \geq l$, we have $\omega(xy) = \omega(x) + \omega(y) > 0$. Hence $1 + xy \in 1 + \mathfrak{m}_L$ and therefore $\widetilde{a}(1 + xy) \in T^a(K)_b^+$. Moreover, $\omega\left(\frac{xy^2}{1+xy}\right) = \omega(x) + 2\omega(y) \geq 1 + \omega(y)$ and $\omega\left(\frac{x^2y}{1+xy}\right) = 2\omega(x) + \omega(y) \geq 1 + \omega(x)$. Hence $x_{-a}\left(\frac{xy^2}{1+xy}\right) \in U_{-a,l+1}$ and $x_a\left(\frac{-x^2y}{1+xy}\right) \in U_{a,l'+1}$. As soon as we know (2), this will prove (3).

(2) By the calculation in (1), we have $U_{a,l'} \cdot U_{-a,l} \subset U_{-a,l} \cdot T^a(K)_b^+ \cdot U_{a,l'}$. By Lemma 2.2.1, we have that $T^a(K)_b^+ \cdot U_{-a,l} = U_{-a,l} \cdot T^a(K)_b^+$ and $U_{a,l'} \cdot T^a(K)_b^+ = T^a(K)_b^+ \cdot U_{a,l'}$. Hence $U_{-a,l} \cdot T^a(K)_b^+ \cdot U_{a,l'} \cdot U_{-a,l} \cdot T^a(K)_b^+ \cdot U_{a,l'} \subset U_{-a,l} \cdot T^a(K)_b^+ \cdot U_{a,l'}$. Thus $U_{-a,l} \cdot T^a(K)_b^+ \cdot U_{a,l'}$ is stable by multiplication and inverse. □

2.2.3 Proposition. *Assume that $p \neq 2$ and $\Gamma_a = \Gamma_L = \mathbb{Z}$. Let $l \in \mathbb{Z} = \Gamma_a$. Let H be the subgroup $H = U_{a,l} T^a(K)_b^+ U_{-a,-l+1}$ of $G^a(K)$. Then $H^p \subset [H, H]$ and the derived group $[H, H]$ contains the subgroups $U_{a,l+1}$, $U_{-a,-l+2}$ and $T^a(K)_b^+$.*

Proof. Denote by ϖ a uniformizer of L. We firstly show that $T^a(K)_b^+$ is contained in $[H, H]$. For any $t \in 1 + \mathfrak{m}_L$, $t \neq 1$ and any $u \in L \setminus \{0\}$, one can check the following equalities inside SL_2:

$$\left[\begin{pmatrix} t & \frac{tu}{1-t^2} \\ 0 & \frac{1}{t} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -\frac{(1-t^2)^2}{t^2u} & 1 \end{pmatrix}\right] = \begin{pmatrix} t^2 & u \\ 0 & \frac{1}{t^2} \end{pmatrix} \tag{1}$$

$$\left[\begin{pmatrix} 1 & \frac{(t^2-1)^2}{t^2v} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{t} & 0 \\ \frac{tv}{t^2-1} & t \end{pmatrix}\right] = \begin{pmatrix} t^2 & 0 \\ v & \frac{1}{t^2} \end{pmatrix} \tag{2}$$

For any $t - 1 = s \in \varpi\mathcal{O}_L$, we have $\omega(1 + t) = \omega(2 + s) = 0$ because $p \neq 2$, and $\omega(t) = 0$. Hence, for any $u \in \varpi^{l+1}\mathcal{O}_L$, we have the following:

$$\begin{aligned} \omega\left(\frac{tu}{1-t^2}\right) &= \omega(t) + \omega(u) - \omega(1+t) - \omega(1-t) \\ &= \omega(u) - \omega(s) \\ \omega\left(-\frac{(1-t^2)^2}{t^2u}\right) &= 2\omega(s) - \omega(u) \end{aligned}$$

Moreover, we have:

$$\begin{pmatrix} t^2 & u \\ 0 & t^{-2} \end{pmatrix} \begin{pmatrix} t^2 & -t^{-4}u \\ 0 & t^{-2} \end{pmatrix} = \begin{pmatrix} t^4 & 0 \\ 0 & t^{-4} \end{pmatrix} \tag{3}$$

Let $t = 1+s \in 1+\varpi\mathcal{O}_L$. Set $u = \varpi^{l+\omega(s)}$ so that $\omega\left(\frac{tu}{1-t^2}\right) = l$ and $\omega\left(-\frac{(1-t^2)^2}{t^2u}\right) \geq -l+1$. Hence, $\pi\begin{pmatrix} t & \frac{tu}{1-t^2} \\ 0 & \frac{1}{t}\end{pmatrix} \in H$ and $\pi\begin{pmatrix} 1 & 0 \\ -\frac{(1-t^2)^2}{t^2u} & 1\end{pmatrix} \in H$. Thus, according to the equation (1), we get $\pi\begin{pmatrix} t^2 & u \\ 0 & t^{-2}\end{pmatrix} \in [H,H]$. Similarly, substituting u by $-t^4u$, we get $\pi\begin{pmatrix} t^2 & -t^{-4}u \\ 0 & t^{-2}\end{pmatrix} \in [H,H]$. As a consequence, for any $t \in 1+\varpi\mathcal{O}_L$, we have $\widetilde{a}(t^4) \in [H,H]$ according to the equation (3). The group homomorphism $\left\{\begin{array}{ccc} 1+\mathfrak{m}_L & \to & 1+\mathfrak{m}_L \\ t & \mapsto & t^2 \end{array}\right.$ is surjective. Indeed, since $p \neq 2$, for any $y \in \mathfrak{m}_L$, one can apply Hensel's Lemma to polynomial $Q = X^2+2X-y$. This polynomial has a root $x \in \mathfrak{m}_L$ so that $Q(x) = (1+x)^2-(1+y) = 0$. Hence $\widetilde{a}(t) = \widetilde{a}(s^4) \in [H,H]$ for some $s \in 1+\mathfrak{m}_L$.

As a consequence, the elements:

$$x_a(u) = \widetilde{a}(t^{-2}) \cdot \left[\widetilde{a}(t)x_a\left(\frac{t^2u}{1-t^2}\right), x_{-a}\left(\frac{(1-t^2)^2}{t^4u}\right)\right]$$

where $u \in \varpi^{l+1}\mathcal{O}_L$ and $t = 1+\varpi^{\omega(u)-l}$, are in $[H,H]$. Indeed, $\omega(u)-l \geq (l+1)-l = 1$ so that t is in $1+\mathfrak{m}_L$. Moreover, $\omega\left(\frac{t^2u}{1-t^2}\right) = \omega(u)-\omega(1-t^2) = \omega(u)-(\omega(u)-l) = l$ and $\omega\left(\frac{(1-t^2)^2}{t^4u}\right) = 2\omega(1-t^2)-\omega(u) = \omega(u)-2l \geq 1-l$. Hence, the group $[H,H]$ contains $U_{a,l+1}$.

Similarly, it contains $U_{-a,(-l+1)+1} = U_{a,-l+2}$, using the equation (2) instead of (1).

It remains to prove that $H^p \subset [H,H]$. Let $g \in H = U_{-a,-l+1}T^a(K)_b^+U_{a,l}$ and write $g = x_{-a}(v)\widetilde{a}(t)x_a(u)$. Consider the quotient morphism $\pi : H \to H/[H,H]$. Then $\pi(g^p) = \pi(g)^p = \Big(\pi\big(x_{-a}(v)\big)\pi\big(\widetilde{a}(t)\big)\pi\big(x_a(u)\big)\Big)^p$. Since $H/[H,H]$ is commutative, we have $\pi(g^p) = \pi\big(x_{-a}(v)\big)^p\pi\big(\widetilde{a}(t)\big)^p\pi\big(x_a(u)\big)^p = \pi\big(x_{-a}(pv)\big)\pi\big(\widetilde{a}(t^p)\big)\pi\big(x_a(pu)\big)$. We have seen that $\widetilde{a}(t) \in [H,H]$ so that $\widetilde{a}(t^p) \in [H,H]$. Moreover, because $\omega(pu) > \omega(u)$ and $\omega(pv) > \omega(v)$, we have $x_{-a}(pv) \in U_{-a,-l+2} \subset [H,H]$ and $x_a(pu) \in U_{a,l+1} \subset [H,H]$. Thus $\pi(g^p) = 1$. Hence $g^p \in [H,H]$. $\square$

2.3 The non-reduced case

Let $a \in \Phi$ be a multipliable root of Φ arising from an absolute root $\alpha \in \widetilde{\Phi}$.

In this paragraph, we denote by $L = L_\alpha = L_a$ and $L_2 = L_{\alpha+{}^\tau\alpha} = L_{2a}$, where $\tau = \tau_a$ is the non trivial element of $\mathrm{Gal}(L/L_2)$. In order to simplify notations, for any $x \in L$, we denote ${}^\tau x = \tau(x)$. Denote by h the L_2-Hermitian form:

$$\begin{array}{cccc} h: & L\times L\times L & \to & L \\ & (x_{-1},x_0,x_1) & \mapsto & \displaystyle\sum_{i=-1}^{1} x_{-i}\tau(x_i) \end{array}$$

Recall that the universal covering is a central K-isogeny $\pi : R_{L/K}(SU(h)) \to G^a$, from which we compute, inside $\mathrm{SU}(h)$, relations between elements of U_a, U_{-a} and T thanks to parametrizations x_a, x_{-a} and $\widetilde{a}$.

Denote by $T^a = T \cap G^a$ and $T^a(K)_b^+$ the image of the group homomorphism: $\widetilde{a} : 1 + \mathfrak{m}_L \to T^a(K)$. When G^a is simply-connected, then $T^a(K)_b^+ = T(K)_b^+ \cap T^a(K)$. For any $l \in \mathbb{N}^*$, set $T^a(K)_b^l = \widetilde{a}\left(1 + \mathfrak{m}_L^l\right)$. Normalize ω by $\Gamma_a = \Gamma_{-a} = \frac{1}{2}\mathbb{Z}$, so that $\Gamma_L = \mathbb{Z}$ and $\Gamma_{L_2} = 2\mathbb{Z}$ or $\mathbb{Z}$ depending on whether the extension L/L_2 is ramified or not. The analogue to Proposition 2.2.3, in the non-reduced case, is the following:

2.3.1 Proposition. *Assume that $p \geq 5$. Let $l \in \Gamma_a = \frac{1}{2}\mathbb{Z}$. Let H be the subgroup $H = U_{-a,-l}T^a(K)_b^+U_{a,l+\frac{1}{2}}$ of $G(K)$.*

If L/L_2 is not ramified, then there exists $l'' \in \mathbb{N}^*$ such that $[H,H]$ contains the following subgroups $T^a(K)_b^{l''}$, $U_{-a,-l+1}$ and $U_{a,l+\frac{3}{2}}$.

If L/L_2 is ramified, then there exists $l'' \in \mathbb{N}^*$ such that $[H,H]$ contains the following subgroups $T^a(K)_b^{l''}$, $U_{-a,-l+\frac{3}{2}}$ and $U_{a,l+2}$.

Precisely, up to exchanging a with $-a$, we can assume $l \in \Gamma'_a = \mathbb{Z}$ and, in this case, we get:

$$l'' = \max(1 + 2\varepsilon, 3 + \varepsilon)$$

where

$$\varepsilon = \begin{cases} 1 & \text{if } L/L_2 \text{ is ramified and } l \in 2\mathbb{Z}+1 = \Gamma_L \setminus \Gamma_{L_2} \\ 0 & \text{otherwise} \end{cases}$$

Moreover, if K is of characteristic p or if the extension $K/\mathbb{Q}_p$ is ramified (so that $\omega(p) \geq 2\omega(\varpi_K)$), we have $H^p \subset [H,H]$.

2.3.2 Remark. Since the maximal pro-p subgroups are pairwise conjugate by [Loi16, 1.2.1], by the choice of a maximal pro-p subgroup corresponding to a suitable alcove, we can assume later that $\varepsilon = 0$. Such a choice will be done in Section 3.1.2. Moreover, because of the lack of rigidity, computations in the rank 1 case give large inequalities for the commutator group. In fact, when the rank is ≥ 2, we can make a stronger assumption, to get a more precise computation of the Frattini subgroup, as stated in Proposition 2.3.11.

In order to simplify notation, denote by $H(L, L_2)$ the rational points of the K-group $H(L, L_2)$, instead of $H(L, L_2)(K)$. For any $(x, y), (u, v) \in H(L, L_2)$ and for any $t \in 1 + \varpi_L\mathcal{O}_L$, up to precomposing by π, we have the following matrix realization:

$$\widetilde{a}(t) = \begin{pmatrix} t & 0 & 0 \\ 0 & t^{-1}\,{}^{\tau}t & 0 \\ 0 & 0 & {}^{\tau}t^{-1} \end{pmatrix}$$

$$x_a(x, y) = \begin{pmatrix} 1 & -{}^{\tau}x & -y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \qquad x_{-a}(u, v) = \begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ -v & -{}^{\tau}u & 1 \end{pmatrix}$$

We want to obtain some unipotent elements, and some semi-simple elements, by multiplying suitable commutators and p-powers of elements in H, as we have done, previously, in the reduced case. In particular, in Lemma 2.3.4 we bound explicitely, thanks to these parametrizations, the group generated by commutators of elements of the torus and unipotent elements in a given root group. In Lemma 2.3.6, we provide an explicit formula for the commutators of unipotent elements taken in opposite root groups, in terms of the parametrizations. Finally, thanks to Lemma 2.3.10, we invert such a commutation relation. At last, we prove Proposition 2.3.1 thanks to these lemmas.

The following lemma provides the existence of elements with minimal valuation, used in the parametrization of coroots.

2.3.3 Lemma. *Let L/K be a quadratic Galois extension of local fields and $\tau \in \mathrm{Gal}(L/K)$ be the non-trivial element. Let ϖ_L be a uniformizer of the local ring $\mathcal{O}_L$. Denote by p the residue characteristic and assume that $p \neq 2$.*

(1) *For any $t \in 1 + \mathfrak{m}_L$, we have $\omega\left(t^2 - {}^\tau t\right) \geq \omega\left(\varpi_L\right)$ and $\omega\left(t{}^\tau t - 1\right) \geq \omega\left(\varpi_L\right)$.*

(2) *If the extension L/K is unramified, then there exists $t \in 1 + \mathfrak{m}_L$ such that $\omega\left(t{}^\tau t - 1\right) = \omega\left(t^2 - {}^\tau t\right) = \omega\left(\varpi_L\right)$.*

(3) *If the extension L/K is ramified, then for any $t \in 1 + \mathfrak{m}_L$, we have the inequality $\omega\left(t{}^\tau t - 1\right) \geq 2\omega\left(\varpi_L\right)$. If $p \geq 5$, then there exists $t \in 1 + \mathfrak{m}_L$ such that $\omega\left(t{}^\tau t - 1\right) = 2\omega\left(t^2 - {}^\tau t\right) = 2\omega\left(\varpi_L\right)$.*

Proof. (1) Write $t = 1 + s$ with $\omega(s) \geq \omega(\varpi_L)$. Then $\omega(t^2 - {}^\tau t) = \omega(2s + s^2 - {}^\tau s) \geq \omega(s)$ and $\omega(t{}^\tau t - 1) = \omega(s + {}^\tau s + s{}^\tau s) \geq \omega(s)$.

(2) If L/K is unramified, one can choose a uniformizer $\varpi_L \in \mathcal{O}_L \cap K$. Let $t = 1 + \varpi_L$, so that $t^2 - {}^\tau t = \varpi_L + \varpi_L^2$. Since $p \neq 2$, then $\omega(2) = 0$. Hence $\omega\left(t{}^\tau t - 1\right) = \omega\left(2\varpi_L + \varpi_L^2\right) = \omega\left(\varpi_L\right)$.

(3) If L/K is ramified, the inequality $\omega\left(t{}^\tau t - 1\right) \geq \omega\left(\varpi_L\right)$ is never an equality because $t{}^\tau t - 1 \in K$. Consequently, $\omega\left(t{}^\tau t - 1\right) \geq 2\omega\left(\varpi_L\right)$. Remark that $\omega\left(\varpi_L + {}^\tau\varpi_L\right) \geq 2\omega\left(\varpi_L\right) = \omega\left(\varpi_L{}^\tau\varpi_L\right)$. Define $t = 1 + \varpi_L$, so that $t^2 - {}^\tau t = 2\varpi_L - {}^\tau\varpi_L + \varpi_L^2$.

By contradiction, if we had $\omega\left(2\varpi_L - {}^\tau\varpi_L\right) \geq 2\omega\left(\varpi_L\right)$, then, by triangle inequality, we would get $\omega\left(3\varpi_L\right) \geq \min\Big(\omega\left(\varpi_L + {}^\tau\varpi_L\right), \omega\left(2\varpi_L - {}^\tau\varpi_L\right)\Big) \geq 2\omega\left(\varpi_L\right)$. When $p \neq 3$, we have $\omega\left(3\varpi_L\right) = \omega\left(\varpi_L\right)$. Hence, there is a contradiction with $\omega\left(\varpi_L\right) > 0$. As a consequence, $\omega\left(2\varpi_L - {}^\tau\varpi_L\right) = \omega\left(\varpi_L\right)$ and $\omega(t^2 - {}^\tau t) = \omega(\varpi_L)$, for any uniformizer $\varpi_L \in \mathcal{O}_L$.

Define $\varpi'_L = \varpi_L + \varpi_L{}^\tau\varpi_L$. This element $\varpi'_L \in \mathcal{O}_L$ is also a uniformizer. Define $t' = 1 + \varpi'_L$. We have seen that $\omega\left(t'^2 - {}^\tau t'\right) = \omega\left(\varpi_L\right)$.

Claim: Either t or t' satisfies the desired equalities.

Indeed, we have $t{}^\tau t - 1 = \varpi_L + {}^\tau\varpi_L + \varpi_L{}^\tau\varpi_L$ and $t'{}^\tau t' - 1 = \varpi_L + {}^\tau\varpi_L + 3\varpi_L{}^\tau\varpi_L + \mathrm{Tr}_{L/K}\left(\varpi_L^2{}^\tau\varpi_L\right) + N_{L/K}\left(\varpi_L\right)^2$.

By contradiction, assume that we have $\omega\left(\varpi_L + {}^\tau\varpi_L + \varpi_L{}^\tau\varpi_L\right) > 2\omega\left(\varpi_L\right)$ and $\omega\left(\varpi_L + {}^\tau\varpi_L + 3\varpi_L{}^\tau\varpi_L\right) > 2\omega\left(\varpi_L\right)$. Then, by triangle inequality, we get $\omega\left(2\varpi_L{}^\tau\varpi_L\right) > 2\omega\left(\varpi_L\right)$. Since $p \neq 2$, we have $\omega\left(2\varpi_L{}^\tau\varpi_L\right) = 2\omega\left(\varpi_L\right)$ and there is a contradiction.

Hence, we have, at least, $\omega\left(\varpi_L + {}^\tau\varpi_L + \varpi_L{}^\tau\varpi_L\right) = 2\omega\left(\varpi_L\right)$, or $\omega\left(\varpi_L + {}^\tau\varpi_L + 3\varpi_L{}^\tau\varpi_L\right) = 2\omega\left(\varpi_L\right)$. So, at least one of the two following equalities $\omega\left(t{}^\tau t - 1\right) = 2\omega\left(\varpi_L\right)$ or $\omega\left(t'{}^\tau t' - 1\right) = 2\omega\left(\varpi_L\right)$ is satisfied. Hence t or t' is suitable. □

Denote by $H(L,L_2)_l = \left\{(u,v) \in H(L,L_2),\ \frac{1}{2}\omega(v) \geq l\right\}$ the filtered subgroup of $H(L,L_2)$. Remark that $H(L,L_2)_l$ can be seen as the integral points of a $\mathcal{O}_K$-model of the K-group scheme $H(L,L_2)$, namely the group scheme $\mathcal{H}^l$ defined by [Lan96, 4.23]. Recall that for any $l \in \mathbb{R}$, we have $H(L,L_2)_l \simeq U_{a,l}$, by definition of the filtration on root groups, through the isomorphism $(u,v) \mapsto x_a(u,v)$. Recall that we also have an isomorphism $\widetilde{a} : 1 + \mathfrak{m}_L \simeq T^a(K)_b^+$.

2.3.4 Lemma. *Let $l \in \Gamma_a = \frac{1}{2}\mathbb{Z}$. Then we have:*

$$\left[T^a(K)_b^+, U_{a,l}\right] \subset \left[T(K)_b^+, U_{a,l}\right] \subset U_{a,l+\frac{1}{2}}$$

Moreover, if L/L_2 is unramified, we have:

$$U_{a,l+1} \subset \left[T^a(K)_b^+, U_{a,l}\right]$$

If L/L_2 is ramified and $p \geq 5$, we have:

$$U_{a,l+\frac{3}{2}} \subset \left[T^a(K)_b^+, U_{a,l}\right]$$

Proof. Let $t \in T(K)_b^+$. Then $\alpha(t)$ belongs to the maximal pro-p subgroup of $\mathcal{O}_L^\times$. Thus $1 - \alpha(t) \in \mathfrak{m}_L$ and $1 - {}^\tau\alpha(t) \in \mathfrak{m}_L$. Let $(u,v) \in H(L,L_2)$. Then $t x_a(u,v) t^{-1} = x_a\big(\alpha(t)u, \alpha(t){}^\tau\alpha(t)v\big)$ by definition. Thus

$$\begin{aligned}
[t, x_a(u,v)] =& x_a(\alpha(t)u, \alpha(t){}^\tau\alpha(t)v)x_a(-u, {}^\tau v)\\
=& x_a\Big(\big(\alpha(t)-1\big)u, -{}^\tau\alpha(t){}^\tau u u + \alpha(t){}^\tau\alpha(t)v + {}^\tau v\Big)\\
=& x_a\Big(\big(\alpha(t)-1\big)u, \big(1 - {}^\tau\alpha(t)\big){}^\tau v + {}^\tau\alpha(t)\big(\alpha(t)-1\big)v\Big) \quad \text{since } {}^\tau u u = v + {}^\tau v
\end{aligned}$$

Hence

$$\begin{aligned}
\varphi_a([t, x_a(u,v)]) =& \frac{1}{2}\omega\Big(\big(1 - {}^\tau\alpha(t)\big){}^\tau v + {}^\tau\alpha(t)\big(\alpha(t)-1\big)v\Big)\\
\geq& \frac{1}{2}\min\Big(\omega\big(1 - {}^\tau\alpha(t)\big) + \omega({}^\tau v), \omega({}^\tau\alpha(t)) + \omega\big(\alpha(t)-1\big) + \omega(v)\Big)\\
\geq& \frac{1}{2}\Big(\omega(\varpi_L) + \omega(v)\Big) = \frac{1}{2} + \varphi_a(x_a(u,v))
\end{aligned}$$

It proves that $[T^a(K)_b^+, U_{a,l}] \subset [T(K)_b^+, U_{a,l}] \subset U_{a,l+\frac{1}{2}}$.

For any $t \in 1 + \varpi_L \mathcal{O}_L \simeq T^a(K)_b^+$ and all $(u,v) \in H(L, L_2)_l$, we have:

$$\left[\begin{pmatrix} 1 & -{}^\tau u & -v \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} t & 0 & 0 \\ 0 & \frac{{}^\tau t}{t} & 0 \\ 0 & 0 & \frac{1}{{}^\tau t} \end{pmatrix}\right] = \begin{pmatrix} 1 & -{}^\tau U & -V \\ 0 & 1 & U \\ 0 & 0 & 1 \end{pmatrix}$$

where $U = \left(1 - \frac{{}^\tau t^2}{t}\right) u$ and $V = \left(1 - \frac{{}^\tau t^2}{t}\right) v + \left(t\,{}^\tau t - \frac{{}^\tau t^2}{t}\right) {}^\tau v$. One can check that $(U,V) \in H(L, L_2)$. We have:

$$\begin{array}{rcll} \omega(V) & \geq & \min\left(\omega\left(t - {}^\tau t^2\right) + \omega(v) - \omega(t), \omega\left(\frac{{}^\tau t}{t}\right) + \omega\left(t^2 - {}^\tau t\right) + \omega\left({}^\tau v\right)\right) & \\ & & & \text{by the triangle inequality} \\ & = & \omega(v) + \omega\left(t^2 - {}^\tau t\right) & \text{because } \tau \text{ preserves the valuation} \\ & \geq & 2l + 1 & \text{by lemma 2.3.3(1)} \end{array}$$

From this inequality, we deduce $(U,V) \in H(L, L_2)_{l+\frac{1}{2}}$, hence we have $\left[U_{a,l}, T(K)_b^+\right] \subset U_{a,l+\frac{1}{2}}$.

Conversely, let $l' \in \frac{1}{2}\mathbb{Z}$. Let $(U,V) \in H(L, L_2)_{l'}$. We want elements $t \in 1 + \mathfrak{m}_L$ and $(u,v) \in H(L, L_2)$ such that $[\widetilde{a}(t), x_a(u,v)] = x_a(U,V)$ and so that $\omega(v)$ is as big as possible.

Choose t satisfying the equalities (2) or (3) in Lemma 2.3.3 applied to the extension of local fields L/L_2. Let $u = \frac{t}{t - {}^\tau t^2} U$. We seek $X, Y \in \mathcal{O}_K(t, {}^\tau t)$ such that $\left(1 - \frac{{}^\tau t^2}{t}\right) v + \left(t\,{}^\tau t - \frac{{}^\tau t^2}{t}\right) {}^\tau v = V$ where we set $v = XV + Y\,{}^\tau V$. It suffices to find X, Y such that:

$$\begin{cases} \left(1 - \frac{{}^\tau t^2}{t}\right) X + \left(t\,{}^\tau t - \frac{{}^\tau t^2}{t}\right) {}^\tau Y & = & 1 \\ \left(1 - \frac{{}^\tau t^2}{t}\right) Y + \left(t\,{}^\tau t - \frac{{}^\tau t^2}{t}\right) {}^\tau X & = & 0 \end{cases}$$

The unique solution of this linear system is:

$$\begin{cases} X & = & \frac{1}{(1 - t\,{}^\tau t)\left(1 - \frac{{}^\tau t^2}{t}\right)} \\ Y & = & \frac{\frac{{}^\tau t^2}{t}}{(1 - t\,{}^\tau t)\left(1 - \frac{{}^\tau t^2}{t}\right)} \end{cases}$$

so that:

$$v = XV + Y\,{}^\tau V = \frac{V + \frac{{}^\tau t^2}{t}\,{}^\tau V}{(1 - t\,{}^\tau t)\left(1 - \frac{{}^\tau t^2}{t}\right)}$$

satisfies $(u,v) \in H(L, L_2)$.

By a matrix computation, and because t, u, v have been chosen for this, we can check that $[x_a(u,v), \widetilde{a}(t)] = x_a(U,V)$. Moreover, the valuation gives us $\omega(v) \geq \omega(V) - \omega(1 - t\,{}^\tau t) - \omega(t - {}^\tau t^2)$ because $\omega\left(V + \frac{{}^\tau t^2}{t}\,{}^\tau V\right) \geq \omega(V)$.

When L/L_2 is unramified, by 2.3.3(2), this gives us $\omega(v) \geq 2l' - 2$. From this inequality, we deduce $(u, v) \in H(L, L_2)_{l'-1}$, hence:

$$\left[U_{a,l'-1}, T^a(K)_b^+\right] \supset U_{a,l'}$$

When L/L_2 is ramified, by 2.3.3(3), this gives us $\omega(v) \geq 2l' - 3$. From this inequality, we deduce $(u, v) \in H(L, L_2)_{l'-\frac{3}{2}}$, hence:

$$\left[U_{a,l'-\frac{3}{2}}, T^a(K)_b^+\right] \supset U_{a,l'}$$

□

2.3.5 Remark. These inequalities could be refined, with a deeper study on the arithmetic properties of the local fields. As an example, when L/L_2 is ramified, and $l \notin \mathbb{Z}$, we obtain $\left[T^a(K)_b^+, U_{a,l}\right] \subset U_{a,l+1}$.

2.3.6 Lemma (Commutation of opposite root groups)**.** *Let $l, l' \in \Gamma_a = \frac{1}{2}\mathbb{Z}$ be such that $l + l' > 0$. Let $(x, y) \in H(L, L_2)_l$ and $(u, v) \in H(L, L_2)_{l'}$. We have $[x_{-a}(x, y), x_a(u, v)] = x_{-a}(X, Y)\widetilde{a}(T)x_a(U, V)$ where:*

$$\left\{\begin{array}{lcl} T & = & 1 - {}^\tau u x + v y \\ U & = & \frac{1}{{}^\tau T}\left(u^2\,{}^\tau x - {}^\tau v x - u\,{}^\tau v\,{}^\tau y\right) \\ V & = & \frac{1}{T}\left(u v\,{}^\tau x - {}^\tau u\,{}^\tau v x + v\,{}^\tau v y\right) \\ X & = & \frac{1}{T}\left({}^\tau u x^2 - u y - v x y\right) \\ Y & = & \frac{1}{T}\left({}^\tau x u y - {}^\tau u x\,{}^\tau y + v y\,{}^\tau y\right) \end{array}\right.$$

Moreover, $\omega(V) \geq \lceil 3l' + l \rceil$ and $\omega(Y) \geq \lceil l' + 3l \rceil$.
Consequently, $U_{-a,\frac{\lceil 3l+l' \rceil}{2}} T^a(K)_b^+ U_{a,\frac{\lceil l+3l' \rceil}{2}}$ is a group and

$$\begin{array}{lcl} \left[U_{-a,l}, U_{a,l'}\right] & \subset & U_{-a,\frac{\lceil 3l+l' \rceil}{2}} T^a(K)_b^+ U_{a,\frac{\lceil l+3l' \rceil}{2}} \\ & \subset & U_{-a,l+\frac{1}{2}} T^a(K)_b^+ U_{a,l'+\frac{1}{2}} \end{array}$$

Proof. Because τ preserves ω, we have the following in $H(L, L_2)$:

$$2\omega(u) = \omega(u\,{}^\tau u) = \omega(v + {}^\tau v) \geq \omega(v)$$

Hence, we have:

$$\omega(x) + \omega(u) \geq \frac{1}{2}\big(\omega(y) + \omega(v)\big) \geq l + l' > 0$$

By a matrix computation in $\mathrm{SU}(h)$, we have:

$$\begin{pmatrix} 1 & -{}^\tau u & -v \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ -y & -{}^\tau x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ X_0 & 1 & 0 \\ -Y_0 & -{}^\tau X_0 & 1 \end{pmatrix} \begin{pmatrix} T & 0 & 0 \\ 0 & \frac{{}^\tau T}{T} & 0 \\ 0 & 0 & \frac{1}{{}^\tau T} \end{pmatrix} \begin{pmatrix} 1 & -{}^\tau U_0 & -V_0 \\ 0 & 1 & U_0 \\ 0 & 0 & 1 \end{pmatrix}$$

where

$$\begin{cases} T &= 1 - {}^\tau ux + vy \\ U_0 &= \frac{1}{{}^\tau T}(u - {}^\tau vx) \\ V_0 &= \frac{1}{T} v \\ X_0 &= \frac{1}{T}(x - uy) \\ Y_0 &= \frac{1}{T} y \end{cases}$$

Because $\omega({}^\tau ux) \geq \frac{1}{2}\omega(vy) > 0$, we get $T \in 1 + \mathfrak{m}_L$. Hence $\frac{1}{T} \in \mathcal{O}_L^\times$ is well-defined. It follows:

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ -x & 1 & 0 \\ -{}^\tau y & {}^\tau x & 1 \end{pmatrix}, \begin{pmatrix} 1 & -{}^\tau u & -v \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 & 0 \\ X & 1 & 0 \\ -Y & -{}^\tau X & 1 \end{pmatrix} \begin{pmatrix} T & 0 & 0 \\ 0 & \frac{{}^\tau T}{T} & 0 \\ 0 & 0 & \frac{1}{{}^\tau T} \end{pmatrix} \begin{pmatrix} 1 & -{}^\tau U & -V \\ 0 & 1 & U \\ 0 & 0 & 1 \end{pmatrix}$$

where

$$\begin{cases} T &= 1 - {}^\tau ux + vy \\ U &= \frac{1}{{}^\tau T}\left(u^2\, {}^\tau x - {}^\tau vx - u\, {}^\tau v\, {}^\tau y\right) \\ V &= \frac{1}{T}\left(uv\, {}^\tau x - {}^\tau u\, {}^\tau vx + v\, {}^\tau vy\right) \\ X &= \frac{1}{T}\left({}^\tau ux^2 - uy - vxy\right) \\ Y &= \frac{1}{T}\left({}^\tau xuy - {}^\tau ux\, {}^\tau y + vy\, {}^\tau y\right) \end{cases}$$

We have

$$\begin{aligned} \omega(V) &\geq \min\left(\omega(uv\, {}^\tau x), \omega({}^\tau u\, {}^\tau vx), \omega(v\, {}^\tau vy)\right) \\ &\geq \omega(v) + \min\left(\omega(u) + \omega(x), \omega(v) + \omega(y)\right) \\ &\geq 2l' + l + l' \end{aligned}$$

Because $\omega(V) \in \mathbb{Z}$, we have in fact $\omega(V) \geq \lceil 3l' + l \rceil \geq 2l' + 1$.

We proceed in the same way to find a lower bound of $\omega(Y)$.

We got that for any $l, l' \in \frac{1}{2}\mathbb{Z}$ with $l + l' > 0$, any $u \in U_{-a,l}$ and $v \in U_{a,l'}$, the commutator $[u, v]$ belongs to the set $H = U_{-a, \frac{\lceil 3l + l' \rceil}{2}} T^a(K)_b^+ U_{a, \frac{\lceil l + 3l' \rceil}{2}} \subset U_{-a,l} T^a(K)_b^+ U_{a,l'}$. It remains to prove that H is a group. Since $T^a(K)_b^+$ is contained in $T(K)_b$, it normalizes the valued root groups. Thus H is stable by left (resp. right) multiplication by elements in $T^a(K)_b^+$ and in $U_{-a, \frac{\lceil 3l + l' \rceil}{2}}$ (resp. $U_{a, \frac{\lceil l + 3l' \rceil}{2}}$). Let $h \in H$ and write it $h = utv$ with $u \in U_{-a, \frac{\lceil 3l + l' \rceil}{2}}$, $t \in T^a(K)_b^+$ and $v \in U_{a, \frac{\lceil l + 3l' \rceil}{2}}$. For any $w \in U_{a, \frac{\lceil l + 3l' \rceil}{2}}$, we have $wh = wutv = u[u^{-1}, w]wtv$. Since $u \in U_{-a,l}$ and $w \in U_{a,l'}$, We have shown that $[u^{-1}, w] \in H$. By left multiplication, we have $u[u^{-1}, w] \in H$. Finally, by right multiplication, we get $wh \in H$, so that H is a group and we get the desired inclusion. $\square$

In order to compute a derived group in terms of root groups, we would like to invert the above equations. Precisely, given a $t \in 1 + \mathfrak{m}_L^{l''}$, we seek elements $(u, v), (x, y) \in H(L, L_2)$ with prescribed valuations $l, l' \in \frac{1}{2}\mathbb{Z}$ such that $t = 1 - {}^\tau ux + vy$. The existence of such $(u, v), (x, y)$ is not guaranteed if l'' is not large enough. Firstly, we seek an element $(u, v) \in H(L, L_2)_l$ such that $\omega\left(\mathrm{Tr}(u)\right)$ is minimal.

2.3.7 Lemma. *Let L/K be a quadratic Galois extension of local fields with residue characteristic $p \neq 2$ and a discrete valuation $\omega : L^\times \to \mathbb{Z}$. There exists a uniformizer ϖ_L in $\mathcal{O}_L$ such that $\mathrm{Tr}_{L/K}(\varpi_L)$ is a uniformizer of $\mathcal{O}_K$.*

Proof. If L/K is unramified, we can choose a uniformizer ϖ_L of $\mathcal{O}_L$ in $\mathcal{O}_K$. Because $p \neq 2$, the element $\mathrm{Tr}_{L/K}(\varpi_L) = 2\varpi_L$ is a uniformizer in $\mathcal{O}_K$.

If L/K is ramified, let ϖ' be a uniformizer of $\mathcal{O}_L$. We know that $\omega\left(\mathrm{Tr}_{L/K}(\varpi')\right) \geq \min\left(\omega\left(\varpi'\right), \omega\left({}^\tau\varpi'\right)\right) = 1$. This is never an equality because $\Gamma_K = \omega(K^\times) = 2\mathbb{Z}$.

If $\omega\left(\mathrm{Tr}_{L/K}(\varpi')\right) = 2$, then we set $\varpi_L = \varpi'$. Otherwise, we set $\varpi_L = \varpi' + N_{L/K}(\varpi')$. Thus, ϖ_L is a uniformizer because $\omega\left(N_{L/K}(\varpi')\right) = 2 > 1 = \omega(\varpi')$. Moreover, $\mathrm{Tr}_{L/K}(\varpi_L) = \mathrm{Tr}_{L/K}(\varpi') + 2N_{L/K}(\varpi')$. Because $\omega\left(\mathrm{Tr}_{L/K}(\varpi')\right) > \omega\left(2N_{L/K}(\varpi')\right) = 2$, we get the result. □

2.3.8 Lemma. *Assume that $p \neq 2$ and let $l \in \Gamma_L = \mathbb{Z}$.*

If L/L_2 is unramified, set $\varepsilon = 0$.

If L/L_2 is ramified, set $\varepsilon = \begin{cases} 0 & \textit{if } l \in \Gamma_{L_2} = 2\mathbb{Z} \\ 1 & \textit{otherwise} \end{cases}$

There exists $u \in L$ such that:

(a) $\omega(u) = l$;

(b) $\omega\left(\mathrm{Tr}_{L/L_2}(u)\right) = l + \varepsilon$;

(c) $\left(u, \frac{1}{2}u^\tau u\right) \in H(L, L_2)_l$.

Proof. Let ϖ_L be a uniformizer of $\mathcal{O}_L$ such that $\varpi_{L_2} = \mathrm{Tr}_{L/L_2}(\varpi_L)$ is a uniformizer of $\mathcal{O}_{L_2}$, such a uniformizer exists by Lemma 2.3.7. Define $u = (\varpi_L)^\varepsilon \cdot (\varpi_{L_2})^{\frac{l-\varepsilon}{\omega(\varpi_{L_2})}}$.

(a) $\omega(u) = \varepsilon\omega(\varpi_L) + \frac{l-\varepsilon}{\omega(\varpi_{L_2})}\omega(\varpi_{L_2}) = l$.

(b) We have:

$$\begin{aligned} \mathrm{Tr}_{L/L_2}(u) &= \mathrm{Tr}_{L/L_2}\left((\varpi_L)^\varepsilon\right) \cdot (\varpi_{L_2})^{\frac{l-\varepsilon}{\omega(\varpi_{L_2})}} \\ &= \begin{cases} (\varpi_{L_2})^{\frac{l-\varepsilon}{\omega(\varpi_{L_2})}+\varepsilon} & \text{if } \varepsilon = 1 \\ 2(\varpi_{L_2})^{\frac{l-\varepsilon}{\omega(\varpi_{L_2})}} & \text{if } \varepsilon = 0 \end{cases} \end{aligned}$$

Hence $\omega\left(\mathrm{Tr}_{L/L_2}(u)\right) = \left(\frac{l-\varepsilon}{\omega(\varpi_{L_2})} + \varepsilon\right)\omega(\varpi_{L_2}) = l - \varepsilon + \varepsilon\omega(\varpi_{L_2}) = l + \varepsilon$.

(c) We have $N_{L/L_2}(u) = u^\tau u = \mathrm{Tr}\left(\frac{1}{2}u^\tau u\right)$. □

As a consequence, we got an element (u, v) such that $\mathrm{Tr}_{L/L_2}(u)$ is minimal. Secondly, we seek an element $(x, y) \in H(L, L_2)_{l'}$ such that $t = 1 - {}^\tau u x + v y$. This is a quadratic problem. That is why we recall the following lemma on the existence of square root.

2.3.9 Lemma. *Let L be a local field of residue characteristic $p \neq 2$. For all $a \in \mathfrak{m}_L$, there exists $b \in \mathfrak{m}_L$ such that $(1+b)^2 = 1+a$ and $\omega(a) = \omega(b)$.*

Proof. Let $a \in \mathfrak{m}_L$. By Hensel's Lemma, the polynomial $X^2 - 1 - a$ admits exactly two roots $1+b$ and $-1+b'$ in $\mathcal{O}_L$, with $b, b' \in \mathfrak{m}_L$ since 1 and -1 are two distinct roots in κ_L of the polynomial $X^2 - 1$. Moreover $\omega(a) = \omega\big((1+b)^2 - 1\big) = \omega(b) + \omega(2+b)$. Since $p \neq 2$, we have $\omega(2+b) = 0$. Hence, $\omega(a) = \omega(b)$. □

We provide a solution $(x, y) \in H(L, L_2)_{l'}$ of $t = 1 - {}^{\tau}ux + vy$ for a suitable value l'' such that $t \in 1 + \mathfrak{m}_L^{l''}$.

2.3.10 Lemma. *Assume that $p \neq 2$. Let $l, l' \in \Gamma_a$ be such that $l + l' > 0$ and $l \in \Gamma'_a = \mathbb{Z}$. Define $\varepsilon \in \{0, 1\}$ as in Lemma 2.3.8. Define*

$$l'' = \max\big(1 + 2\varepsilon, \varepsilon + 2l + 2l'\big) \in \mathbb{N}^*$$

For any $w \in \mathfrak{m}_L^{l''}$, there exist $(u, v) \in H(L, L_2)_l$ and $(x, y) \in H(L, L_2)_{l'}$ such that ${}^{\tau}ux - vy = w$.

Proof. In order to simplify notation in this proof, we denote by T the field trace operator $\mathrm{Tr}_{L/L_2} : L \to L_2$.

Let $w \in (\mathfrak{m}_L)^{l''}$. Choose $u \in L$ satisfying the properties (a),(b) and (c) of Lemma 2.3.8 and set $v = \frac{1}{2}u{}^{\tau}u$. We seek an element $(x, y) \in H(L, L_2) \cap (L_2 \times L)$ such that ${}^{\tau}ux - vy = w$, which is equivalent to

$$\begin{cases} y = \frac{-w + {}^{\tau}ux}{v} \\ x^2 = T(y) = -T\left(\frac{w}{v}\right) + xT\left(\frac{{}^{\tau}u}{v}\right) \end{cases}$$

because $v \neq 0$ (otherwise property (a) would be contradicted).

Denote $\delta = 4\frac{T\left(\frac{w}{v}\right)}{T\left(\frac{{}^{\tau}u}{v}\right)^2}$. We have $T\left(\frac{{}^{\tau}u}{v}\right) = 2\frac{T(u)}{u{}^{\tau}u}$ by definition of $v = \frac{1}{2}u{}^{\tau}u \in L_2$ and by L_2-linearity of T. Hence $\omega\left(T\left(\frac{{}^{\tau}u}{v}\right)\right) = \omega\left(T(u)\right) - 2\omega(u) = -l + \varepsilon$. We have $\omega\left(T\left(\frac{w}{v}\right)\right) \geq \omega(w) - \omega(v) \geq l'' - 2l$. Hence $\omega(\delta) = \omega\left(T\left(\frac{w}{v}\right)\right) - 2\omega\left(T\left(\frac{{}^{\tau}u}{v}\right)\right) \geq l'' - 2\varepsilon \geq 1$. By Lemma 2.3.9, there exists $b \in \mathfrak{m}_{L_2}$ such that $(1+b)^2 = 1 - \delta$ and $\omega(b) = \omega(\delta)$. We denote $\sqrt[2]{1-\delta} = 1 + b$. Hence $\sqrt[2]{1-\delta} \in 1 + \delta\mathcal{O}_{L_2}$ is well-defined and $\omega\left(\sqrt[2]{1-\delta} - 1\right) = \omega(\delta)$.

Set $x = \frac{1}{2}T\left(\frac{{}^{\tau}u}{v}\right)\left(1 - \sqrt[2]{1-\delta}\right) \in L_2$ and set $y = \frac{w - {}^{\tau}ux}{v} \in L$. We have $x^2 = T(y)$. Moreover, $\omega(x) = \omega(\delta) + \varepsilon - l$. We check the valuation of y:

$$\begin{aligned}
\omega(y) &\geq \min\big(\omega(w), \omega(u) + \omega(x)\big) - \omega(v) \\
&\geq \min\big(l'', \omega(\delta) + \varepsilon\big) - 2l \\
&\geq \min\big(l'', l'' - 2\varepsilon + \varepsilon\big) - 2l \\
&= l'' - \varepsilon - 2l \\
&\geq 2l'
\end{aligned}$$

Hence $(u, v) \in H(L, L_2)_l$ and $(x, y) \in H(L, L_2)_{l'}$ are suitable. □

Finally, we can combine Lemmas 2.3.4, 2.3.6 and 2.3.10 in order to prove Proposition 2.3.1.

Proof of Proposition 2.3.1. Up to exchanging a and $-a$, one can suppose $l \in \Gamma'_a = \mathbb{Z} = \Gamma_L$. We let $l'' = \max(1+2\varepsilon, 3+\varepsilon)$ where

$$\varepsilon = \begin{cases} 1 & \text{if } L/L_2 \text{ is ramified and } l \in 2\mathbb{Z}+1 = \Gamma_L \setminus \Gamma_{L_2} \\ 0 & \text{otherwise} \end{cases}$$

By Lemma 2.3.4, we get $U_{-a,-l+1} \subset [H,H]$ and $U_{a,l+\frac{3}{2}} \subset [H,H]$ when L/L_2 is unramified; we get $U_{-a,-l+\frac{3}{2}} \subset [H,H]$ and $U_{a,l+2} \subset [H,H]$ when L/L_2 is ramified.

Let $t \in T^a(K)_b^{l''}$ and write it $t = \widetilde{a}(1+w)$ where $w \in (\mathfrak{m}_L)^{l''}$. Set $l_0 = -l + \frac{1}{2} \in \mathbb{Z}$ and $l'_0 = l+1$. By Lemma 2.3.10, there exist $(u,v) \in H(L,L_2)_{l_0}$ and $(x,y) \in H(L,L_2)_{l'_0}$ such that $-w = {}^\tau u x - v y$.

We use the commutation relation of opposite root groups 2.3.6. Let:

$$\begin{cases} T & = & 1+w \\ U & = & \frac{1}{{}^\tau T}\left(u^2 {}^\tau x - {}^\tau v x - u {}^\tau v {}^\tau y\right) \\ V & = & \frac{1}{T}\left(u v {}^\tau x - {}^\tau u {}^\tau v x + v {}^\tau v y\right) \\ X & = & \frac{1}{T}\left({}^\tau u x^2 - u y - v x y\right) \\ Y & = & \frac{1}{T}\left({}^\tau x u y - {}^\tau u x {}^\tau y + v y {}^\tau y\right) \end{cases}$$

By Lemma 2.3.6, we have $[x_{-a}(x,y), x_a(u,v)] = x_{-a}(X,Y)\widetilde{a}(T)x_a(U,V)$ with $\omega(V) \geq \lceil 3l'_0 + l_0 \rceil$ and $\omega(Y) \geq \lceil l'_0 + 3l_0 \rceil$.

Because $l \in \mathbb{Z}$, we have $\frac{1}{2}\lceil 3l'_0 + l_0 \rceil = l+2$ and $\frac{1}{2}\lceil l'_0 + 3l_0 \rceil = -l + \frac{3}{2}$. Hence $x_{-a}(X,Y) \in [T^a(K)_b^+, U_{-a,-l}]$ and $x_a(U,V) \in [T^a(K)_b^+, U_{a,l+\frac{1}{2}}]$ by Lemma 2.3.4. Because $\widetilde{a}(1+w) = x_{-a}(X,Y)^{-1}\,[x_{-a}(x,y), x_a(u,v)]\,x_a(U,V)^{-1} \in [H,H]$, we get $T^a(K)_b^{l''} \subset [H,H]$.

It remains to check the inclusion $H^p \subset [H,H]$ when K is of positive characteristic or is a ramified extension of $\mathbb{Q}_p$. Let $g \in H$ and write it as a product $g = x_{-a}(x,y)\widetilde{a}(1+w)x_a(u,v)$ with $w \in \mathfrak{m}_L$ and $(x,y) \in H(L,L_2)_{-l}$, $(u,v) \in H(L,L_2)_{l+\frac{1}{2}}$ so that $\omega(y) \geq -2l$ and $\omega(v) \geq 2l+1$. Consider the quotient homomorphism $\pi : H \to H/[H,H]$. Then

$$\pi(g^p) = \pi\Big(x_{-a}(x,y)^p\Big)\pi\Big(\widetilde{a}(1+w)^p\Big)\pi\Big(x_a(u,v)^p\Big).$$

with

$$\begin{aligned} x_{-a}(x,y)^p &= x_{-a}\left(px, py + \frac{p(p-1)}{2} x {}^\tau x\right) \\ \widetilde{a}(1+w)^p &= \widetilde{a}\left(1 + \sum_{k=1}^{p} \binom{p}{k} w^k\right) \\ x_a(u,v)^p &= x_a\left(pu, pv + \frac{p(p-1)}{2} u {}^\tau u\right) \end{aligned}$$

If K is of positive characteristic, then inside $H/[H,H]$, the element $x_a(u,v)^p$ is the neutral element in characteristic $p \neq 2$ and the same is for the element $x_{-a}(x,y)^p$. We have $(1+w)^p = 1+w^p$ with $\omega(w^p) \geq p \geq 5 \geq l''$. Hence $\widetilde{a}(1+w)^p \in T^a(K)_b^{l''} \subset [H,H]$. Thus $g^p \in [H,H]$.

If K is of characteristic 0 and the extension $K/\mathbb{Q}_p$ is ramified, then we have:

$$\begin{aligned}
\varphi_a\left(x_a(u,v)^p\right) =& \frac{1}{2}\omega\left(pv + \frac{p(p-1)}{2}u^\tau u\right)\\
\geq& \frac{1}{2}\left(\omega(p) + \min\left(\omega(v), \underbrace{\omega\left(\frac{p-1}{2}\right)}_{=0} + \underbrace{2\omega(u)}_{\geq \omega(v)}\right)\right)\\
\geq& \omega(\varpi_K) + l \qquad \text{since } \omega(p) \geq 2\omega(\varpi_K) \text{ and } \frac{1}{2}\omega(v) \geq l
\end{aligned}$$

If L/L_2 is unramified, then $\omega(\varpi_K) + l = 1 + l$ and in this case $x_a(u,v)^p \in U_{a,l+1} \subset [H,H]$. If L/L_2 is ramified, then $\omega(\varpi_K) + l = 2 + l$ and in this case $x_a(u,v)^p \in U_{a,l+2} \subset U_{a,l+\frac{3}{2}} \subset [H,H]$. The same is true for $x_{-a}(x,y)$. Finally, for $\widetilde{a}(1+w)^p$, we have:

$$\begin{aligned}
\omega\left((1+w)^p - 1\right) \geq& \min_{1\leq k\leq p}\left(\omega(\binom{p}{k}) + k\omega(w)\right)\\
\geq& \omega(p) + 1\\
\geq& 2\omega(\varpi_K) + 1\\
\geq& 2(1+\varepsilon) + 1 = 3 + 2\varepsilon\\
\geq& l'' = \max(1+2\varepsilon, 3+\varepsilon)
\end{aligned}$$

Hence $\widetilde{a}(1+w)^p \in T^a(K)_b^{l''} \subset [H,H]$ and we finally obtain that $g^p \in [H,H]$ for any $g \in H$. $\square$

In the case of higher rank, we obtain in Proposition 4.1.3 some inclusions of the form $U_{a,l_a} \subset [H,H]$ with a suitable value l_a, by commuting some root groups corresponding to non-collinear roots. Hence, it is useful to do a further assumption on subgroups contained in $[H,H]$.

2.3.11 Proposition. *Assume that $p \geq 5$. Let $l \in \Gamma_a = \frac{1}{2}\mathbb{Z}$. Let P be a compact open subgroup of $G(K)$ containing $H = U_{-a,-l}T^a(K)_b^+U_{a,l+\frac{1}{2}}$. Let $l'' = 1 + 2\varepsilon$ where*

$$\varepsilon = \begin{cases} 1 & \text{if } L/L_2 \text{ is ramified and } l \in 2\mathbb{Z}+1 = \Gamma_L \setminus \Gamma_{L_2} \\ 0 & \text{otherwise} \end{cases}$$

If $[P,P]$ contains $U_{a,l+1}$ and $U_{-a,-l+\frac{1}{2}}$, then $T^a(K)_b^{l''} \subset [P,P]$

Proof. In the above proof, up to exchanging a and $-a$ so that $l \in \mathbb{Z} + \frac{1}{2}$ and $l' \in \mathbb{Z}$, we can replace the equalities $l'_0 = l+1$ and $l_0 = -l+\frac{1}{2}$ by $l'_0 = l+\frac{1}{2} \in \mathbb{Z}$ and $l_0 = -l$. Indeed, in this case we obtain $\lceil 3l_0 + l'_0 \rceil = \lceil -2l + \frac{1}{2} \rceil = -2l + 1$, so that $U_{-a, \frac{1}{2}\lceil 3l_0 + l'_0 \rceil} \subset [P, P]$ by the additional assumption. In the same way, $\lceil 3l'_0 + l_0 \rceil = 2l + 2$ so that $U_{a, \frac{1}{2}\lceil l'_0 + 3l_0 \rceil} \subset [P, P]$. As a consequence, we can conclude as before. □

To conclude this section, we compute the commutation relation between elements of the same root group. This is non-trivial because, in the non-reduced case, the root group is non-commutative. This will be useful in order to understand the action of a maximal pro-p subgroup on the Bruhat-Tits building.

2.3.12 Lemma (Computation of the derived group of a valued root group: specificity on the non-reduced case)**.** *Let $l, l' \in \Gamma_a = \frac{1}{2}\mathbb{Z}$. In general, we have $\left[U_{a,l}, U_{a,l'}\right] \subset U_{2a, \lceil l \rceil + \lceil l' \rceil}$.*

If L/L_2 is unramified and $p \neq 2$, then $[U_{a,l}, U_{a,l}] = U_{2a, 2\lceil l \rceil}$.

If L/L_2 is ramified and $p \neq 2$, then $[U_{a,l}, U_{a,l}] = U_{2a, 2\lceil l \rceil + 1}$.

Proof. Let $(u,v),(x,y) \in H(L, L_2)$. In matrix-wise terms, we have

$$\left[\begin{pmatrix} 1 & -{}^\tau x & -y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -{}^\tau u & -v \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 & x^\tau u - u^\tau x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We deduce that $[x_a(x,y), x_a(u,v)] = x_a\left(0, u^\tau x - x^\tau u\right)$.

If $\omega(y) \geq 2l$, then $\omega(x) \geq \lceil l \rceil$ because $\omega(x) \in \Gamma_L = \mathbb{Z}$. Likewise, if $\omega(v) \geq 2l'$, then $\omega(u) \geq \lceil l' \rceil$. Hence $\omega\left(x^\tau u - u^\tau x\right) \geq \omega(u) + \omega(x) \geq \lceil l \rceil + \lceil l' \rceil$. We obtain $\left[U_{a,l}, U_{a,l'}\right] \subset U_{2a, \lceil l \rceil + \lceil l' \rceil}$.

Conversely, we show that any element of $U_{2a, 2\lceil l \rceil}$ can be written as the commutator of two suitable elements in $U_{a,l}$. For that, it suffices to show that for any $w \in L^0$ with $\omega(w) \geq 2\lceil l \rceil$, there exist $(u,v),(x,y) \in H(L, L_2)_l$ such that $w = x^\tau u - u^\tau x$.

We firstly consider the case of a unramified extension L/L_2 with $p \neq 2$. In this case, we have $\Gamma'_{2a} = \Gamma_{2a} = \mathbb{Z}$ by Lemma 2.1.10. Hence, there exists $\lambda_0 \in L^0$ with $\omega(\lambda_0) = 0$. Let $\varpi \in \mathcal{O}_{L_2}$ be a uniformizer. Set $x = \lambda_0 \varpi^{\lceil l \rceil}$ and set $y = \frac{1}{2} x^\tau x$ so that $(x,y) \in H(L, L_2)_l$. Let $w \in L^0$ with $\omega(w) \geq 2\lceil l \rceil$. Then $u = \frac{w}{x - {}^\tau x} \in L_2$. Indeed, ${}^\tau u = \frac{{}^\tau w}{{}^\tau x - x} = \frac{-w}{-(x - {}^\tau x)} = u$. Moreover, $\omega(x - {}^\tau x) = \omega\left((\lambda_0 - {}^\tau \lambda_0)\varpi^{\lceil l \rceil}\right) = \omega(2\lambda_0) + \omega(\varpi^{\lceil l \rceil}) = \lceil l \rceil$ because $p \neq 2$. Hence $\omega(u) = \omega(w) - \omega(x - {}^\tau x) = \lceil l \rceil$. Set $v = \frac{1}{2} u^\tau u = \frac{u^2}{2}$ so that $(u,v) \in H(L, L_2)_l$. We have $x^\tau u - u^\tau x = u(x - {}^\tau x) = w$.

We secondly consider the case of a ramified extension L/L_2 with $p \neq 2$. In this case, $\Gamma'_{2a} = \Gamma_{2a} = 2\mathbb{Z} + 1$ by Lemma 2.1.10. Thus $U_{2a, 2\lceil l \rceil} = U_{2a, 2\lceil l \rceil + 1}$. Moreover, there exists $\lambda_0 \in L^0$ with $\omega(\lambda_0) = 1$. Let $\varpi \in \mathcal{O}_{L_2}$ be a uniformizer.

If $\lceil l \rceil \in 2\mathbb{Z}$, we set $x = \lambda_0 \varpi^{\frac{\lceil l \rceil}{2}}$ and $y = \frac{1}{2} x^\tau x$ so that $(x, y) \in H(L, L_2)_l$.

Otherwise, $\lceil l \rceil \in 2\mathbb{Z} + 1$. We set $x = \lambda_0 \varpi^{\frac{\lceil l \rceil - 1}{2}}$ and $y = \frac{1}{2} x^\tau x$ so that $(x, y) \in H(L, L_2)_l$.

Let $w \in L^0$ with $\omega(w) \geq 2\lceil l \rceil + 1$. Then, as before, we get $u = \frac{w}{x - {}^\tau x} \in L_2$. Moreover, $\omega\left(\lambda_0 - {}^\tau\lambda_0\right) = \omega(2\lambda_0) = 1$ because $p \neq 2$. Hence, we obtain the inequalities $\omega(x) \geq \lceil l \rceil$ and $\omega\left(x - {}^\tau x\right) \leq \lceil l \rceil + 1$. Hence $\omega(u) = \omega(w) - \omega(x - {}^\tau x) \geq \lceil l \rceil$. We set $v = \frac{1}{2} u^\tau u = \frac{u^2}{2}$ so that $(u, v) \in H(L, L_2)_l$. We get $x^\tau u - u^\tau x = u(x - {}^\tau x) = w$. □

3 Bruhat-Tits theory for quasi-split semi-simple groups

In Bruhat-Tits theory, a building is attached to a reductive group in two steps. The first step, in [BT84, §4], corresponds to split and quasi-split groups. The second step in [BT84, §5] is an étale descent to the base field. In order to describe some subgroups in terms of the action on the Bruhat-Tits building, in Section 3.1, we recall how the simplicial structure of the building is defined thanks to the valuation of root groups. Then, in Section 3.2, we consider the action of the group $G(K)$ on its Bruhat-Tits building $X(G, K)$. In this section, K is a local field and G is an absolutely simple simply-connected quasi-split K-group.

3.1 Numerical description of walls and alcoves

The Bruhat-Tits building of (G, K) is obtained by gluing together affine spaces, called apartments, having the same given simplicial structure. This consists in defining the building as $X(G, K) = G(K) \times \mathbb{A}/\sim$, where $\mathbb{A}$ is a suitable affine space, called the standard apartment, see [Lan96, §9]. The apartments are glued together along hyperplanes called walls, that we will describe as zero sets of affine functions thanks to the sets of values defined in Section 2.1.5. In Section 3.1.1, we recall how we deduce the simplicial structure of an apartment from the definition of walls. More precisely, we define an "affinisation" of the spherical root system following the Bruhat-Tits method. In Lemma 3.1.14, we check that this construction coincide with the affine root system defined by Tits in [Tit79]. In Section 3.1.2, we describe, thanks to the sets of values, a well-chosen alcove, which is the candidate to be a fundamental domain of the action of $G(K)$ on $X(G, K)$. In Section 3.1.3, we will study locally the building in the neighbourhood of an alcove.

3.1.1 Walls of an apartment of the Bruhat-Tits building

In [Lan96, §1], we define a simplicial structure for apartments as follows. Firstly, we let $\mathbb{A} = \mathbb{A}(G, S, K)$ be the unique affine space under $V = X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ together with a suitable group homomorphism $\nu : \mathcal{N}_G(S)(K) \to \mathrm{Aff}(\mathbb{A})$.

Secondly, each relative root $a \in \Phi \subset X^*(S)$ induces a linear form on V deduced by linearity from the dual pairing $X_*(S) \times X^*(S) \to \mathbb{Z}$. Hence, up to choice of a special vertex defining an origin $\mathcal{O} \in \mathbb{A}$ of the affine space, each relative root induces an affine map on $\mathbb{A}$.

Thirdly, from the spherical root system (where each root is seen as a linear form), we define an "affinisation". Hence, each affine map $\theta(a,l) = a(\cdot - \mathcal{O}) + l : \mathbb{A} \to \mathbb{R}$, where $a \in \Phi$ and $l \in \mathbb{R}$, determinates a unique **half-apartment** denoted by:

$$D(a,l) = \{x \in \mathbb{A},\ \theta(a,l)(x) > 0\}$$

whose border (an affine subspace of codimension one) is denoted by $\mathcal{H}_{a,l} = \{x \in \mathbb{A},\ \theta(a,l)(x) = 0\}$. When $l \in \Gamma'_a$, the affine map $\theta(a,l)$ is called an **affine root**. In Lemma 3.1.14, we will see that the set of affine roots is the affine root system of [Tit79, 1.6].

For each affine root $\theta(a,l)$, the corresponding $\mathcal{H}_{a,l}$ is called a **wall** of $\mathbb{A}$. The walls induce a structure of poly-simplicial complex on $\mathbb{A}$: a connected component of $\mathbb{A} \setminus \bigcup_{a\in\Phi,\ l\in\Gamma'_a} \mathcal{H}_{a,l}$ is called an **alcove**. It is a simplex of maximal dimension. More generally, we define an equivalence relation on points on $\mathbb{A}$ by $x \sim y$ if, for any affine root θ, the real numbers $\theta(x)$ and $\theta(y)$ have the same sign or are both equal to zero. That means $x \sim y$ if, and only if, x and y always are in the same half-apartment. An equivalence class is called a **facet**; alcoves are the facets of maximal dimension. The set of facets constitutes a partition of $\mathbb{A}$. Finally, the affine space $\mathbb{A}$ together with the affine root system $\{\theta(a,l),\ a \in \Phi \text{ and } l \in \Gamma'_a\}$ and the structure of poly-simplicial complex deduced from the walls is called the **standard apartment**.

3.1.1 Notation. For any non-empty bounded subset Ω of $\mathbb{A}$, according to [BT72, §6 & §7], [BT84, §4] and [Lan96, §5], we denote:

- for any relative root $a \in \Phi$ [BT72, 6.4.2 & 7.1.1]:
  $$\begin{array}{rcl} f_\Omega(a) & = & \inf\{l \in \mathbb{R},\ a(x) + l \geq 0\ \forall x \in \Omega\} \\ & = & \sup\{-a(x),\ x \in \Omega\} \text{ [Lan96, §5]} \end{array} ;$$
  $U_{a,\Omega} = U_{a,f_\Omega(a)}$ for any relative root $a \in \Phi$;
  $$\begin{array}{rcl} f'_\Omega(a) & = & \inf\left\{l \in \Gamma'_a,\ l \geq f_\Omega(a) \text{ or, if } \tfrac{a}{2} \in \Phi,\ \tfrac{1}{2}l \geq f_\Omega(\tfrac{a}{2})\right\} \text{ [BT84, 4.5.2]} \\ & = & \inf\left\{l \in \Gamma'_a,\ l \geq f_\Omega(a)\right\} \text{ because, by definition } f_\Omega(\tfrac{a}{2}) = \tfrac{1}{2} f_\Omega(a) \\ & = & \inf\left\{l \in \Gamma'_a,\ U_{a,l} \subset U_{a,f_\Omega(a)}\right\} \text{ by [BT72, 6.4.9 (i)];} \end{array}$$

- U_Ω the subgroup of $G(K)$ generated by the groups $U_{a,\Omega}$ where $a \in \Phi$ [BT72, 6.4.2];

- $\widehat{N_\Omega} = \{n \in \mathcal{N}_G(S)(K),\ \forall x \in \Omega,\ n \cdot x = x\}$ [BT72, 7.1.10];

- $P_\Omega = U_\Omega \cdot T(K)_b$, (we recall that $T(K)_b$ normalizes U_Ω) [BT72, 7.1.1];

- $\widehat{P_\Omega}$ the subgroup of $G(K)$ generated by U_Ω and $\widehat{N_\Omega}$ [BT72, 7.1.10].

f'_Ω is called the optimized of f_Ω. According to [BT84, 4.5.2], it satisfies $U_{a,f'_\Omega(a)} = U_{a,f_\Omega(a)}$ so that $f'_\Omega(a) = \sup\{l \in \mathbb{R},\ U_{a,l} = U_{a,f_\Omega(a)}\}$ since ω is discrete.

Moreover, because G is a (quasi-split) semi-simple K-group, the group $\widehat{P}_\Omega$ can be realized as the integral points of some suitable model $\mathfrak{G}_\Omega$ of G (denoted by $\widehat{\mathfrak{G}}_\Omega$ in [BT84]). According to [BT84, 4.6.32], because G is simply connected, if Ω is contained in a panel of $X(G,K)$, then we have $\widehat{\mathfrak{G}}_\Omega = \mathfrak{G}^\circ_\Omega$. According to [BT84, 4.6.28], the group $\widehat{P}_\Omega = \mathfrak{G}^\circ_\Omega(\mathcal{O}_K)$ is the connected pointwise stabilizer in $G(K)$ of the subset $\Omega \subset X(G,K)$ (indeed, we have $G = G^1$ since G is semi-simple, see definition [BT84, 4.2.16]).

Recall that Φ and $\widetilde{\Phi}$ are irreducible since G is absolutely simple. From the dual pairing, each relative root $a \in \Phi$ can be realized geometrically in the Euclidean dual space V^*. By [Bou81, VI.1.4 Prop. 12], there are exactly one or two values for the length of a root if Φ is reduced; and by [Bou81, VI.4.14] there are three values if Φ is non-reduced. Moreover, when the irreducible root system Φ is non-reduced, it is entirely determined, up to isomorphism, by its rank. We say that a root $a \in \Phi$ is a **long root** if its length is maximal, and is a **short root** otherwise. More precisely, if Φ is a reduced non-simply laced root system, the ratio between the length of a long root and the length of a short root is exactly $\sqrt{d'}$ where the integer $d' \in \{1,2,3\}$ has been defined in 2.1.3 considering the smallest extension of K splitting G. Note that if $d' \geq 3$, then by [BT84, 4.1.16], the root system $\widetilde{\Phi}$ is of type D_4 and the root system Φ, of type G_2, is reduced.

3.1.2 Proposition. *Let d and L' as in 2.1.3.*

(1) If $d = 1$, every root $a \in \Phi$ has $L_a = L' = \widetilde{K} = K$ as splitting field (up to isomorphism, in the sense of 2.1.4).

(2) If $d \geq 2$ and Φ is reduced, every short root has L' as splitting field; every long root has K as splitting field.

(3) If $d = 2$ and Φ is non-reduced, every non-divisible root has L' as splitting field; every divisible root has K as splitting field.

Proof. (1) If $d = 1$, then $\Sigma = \Sigma_a$ for any root $a \in \Phi = \widetilde{\Phi}$. Hence, we have the equality of the corresponding fixed fields $\widetilde{K} = K = L_a = L'$.

Suppose now that $d \geq 2$. Because $\mathrm{Dyn}(\widetilde{\Delta})$ has a non-trivial symmetry, all the absolute roots have the same length in the geometric realisation in $\widetilde{V}^*$ defined in 2.1.2. Let a be a relative root, seen as orbit, which contain several absolute roots. In the geometric realization, the orbit a can be geometrically realized as the orthogonal projection of its absolute roots. Hence, the length of the orbits having several roots is shorter than that of the orbits having only one root.

Let $a \in \Phi$ be a relative root and let $\alpha \in \widetilde{\Phi}$ be an absolute root so that the relative root $a = \alpha|_S$ is its orbit for the Galois action.

(2) If $d \geq 2$ and Φ is reduced. If a is short, then Σ does not fix α. Moreover, we observe that for $d = 6$ (hence $\widetilde{\Phi}$ is of type D_4), the stabilizer of α in $\Sigma \simeq \mathfrak{S}_3$ has index 3. Hence L_α is a separable extension of K of degree 3 if $d \geq 3$ and of degree 2 otherwise, hence isomorphic to L'. Thus $L' = L_a$. If a is long, then Σ is the stabilizer of α. Hence $K = L_a$.

(3) If $d = 2$ and Φ is non-reduced. If a is divisible, then a is a long root. Hence Σ is the stabilizer of α. Thus $K = L_a$. Otherwise, a is a short root. Hence the stabilizer of α is trivial. Thus $L' = \widetilde{K} = L_a$. □

3.1.2 Description of an alcove by its panels

An alcove is the candidate to be a fundamental domain of the action of $G(K)$ on its Bruhat-Tits building $X(G, K)$.

3.1.3 Definition. A **panel** is a facet of $X(G, S)$ of codimension 1.

We want to describe precisely, thanks to some relative roots and their sets of values, walls bounding a given alcove. To do this, we may have to consider a dual root system, which appears to be necessary in some ramified cases.

Firstly, we define a dual root system of Φ by a suitable normalisation of the canonical dual root system in Lie considerations.

3.1.4 Notation. We consider a geometric realization of Φ_{nd} in the Euclidean space $\big(V^*, (\cdot|\cdot)\big)$. For each root $a \in \Phi_{\text{nd}}$, we set $\lambda_a = \frac{\mu^2}{(a|a)} \in \{1, d'\}$ and $a^D = \lambda_a a \in V$ where μ is the length of a long root in Φ_{nd}, so that $a^D = a$ for any long root of Φ_{nd}. The set $\Phi_{\text{nd}}^D = \{a^D,\ a \in \Phi_{\text{nd}}\}$ is a root system, because it is proportional (by a factor $\frac{\mu^2}{2}$) to the dual root system $\Phi_{\text{nd}}^\vee$ of [Bou81, VI.1.1 Prop. 2]. In particular, $\Phi_{\text{nd}}^D = \Phi_{\text{nd}}$ if, and only if, Φ is a simply laced root system (type A, D, or E). Moreover, by [Bou81, VI.1.5 Rem.(5)], if Δ is a basis of Φ, then $\Delta^D = \{a^D,\ a \in \Delta\}$ is a basis of Φ_{nd}^D.

Whereas $\Phi^\vee$ and Φ^D are constructions strictly in terms of Lie theory, we have found it was more convenient to introduce the following root system Φ^δ which takes into account the splitting field extensions of root groups.

3.1.5 Definition. For any non-divisible root $a \in \Phi_{\text{nd}}$, we denote by $\delta_a \in \{1, d'\}$ the order of the quotient group Γ_a / Γ_K (resp. $\Gamma_a / \Gamma_{L'}$) if Φ is reduced (resp. non-reduced), by $a^\delta = \delta_a a$ and by $\Phi_{\text{nd}}^\delta = \{a^\delta,\ a \in \Phi_{\text{nd}}\}$. We denote by $\Delta^\delta = \{a^\delta,\ a \in \Delta\}$. We will see below that $\Phi_{\text{nd}}^\delta = \Phi_{\text{nd}}$ or Φ_{nd}^D.

3.1.6 Notation. In the following, we denote by:

- h the highest root of Φ with respect to the chosen basis Δ;
- $\theta \in \Phi_{\text{nd}}$ the root such that θ^δ is the highest root of Φ_{nd}^δ with respect to the basis Δ^δ.

3.1.7 Proposition. *Let $\mathbf{c}_{\mathrm{af}}$ be the intersection of half-apartments $D(a,0)$ for $a \in \Phi^+$ and $D(b,0^+)$ for $b \in \Phi^-$. Then $\mathbf{c}_{\mathrm{af}}$ is an alcove and any panel of $\mathbf{c}_{\mathrm{af}}$ is contained in one of the walls $\mathcal{H}_{a,0}$ for $a \in \Delta$, or $\mathcal{H}_{-\theta,0^+}$.*

If Φ is reduced, we have:

- *$\Phi^\delta = \Phi$ when L'/K is unramified;*
- *$\Phi^\delta = \Phi^D$ when L'/K is ramified.*

If Φ is non-reduced, we have $\Phi^\delta_{\mathrm{nd}} = \Phi^D_{\mathrm{nd}} = \Phi_{\mathrm{nm}}$, so that $h = 2\theta$.

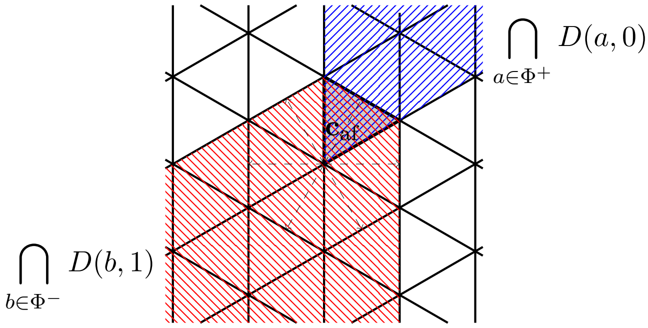
Proof. Note that if a is multipliable and $2l \in \Gamma'_{2a}$, it is possible that $\mathcal{H}_{2a,2l} = \mathcal{H}_{a,l}$ be a wall even if $l \notin \Gamma'_a$. Moreover, we have $\Gamma_a = \Gamma'_a \cup \frac{1}{2}\Gamma'_{2a}$ in this case. Otherwise, if a is non-multipliable and non divisible, we have $\Gamma_a = \Gamma'_a$ by Lemma 2.1.9. In fact, the walls of $\mathbb{A}$ are described by the various $a \in \Phi_{\mathrm{nd}}$ and $l \in \Gamma_a$.

According to [BT84, 4.2.23], we can classify the scalings to describe the various alcoves for an absolutely simple group G. In a similar way, there exists a classification of (quasi-split) absolutely almost-simple groups over a local field, provided by Tits in [Tit79, §4]. Here, we reduce the discussion to three types of behaviours.

First case: Φ is reduced and L'/K is unramified. These groups are the residually split groups named A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4 and G_2; and the non-residually split groups named ${}^2A'_{2n-1}$, ${}^2D_{n+1}$, ${}^3D_4$ and ${}^2E_6$ in the Tits tables [Tit79, 4.2, 4.3]. These correspond respectively to scalings, classified in [BT72, 1.4.6], of type A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4 and G_2; and C_n, B_n, F_4 and G_2.

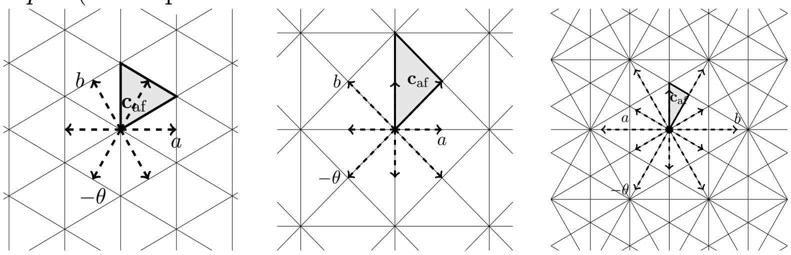
Let a be a relative root. Because Φ is reduced, $\Gamma_a = \Gamma_{L_a}$ by Lemma 2.1.9. Because L'/K is unramified, we have $\Gamma_{L'} = \Gamma_K$. Hence, by Proposition 3.1.2, we have $\Gamma_a = \Gamma_K$. Hence $\Phi^\delta = \Phi$ and $h = \theta$.

In order to simplify notations, we normalize the valuation ω so that $\Gamma_{L'} = \mathbb{Z} = \Gamma_K$ and $0^+ = 1$. By definition of alcoves as connected components, we can define an alcove as the intersection of all the various half-apartments $D(a,l)$ and $D(b,l^+)$ where $a \in \Phi^+$, $b \in \Phi^-$ and $l \in \mathbb{R}^+$. Because $D(a,l) \subset D(a,l')$ for any $l < l'$, we are in fact considering the finite intersection of all the various half-apartments $D(a,0)$ and $D(b,1)$ where $a \in \Phi^+$ and $b \in \Phi^-$. We call it "the" fundamental alcove, denoted by $\mathbf{c}_{\mathrm{af}}$.



By [Bou81, VI.2.2 Prop. 5], its panels are exactly contained inside the walls $\mathcal{H}_{a,0}$, where $a \in \Delta$, and $\mathcal{H}_{-h,1} = \mathcal{H}_{-\theta,0^+}$.

3.1.8 Example (The apartments and their fundamental alcoves in dimension 2).



Type A_2 Type C_2 Type G_2

Second case: Φ is reduced and L'/K is ramified. These groups are the residually split groups named B-C_n, C-B_n, F_4^I and G_2^I in the Tits tables [Tit79, 4.2]. These correspond respectively to scalings, classified in [BT72, 1.4.6], of type B-C_n, C-B_n, F_4^I and G_2^I.

Because L'/K is ramified, $d' \in \{2,3\}$, hence Φ is a non-simply laced root system. Moreover, we have $d'\Gamma_{L'} = \Gamma_K$. Let a be a relative root. Because Φ is reduced, $\Gamma_a = \Gamma_{L_a}$ by Lemma 2.1.9. By Proposition 3.1.2, if a is a long root, $\Gamma_a = \Gamma_K$; if a is a short root, $\Gamma_a = \Gamma_{L'}$. Thus, $\delta_a = \lambda_a$. Hence $\Phi^\delta = \Phi^D$ and for any root $a \in \Phi$, we have $\delta_a \Gamma_a = \Gamma_K$.

In order to simplify notations, we normalize the valuation ω so that $\Gamma_{L'} = \mathbb{Z}$. The intersection of all the various half-apartments $D(a,0)$ and $D(b,0^+)$ where $a \in \Phi^+$ and $b \in \Phi^-$ is exactly an alcove. If $b \in \Phi^-$ is short, then $\Gamma_b = \Gamma_{L'}$ so that $D(b,0^+) = D(b,1)$; if $b \in \Phi^-$ is long, then $\Gamma_b = \Gamma_K$ so that $D(b,0^+) = D(b,d')$. We call it "the" fundamental alcove, denoted by $\mathbf{c}_{\mathrm{af}}$.

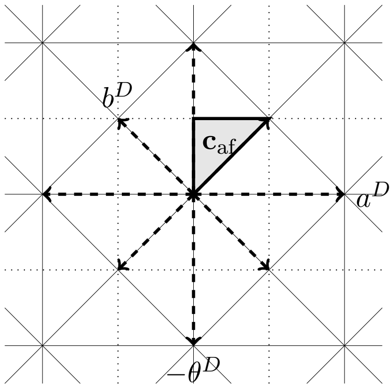
Its panels are exactly contained inside the walls $\mathcal{H}_{a,0}$, where $a \in \Delta$, and $\mathcal{H}_{-\theta,1} = \mathcal{H}_{-\theta,0^+}$. Indeed, let $a \in \Phi$ and $l \in \mathbb{R}$. Let $l^D = \delta_a l$ so that for any $x \in \mathbb{A}$:

$$a(x - \mathcal{O}) + l = 0 \Leftrightarrow a^D(x - \mathcal{O}) + l^D = 0$$

By definition, the set $\mathcal{H}_{a,l}$ is a wall of $\mathbb{A}$ if, and only if, $l \in \Gamma_a$; hence if, and only if, $l^D \in \Gamma_K$. Thus, the panels of $\mathbf{c}_{\mathrm{af}}$ are contained in the walls $\mathcal{H}_{a^D,l^D}$

described in the first case. Because the highest root θ^D is a long root in Φ^D by [Bou81, VI.1.8 Prop. 25 (iii)], hence θ is a short root in Φ and $\delta_\theta = d'$.

3.1.9 Remark. The ramification has the effect of adding some walls in the direction corresponding to short roots. For instance, if $d = 2$ and if the absolute root system $\widetilde{\Phi}$ is of type A_3, then the relative root system is of type C_2 and we obtain the following picture where we print the "added" walls with dotted lines, and the root system Φ^D instead of Φ:



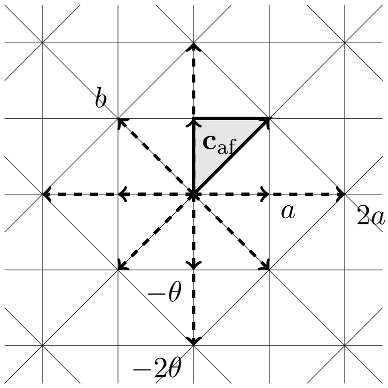
Third case: Φ is non-reduced. These groups are named $C\text{-}BC_n$ and ${}^2A'_{2n}$ in the Tits tables [Tit79, 4.2, 4.3]. These correspond respectively to scalings, classified in [BT72, 1.4.6], of type $C\text{-}BC_n^{III}$ and $C\text{-}BC_n^{IV}$.

Because Φ is non-reduced, $d = d' = 2$. In order to simplify notations, we normalize the valuation ω so that $\Gamma_{L'} = \mathbb{Z}$. Let a be a non-divisible relative root. If a is multipliable, by Lemma 2.1.10, we have $\Gamma_a = \frac{1}{2}\Gamma_{L'}$ and $\delta_a = \lambda_a = 2$ since a is short in Φ_{nd}; if a is non-multipliable, by Lemma 2.1.9, and by Proposition 3.1.2, we have $\Gamma_a = \Gamma_{L_a} = \Gamma_{L'}$ and $\delta_a = \lambda_a = 1$ since a is long in Φ_{nd}. Thus, $\delta_a \Gamma_a = \Gamma_{L'}$ and $\Phi^\delta_{\mathrm{nd}} = \Phi^D_{\mathrm{nd}} = \Phi_{\mathrm{nm}}$.

As above, one can see that the intersection of all the various following half-apartments: $D(a, 0)$ where $a \in \Phi^+_{\mathrm{nd}}$, $D(b, 1)$ where $b \in \Phi^-_{\mathrm{nd}}$ is non-multipliable, and $D(b', \frac{1}{2})$ where $b' \in \Phi^-_{\mathrm{nd}}$ is multipliable, is exactly an alcove. We call it "the" fundamental alcove, denoted by $\mathbf{c}_{\mathrm{af}}$. Its panels are exactly contained inside the walls $\mathcal{H}_{a,0}$, where $a \in \Delta$, and $\mathcal{H}_{-\theta,\frac{1}{2}} = \mathcal{H}_{-\theta,0^+}$.

Indeed, we proceed in the same way as in the previous case, with the reduced root system Φ^D_{nd}. $\square$

3.1.10 Example ($\widetilde{\Phi}$ of type A_4 and Φ of type BC_2).



3.1.3 Counting alcoves of a panel residue

Because a maximal pro-p subgroup P fixes an alcove $\mathbf{c}$, it acts on the set of alcoves which are adjacent to $\mathbf{c}$. We want to describe this set of alcoves.

3.1.11 Definition. Let F be a panel. The **panel residue** with respect to F, denoted by E_F, is the set of the alcoves whose closure contains F.

The **combinatorial unit ball** centered in $\mathbf{c}$, denoted by $B(\mathbf{c}, 1)$, is the union of all the panel residues with respect to a panel F in the closure of $\mathbf{c}$.

We say that two alcoves are **adjacent** if they have a common panel.

In what follows, we provide a reformulation and a proof of [Tit79, 1.6].

3.1.12 Proposition. *Let $a \in \Phi$ and $l \in \Gamma_a$. The group U_{a,l^+} is a normal subgroup of $U_{a,l}$. We denote by $X_{a,l} = U_{a,l}/U_{a,l^+}$ the quotient group.*

If a is non-multipliable, then there exists a canonical κ_{L_a}-vector space structure on $X_{a,l}$ of dimension 1.

If a is multipliable, then there exists a canonical group embedding $X_{2a,2l} \to X_{a,l}$; so that we have the inclusion $[X_{a,l}, X_{a,l}] \subset X_{2a,2l}$ and, in particular, $X_{2a,2l}$ is a normal subgroup of $X_{a,l}$. There exists a canonical κ_{L_a}-vector space structure on the quotient group $X_{a,l}/X_{2a,2l}$ of dimension 0 or 1.

Proof. Suppose that a is non-multipliable, then $U_a(K)$ is commutative. Hence U_{a,l^+} is a normal subgroup of $U_{a,l}$ and the quotient group $X_{a,l}$ is commutative. We define a $\mathcal{O}_{L_a}$-module structure on $X_{a,l}$ by:

$$\forall x \in \mathcal{O}_{L_a},\ \forall y \in L_a \text{ such that } \omega(y) \geq l,\ x \cdot x_a(y) U_{a,l^+} = x_a(xy) U_{a,l^+}$$

For any $x \in \varpi_{L_a}\mathcal{O}_{L_a}$ and any $y \in L_a$ such that $\omega(y) \geq l$, we have $\omega(xy) \geq l^+$, hence $xU_{a,l} \subset U_{a,l^+}$. This provides a $\kappa_{L_a} = \mathcal{O}_{L_a}/\varpi_{L_a}\mathcal{O}_{L_a}$-vector space structure on $X_{a,l}$. We check that this vector space is of dimension 1: for any $y, y' \in L_a$ such that $\omega(y) = \omega(y') = l$, since y is invertible, we have $x = y^{-1}y' \in \mathcal{O}_{L_a}$. Moreover, such elements y, y' exist by definition of Γ_a.

Suppose now that a is multipliable. By Lemma 2.3.12 applied to $l, l^+ \in \Gamma_a$, we get that U_{a,l^+} is a normal subgroup of $U_{a,l}$.

The normal subgroup $U_{2a,2l^+}$ of $U_{2a,2l}$ is the kernel of the canonical group homomorphism $U_{2a,2l} \to X_{a,l}$. Hence we deduce a group embedding $X_{2a,2l} \to X_{a,l}$. Passing to the quotient the formula of Lemma 2.3.12, we get $[X_{a,l}, X_{a,l}] \subset X_{2a,2l}$. In particular, the inclusion $[X_{a,l}, X_{2a,2l}] \subset [X_{a,l}, X_{a,l}] \subset X_{2a,2l}$ shows that $X_{2a,2l}$ is a normal subgroup of $X_{a,l}$.

In particular, the group $X_{a,l}/X_{2a,2l}$ is commutative. There exist an $\mathcal{O}_{L_a}$-module structure given by:

$$\forall x \in \mathcal{O}_{L_a},\ \forall (y,y') \in H(L_a, L_{2a}) \text{ such that } \omega(y') \geq 2l,$$
$$x \cdot x_a(y,y')U_{a,l^+}U_{2a,2l} = x_a(xy, x^\tau x y')U_{a,l^+}U_{2a,2l}$$

For any $x \in \varpi_{L_a}\mathcal{O}_{L_a}$ and any $(y,y') \in H(L_a, L_{2a})$ such that $\omega(y') \geq 2l$, we have $\omega(x^\tau x y') \geq 2(l^+)$. This defines a κ_{L_a}-vector-space structure on $X_{a,l}/X_{2a,2l}$. This vector-space is of dimension at most 1. Indeed, if there exist elements $(y,y'), (z,z') \in H(L_a, L_{2a})$ such that $\omega(y') = \omega(z') = 2l$, then we can set $x = y^{-1}z \in \mathcal{O}_{L_a}$ because y is invertible. Hence, we have $x_a(z,z') \in x \cdot x_a(y,y')U_{2a,2l}$. □

If a is a non-multipliable root, we set $X_{2a,2l} = 0$ and $\kappa_{L_{2a}} = \kappa_{L_a}$. Hence, the dimension $d(a,l) = \dim_{\kappa_{L_{2a}}} X_{a,l}/X_{2a,2l}$ has a sense for any root $a \in \Phi$.

3.1.13 Remark. Let F be a panel contained in a wall $\mathcal{H}_{a,l}$ corresponding to an affine root $\theta(a,l)$. Denote $q = \mathrm{Card}(\kappa_{L_{2a}})$. The panel residue E_F contains $1 + \mathrm{Card}(X_{a,l}) = 1 + q^{d(\frac{a}{2},\frac{l}{2})+d(a,l)+d(2a,2l)}$ elements. This is a consequence of Lemma 3.2.6.

Indeed, let $\mathbf{c}$ be an alcove whose F is a panel residue in the wall $\mathcal{H}_{a,l}$ with $f_{\mathbf{c}}(a) = l$. Then $\mathrm{Card}(E_F) = 1 + \mathrm{Card}(X_{a,l})$ when a is non-divisible and $\mathrm{Card}(E_F) = 1 + \mathrm{Card}(X_{\frac{a}{2},\frac{l}{2}})$ when a is divisible since $\mathcal{H}_{\frac{a}{2},\frac{l}{2}}$ is the same wall. If a is non-divisible, then $d(\frac{a}{2},\frac{l}{2}) = 0$, $q^{d(a,l)} = \mathrm{Card}\left(X_{a,l}/X_{2a,2l}\right)$ and $q^{d(2a,2l)} = \mathrm{Card}(X_{2a,2l})$ since $\kappa_{L_{2a}} = \kappa_{L_{4a}}$. Thus, $\mathrm{Card}(E_F) = 1 + \mathrm{Card}(X_{a,l}) = 1 + q^{d(a,l)+d(2a,2l)}$. If a is divisible, then $\kappa_{L_a} = \kappa_{L_{2a}}$ and $q^{d(\frac{a}{2},\frac{l}{2})} = \mathrm{Card}(X_{\frac{a}{2},\frac{l}{2}}/X_{a,l})$, $q^{d(a,l)} = \mathrm{Card}(X_{a,l})$ and $d(2a,2l) = 0$. Thus, $\mathrm{Card}(E_F) = 1 + \mathrm{Card}(X_{\frac{a}{2},\frac{l}{2}}) = 1 + q^{d(\frac{a}{2},\frac{l}{2})+d(a,l)}$.

The following lemma states that the affine root systems defined in [BT72, 6.2.6] and in [Tit79, 1.6] are the same.

3.1.14 Lemma. *Let $a \in \Phi$ be a root and $l \in \mathbb{R}$. Then $d(a,l) > 0$ if, and only if, $l \in \Gamma'_a$.*

Proof.

$$\begin{aligned}
l \in \Gamma'_a \quad &\Leftrightarrow \quad \exists \mathbf{u} \in U_a(K),\ \varphi_a(\mathbf{u}) = l = \sup \varphi_a(\mathbf{u}U_{2a}(K)) \\
&\Leftrightarrow \quad \exists \mathbf{u} \in U_a(K),\ \varphi_a(\mathbf{u}) = l \text{ and } \forall \mathbf{u}'' \in U_{2a}(K),\ \varphi_a(\mathbf{u}\mathbf{u}'') < l^+ \\
&\Leftrightarrow \quad U_{a,l} \neq U_{a,l^+} \text{ and } \exists \mathbf{u} \in U_{a,l},\ \forall \mathbf{u}'' \in U_{2a}(K),\ \mathbf{u}\mathbf{u}'' \notin U_{a,l^+} \\
&\Leftrightarrow \quad X_{a,l} \neq 0 \text{ and } X_{a,l} \neq X_{2a,2l} \\
&\Leftrightarrow \quad d(a,l) \neq 0
\end{aligned}$$

□

This affine root system is an affinisation of the spherical root system. It can be obtained by adding affine reflections corresponding to elements $\mathbf{m}(u) = u'uu''$ where for any $u \in U_a(K) \setminus \{1\}$, there exist $u', u'' \in U_{-a}K$ uniquely determined such that $\mathbf{m}(u) \in \mathcal{N}_G(S)(K)$.

3.2 Action on a combinatorial unit ball

We consider a maximal pro-p-subgroup $P = P_{\mathbf{c}}^+$ of $G(K)$ for an alcove $\mathbf{c} \subset \mathbb{A}$. For any $a \in \Phi$, if there exists a wall $\mathcal{H}_{a,l}$ bounding $\mathbf{c}$, we denote by $F_{\mathbf{c},a}$ the panel of $\mathbf{c}$ contained in $\mathcal{H}_{a,l}$. Let $E_{\mathbf{c},a} = E_{F_{\mathbf{c},a}}$ be the panel residue of $F_{\mathbf{c},a}$. We want to study the action of the derived group and of the Frattini subgroup of P on the Bruhat-Tits building $X(G,K)$ of G over K. For this, we consider the action, on each set $E_{\mathbf{c},a}$, of the various valued root groups $U_{a,\mathbf{c}}$ and of the group $T(K)_b^+$.

3.2.1 Lemma. *Let $\mathbf{c}_1$ and $\mathbf{c}_2$ be two adjacent alcoves of the apartment $\mathbb{A}$ along a wall directed by a root $a \in \Phi$. If $b \in \Phi \setminus \mathbb{R}a$, then $f'_{\mathbf{c}_1}(b) = f'_{\mathbf{c}_2}(b)$ where f' is defined in 3.1.1. In particular, we have $U_{b,\mathbf{c}_1} = U_{b,\mathbf{c}_2}$.*

Proof. In order that $f'_{\mathbf{c}_1}(b) \neq f'_{\mathbf{c}_2}(b)$, it is necessary and sufficient that there exists a wall directed by b separating the alcoves $\mathbf{c}_1$ and $\mathbf{c}_2$ in two opposed half-apartments. The alcoves $\mathbf{c}_1$ and $\mathbf{c}_2$ contain a panel contained in a wall directed by a. This wall is the only one separating the alcoves in two opposed half-apartments. Hence, if $f'_{\mathbf{c}_1}(b) \neq f'_{\mathbf{c}_2}(b)$, then a and b are collinear. □

Note that, as we have assumed that G is simply-connected, then by [BT84, 4.6.32], stabilizing a facet is equivalent to pointwise fixing it. Thus, we consider $E_{\mathbf{c},a}$ as finite set of alcoves.

3.2.2 Proposition. *Let $a \in \Phi = \Phi(G,S)$ be a relative root such that there exists a wall $\mathcal{H}_{a,l}$ bounding $\mathbf{c}$. If a is non-multipliable or if the quadratic extension L_a/L_{2a} is ramified, then the Frattini subgroup $\mathrm{Frat}(P)$ fixes $E_{\mathbf{c},a}$ pointwise.*

As a consequence, if Φ is a reduced root system or if the extension $\widetilde{K}/K$ is ramified, then $\mathrm{Frat}(P)$ fixes pointwise the simplicial closure $\mathrm{cl}(B(\mathbf{c},1))$ of the combinatorial unit ball.

In general, denoting by Q_a the pointwise stabilizer of $E_{\mathbf{c},a}$ in P, we have the group inclusion

$$\mathrm{Frat}(P) \subset \begin{cases} Q_a U_{2a,\mathbf{c}} & \text{if } a \text{ is multipliable, } L_a/L_{2a} \text{ is unramified and } f_{\mathbf{c}}(a) \in \Gamma'_a \\ Q_a & \text{otherwise.} \end{cases}$$

The rest of this section consists in proving the above proposition.

Let $\mathbf{c}'$ be an alcove of $\mathbb{A}$ adjacent to $\mathbf{c}$. In particular, we have $\mathbf{c}' \in B(\mathbf{c},1)$. Write $a' + r'$, with $a' \in \Phi$ and $r' \in \Gamma_{a'}$, the affine root directing the wall separating the alcoves $\mathbf{c}$ and $\mathbf{c}'$. If a' is divisible, we set $a = \frac{1}{2}a'$ and $r = \frac{1}{2}r'$.

Remark that we still have $r \in \Gamma_a$ but $a + r$ may or may not be an affine root according to r is an element of Γ'_a or not. Otherwise, we set $a = a'$ and $r = r'$. We also have the following definition of r by the equality $r = f_{\mathbf{c}}(a) = f'_{\mathbf{c}}(a)$ by [Lan96, 7.7]. Up to exchanging a and $-a$, one can assume that $f_{\mathbf{c}'}(a) = f_{\mathbf{c}}(a)^+ > f_{\mathbf{c}}(a)$ and that $f_{\mathbf{c}'}(-a) < f_{\mathbf{c}}(-a) = f_{\mathbf{c}'}(-a)^+$.

The group P acts on the finite set of alcoves $E_{\mathbf{c},a}$ and fixes $\mathbf{c}$. Hence, it acts on the set of alcoves $E'_{\mathbf{c},a} = E_{\mathbf{c},a} \setminus \{\mathbf{c}\}$. Denote by Q_a the kernel of this action. We will show that the quotient group P/Q_a is isomorphic to $U_{a,r}/U_{a,r^+}$.

3.2.3 Lemma. *The group $U_{a,\mathbf{c}}$ acts transitively on the set $E'_{\mathbf{c},a}$.*

Proof. By construction of the building, the subgroup $P_{\mathbf{c}}$ acts transitively on the set of apartments containing $\mathbf{c}$ [Lan96, 9.7 (i)]. Because the action preserves the type of facets, we obtain $E'_{\mathbf{c},a} = P_{\mathbf{c}} \cdot \mathbf{c}'$.

Write $P_{\mathbf{c}} = U_{a,\mathbf{c}} \cdot \prod_{b\in\Phi^+_{\mathrm{nd}}\setminus(a)} U_{\mathbf{c},b} \cdot U_{-\Phi^+,\mathbf{c}} \cdot T(K)_b$ [BT72, 7.1.8]. The group $T(K)_b$ fixes $\mathbb{A}$ pointwise [Lan96, 9.8], hence it also fixes $\mathbf{c}'$. For any $b \in \Phi \setminus \mathbb{R}a$, by Lemma 3.2.1 we have $U_{b,\mathbf{c}} = U_{b,\mathbf{c}'}$. Hence $U_{b,\mathbf{c}}$ fixes $\mathbf{c}'$. Since we assumed that $f_{\mathbf{c}'}(-a) < f_{\mathbf{c}}(-a)$, we have $U_{-a,\mathbf{c}} \subset U_{-a,\mathbf{c}'}$. Hence $U_{-a,\mathbf{c}}$ fixes $\mathbf{c}'$. As a consequence $E'_{\mathbf{c},a} = U_{a,\mathbf{c}} \cdot \mathbf{c}'$, because the valued root groups $U_{\mathbf{c},b}$ and the group $T(K)_b$ fix $\mathbf{c}'$. ☐

3.2.4 Lemma. *Let $g \in P$ be an element fixing $\mathbf{c}'$. If for every $v \in U_{a,\mathbf{c}}$ the element $[v, g]$ fixes $\mathbf{c}'$, then g fixes $E_{\mathbf{c},a}$.*

Proof. Let $\mathbf{c}'' \in E'_{\mathbf{c},a}$. By Lemma 3.2.3, there exists an element $v \in U_{a,\mathbf{c}}$ such that $\mathbf{c}'' = v\mathbf{c}'$. We do the following computation:

$$\begin{array}{rcll} g \cdot \mathbf{c}'' & = & gv \cdot \mathbf{c}' & \\ & = & v[v^{-1}, g]g \cdot \mathbf{c}' & \\ & = & v[v^{-1}, g] \cdot \mathbf{c}' & \text{because } g \text{ fixes } \mathbf{c}' \\ & = & v\mathbf{c}' & \text{because } [v^{-1}, g] \text{ fixes } \mathbf{c}' \\ & = & \mathbf{c}'' & \end{array}$$

Since this is true for any $\mathbf{c}'' \in E'_{a,\mathbf{c}}$, we conclude that g fixes $E_{a,\mathbf{c}}$. ☐

Hence, to show that $g \in [P, P]$ fixes $E_{\mathbf{c},a}$, it suffices to verify that $[U_{a,\mathbf{c}}, g]$ fixes $\mathbf{c}'$. We are reduced to compute commutators. Recall that the group $U_{a,f_{\mathbf{c}}(a)^+} = U_{a,\mathbf{c}'}$ fixes $\mathbf{c}'$.

3.2.5 Lemma. *The following groups:*

1. $U_{a,f_{\mathbf{c}}(a)^+}$

2. $T(K)_b^+$

3. $U_{b,\mathbf{c}}$ *where* $b \in \Phi \setminus \mathbb{R}a$

4. $U_{-a,\mathbf{c}}$

fix the panel residue $E_{\mathbf{c},a}$.

Proof. (1) Let $u \in U_{a,f_{\mathbf{c}}(a)^+}$. Then u fixes $\mathbf{c}'$. Let $v \in U_{a,\mathbf{c}}$.

If a is non-multipliable, then $[v,u]=1$ because the root group $U_a(K)$ is commutative.

If a is multipliable, by Lemma 2.3.12, we know that $[v^{-1},u] \in U_{2a,\lceil f_{\mathbf{c}}(a)^+\rceil + \lceil f_{\mathbf{c}}(a)\rceil}$. Since $\lceil f_{\mathbf{c}}(a)^+\rceil + \lceil f_{\mathbf{c}}(a)\rceil > 2f_{\mathbf{c}}(a)$, we deduce that $[v^{-1},u] \in U_{a,f_{\mathbf{c}}(a)^+} = U_{a,f_{\mathbf{c}'}(a)}$ fixes $\mathbf{c}'$.

Applying Lemma 3.2.4, we obtain that u fixes $E_{\mathbf{c},a}$.

(2) Let $t \in T(K)_b^+$. The element t fixes $\mathbf{c}'$ because $T(K)_b$ fixes the apartment $\mathbb{A}$. By Lemmas 2.2.1 and 2.3.4, we know that $[T(K)_b^+, U_{a,\mathbf{c}}] \subset U_{a,f_{\mathbf{c}}(a)^+} = U_{a,\mathbf{c}'}$. Hence $[v,t] \in U_{a,\mathbf{c}'}$ fixes $\mathbf{c}'$ for any $v \in U_{a,\mathbf{c}}$. We deduce from Lemma 3.2.4 that $T(K)_b^+$ fixes $E_{\mathbf{c},a}$.

(3) Let $g \in U_{b,\mathbf{c}}$ and $v \in U_{a,\mathbf{c}}$. By Lemma 3.2.1, we get $U_{b,\mathbf{c}} = U_{b,\mathbf{c}'}$. Hence $g \cdot \mathbf{c}' = \mathbf{c}'$. By quasi-concavity of the functions f' applied in the case where a and b are not collinear, we get by [BT84, 4.5.3]:

$$[v^{-1},g] \in \prod_{m,n\in\mathbb{N}^*,\ ma+nb\in\Phi} U_{ma+nb,f'_{\mathbf{c}}(ma+nb)}$$

Applying again Lemma 3.2.1, we get $U_{ma+nb,\mathbf{c}} = U_{ma+nb,\mathbf{c}'}$. Thus $[v,g]$ fixes $\mathbf{c}'$ for any v, hence, by Lemma 3.2.4, the element g fixes $E_{\mathbf{c},a}$.

(4) Let $u \in U_{-a,\mathbf{c}}$ and $v \in U_{a,\mathbf{c}}$. Since $f_{\mathbf{c}'}(-a) < f_{\mathbf{c}}(-a)$, we get $U_{-a,\mathbf{c}} \subset U_{-a,\mathbf{c}'}$. Hence u fixes $\mathbf{c}'$.

According to whether a is multipliable or not, we know that $[v,u] \subset U_{-a,f_{\mathbf{c}}(-a)^+} T(K)_b^+ U_{a,f_{\mathbf{c}}(a)^+}$, by applying either Lemma 2.3.6 or Lemma 2.2.2. The groups $U_{a,f_{\mathbf{c}}(a)^+}$, $T(K)_b^+$, and $U_{-a,f_{\mathbf{c}}(-a)^+} \subset U_{-a,f_{\mathbf{c}}(-a)}$ fix $\mathbf{c}'$. Thus, the commutator $[v,u]$ fixes $\mathbf{c}'$ because it can be written as the product of three such elements. Applying lemma 3.2.4, we conclude that u fixes $E_{\mathbf{c},a}$. $\square$

In fact, we have seen that the action of an element of $U_{a,\mathbf{c}}$ either acts trivially on $E'_{\mathbf{c},a}$ or does not fix any point of $E'_{\mathbf{c},a}$. That gives:

3.2.6 Lemma. *The group $X_{a,f_{\mathbf{c}}(a)}$, defined in Proposition 3.1.12, acts simply transitively on the set $E'_{\mathbf{c},a}$.*

Proof. From the equalities $f_{\mathbf{c}}(a)^+ = f_{\mathbf{c}'}(a) = f_{\mathbf{c}\cup\mathbf{c}'}(a)$, we deduce $U_{a,f_{\mathbf{c}}(a)^+} = U_{a,\mathbf{c}\cup\mathbf{c}'} = U_{a,\mathbf{c}} \cap U_{a,\mathbf{c}'}$. Thus, $U_{a,\mathbf{c}'}$ is the stabilizer in $U_{a,\mathbf{c}}$ of $\mathbf{c}'$. Hence $U_{a,f_{\mathbf{c}}(a)^+}$ is the kernel of this action (Lemma 3.2.5 (1)). We know that $U_{a,\mathbf{c}}$ acts transitively on $E'_{\mathbf{c},a}$ (Lemma 3.2.3). Hence the quotient group $X_{a,f_{\mathbf{c}}(a)} = U_{a,\mathbf{c}}/U_{a,f_{\mathbf{c}}(a)^+}$ acts simply transitively on $E'_{\mathbf{c},a}$. $\square$

3.2.7 Proposition. *Let $\mathbf{c},\mathbf{c}'$ be two alcoves of $\mathbb{A}$ adjacent along a panel $F_{\mathbf{c},a}$, where the root a is so that $f_{\mathbf{c}}(a)^+ = f_{\mathbf{c}'}(a) = f_{\mathbf{c}\cup\mathbf{c}'}(a)$. Let the maximal pro-p-subgroup P of the pointwise stabilizer of $\mathbf{c}$ in $G(K)$ be acting on the panel* *residue $E_{\mathbf{c},a}$ and Q_a be the pointwise stabilizer in P of this action. Then Q_a is the (unique) maximal pro-p subgroup of the pointwise stabilizer in $G(K)$ of $\mathbf{c} \cup \mathbf{c}'$.*

Proof. From the equalities $f_{\mathbf{c}}(a)^+ = f_{\mathbf{c}'}(a) = f_{\mathbf{c}\cup\mathbf{c}'}(a)$, we deduce $U_{a,f_{\mathbf{c}}(a)^+} = U_{a,\mathbf{c}\cup\mathbf{c}'}$. For any root $b \in \Phi_{\mathrm{nd}} \setminus \mathbb{R}a$, by Lemma 3.2.1, we get $f'_{\mathbf{c}}(b) = f'_{\mathbf{c}'}(b) = f'_{\mathbf{c}\cup\mathbf{c}'}(b)$. Hence $U_{b,f_{\mathbf{c}}(b)} = U_{b,\mathbf{c}\cup\mathbf{c}'}$. Finally, because we have assumed $f'_{\mathbf{c}'}(-a) < f'_{\mathbf{c}}(-a)$, we get the equality of groups $U_{-a,\mathbf{c}\cup\mathbf{c}'} = U_{-a,f'_{\mathbf{c}}(-a)} \cap U_{-a,f'_{\mathbf{c}'}(-a)} = U_{-a,\max(f'_{\mathbf{c}}(-a),f'_{\mathbf{c}'}(-a))} = U_{-a,\mathbf{c}}$. From this, we deduce the equality of groups:

$$U_{a,f_{\mathbf{c}}(a)^+}\left(\prod_{b\in\Phi_{\mathrm{nd}}\setminus\{a\}} U_{b,\mathbf{c}}\right) T(K)_b^+ U_{-\Phi^+,\mathbf{c}} = U_{\Phi^+,\mathbf{c}\cup\mathbf{c}'} T(K)_b^+ U_{-\Phi^+,\mathbf{c}\cup\mathbf{c}'}$$

We denote this group by $P^+_{\mathbf{c}\cup\mathbf{c}'}$ because one could show (as in [Loi16, 3.2.9]) that it is the (unique because of simply connectedness assumption on G) maximal pro-p subgroup of the pointwise stabilizer in $G(K)$ of $\mathbf{c} \cup \mathbf{c}'$.

By Lemma 3.2.5, the subgroup Q_a contains the subgroup $P^+_{\mathbf{c}\cup\mathbf{c}'}$. Conversely, $Q_a \subset P \cap P_{\mathbf{c}'} \subset P_{\mathbf{c}\cup\mathbf{c}'}$. Since Q_a is a subgroup of P, it is a pro-p-group. Thus $Q_a = P^+_{\mathbf{c}\cup\mathbf{c}'}$ by maximality. Moreover Q_a is a normal subgroup of P being the kernel of the action of P on $E_{\mathbf{c},a}$ and the quotient $P/P^+_{\mathbf{c}\cup\mathbf{c}'} = P/Q_a$ is isomorphic to $U_{a,f_{\mathbf{c}}(a)}/U_{a,f_{\mathbf{c}}(a)^+} = X_{a,f_{\mathbf{c}}(a)}$. □

If the quotient group $X_{a,f_{\mathbf{c}}(a)}$ is not commutative, that can happen when the root a is multipliable, then the Frattini subgroup of P is different from Q_a. Thus, we have to enlarge Q_a onto a group Q'_a so that the quotient P/Q'_a becomes a commutative group so that Q'_a contains the Frattini subgroup of P.

Proof of Proposition 3.2.2. We define a subgroup Q'_a by $Q'_a = Q_a U_{2a,2f_{\mathbf{c}}(a)}$ if a is multipliable, L_a/L_{2a} is unramified and $f_{\mathbf{c}}(a) \in \Gamma'_a$; and by $Q'_a = Q_a$ otherwise. We show that the quotient group P/Q'_a can be endowed with a vector space structure.

If a is non-multipliable, then by Proposition 3.1.12, we know that the quotient group $P/Q'_a = P/Q_a = X_{a,f_{\mathbf{c}}(a)}$ is a κ_{L_a}-vector space (of dimension 1). Now, suppose that a is multipliable.

If $f_{\mathbf{c}}(a) \notin \Gamma'_a$, then by Lemma 3.1.14, we know that $d(a,l) = 0$ so that $P/Q'_a = P/Q_a = X_{a,l} = X_{2a,2l}$ is a $\kappa_{L_{2a}}$-vector space of dimension 1 since $2a$ is non-multipliable. Now, suppose that $f_{\mathbf{c}}(a) \in \Gamma'_a$.

If L_a/L_{2a} is ramified. By Lemma 2.1.10, we know that $2f_{\mathbf{c}}(a) \notin \Gamma_{2a} = \Gamma'_{2a}$. Hence by Lemma 3.1.14, we know that $X_{2a,2f_{\mathbf{c}}(a)} = 0$. Then, by Proposition 3.1.12, we know that $P/Q'_a = P/Q_a = X_{a,f_{\mathbf{c}}(a)} = X_{a,f_{\mathbf{c}}(a)}/X_{2a,2f_{\mathbf{c}}(a)}$ is a κ_{L_a}-vector space of dimension 1.

Finally, the remaining case is L_a/L_{2a} unramified, $f_{\mathbf{c}}(a) \in \Gamma'_a$. It this case, because $Q'_a = Q_a U_{2a,2f_{\mathbf{c}}(a)}$, we have $P/Q'_a = X_{a,f_{\mathbf{c}}(a)}/X_{2a,2f_{\mathbf{c}}(a)}$ which is a κ_{L_a}-vector space of dimension 1 by Proposition 3.1.12.

As a consequence, on the one hand, the group P/Q'_a is commutative; hence $[P,P] \subset Q'_a$. On the other hand, the group P/Q'_a is of exponent p; hence $P^p \subset Q'_a$. We get $P^p[P,P] \subset Q'_a$. Because $G(K)$ acts continuously on $X(G,K)$, the group Q_a is an open subgroup of P as the kernel of the action of P on $E_{\mathbf{c},a}$. Moreover, the group $Q_a U_{2a,2f_{\mathbf{c}}(a)}$ is still open, hence closed. Hence $\mathrm{Frat}(P) = \overline{P^p[P,P]} \subset Q'_a$.

If Φ is a reduced root system or if the extension L'/K is ramified, then for any root $a \in \Phi$ corresponding to a panel of $\mathbf{c}$, we get that $\mathrm{Frat}(P)$ fixes $E_{\mathbf{c},a}$ pointwise and so it fixes the combinatorial ball of radius 1 centered in $\mathbf{c}$, denoted by $B(\mathbf{c},1)$, which is the union of all the $E_{\mathbf{c},a}$. By continuity of the action, the group $\mathrm{Frat}(P) = \overline{P^p[P,P]}$ fixes pointwise the simplicial closure of $B(\mathbf{c},1)$. □

3.2.8 Remark. In fact, when $\widetilde{K}/K$ is unramified and the irreducible root system Φ is non-reduced, for any multipliable root $a \in \Phi$, according to Lemma 2.1.10, we have either $f_{\mathbf{c}}(a) \in \Gamma'_a$ or $f_{\mathbf{c}}(-a) = -f_{\mathbf{c}}(a)^+ \in \Gamma'_a$.

3.2.9 Remark. Though the bounded torus $T(K)_b$ fixes pointwise the apartment $\mathbb{A}$, its action on the 1-neighbourhood of this apartement is, in general, non-trivial. For instance, assume that Φ is a reduced root system and choose a spherical root $a \in \Phi$ directing a wall bordering the alcove $\mathbf{c}$. The action of $T(K)_b$ on $E_{\mathbf{c},a}$ corresponds to the action of a subgroup of $\kappa_{L_a}^{\times 2} \subset \kappa_{L_a}^{\times}$. Let us explain this.

Normalise ω so that $\Gamma_{L_a} = \Gamma_a = \mathbb{Z}$ and choose ϖ as a uniformizer of $\mathcal{O}_{L_a}$. By simple transitivity of the action of $X_{a,f_{\mathbf{c}}(a)}$ (Lemma 3.2.6), we identify $E'_{\mathbf{c},a}$ with $U_{a,f_{\mathbf{c}}(a)}/U_{a,f_{\mathbf{c}}(a)+}$ by $\mathbf{c}'' = x_a(u) \cdot \mathbf{c}' \mapsto x_a(u) \mod U_{a,f_{\mathbf{c}}(a)+}$. Because $X_{a,f_{\mathbf{c}}(a)}$ is isomorphic to κ_{L_a} (Proposition 3.1.12), we identify $X_{a,f_{\mathbf{c}}(a)}$ with κ_{L_a} by $x_a(u) \mod U_{a,f_{\mathbf{c}}(a)+} \mapsto u\varpi^{-f_{\mathbf{c}}(a)} \mod \mathfrak{m}_{L_a}$. From the parametrization $\widetilde{a} : \mathbb{G}_m(\mathcal{O}_{L_a}) \to T^a(K)_b = T(K)_b \cap \langle U_{-a}(K), U_a(K)\rangle$, the action of $T(K)_b$ on $E'_{\mathbf{c},a}$ correspond to the action of $\kappa_{L_a}^{\times}$ on κ_{L_a} given by $z \cdot u = a\big(\widetilde{a}(z)\big)u = z^2 u$. Indeed, for any $z \in L_a^{\times}$, we have:

$$\begin{aligned}
\widetilde{a}(z) \cdot \mathbf{c}'' &= \widetilde{a}(z) x_a(u) \cdot \mathbf{c}' && \text{for some } u \in U_{a,\mathbf{c}}\\
&= \widetilde{a}(z) x_a(u) \widetilde{a}(z^{-1}) \underbrace{\widetilde{a}(z) \cdot \mathbf{c}'}_{=\mathbf{c}'}\\
&= x_a\Big(a\big(\widetilde{a}(z)\big)u\Big) \cdot \mathbf{c}'
\end{aligned}$$

3.2.10 Corollary (of Proposition 3.2.2)**.** *For any non divisible relative root $a \in \Phi_{\mathrm{nd}}$,*

- *if $a \notin \Delta \cup \{-\theta\}$, we set $V_{a,\mathbf{c}} = U_{a,\mathbf{c}}$;*
- *if $a \in \Delta \cup \{-\theta\}$ is non-multipliable, we set $V_{a,\mathbf{c}} = U_{a,f_{\mathbf{c}}(a)^+}$;*
- *if $a \in \Delta \cup \{-\theta\}$ and if a is multipliable, and either L_a/L_{2a} is ramified or $f_{\mathbf{c}}(a) \notin \Gamma'_a$, we set $V_{a,\mathbf{c}} = U_{a,f_{\mathbf{c}}(a)^+}$;*
- *if $a \in \Delta \cup \{-\theta\}$ and if a is multipliable, the extension L_a/L_{2a} is unramified and $f_{\mathbf{c}}(a) \in \Gamma'_a$, we set $V_{a,\mathbf{c}} = U_{a,f_{\mathbf{c}}(a)^+} U_{2a,2f_{\mathbf{c}}(a)} = U_{a,f_{\mathbf{c}}(a)^+} U_{2a,\mathbf{c}}$.*

We have the following:

$$\mathrm{Frat}(P) \subset \prod_{a \in \Phi^-_{\mathrm{nd}}} V_{a,\mathbf{c}} \cdot T(K)_b^+ \cdot \prod_{a \in \Phi^+_{\mathrm{nd}}} V_{a,\mathbf{c}} = T(K)_b^+ \prod_{a \in \Phi_{\mathrm{nd}}} V_{a,\mathbf{c}}$$

Proof. Since $\mathrm{Frat}(P) \subset P$, any element $g \in \mathrm{Frat}(P)$ can be written is a unique way as

$$g = t \prod_{a \in \Phi_{\mathrm{nd}}} u_a$$

with $t \in T(K)_b^+$ and $u_a \in U_{a,\mathbf{c}}$ for any $a \in \Phi_{\mathrm{nd}}$. It suffices to check that $u_a \in V_{a,\mathbf{c}}$ for any $a \in \Delta \cup \{-\theta\}$. Let $a \in \Delta \cup \{-\theta\}$ and consider the projection morphism $\pi_{Q_a} : P \to P/Q_a$. By Proposition 3.2.2, we have the inclusion $\mathrm{Frat}(P) \subset Q_a U_{2a,\mathbf{c}}$ when a is multipliable, the extension L_a/L_{2a} is unramified and $f_{\mathbf{c}}(a) \in \Gamma'_a$; we have the inclusion $\mathrm{Frat}(P) \subset Q_a$ otherwise. Thus $\pi_{Q_a}(g) = \pi_{Q_a}(u_a) \in V_{a,\mathbf{c}}/U_{a,f_{\mathbf{c}}(a)^+}$. Hence $u_a \in V_{a,\mathbf{c}}$. □

3.2.11 Proposition. *We assume that Φ is a reduced root system. The group $Q = T(K)_b^+ \prod_{a \in \Phi} V_{a,\mathbf{c}}$ is the maximal pro-p subgroup of the pointwise stabilizer in $G(K)$ of $\mathrm{cl}(B(\mathbf{c}, 1))$.*

Proof. Denote by $\mathrm{cl}\,(B(\mathbf{c}, 1))$ the simplicial closure of the combinatorial ball of radius 1. Set $\Omega = \mathrm{cl}(B(\mathbf{c}, 1)) \cap \mathbb{A}$. Denote by $\widehat{P}_{B(\mathbf{c},1)}$ (resp. $\widehat{P}_\Omega$) the pointwise stabilizer in $G(K)$ of $\mathrm{cl}\,(B(\mathbf{c}, 1))$ (resp. Ω). By [Lan96, 9.3 and 8.10], we can write $\widehat{P}_\Omega = T(K)_b \prod_{a \in \Phi} U_{a,\Omega}$.

By Lemma 3.2.5, we get that Q fixes $\mathrm{cl}(B(\mathbf{c}, 1))$ pointwise so that $Q \subset \widehat{P}_{B(\mathbf{c},1)} \subset \widehat{P}_\Omega$. Because $U_{a,\Omega} = V_{a,\mathbf{c}}$, we have that $\widehat{P}_\Omega / Q \simeq T(K)_b / T(K)_b^+$, so that Q is a maximal pro-p-subgroup of $\widehat{P}_\Omega$, therefore of $\widehat{P}_{B(\mathbf{c},1)}$.

It remains to show that it is the only one, in other words that Q is normal in $\widehat{P}_{B(\mathbf{c},1)}$. But since $T(K)_b$ normalises Q, this gives the result. □

4 Computation in higher rank

As before, G is an absolutely simple quasi-split simply-connected K-group and P is a maximal pro-p subgroup of $G(K)$. By a geometrical analysis, we provided, in Proposition 3.2.11, a description of the Frattini subgroup

Frat(P) as a subgroup of the (unique) maximal pro-p subgroup Q of a well-described stabilizer in $G(K)$. We now want to provide a large enough subset of Frat(P), so that this subset generates Q, and thus Frat(P). We provide unipotent elements of Frat(P) by finding some values $l_a \in \mathbb{R}$ with $a \in \Phi$ such that the valued root groups U_{a,l_a} are subgroups of $[P,P] \subset \mathrm{Frat}(P)$. In the rank-1 case treated in Section 2, we have already found some values l_a. In higher rank, we can improve these values for most roots; more precisely, for all roots which are not corresponding to panels of the (unique) alcove stabilized by P. In Section 4.1, we invert most commutation relations providing bounds of valuations of root groups. In Section 4.2, we combine those inversions in the whole root system.

4.1 Commutation relations between root groups of a quasi-split group

We consider both the split semi-simple $\widetilde{K}$-group $\widetilde{G} = G_{\widetilde{K}}$ and the quasi-split K-group G. A **Chevalley-Steinberg system** of $(G, \widetilde{K}, K)$ is the datum of morphisms: $\widetilde{x}_\alpha : \mathbb{G}_{a,\widetilde{K}} \to \widetilde{U}_\alpha$ parametrizing the various root groups of $\widetilde{G}$, and satisfying some axioms of compatibility, given in [BT84, 4.1.3], taking in account the commutation relations of absolute root groups and the $\mathrm{Gal}(\widetilde{K}/K)$-action on root groups. Note that despite the morphisms parametrize root groups of $\widetilde{G}$, a Chevalley-Steinberg system also depends on the quasi-split group G because of the relations between the $\widetilde{x}_\alpha$ where $\alpha \in \widetilde{\Phi}$. According to [BT84, 4.1.3], a quasi-split group always admits a Chevalley-Steinberg system.

According to [Bor91, 14.5], given an ordering on $\widetilde{\Phi}$, there exist constants $(c_{r,s;\alpha,\beta})_{r,s\in\mathbb{N}^*;\alpha,\beta\in\widetilde{\Phi}}$ in $\widetilde{K}$ for $r\alpha + s\beta \in \widetilde{\Phi}$, uniquely determined by the Chevalley-Steinberg system $(\widetilde{x}_\alpha)_{\alpha\in\widetilde{\Phi}}$, so that we have the following relations:

$$[\widetilde{x}_\alpha(u), \widetilde{x}_\beta(v)] = \prod_{\substack{r,s\in\mathbb{N}^* \\ r\alpha+s\beta\in\widetilde{\Phi}}} \widetilde{x}_{r\alpha+s\beta}(c_{r,s;\alpha,\beta} u^r v^s)$$

for any non-collinear roots $\alpha, \beta \in \widetilde{\Phi}$ and any parameters $u, v \in \widetilde{K}$. These constants are called the **structure constants**. There is some flexibility in the choice of a Chevalley-Steinberg system, so that we can choose $c_{r,s;\alpha,\beta}$ in $\mathbb{Z}1_{\widetilde{K}}$ where $1_{\widetilde{K}}$ denotes the identity element of $\widetilde{K}^\times$. More precisely, because $\widetilde{G}$ is split, it comes from a base change of a $\mathbb{Z}$-reductive group [DG, XXV 1.3]. In this case, one can determinate the $c_{r,s;\alpha,\beta}$, up to sign, to be some coefficients of a Cartan matrix [DG, XXIII 6.4]. More precisely, we have:

4.1.1 Lemma. *Let $\alpha, \beta \in \widetilde{\Phi}$ be two (non-collinear) roots such that $\alpha+\beta \in \widetilde{\Phi}$.*
If $\widetilde{\Phi}$ is of type A_n, D_n, or E_n, then $c_{1,1;\alpha,\beta} \in \{\pm 1_{\widetilde{K}}\}$.
If $\widetilde{\Phi}$ is of type B_n, C_n, or F_4, then $c_{1,1;\alpha,\beta} \in \{\pm 1_{\widetilde{K}}, \pm 2 \cdot 1_{\widetilde{K}}\}$.

If $\widetilde{\Phi}$ is of type G_2, then $c_{1,1;\alpha,\beta} \in \{\pm 1_{\widetilde{K}}, \pm 2 \cdot 1_{\widetilde{K}}, \pm 3 \cdot 1_{\widetilde{K}}\}$.

In the quasi-split case, given two non-collinear relative roots $a, b \in \Phi$, there exist commutation relations between the corresponding root groups in terms of the parametrizations $(x_a)_{a\in\Phi}$. These commutation relations can be completely computed in the irreducible root system $\Phi(a,b) = \Phi \cap (\mathbb{R}a \oplus \mathbb{R}b)$ of rank 2. Hence $\Phi(a,b)$ is of type A_2, C_2, BC_2 or G_2, and we can assume that a is shorter or has the same length as b. The various commutation relations are written down in [BT84, Annexe A] where Bruhat and Tits consider the angles between roots. Here, we follow another description in terms of length of roots, as in [PR84, §1].

We recall that, according to Section 2.1.2, the Galois group $\mathrm{Gal}(\widetilde{K}/K)$ acts on the absolute roots $\widetilde{\Phi}$ and that the relative roots Φ can be seen as the orbits for this action. We recall that $d' = [L'/K]$ has been defined in 2.1.3 to be the number of absolute roots in a short root seen as an orbit. We do the following assumptions:

4.1.2 Hypothesis. We assume that:

- the residue characteristic p of K is such that $p > d'$;
- the above structure constants $c_{1,1;\alpha,\beta}$, where $\alpha, \beta \in \widetilde{\Phi}$, are invertible in $\mathcal{O}_K$.

Equivalently, in terms of the relative root system, this is to assume that:

- $p \geq 3$ if the relative root system Φ of the quasi-split absolutely simple K-group G is of type B_n, C_n, BC_n or F_4;
- $p \geq 5$ if Φ is of type G_2.

Proof. Let us prove that both assumptions are equivalent.

If $d' = 1$, then $\widetilde{\Phi} = \Phi$. We have that 2 does not divide the structure constants $c_{1,1;\alpha,\beta}$ (seen in $\mathbb{Z}$), where $\alpha, \beta \in \widetilde{\Phi}$ if and only if, case $(2,1)$ or case $(2,2)$ of [DG, XXIII 6.4] do not arise. This is if, and only if, $\widetilde{\Phi} = \Phi$ is of type A_n, D_n, E_n. We have that 3 does not divide the structure constants $c_{1,1;\alpha,\beta}$ (seen in $\mathbb{Z}$), where $\alpha, \beta \in \widetilde{\Phi}$ if and only if, case $(3,1)$ of [DG, XXIII 6.4] do not arise. This is if, and only if, $\widetilde{\Phi} = \Phi$ is not of type G_2. Hence, in the case $d' = 1$, we have that the structure constants $c_{1,1;\alpha,\beta}$ are invertible in $\mathcal{O}_K$ if, and only if, $p \geq 3$ if the relative root system Φ of the quasi-split absolutely simple K-group G is of type B_n, C_n, or F_4; and $p \geq 5$ if Φ is of type G_2.

If $d' = 2$, then the Galois action on $\widetilde{\Phi}$ induces precisely an involution of the Dynkin diagram. Thus $\widetilde{\Phi}$ must be of type A_n, D_n or E_6 and Φ is of type B_n, C_n, BC_n or F_4. In that case, the structure constants $c_{1,1;\alpha,\beta} \in \{\pm 1\}$ are invertible in $\mathcal{O}_K$ and the condition $p > d' = 2$ is equivalent to $p \geq 3$.

If $d' = 3$, then the Galois action on $\widetilde{\Phi}$ induces a symmetry of order 3 of the Dynkin diagram. Thus $\widetilde{\Phi}$ is of type D_4 and Φ is of type G_2. In that case, the structure constants $c_{1,1;\alpha,\beta} \in \{\pm 1\}$ are invertible in $\mathcal{O}_K$ and the condition $p > d' = 3$ is equivalent to $p \geq 5$. □

4.1.3 Proposition. *Let $a, b, c \in \Phi$ be relative roots such that $c = a + b$ and, at least, one of the two roots a, b is non-multipliable. Let $l_a \in \Gamma_a$, $l_b \in \Gamma_b$ and $l_c \in \mathbb{R}$ be values such that $l_c = l_a + l_b$.*

Let $u \in U_{c,l_c}$. If Hypothesis 4.1.2 is satisfied, then there exist elements $v \in U_{a,l_a}$, $v' \in U_{b,l_b}$ and $v'' \in \prod_{\substack{r,s \in \mathbb{N}^* \\ r+s>2}} U_{ra+sb, rl_a+sl_b}$ such that $u = [v, v']v''$.

Proof. If u is the identity element, the statement is clear. From now on, we assume that u is not the identity element. We choose $\alpha \in a$ and $\beta \in b$. In this proof, length of root is considered in the irreducible (possibly non-reduced) root system $\Phi(a, b)$ of rank 2.

In the below various cases, we always follow the same sketch of proof. Firstly, we recall the splitting field of the roots a, b and $c = a+b$ computed in Proposition 3.1.2. Secondly, we recall the commutation relation between U_a and U_b, provided by [BT84, A.6] and we draw the relative roots that appear in the writing of this commutation relation. Thirdly, given a non-trivial unipotent element $u \in U_{c,l_c}$, we use the parametrisation of root groups, defined in Section 2.1.3, to provide suitable elements $v \in U_{a,l_a}$ and $v' \in U_{b,l_b}$. Finally, we check that $v'' = [v, v']^{-1}u$ is suitable.

Case $d' = 1$ or the relative roots a, b, c are long:

By Proposition 3.1.2, we have $L_a = L_b = L_c = K$.

By [BT84, A.6], we have the following commutation relation:

$$\forall y \in L_a,\ z \in L_b,\ [x_a(y), x_b(z)] = \prod_{r,s \in \mathbb{N}^*} x_{ra+sb}(c_{r,s;\alpha,\beta} y^r z^s)$$

where the structure constant $c_{r,s;\alpha,\beta}$ is the integer $C^{r\alpha+s\beta}_{\alpha,\beta}$ of [BT84, A.6 a)].

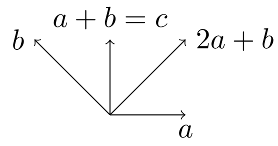
There exists a parameter $x \in L_c$ such that $u = x_c(x)$ and $\omega(x) \geq l_c$. We choose $y \in L_a$ such that $\omega(y) = l_a$. This is possible because $l_a \in \Gamma_a = \Gamma_{L_a}$ by Lemma 2.1.9. We set $z = c^{-1}_{1,1;\alpha,\beta} x y^{-1} \in L_b$. Then $\omega(z) = \omega(x) - \omega(y) \geq l_c - l_a = l_b$ satisfies $x = c_{1,1;\alpha,\beta} y z$. Then, we set $v = x_a(y)$, $v' = x_b(z)$ and $(v'')^{-1} = \prod_{r,s \in \mathbb{N}^*,\ r+s>2} x_{ra+sb}(c_{r,s;\alpha,\beta} y^r z^s)$. For any pair of non-negative integers (r, s) such that $r+s > 2$ and $ra+sb$ is a root, we get $\omega(c_{r,s;\alpha,\beta} y^r z^s) \geq r\omega(y) + s\omega(z) \geq rl_a + sl_b$. Hence $v'' \in \prod_{r,s \in \mathbb{N}^*; r+s>2} U_{ra+sb, rl_a+sl_b}$. Thus $[v, v'] = u(v'')^{-1}$.

Case $d' = 2$, the roots a, c are short, b is long and non-divisible:

By Proposition 3.1.2, we have $L_b = L_{2a+b} = K$ and $L_a = L_c = L'$.

By [BT84, A.6.b], there exist $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ such that we have the following commutation relation:

$$\begin{aligned} &\forall y \in L_a,\ \forall z \in L_b, \\ &\quad \Big[x_a(y), x_b(z)\Big] = x_{a+b}\Big(\varepsilon_1 yz\Big) \\ &\qquad\qquad\qquad\qquad\ x_{2a+b}\Big(\varepsilon_2 y{}^{\tau}yz\Big) \end{aligned}$$



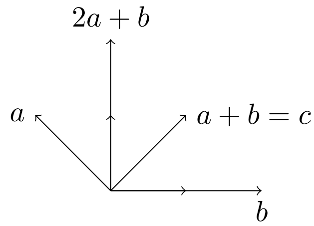
There exists a parameter $x \in L_c$ such that $u = x_c(x)$ and $\omega(x) \geq l_c$. We choose $z \in L_b$ such that $\omega(z) = l_b$. This is possible because $l_b \in \Gamma_b = \Gamma_{L_b}$. We set $y = \varepsilon_1 x z^{-1} \in L' = L_a$. Then $\omega(y) = \omega(x) - \omega(z) \geq l_c - l_b = l_a$ and $x = \varepsilon_1 yz$. The root $2a+b$ is non-divisible and we get $\omega(y{}^{\tau}yz) = 2\omega(y) + \omega(z) \geq 2l_a + l_b$. Then, we set $v = x_a(y)$, $v' = x_b(z)$ and $v'' = x_{2a+b}(-\varepsilon_2 y{}^{\tau}yz)$. Hence $v'' \in U_{2a+b, 2l_a+l_b}$. Thus $u = [v, v']v''$.

Case $d' = 2$, the roots a, c are short, b is long and divisible:

By Proposition 3.1.2, we have $L_a = L_c = L'$.

By [BT84, A.6.c], there exist $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ such that we have the following commutation relation:

$$\begin{aligned} &\forall y \in L_a,\ \forall z \in L^0_{\frac{b}{2}}, \\ &\quad \Big[x_a(y), x_{\frac{b}{2}}(0, z)\Big] = x_{a+b}\Big(\varepsilon_1 yz\Big) \\ &\qquad\qquad\qquad\qquad\qquad\ x_{a+\frac{b}{2}}\Big(0, \varepsilon_2 y{}^{\tau}yz\Big) \end{aligned}$$



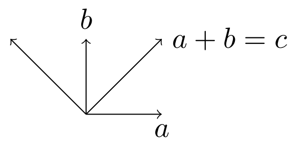
There exists a parameter $x \in L_c$ such that $u = x_c(x)$ and $\omega(x) \geq l_c$. By Lemma 2.1.10, we have $l_b \in \Gamma_b = \omega(L'^{0\times})$. Hence, we can choose $z \in L^0_{\frac{b}{2}} = L'^0$ such that $\omega(z) = l_b$. We set $y = \varepsilon_1 x z^{-1} \in L_a = L'$. Then $\omega(y) = \omega(x) - \omega(z) \geq l_c - l_b = l_a$ and $x = \varepsilon_1 yz$. The root $2a+b$ is divisible and we can check that $\omega(\varepsilon_2 y{}^{\tau}yz) = 2\omega(y) + \omega(z) \geq 2l_a + l_b$. Then, we set $v = x_a(y)$, $v' = x_b(z)$ and $v'' = x_{a+\frac{b}{2}}(0, -\varepsilon_2 y{}^{\tau}yz)$. Thus $u = [v, v']v''$.

Case $d' = 2$, the roots a, b are short, c is long and non-divisible:

By Proposition 3.1.2, we have $L_a = L_b = L'$ and $L_c = K$.

By [BT84, A.6.b], there exists $\varepsilon \in \{\pm 1\}$ such that we have the following commutation relation:

$$\begin{aligned} &\forall y \in L_a,\ \forall z \in L_b, \\ &\quad \Big[x_a(y), x_b(z)\Big] = x_{a+b}\Big(\varepsilon(yz + {}^{\tau}y{}^{\tau}z)\Big) \end{aligned}$$



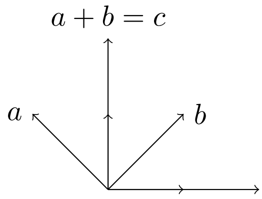
There exists a parameter $x \in L_c$ such that $u = x_c(x)$ and $\omega(x) \geq l_c$. We choose $z \in L_b = L'$ such that $\omega(z) = l_b$. This is possible because $l_b \in \Gamma_b$. We set $y = \frac{\varepsilon}{2} x z^{-1} \in L_a = L'$. This makes sense because p does not divide $d' = 2$, hence $2 \in \mathcal{O}_K^\times$. Then $\omega(y) = \omega(x) - \omega(z) \geq l_c - l_b = l_a$ and $\varepsilon \mathrm{Tr}(yz) = \frac{x}{2} + \frac{{}^{\tau}x}{2} = x$ because $x \in K$. Then, we set $v = x_a(y)$, $v' = x_b(z)$ and $v'' = 1$. Thus $u = [v, v']v''$.

Case $d' = 2$, the roots a, b are short, c is long and divisible:

By Proposition 3.1.2, we have $L_a = L_b = L_{\frac{c}{2}} = L'$.

By [BT84, A.6.c], there exists $\varepsilon \in \{\pm 1\}$ such that we have the following commutation relation:

$$\forall y \in L_a,\ \forall z \in L_b,\quad \Big[x_a(y), x_b(z)\Big] = x_{\frac{a+b}{2}}\Big(0, \varepsilon(yz - {}^{\tau}y^{\tau}z)\Big)$$



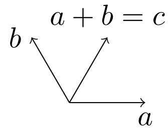
There exists a parameter $x \in L^0_{\frac{c}{2}} = L'^0$ such that $u = x_{\frac{c}{2}}(0,x)$ and $\omega(x) \geq l_c$. We choose $z \in L_b = L'$ such that $\omega(z) = l_b$. This is possible because $l_b \in \Gamma_b$. We set $y = \frac{\varepsilon}{2} x z^{-1} \in L_a = L'$. This is possible because p does not divide $d' = 2$, hence $2 \in \mathcal{O}_K^{\times}$. Then $\omega(y) = \omega(x) - \omega(z) \geq l_c - l_b = l_a$ and $\varepsilon\left(yz - {}^{\tau}y^{\tau}z\right) = \frac{x - {}^{\tau}x}{2} = x$ because $x + {}^{\tau}x = 0$. Then, we set $v = x_a(y)$, $v' = x_b(z)$ and $v'' = 1$. Thus $u = [v, v']v''$.

Case $d' = 2$, the roots a, b, c are short, a, b are non-multipliable:

By Proposition 3.1.2, we have $L_a = L_b = L_c = L'$.

By [BT84, A.6.b], there exists $\varepsilon \in \{\pm 1\}$ such that we have the following commutation relation:

$$\forall y \in L_a,\ \forall z \in L_b,\quad \Big[x_a(y), x_b(z)\Big] = x_{a+b}\Big(\varepsilon yz\Big)$$



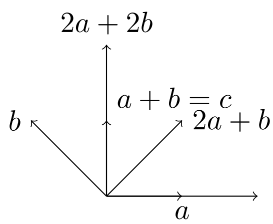
There exists a parameter $x \in L_c$ such that $u = x_c(x)$ and $\omega(x) \geq l_c$. We choose $z \in L_b = L'$ such that $\omega(z) = l_b$. We set $y = \varepsilon x z^{-1} \in L_a = L'$. Then $\omega(y) = \omega(x) - \omega(z) \geq l_c - l_b = l_a$ and $x = \varepsilon yz$. Then, we set $v = x_a(y)$, $v' = x_b(z)$ and $v'' = 1$. Thus $u = [v, v']v''$.

Case $d' = 2$, the roots a, b, c are short, b is non-multipliable and a is multipliable:

By Proposition 3.1.2, we have $L_a = L_b = L_c = L'$.

By [BT84, A.6.c], there exist $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ such that we have the following commutation relation:

$$\forall (y, y') \in H(L_a, L_{2a}),\ \forall z \in L_b,\quad \Big[x_a(y,y'), x_b(z)\Big] = x_{a+b}\Big(\varepsilon_1 yz, {}^{\tau}y'z^{\tau}z\Big)\, x_{2a+b}\Big(\varepsilon_2 zy'\Big)$$



There exists a parameter $(x, x') \in H(L_c, L_{2c})$ such that $u = x_c(x, x')$ and $\omega(x') \geq 2l_c$. We choose $z \in L_b$ such that $\omega(z) = l_b$. This is possible because $l_b \in \Gamma_b$. We set $y = \varepsilon_1 x z^{-1} \in L'$ and $y' = {}^{\tau}x' z^{-1} {}^{\tau}z^{-1}$. Then $y^{\tau}y = y' + {}^{\tau}y'$ and $\omega(y') = \omega(x') - 2\omega(z) \geq 2l_c - 2l_b = 2l_a$. This implies $(y, y') \in H(L_a, L_{2a})_{l_a}$. Moreover $(x, x') = (\varepsilon_1 yz, {}^{\tau}y'z^{\tau}z)$. The root $2a + b$ is non-multipliable, non-divisible, and we can check that $\omega(\varepsilon_2 z y') = \omega(y') + \omega(z) \geq 2l_a + l_b$. Then, we set $v = x_a(y, y')$, $v' = x_b(z)$ and $v'' = x_{2a+b}(-\varepsilon_2 {}^\tau x' {}^\tau z^{-1})$. Thus $u = [v, v']v''$.

Case $d' = 2$, the roots a, b, c are short and a, b are multipliable:

This case where a and b are both multipliable is the only one excluded by the third assumption. It is considered in Remark 4.1.4.

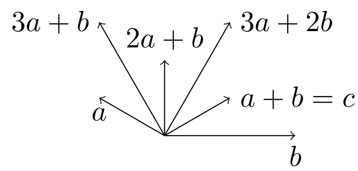
From now on, we assume $d' = 3$. This occurs only for the trialitarian D_4.

Case $d' = 3$, the roots a, c are short and b is long:

By Proposition 3.1.2, we have $L_a = L_c = L_{2a+b} = L'$ and $L_b = L_{3a+b} = L_{3a+2b} = K$.

We denote by $\tau \in \Sigma$ an element representing an element of order 3 in the Galois group. For any $y \in L'$, we denote $\Theta(y) = {}^\tau y {}^{\tau^2} y$ and $\mathrm{N}(y) = y\Theta(y)$. By [BT84, A.6.d], there exist an integer $\eta \in \{1, 2\}$ and four signs $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{-1, 1\}$ such that we have the following commutation relation:

$$\forall y \in L_a,\ \forall z \in L_b,$$
$$\begin{aligned} \Big[x_a(y), x_b(z)\Big] = \; & x_{a+b}\Big(\varepsilon_1 y z\Big) \\ & x_{2a+b}\Big(\varepsilon_2 \Theta(y) z\Big) \\ & x_{3a+b}\Big(\varepsilon_3 \mathrm{N}(y) z\Big) \\ & x_{3a+2b}\Big(\varepsilon_4 \eta \mathrm{N}(y) z^2\Big) \end{aligned}$$



There exists a parameter $x \in L_c = L'$ such that $u = x_c(x)$ and $\omega(x) \geq l_c$. We choose $z \in L_b = K$ such that $\omega(z) = l_b$. This is possible because $l_b \in \Gamma_b$. We set $y = \varepsilon_1 x z^{-1} \in L_a = L'$. Then $\omega(y) = \omega(x) - \omega(z) \geq l_c - l_b = l_a$ and $x = \varepsilon_1 y z$. The root $2a + b$ is short and the parameter $\varepsilon_2 \Theta(y) z \in L'$ satisfies $\omega(\varepsilon_2 {}^\tau y {}^{\tau^2} y z) = 2\omega(y) + \omega(z) \geq 2l_a + l_b$. The root $3a + b$ is long and the parameter $\varepsilon_3 \mathrm{N}(y) z \in K$ satisfies $\omega(\varepsilon_3 y {}^\tau y {}^{\tau^2} y z) = 3\omega(y) + \omega(z) \geq 3l_a + l_b$. The root $3a + 2b$ is long and the parameter $\eta \varepsilon_4 z^2 \mathrm{N}(y) \in K$ satisfies $\omega(\eta \varepsilon_4 z^2 y {}^\tau y {}^{\tau^2} y) = \omega(\eta) + 3\omega(y) + 2\omega(z) \geq 3l_a + 2l_b$.

Then we set $v = x_a(y)$, $v' = x_b(z)$ and

$$v'' = x_{3a+2b}\Big(-\eta\varepsilon_4 \mathrm{N}(y) z^2\Big) x_{3a+b}\Big(-\varepsilon_3 \mathrm{N}(y) z\Big) x_{2a+b}\Big(-\varepsilon_2 \Theta(y) z\Big)$$

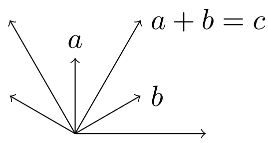
Hence $v'' \in U_{2a+b, 2l_a+l_b} U_{3a+b, 3l_a+l_b} U_{3a+2b, 3l_a+2l_b}$. Thus $u = [v, v']v''$

Case $d' = 3$, the roots a, b are short and c is long:

By Proposition 3.1.2, we have $L_a = L_b = L'$ and $L_c = K$.

We denote by $\tau \in \Sigma$ an element representing an element of order 3 in the Galois group Σ. For any $y \in L'$, we denote $\mathrm{Tr}(y) = y + {}^\tau y + {}^{\tau^2} y$. By [BT84, A.6.d], there exists a sign $\varepsilon \in \{-1, 1\}$ such that:

$$\forall y \in L_a,\ \forall z \in L_b,$$
$$\Big[x_a(y), x_b(z)\Big] = x_{a+b}\Big(\varepsilon \mathrm{Tr}(yz)\Big)$$



There exists a parameter $x \in L_c = K$ such that $u = x_c(x)$ and $\omega(x) \geq l_c$. We choose $z \in L_b = L'$ such that $\omega(z) = l_b$. This is possible because

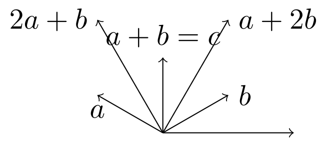
$l_b \in \Gamma_b$. We set $y = \frac{\varepsilon}{3} x z^{-1} \in L_a = L'$. This is possible because p does not divide $3 = d'$, hence $3 \in \mathcal{O}_K^\times$. Then $\omega(y) = \omega(x) - \omega(z) \geq l_c - l_b = l_a$ and $x = \varepsilon \mathrm{Tr}(yz)$. Then, we set $v = x_a(y)$, $v' = x_b(z)$ and $v'' = 1$. Thus $u = [v, v']v''$

Case $d' = 3$ and the roots a, b, c are short:

By Proposition 3.1.2, we have $L_a = L_b = L_c = L'$ and $L_{2a+b} = L_{a+2b} = K$.

We denote by $\tau \in \Sigma$ an element representing an element of order 3 in the Galois group Σ. For any $y \in L'$, we denote $\Theta(y) = {}^\tau y {}^{\tau^2} y \in L'$ and $\mathrm{Tr}(y) = y + {}^\tau y + {}^{\tau^2} y \in K$ and $\mathrm{N}(y) = y\Theta(y) \in K$. For any $y, z \in L'$, we denote $(y * z) = \Theta(y+z) - \Theta(y) - \Theta(z) = {}^\tau y {}^{\tau^2} z + {}^{\tau^2} y {}^\tau z$. By [BT84, A.6.d], there exist three signs $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}$ such that we have the following commutation relation:

$$\begin{aligned} &\forall y \in L_a,\ \forall z \in L_b, \\ &\Big[x_a(y), x_b(z)\Big] = x_{a+b}\Big(\varepsilon_1 (y * z)\Big) \\ &\qquad x_{2a+b}\Big(\varepsilon_2 \mathrm{Tr}\big(\Theta(y) z\big)\Big) \\ &\qquad x_{a+2b}\Big(\varepsilon_3 \mathrm{Tr}\big(y \Theta(z)\big)\Big) \end{aligned}$$



There exists a parameter $x \in L_c = K$ such that $u = x_c(x)$ and $\omega(x) \geq l_c$. We choose $z \in L_b = L'$ such that $\omega(z) = l_b$, this is possible because $l_b \in \Gamma_b$. Because p does not divide 2, hence $2 \in \mathcal{O}_K^\times$, we can define:

$$y = \frac{\varepsilon_1}{2} \cdot \frac{\mathrm{Tr}(xz) - 2xz}{\Theta(z)} = \frac{\varepsilon_1}{2\mathrm{N}(z)} \left(z \mathrm{Tr}(xz) - 2xz^2 \right)$$

so that $(y * z) = \varepsilon_1 x$. Indeed:

$$\begin{aligned} (y * z) &= \tfrac{\varepsilon_1}{2\mathrm{N}(z)} \left({}^\tau z \mathrm{Tr}(xz) - 2 {}^\tau x {}^\tau z^2 \right) {}^{\tau^2} z + \tfrac{\varepsilon_1}{2\mathrm{N}(z)} \left({}^{\tau^2} z \mathrm{Tr}(xz) - 2 {}^{\tau^2} x {}^{\tau^2} z^2 \right) {}^\tau z \\ &= \tfrac{\varepsilon_1 \Theta(z)}{2\mathrm{N}(z)} \left(\mathrm{Tr}(xz) - 2 {}^\tau x {}^\tau z + \mathrm{Tr}(xz) - 2 {}^{\tau^2} x {}^{\tau^2} z \right) \\ &= \tfrac{\varepsilon_1}{2z} (2xz) \end{aligned}$$

Then we have:

$$\begin{aligned} \omega(y) = &\ \omega\big(\mathrm{Tr}(xz) - 2xz\big) - \omega\big(\Theta(z)\big) \\ \geq &\ \min\Big(\omega\big(\mathrm{Tr}(xz)\big), \omega(x) + \omega(z)\Big) - 2\omega(z) \\ \geq &\ \big(\omega(x) + \omega(z)\big) - 2\omega(z) \\ = &\ \omega(x) - \omega(z) \\ \geq &\ l_c - l_b = l_a \end{aligned}$$

In fact, we get $\omega(y) = \omega(x) - \omega(z)$ because we deduce the inequality $\omega(x) \geq \omega(y) + \omega(z)$ from the formula $x = \varepsilon_1 (y * z)$. The root $2a + b$ is long and we can check that the parameter $\varepsilon_2 \mathrm{Tr}\big(\Theta(y) z\big) \in K$ satisfies $\omega\Big(\varepsilon_2 \mathrm{Tr}\big(\Theta(y) z\big)\Big) \geq 2\omega(y) + \omega(z) = 2l_a + l_b$. The root $a + 2b$ is long and we can check that the parameter $\varepsilon_3 \mathrm{Tr}\big(y\Theta(z)\big) \in K$ satisfies $\omega\Big(\varepsilon_3 \mathrm{Tr}\big(y\Theta(z)\big)\Big) \geq \omega(y) + 2\omega(z) = l_a + 2l_b$. Then, we set $v = x_a(y)$, $v' = x_b(z)$ and

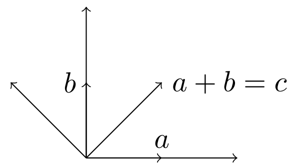
$$v'' = x_{a+2b}\Big(- \varepsilon_3 \mathrm{Tr}\big(y\Theta(z)\big)\Big) x_{2a+b}\Big(- \varepsilon_2 \mathrm{Tr}\big(\Theta(y)z\big)\Big)$$

Hence $v'' \in U_{2a+b,2l_a+l_b} U_{a+2b,l_a+2l_b}$. Thus $u = [v, v']v''$.

All the cases except the excluded one, where a, b both are multipliable, have been treated. $\square$

4.1.4 Remark. In the excluded case, by [BT84, A.6.c], there exists a sign $\varepsilon \in \{\pm 1\}$ such that we have the following commutation relation:

$$\begin{aligned} &\forall (y, y') \in H(L_a, L_{2a}), \\ &\forall (z, z') \in H(L_b, L_{2b}), \\ &\Big[x_a(y, y'), x_b(z, z')\Big] \quad = \quad x_{a+b}\Big(\varepsilon y z\Big) \end{aligned}$$



There exists a parameter $x \in L_c = L'$ such that $u = x_c(x)$ and $\omega(x) \geq l_c$. The problem is that, for a multipliable root $a \in \Phi$, the set of values Γ_a does not control completely the valuation of the first term y of a parameter $(y, y') \in H(L_a, L_{2a})$. One can show that, when $l_a \notin \Gamma'_a$, we get $\omega(y) > l_a$. Hence the inclusion $[U_{a,l_a}, U_{b,l_b}] \subset U_{a+b,l_a+l_b}$ is not, in general, an equality.

4.2 Generation of unipotent elements thanks to commutation relations between valued root groups

In Corollary 3.2.10, we obtained that $\mathrm{Frat}(P)$ is a subgroup of a pro-p group Q written in terms of valued root groups. We want to get an equality when it is possible. It suffices to provide a generating system of the biggest group consisting of p-powers and commutators of elements chosen in P. In a general consideration of a compact open subgroup H of $G(K)$, in Section 4.2.1, we do an induction on the positive roots from the highest to the simple roots to provide bounds of valued root groups contained in $[H, H]$; in Section 4.2.2, we furthermore consider the length of roots to provide bounds for the whole root system. In Section 4.2.3, we go back to the situation of the Frattini subgroup $\mathrm{Frat}(P) = \overline{P^p[P,P]} \supset [P, P]$.

In order to do an induction on the set of relative roots, the following lemma in Lie combinatorics explains how to get, step by step, all the roots as a linear combination with integer coefficients of the lowest root and the simple roots.

4.2.1 Lemma. *Let Φ be an irreducible root system of rank greater or equal to 2 and Δ be a basis of simple roots in Φ, associated to an order Φ^+. Let h be the highest root for this order.*

(1) Let $\beta \in \Phi^+ \setminus (\Delta \cup 2\Delta)$ be a positive root which is not the multiple of a simple root. Then, there exists a simple root $\alpha \in \Delta$ and a positive root

$\beta' \in \Phi^+$ such that $\beta = \alpha + \beta'$, the roots α, β' are not collinear and the positive root β' is non-multipliable.

(2) *Let $\gamma \in \Phi^- \setminus \{-h\}$. There exists a positive root $\beta \in \Phi^+$ and a negative root $\gamma' \in \Phi^-$ such that $\gamma = \beta + \gamma'$ and the roots β, γ' are not collinear.*

(3) *Let $\alpha \in \Delta \cup 2\Delta$. There exists a simple root $\beta \in \Delta$ such that $\alpha + \beta$ is a positive root and the roots $\alpha + \beta \in \Phi^+$ and $-\beta$ are not collinear.*

Proof. (1) Let $\beta \in \Phi^+ \setminus (\Delta \cup 2\Delta)$ be a positive non-simple root. Then, by [Bou81, VI.1.6], there is a simple root α and a positive root $\beta' \in \Phi^+$ such that $\beta = \alpha + \beta'$. Moreover, β' and α are not collinear because we assumed that β is not the multiple of a simple root.

Assume that Φ is non reduced of rank at least 2 and that β' is multipliable. We denote (using notations for vectors ε_i as in [Bou81, VI.4.14]) by $a_1 = \varepsilon_1 - \varepsilon_2, \dots, a_{l-1} = \varepsilon_{l-1} - \varepsilon_l$, $a_l = \varepsilon_l$ a basis of Φ with a_l the unique multipliable simple root. Then, there exists some j such that $\alpha = a_j$ There exists some $k \leqslant l$ such that $\beta' = \varepsilon_k = \sum_{i=k}^{l} a_i$ since β' is multipliable. If $j \neq l$, then $\beta = \alpha + \beta' = \varepsilon_j - \varepsilon_{j+1} + \varepsilon_k$ is a root, which means that $k = j+1$ and $\beta = \varepsilon_j$. Thus, one can replace α, β' by $\alpha = a_l = \varepsilon_l$ and $\beta' = \varepsilon_j - \varepsilon_l$ which is positive, non-multipliable and non-collinear to α. Otherwise, we have $j = l$ so that $\beta = \varepsilon_k + \varepsilon_l$. Hence $k < l$ since $\beta \neq 2a_l = 2\varepsilon_l$. Thus, one can replace α, β' by $\alpha = a_k = \varepsilon_k - \varepsilon_{k+1}$ and $\beta' = \varepsilon_{k+1} + \varepsilon_l$ which is positive, non-multipliable and non-collinear to α.

(2) According to notations of [Bou81, VI.1.3], we denote by V the $\mathbb{R}$-vector space generated by Δ containing Φ and by $(\cdot|\cdot)$ a scalar product which is invariant by the Weyl group.

Let $\gamma \in \Phi^- \setminus \{-h, -\frac{h}{2}\}$. If $(-h|\gamma) > 0$, then the sum $\beta = h + \gamma \in \Phi^+$ is a positive root [Bou81, Corollary of Theorem 1]. Moreover, $-h$ and β are not collinear because we assumed that γ and h are not collinear. Hence β and $\gamma' = -h$ satisfies assertion (2). Otherwise, we necessarily get the equality $(-h|\gamma) = 0$ according to [Bou81, VI.1.8 Proposition 25] and there exists a simple root $\alpha \in \Delta$ such that $(\alpha|\gamma) > 0$, because the roots $\alpha \in \Delta$ form a basis of the Euclidean space V and $-h \neq 0$. The roots γ and α are not collinear because, if they were, we should have $\gamma \in \mathbb{R}_+\alpha$ according to assumption $(\gamma|\alpha) > 0$; and this contradicts $\gamma \in \Phi^-$. Hence $\gamma' = \gamma - \alpha \in \Phi^-$ is a negative root. Thus, γ' and $\beta = \alpha$ satisfies assertion (2).

Let $\gamma = -\frac{h}{2}$. In particular, this happens only if Φ is non-reduced. We can apply the same method inside Φ_{nd}, because the root $-\frac{h}{2}$ is a short root of Φ_{nd}, hence it cannot be collinear to the highest root of Φ_{nd}.

(3) Let $\alpha \in \Delta \cup 2\Delta$. For any $\beta \neq \alpha$ connected to α (resp. $\beta \neq \frac{\alpha}{2}$ connected to $\frac{\alpha}{2}$ when $\alpha \in 2\Delta$) by an edge in $\mathrm{Dyn}(\Delta)$, we have $\alpha + \beta \in \Phi^+$. Thus β satisfies (3). Such a simple root β exists because we assumed that the rank of Φ is ≥ 2. $\square$

4.2.2 Lemma. *Let Φ be an irreducible root system and Δ be a basis of simple roots in Φ, associated to an order Φ^+. Let h be the highest root for this order. For any root $\gamma \in \Phi$, there exist non-negative integers $(n_\alpha(\gamma))_{\alpha\in\Delta}$ such that:*

$$\gamma = -h + \sum_{\alpha\in\Delta} n_\alpha(\gamma)\alpha$$

Proof. By [Bou81, VI.1.8 Proposition 25 (i)], there exist integers $p_\alpha \geq q_\alpha$ for $\alpha \in \Delta$ such that $h = \sum p_\alpha \alpha$ and $-\gamma = \sum q_\alpha \alpha$. Hence $n_\alpha(\gamma) = p_\alpha - q_\alpha \geq 0$. $\square$

We recall the following definition from [BT72, 6.4.3 and 6.4.5]:

4.2.3 Definition. Let $f : \Phi \to \mathbb{R}$ be a map. We say that the map f is **concave** if it satisfies the following axioms:

(C0) $f(2a) \leq 2f(a)$ for any root $a \in \Phi$ such that $2a \in \Phi$;

(C1) $f(a+b) \leq f(a) + f(b)$ for any roots $a, b \in \Phi$ such that $a+b \in \Phi$;

(C2) $0 \leq f(a) + f(-a)$ for any root $a \in \Phi$.

Despite these axioms look like a convexity property, they correspond in fact to a concavity property in terms of valued root groups.

4.2.4 Example. For any non-empty subset $\Omega \subset \mathbb{A}$, the map $f_\Omega : a \mapsto \sup\{-a(x),\ x \in \Omega\}$ is concave [BT84, 4.6.26]. Later, we will apply Propositions 4.2.6 and 4.2.9 to values $l_a = f_{\mathbf{c}_{\mathrm{af}}}(a)$.

4.2.1 Lower bounds for positive root groups

Let $(l_a)_{a\in\Phi}$ be any values in $\mathbb{R}$. We define the following values $(l'_b)_{b\in\Phi^+}$ depending on the l_a, to become bounds for the positive root groups.

4.2.5 Notation. For any positive root $b \in \Phi^+$, we can write uniquely $b = \sum_{\alpha\in\Delta} n_\alpha(b)\alpha$ where $n_\alpha(b) \in \mathbb{N}$ are non-negative integers (not all equal to zero). We define a value $l'_b = \sum_{\alpha\in\Delta} n_\alpha(b) l_\alpha$.

Thanks to Lemma 4.2.1, we do several inductions on various root systems to provide bounds, thanks to Proposition 4.1.3, for the valuations of the valued root groups contained in the Frattini subgroup $\mathrm{Frat}(P)$. The first step, in terms of positive roots, is the following:

4.2.6 Proposition. *Let $(l_a)_{a\in\Phi}$ be values in $\mathbb{R}$. Assume that for every simple root $a \in \Delta$, we have $l_a \in \Gamma_a$.*

(1) Then $l'_b \in \Gamma_b$ for any non-divisible positive root $b \in \Phi^+_{\mathrm{nd}}$.

(2) Assume, moreover, that the map $a \mapsto l_a$ is concave. Then we have $l'_b \geq l_b$ for any positive root $b \in \Phi^+$.

(3) Furthermore, assume that Hypothesis 4.1.2 is satisfied and that the rank of Φ is at least 2. Let H be a (compact open) subgroup of $G(K)$ containing the valued root groups U_{a,l_a} for $a \in \Phi$. Then for any root $b \in \Phi^+ \setminus (\Delta \cup 2\Delta)$, the derived group $[H,H]$ contains the valued root group U_{b,l'_b}.

Proof. (1) We apply Proposition 3.1.2 and Lemmas 2.1.10 and 2.1.9 in the various cases.

First case: Φ is a reduced root system and L'/K is unramified. For any root $b \in \Phi^+$, the set of values Γ_b of b is $\Gamma_{L'} = \Gamma_K$. Hence, the sum $l'_b = \sum_{\alpha\in\Delta} n_\alpha(b) l_a$ is an element of $\Gamma_K = \Gamma_b$.

Second case: Φ is a reduced root system and L'/K is ramified. For any long root of Φ, its set of values is the group $d'\Gamma_{L'} = \Gamma_K$. For any short root of Φ, its set of values is the group $\Gamma_{L'}$. Hence, for any short root $b \in \Phi$, the sum $l'_b = \sum_{\alpha\in\Delta} n_\alpha(b) l_\alpha$ is an element of $\Gamma_{L'} = \Gamma_b$.

Let $b \in \Phi$ be a long relative root arising from an absolute root $\beta \in \widetilde{\Phi}$. Write $\beta = \sum_{\widetilde{\alpha}\in\widetilde{\Delta}} n'_{\widetilde{\alpha}}(\beta)\widetilde{\alpha}$. Hence $n_\alpha(b) = \sum_{\widetilde{\alpha}\in\alpha} n'_{\widetilde{\alpha}}(\beta)$. Moreover, $n'_{\widetilde{\alpha}}(\beta)$ is constant along the class α because β is Σ-invariant and $\alpha = \Sigma\cdot\widetilde{\alpha}$ is an orbit. Hence, for any short simple root α arising from $\widetilde{\alpha}$, we obtain $n_\alpha(b) = d' n'_{\widetilde{\alpha}}(\beta)$. As a consequence, $n_\alpha(b) l_\alpha = n'_{\widetilde{\alpha}}(\beta) d' l_\alpha \in d'\Gamma_{L'} = \Gamma_K$. For any long simple root α, we have $l_\alpha \in \Gamma_K$. Hence, the sum $l'_b = \sum_{\alpha\in\Delta} n_\alpha(b) l_\alpha$ is an element of $\Gamma_K = \Gamma_b$.

Third case: Φ is a non-reduced root system. The set of values of any multipliable root is $\frac{1}{2}\Gamma_{L'}$. The set of values of any non-multipliable, non-divisible root is $\Gamma_{L'}$. Indeed, the splitting field of a non-multipliable, non-divisible root is L' by Proposition 3.1.2 (3). For any multipliable root $b \in \Phi^+$, the sum l'_b is an element of $\frac{1}{2}\Gamma_{L'} = \Gamma_b$. Up to isomorphism, there is a unique non-reduced root system of rank l described in [Bou81, VI.4.14]. We number by $a_1, \dots, a_{l-1}$ the non-multipliable simple roots and by a_l the multipliable simple root. Any non-multipliable non-divisible root $b \in \Phi^+$ can be written as $b = \sum_{j=1}^{l} n_j(b) a_j$ with $n_l(b) \in \{0, 2\}$. We have $n_j(b) l_{a_j} \in \Gamma_{a_j} = \Gamma_{L'}$ for any $j < l$ and $n_l(b) l_{a_l} \in 2\Gamma_{a_l} = \Gamma_{L'}$. Hence the sum l'_b is an element of $\Gamma_{L'} = \Gamma_b$.

(2) For any positive root $b \in \Phi^+$, we apply recursively Lemma 4.2.1(1) to Φ^+ in order to write $b = \sum_{j=1}^{N} a_j$ where $a_j \in \Delta$ are simple roots (possibly with repetitions) and $N \in \mathbb{N}^*$ such that $b_n = \sum_{j=1}^{n} a_j$ is a (positive) root for any $n \in [1, N]$. By induction, we get that $l'_{b_n} \geq l_{b_n}$. Indeed, for any $0 \leq n \leq N-1$, we have $l'_{b_{n+1}} = l'_{b_n} + l_{a_{n+1}} \geq l_{b_n} + l_{a_{n+1}}$ by induction hypothesis; and from the concavity relation (C1), we end the inequality by $l_{b_n} + l_{a_{n+1}} \geq l_{b_n + a_{n+1}} = l_{b_{n+1}}$. Hence, we obtain the inequality $l_b \leq l'_b$.

(3) Consequently, we have the inclusion $U_{b,l'_b} \subset U_{b,l_b}$. We proceed by decreasing strong induction on height in the root system Φ relatively to the basis Δ.

Basis: Let h be the highest root of Φ. For the root group U_{h,l'_h}, we know by Lemma 4.2.1(1) that there exists a simple root $a \in \Delta$ and a positive root

$b \in \Phi^+$ non-collinear to a, and non-multipliable, such that $h = a + b$. Recall that $l'_h = l_a + l'_b$ by definition since a is a simple root. Let $u \in U_{h,l'_h}$. We have the group inclusion $U_{b,l'_b} \subset U_{b,l_b}$.

Since b is non-multipliable, we get $l'_b \in \Gamma_b$ by (1) and we can apply Proposition 4.1.3, to a and b, so that there exist elements $v \in U_{a,l_a}$, $v' \in U_{b,l'_b}$ and $v'' \in \prod_{r,s \in \mathbb{N}^*; r+s>2} U_{ra+sb, rl_a+sl'_b}$ such that $u = [v, v']v''$. But, for any pair of positive integers (r, s) such that $r + s > 2$, the character $ra + sb$ is not a root because this would contradict maximality of height of h. Hence $v'' = 1$. Thus, we get $U_{h,l'_h} \subset [H, H]$.

Inductive step: Let $c \in \Phi^+ \setminus (\Delta \cup 2\Delta)$. By Lemma 4.2.1(1), we write $c = a + b$ where $a \in \Delta$ and $b \in \Phi^+_{\mathrm{nd}}$. Let $u \in U_{c,l'_c}$. Since $l_a \in \Gamma_a$ and $l'_b \in \Gamma_b$ by (1), we know by Proposition 4.1.3, that there exist elements $v \in U_{a,l_a}$, $v' \in U_{b,l'_b}$ and $v'' \in \prod_{r,s \in \mathbb{N}^*; r+s>2} U_{ra+sb, rl_a+sl'_b}$ such that $u = [v, v']v''$. For any pair of positive integers (r, s) such that $r + s > 2$, if the character $ra + sb$ is a root, then we have $rl_a + sl'_b = l'_{ra+sb}$ by definition of l'. Moreover, the height of $ra + sb$ is greater than that of c. By induction hypothesis, the valued root group $U_{ra+sb,l'_{ra+sb}}$ is a subgroup of $[H, H]$, hence $v'' \in [H, H]$. As a consequence, we get $U_{c,l'_c} \subset [H, H]$. □

4.2.2 Lower bounds for negative root groups

In order to get an analogous result for negative roots, doing an induction on height no longer works. In fact, we have to consider length of roots instead of height. We recall that, in Notation 3.1.4, we defined a pure Lie theoretic dual root system Φ^D.

4.2.7 Lemma. *Let Φ be a reduced irreducible non-simply laced root system of rank $l \geq 2$. Let Φ^+ be an ordering on Φ and $\theta \in \Phi$ be the short root such that θ^D is the highest root of Φ^D in the corresponding ordering. Then, any short root $c \in \Phi \setminus \{-\theta\}$ can be written $c = a + b$ where $a, b \in \Phi$ are non-collinear roots such that $a \in \Phi$ is short and $b \in \Phi^+$. In particular, every short root is higher than $-\theta$.*

Proof. We provide these roots case by case thanks to an explicit realization of the root system in $\mathbb{R}^l$ (see [Bou81, VI.4]). Let $(e_i)_{1 \leq i \leq l}$ be the canonical basis of the Euclidean space $\mathbb{R}^l$.

Φ is of type B_l with $l \geq 2$:

Basis: $a_i = e_i - e_{i+1}$ where $1 \leq i < l$ and $a_l = e_l$

Short roots: $\pm e_i$ for $1 \leq i \leq l$ and $\theta = e_1$

For any short root $c \in \Phi \setminus \{-\theta\}$,

- if $c \in \Phi^+$, we write $c = e_i = a + b$ with $1 \leq i \leq l$, $a = -e_j$, $b = e_i + e_j$ and $j \neq i$;
- if $c \in \Phi^-$, we write $c = -e_i = a + b$ with $1 < i \leq l$, $a = -e_1$ and $b = e_1 - e_i$.

Φ is of type C_l with $l \geq 3$:
Basis: $a_i = e_i - e_{i+1}$ where $1 \leq i < l$ and $a_l = 2e_l$
Short roots: $\pm e_i \pm e_j$ where $1 \leq i < j \leq l$ and $\theta = e_1 + e_2$
For any short root $c \in \Phi \setminus \{-\theta\}$,

- if $c = e_i \pm e_j$ where $1 \leq i < j \leq l$, we write $c = a+b$ where $a = -e_i \pm e_j$ and $b = 2e_i$;
- if $c = -e_i \pm e_j$ where $1 < i < j \leq l$, we write $c = a + b$ where $a = -e_1 - e_i$ and $b = e_1 \pm e_j$;
- if $c = -e_1 \pm e_j$ where $2 < j \leq l$, we write $c = a+b$ where $a = -e_1 - e_2$ and $b = e_2 \pm e_j$;
- if $c = -e_1 + e_2$, we write $c = a+b$ where $a = -e_1 - e_3$ and $b = e_2 + e_3$.

Φ is of type F_4:
Basis: $a_1 = e_2 - e_3$, $a_2 = e_3 - e_4$, $a_3 = e_4$ and $a_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$
Highest root: $h = e_1 + e_2 = 2a_1 + 3a_2 + 4a_3 + 2a_4$
Short roots: $\pm e_i$ where $1 \leq i \leq 4$ and $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$ and $\theta = e_1$
For any short root $c \in \Phi \setminus \{-\theta\}$,

- if $c = e_1$, we write $c = a + b$ where $a = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$ and $b = \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$;
- if $c = \pm e_i$ where $1 < i \leq 4$, we write $c = a + b$ where $a = \frac{1}{2}(-e_1 + \pm e_i - e_j - e_k)$ and $b = \frac{1}{2}(e_1 + \pm e_i + e_j + e_k)$ where $\{i, j, k\} = \{2, 3, 4\}$;
- if $c = \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)$, we write $c = a + b$ where $a = \frac{1}{2}(-e_1 \mp e_2 \pm e_3 \pm e_4)$ et $b = e_1 \pm e_2$;
- if $c = \frac{1}{2}(-e_1 \pm e_2 \pm e_3 \pm e_4)$, we write $c = a + b$ where $a = -e_1$ and $b = \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)$.

Φ is of type G_2:
Basis: α, β where α is short and β is long
Highest root: $h = 3\alpha + 2\beta$
We have $\theta = 2\alpha + \beta$. We summarize the choices for the short roots, except $-\theta$, case by case, in the following table:

c	$2\alpha+\beta$	$\alpha+\beta$	α	$-\alpha$	$-\alpha-\beta$
a	α	$-\alpha$	$-\alpha-\beta$	$-2\alpha-\beta$	$-2\alpha-\beta$
b	$\alpha+\beta$	$2\alpha+\beta$	$2\alpha+\beta$	$\alpha+\beta$	α

□

We let $(\delta_c)_{c\in\Phi}$, $\Phi^{\delta}_{\text{nd}}$, θ and h be defined as in Notation 3.1.6. Let $(l_a)_{a\in\Phi}$ be any values in $\mathbb{R}$. We define the following values $(l''_c)_{c\in\Phi}$ depending on the l_a, to become bounds for all the root groups.

4.2.8 Notation. For any non-divisible root $c \in \Phi_{\mathrm{nd}}$, thanks to Lemma 4.2.2 applied in the root system $\Phi^\delta_{\mathrm{nd}}$, we write:

$$c^\delta = -\theta^\delta + \sum_{\alpha\in\Delta} n'_\alpha(c)\alpha^\delta \in \Phi^\delta$$

with $n'_\alpha(c) \in \mathbb{N}$. We define $l''_c \in \mathbb{R}$ by:

$$\delta_c l''_c = \delta_{-\theta} l_{-\theta} + \sum_{\alpha\in\Delta} \delta_\alpha n'_\alpha(c) l_\alpha$$

Furthermore, for any multipliable root $c \in \Phi$, we define $l''_{2c} = 2l''_c$. Note that for any root $c \in \Phi$, there exist integers $n_\alpha(c)$ for $\alpha \in \Delta$, uniquely determined by:

$$c = \sum_{\alpha\in\Delta} n_\alpha(c)\alpha$$

This extends Notation 4.2.5.

These values overestimate the values of valued root groups contained in the derived group $[H, H]$. In particular, this proposition provides values even for simple roots, which were not treated in Proposition 4.2.6. We can remark on an example that, in general, these values are not optimal for positive non-simple roots.

4.2.9 Proposition. *Let $(l_a)_{a\in\Phi}$ be values in $\mathbb{R}$. Assume that for any simple root $a \in \Delta$, we have $l_a \in \Gamma_a$ and that $l_{-\theta} \in \Gamma_{-\theta}$. If Φ is non-reduced, assume furthermore that $l_{-\theta} \in \frac{1}{2}\Gamma_{2\theta}$.*

(1) We have $l''_c \in \Gamma_c$ for any root $c \in \Phi$.

(2) We assume, moreover, that the map $a \mapsto l_a$ is concave. For any root $c \in \Phi$, we have $l''_c \geq l_c$; for any positive root $b \in \Phi^+$, we have $l''_b \geq l'_b \geq l_b$.

(3) We assume, moreover, that the irreducible root system Φ is not of rank 1 and that Hypothesis 4.1.2 is satisfied. Let H be a (compact open) subgroup of $G(K)$ containing the valued root groups U_{a,l_a} for $a \in \Phi$. If G is a trialitarian D_4 (i.e. Φ of type G_2 and $\delta_\theta = 3$), we assume furthermore that $l'_\theta + l_{-\theta} \leq \omega(\varpi_{L'})$. Then the derived group $[H, H]$ contains the valued root groups U_{c,l''_c} for any root $c \in \Phi \setminus \{-\theta, -2\theta\}$.

Proof. (1) If Φ is a reduced root system, then $\Phi^\delta = \Phi$ if the extension L'/K is unramified; and $\Phi^\delta = \Phi^D$ if the extension L'/K is ramified. By Definition 3.1.5, for any root $c \in \Phi$, the integer δ_c is the order of the quotient group Γ_c/Γ_K, so that $\delta_c\Gamma_c = \Gamma_K$. Hence, each term $n'_\alpha(c)\delta_\alpha l_\alpha$ and $\delta_{-\theta}l_{-\theta}$ of the sum belongs to the group Γ_K. Thus $\delta_c l''_c \in \Gamma_K = \delta_c\Gamma_c$, and we obtain $l''_c \in \Gamma_c$ for any root $c \in \Phi$.

If Φ is a non-reduced root system, then the set of values of multipliable roots is $\frac{1}{2}\Gamma_{L'}$ by Lemma 2.1.10 and the set of values of non-multipliable and non-divisible roots is $\Gamma_{L'}$ by Lemma 2.1.9 and Proposition 3.1.2(3). For any non-divisible root $c \in \Phi$, the value $\delta_c l_c$ is an element of $\Gamma_{L'}$ by definition of δ_c. Indeed, if c is multipliable, then $\delta_c = 2$ by definition (see 3.1.5) and $l_c \in \Gamma_c = \frac{1}{2}\Gamma_{L'}$, and if c is non-multipliable and non-divisible, then $\delta_c = 1$ and $l_c \in \Gamma_c = \Gamma_{L'}$. Hence, as a sum of elements in $\Gamma_{L'}$, we get $\delta_c l''_c \in \Gamma_{L'}$. If c is non-multipliable and non-divisible, then $\delta_c = 1$, hence $l''_c \in \Gamma_{L'} = \Gamma_c$. If c is multipliable, then $\delta_c = 2$ hence $l''_c \in \frac{1}{2}\Gamma_{L'} = \Gamma_c$.

Finally, suppose that c is a divisible root and denote by $a_1, \dots, a_l$ the simple roots with a_l the multipliable simple root. Write $\theta = \sum_{i=1}^{l} a_i$. Let $b \in \Phi^+_{\mathrm{nd}}$ be the multipliable root such that $c = \pm 2b$ and write $b = \sum_{i=k}^{l} a_i$ for some $1 \leq k \leq l$. Since $\delta_b = \delta_\theta = 2$, we get

$$c = \pm\delta_b b = -2\theta + \sum_{i=1}^{l} n_i a_i$$

with $n_i \in \{0, 2, 4\}$ for every $1 \leq i \leq l-1$ and $n_l \in \{0, 4\}$. Thus

$$l''_c = 2l''_{\pm b} = \delta_{\pm b} l''_{\pm b} = -2l_{-\theta} + \sum_{i=1}^{l} n_i l_{a_i}.$$

For $i < l$, the simple root a_i is non-multipliable so that $l_{a_i} = \Gamma_{a_i} = \Gamma_{L'}$. Thus $n_i l_{a_i} \in 2\Gamma_{L'}$. For $i = l$, the simple root a_l is multipliable so that $l_{a_l} = \Gamma_{a_l} = \frac{1}{2}\Gamma_{L'}$. Thus $n_l l_{a_l} \in 4\Gamma_{a_l} = 2\Gamma_{L'}$. Hence we get that $l''_c \in \Gamma_{2\theta} + 2\Gamma_{L'}$ by assumption on $l_{-\theta}$ in the non-reduced case. But, according to Lemma 2.1.10, the set $\Gamma_{2\theta} + 2\Gamma_{L'}$ is the set of value of any divisible root. Hence $l''_c \in \Gamma_{2\theta} = \Gamma_c$.

(2) It is more convenient to treat separately the case of divisible roots so that h does not denote here the highest root of Φ but the highest root of Φ_{nd} in this proof.

In the following, for any root $c \in \Phi_{\mathrm{nd}}$, we denote by $n_\alpha(c)$ and $n'_\alpha(c)$ the integers defined in Notation 4.2.8. We furthermore denote by $n^\delta_\alpha(c)$ the integers uniquely determined by the following writing in basis Δ^δ: $c^\delta = \sum_{\alpha\in\Delta} n^\delta_\alpha(c)\alpha^\delta$. From uniqueness, for any $\alpha \in \Delta$, we deduce that $\delta_\alpha n^\delta_\alpha(c) = \delta_c n_\alpha(c)$ and that $n'_\alpha(c) = n^\delta_\alpha(\theta) + n^\delta_\alpha(c) \geq 0$ (it is a non-negative integer).

Let $b \in \Phi^+_{\mathrm{nd}}$ be a non-divisible positive root. In $V^* = \mathrm{Vect}(\Phi)$ we have:

$$\begin{aligned} b^\delta &= -\theta^\delta + \theta^\delta + \sum_{\alpha\in\Delta} n^\delta_\alpha(b)\alpha^\delta \\ &= -\theta^\delta + \sum_{\alpha\in\Delta} \left(n^\delta_\alpha(\theta) + n^\delta_\alpha(b)\right)\alpha^\delta \end{aligned}$$

By definition of l''_b, l'_b, l'_θ, we get:

$$\begin{aligned} \delta_b l''_b &= \delta_\theta l_{-\theta} + \sum_{\alpha\in\Delta} \left(n^\delta_\alpha(b) + n^\delta_\alpha(\theta)\right)\delta_\alpha l_\alpha \\ &= \delta_\theta l_{-\theta} + \left(\sum_{\alpha\in\Delta} \delta_b n_\alpha(b) l_\alpha\right) + \left(\sum_{\alpha\in\Delta} \delta_\theta n_\alpha(\theta) l_\alpha\right) \\ &= \delta_\theta l_{-\theta} + \delta_b l'_b + \delta_\theta l'_\theta \end{aligned}$$

Hence $\delta_b(l''_b - l'_b) = \delta_\theta(l'_\theta + l_{-\theta})$. According to Proposition 4.2.6(2), we have $l'_b \geq l_b$ for all positive roots and, in particular, $l'_\theta \geq l_\theta$. Hence, by axiom (C2), we get $l'_\theta + l_{-\theta} \geq l_\theta + l_{-\theta} \geq 0$. As a consequence, we get $l''_b \geq l'_b \geq l_b$.

Let $b \in \Phi^+$ be a multipliable root. Then $l''_{2b} = 2l''_b \geq l'_{2b} = 2l'_b \geq 2l_b$. By axiom (C0), we have $2l_b \geq l_{2b}$, hence $l''_{2b} \geq l'_{2b} \geq l_{2b}$.

Let $c \in \Phi^-_{\mathrm{nd}}$ be a non-divisible negative root. We want to prove that $l''_c \geq l_c$. We proceed by induction on height in Φ_{nd}.

• **First case:** $\Phi^\delta_{\mathrm{nd}} = \Phi_{\mathrm{nd}}$. Then $\delta_\theta = 1$, $h = \theta$ and $\delta_c = 1$ for any root $c \in \Phi_{\mathrm{nd}}$. By definition, $l''_{-h} = l''_{-\theta} = l_{-\theta} = l_{-h}$.

If $c \neq -\theta$, by Lemma 4.2.1(2), there exist $a \in \Phi_{\mathrm{nd}}$ and $b \in \Phi^+_{\mathrm{nd}}$ such that $c = a + b$. From

$$c = -\theta + \sum_\alpha n'_\alpha(c)\alpha = \left(-\theta + \sum_\alpha n'_\alpha(a)\alpha\right) + \sum_\alpha n_\alpha(b)\alpha = a + b,$$

we deduce $n'_\alpha(c) = n'_\alpha(a) + n_\alpha(b)$. Since $\delta_c = \delta_{-\theta} = \delta_\alpha = \delta_b = 1$, we have

$$\begin{aligned} l''_c =& l_{-\theta} + \sum_{\alpha \in \Delta} n'_\alpha(c) l_\alpha \\ =& \left(l_{-\theta} + \sum_{\alpha \in \Delta} n'_\alpha(a) l_\alpha\right) + \left(\sum_{\alpha \in \Delta} n_\alpha(b) l_b\right) \\ =& l''_a + l'_b \end{aligned}$$

Hence $l''_c = l''_a + l'_b \geq l_a + l'_b$ by induction hypothesis. By axiom (C1) and because $l'_b \geq l_b$, we get $l''_c \geq l_a + l_b \geq l_{a+b} = l_c$.

• **Second case:** $\Phi^\delta_{\mathrm{nd}} = \Phi^D_{\mathrm{nd}} \neq \Phi_{\mathrm{nd}}$. Then $\delta_\theta = d'$.

We proceed by induction on the height of roots. In a first step, we prove the result for all shorts roots of Φ_{nd}, in a second step, for all roots of Φ_{nd}. In the following, h will denote the highest root of Φ_{nd}, so that h is long. The root $\theta \in \Phi_{\mathrm{nd}}$ will always denote the root such that θ^D is the highest root of $\Phi^\delta_{\mathrm{nd}} = \Phi^D_{\mathrm{nd}}$, so that θ is short. Note that in the case of a non-reduced root system, short roots of Φ_{nd} also are short roots of Φ. Recall that, by Lemma 4.2.7, the root $-\theta$ has the smallest height among short roots.

We firstly do the induction, initialized by $l''_{-\theta} = l_{-\theta}$, on height among short roots of Φ_{nd}. Assume that $c \neq -\theta$ is a short root in Φ_{nd}. By Lemma 4.2.7, there exist a short root $a \in \Phi_{\mathrm{nd}}$ and a positive root $b \in \Phi^+_{\mathrm{nd}}$ such that $c = a + b$ so that the height of a is smaller than that of c. Hence $\delta_a = \delta_c = \delta_\theta$. We have

$$\begin{aligned} \delta_\theta b &= \delta_\theta(c - a) = c^\delta - a^\delta \\ &= \left(-\theta^\delta + \sum_\alpha \delta_\alpha n'_\alpha(c)\alpha\right) - \left(-\theta^\delta + \sum_\alpha \delta_\alpha n'_\alpha(a)\alpha\right) \\ &= \sum_\alpha \delta_\alpha \Big(n'_\alpha(c) - n'_\alpha(a)\Big)\alpha. \end{aligned}$$

Hence $\delta_\theta n_\alpha(b) = \delta_\alpha\big(n'_\alpha(c) - n'_\alpha(a)\big)$ for any $\alpha \in \Delta$. Hence, we get:

$$\begin{aligned}
\delta_c l''_c =& \delta_\theta l_{-\theta} + \sum_\alpha \delta_\alpha n'_\alpha(c) l_\alpha \\
=& \Big(\delta_\theta l_{-\theta} + \sum_\alpha \delta_\alpha n'_\alpha(a) l_\alpha\Big) + \sum_\alpha \delta_\alpha \Big(n'_\alpha(c) - n'_\alpha(a)\Big) l_\alpha \\
=& \delta_a l''_a + \delta_\theta l'_b
\end{aligned}$$

Hence $l''_c = l''_a + l'_b \geq l_a + l'_b$ by induction hypothesis. By axiom (C1) and because $l'_b \geq l_b$, we get $l''_c \geq l_a + l_b \geq l_{a+b} = l_c$.

Now we do an induction on height for all roots of Φ_{nd}.

Basis: consider the lowest root $-h$. Because Φ_{nd} is non-simply laced, there exist two short roots $a, b \in \Phi_{\mathrm{nd}}$ such that $-h = a + b$. In particular, $\delta_a = \delta_b = \delta_\theta$. Then:

$$\begin{aligned}
-h =& -\delta_\theta \theta + \sum_\alpha \delta_\alpha n'_\alpha(h) \alpha \\
a =& -\theta + \sum_\alpha \frac{\delta_\alpha}{\delta_a} n'_\alpha(a) \alpha \\
b =& -\theta + \sum_\alpha \frac{\delta_\alpha}{\delta_b} n'_\alpha(b) \alpha \\
(\delta_\theta - 2)\theta =& \sum_\alpha \Big(\delta_\alpha n'_\alpha(h) - \frac{\delta_\alpha}{\delta_a} n'_\alpha(a) - \frac{\delta_\alpha}{\delta_b} n'_\alpha(b)\Big)\alpha \\
=& \sum_\alpha (\delta_\theta - 2) n_\alpha(\theta) \alpha
\end{aligned}$$

Hence, we obtain:

$$\begin{aligned}
l''_{-h} - l''_a - l''_b =& \Big(\delta_\theta l_{-\theta} + \sum_\alpha n'_\alpha(-h)\delta_\alpha l_\alpha\Big) - \Big(l_{-\theta} + \sum_\alpha \frac{\delta_\alpha}{\delta_a} n'_\alpha(a) l_\alpha\Big) \\
& - \Big(l_{-\theta} + \sum_\alpha \frac{\delta_\alpha}{\delta_b} n'_\alpha(b) l_\alpha\Big) \\
=& (\delta_\theta - 2) l_{-\theta} + \sum_\alpha \Big(\delta_\alpha n'_\alpha(-h) - \frac{\delta_\alpha}{\delta_a} n'_\alpha(a) - \frac{\delta_\alpha}{\delta_b} n'_\alpha(b)\Big) l_\alpha \\
=& (\delta_\theta - 2)(l_{-\theta} + l'_\theta)
\end{aligned}$$

Because $\delta_\theta = d' \geq 2$ and $l_{-\theta} + l'_\theta \geq l_{-\theta} + l_\theta \geq 0$, we have $l''_{-h} \geq l''_a + l''_b$. By the case of short roots, we know that $l''_a \geq l_a$ and $l''_b \geq l_b$. Hence, by axiom (C1), we have $l''_{-h} \geq l_a + l_b \geq l_{a+b} = l_{-h}$.

Induction step: we consider the length of a root $c \neq -h$. The case of short roots has been treated. Let $c \neq -h \in \Phi_{\mathrm{nd}}$ be a long root and we assume that $l''_a \geq l_a$ for any root a lower than c in Φ_{nd}. We have $\delta_c = 1$

because c is long, thus $c = c^\delta = -\delta_\theta\theta + \sum_\alpha n'_\alpha(c)\delta_\alpha\alpha$. By Lemma 4.2.1 applied in Φ_{nd}, there exist $a \in \Phi_{\text{nd}}$ and $b \in \Phi^+_{\text{nd}}$ such that $c = a + b$.

If a is long, we have $a = a^\delta = -\delta_\theta\theta + \sum_\alpha n'_\alpha(a)\delta_\alpha\alpha$. Hence, $\delta_\alpha n'_\alpha(c) = \delta_\alpha n'_\alpha(a) + n_\alpha(b)$. As a consequence, $l''_c = l''_a + l'_b$. By induction hypothesis, $l''_a \geq l_a$ because c is strictly higher than a. Hence $l''_c \geq l_a + l'_b \geq l_a + l_b \geq l_{a+b} = l_c$ by axiom (C1).

Otherwise, a is a short root, so that $\delta_a = \delta_\theta = d'$. Hence we have $a = -\theta + \displaystyle\sum_\alpha \frac{\delta_\alpha}{\delta_\theta} n'_\alpha(a)\alpha$. We have:

$$0 = a + b - c = (\delta_\theta - 1)\theta + \sum_\alpha \Big(\frac{\delta_\alpha}{\delta_\theta} n'_\alpha(a) + n_\alpha(b) - n'_\alpha(c)\delta_\alpha\Big)\alpha.$$

By uniqueness of coefficients, for any $\alpha \in \Delta$, we have

$$-(\delta_\theta - 1)n_\alpha(\theta) = \frac{\delta_\alpha}{\delta_\theta} n'_\alpha(a) + n_\alpha(b) - n'_\alpha(c)\delta_\alpha.$$

Hence

$$l''_c - l''_a - l'_b = (\delta_\theta - 1)l_{-\theta} + \sum_\alpha (\delta_\theta - 1)n_\alpha(\theta)l_\alpha = (\delta_\theta - 1)(l_{-\theta} + l'_\theta).$$

Because $l_{-\theta} + l'_\theta \geq l_{-\theta} + l_\theta \geq 0$ by axiom (C2), we obtain $l''_c \geq l''_a + l'_b$. By induction hypothesis, $l''_a \geq l_a$. Hence $l''_c \geq l_a + l_b \geq l_{a+b} = l_c$ by axiom (C1). This finishes the induction.

Finally if c is a multipliable root, then $l''_{2c} = 2l''_c \geq 2l_c \geq l_{2c}$ by axiom (C0). This finishes the proof of (2).

(3) We now establish inclusions $U_{c,l''_c} \subset [H,H]$ of valued root groups, in the order from the longest roots to the shortest roots. According to Φ is a reduced root system or not, there are one, two or three distinct lengths of roots.

Let $c \notin \{-\theta, -2\theta\}$ be a root. Write it as a sum of two non-collinear roots $c = a + b$. We want to apply Proposition 4.1.3, with suitable values $l''_a \in \Gamma_a$, $l'_b \in \Gamma_b$ and $\widehat{l_c} \in \mathbb{R}$ such that $l''_c \geq \widehat{l_c} = l''_a + l'_b$, to prove that $U_{c,l''_c} \subset U_{c,\widehat{l_c}} \subset [H,H]$. Because in 4.1.3, there remains a term v'', we have to be careful in the order of the steps of this proof. We proceed step by step from the longest length to the shortest length of the roots, we treat the case, when it happens, of $c = -h \neq -\theta$ separately, at the end of the study for long roots, where h denotes the highest root of Φ, and the case, when it happens, of a short root $c \neq -\theta = -h$ such that the dual root $-c^D$ is the highest root of Φ^D. Note that when Φ is non-reduced, then we have $c \neq -2\theta = -h$ where h denotes the highest root of Φ.

We denote by $(a,b) = \{ra + sb,\ r,s \in \mathbb{N}\} \cap \Phi$ and by $\Phi(a,b) = (\mathbb{Z}a + \mathbb{Z}b) \cap \Phi$. Be careful that in general, $\Phi(a,b) \neq (\mathbb{R}a + \mathbb{R}b) \cap \Phi$.

• **Case of a divisible root:** Suppose that $c \neq -2\theta = -h$ is a divisible root. Hence Φ is non-reduced and $\delta_{\frac{c}{2}} = \delta_\theta = d' = 2$. By Lemma 4.2.1 applied to Φ_{nm}, there exist non-collinear roots $a, b \in \Phi_{\mathrm{nm}}$ such that $b \in \Phi_{\mathrm{nm}}^+$ and $c = a + b$. Moreover, a, b have to be non-divisible and we have $\delta_a = \delta_b = 1$. We have

$$c = \delta_{\frac{c}{2}} \frac{c}{2} = -\theta^\delta + \sum_{\alpha \in \Delta} n'_\alpha \left(\frac{c}{2}\right) \alpha^\delta = a + b$$

Hence

$$\left(-\delta_\theta \theta + \sum_{\alpha \in \Delta} \delta_\alpha n'_\alpha \left(\frac{c}{2}\right) \alpha\right) = \left(-\delta_\theta \theta + \sum_{\alpha \in \Delta} \delta_\alpha n'_\alpha (a) \alpha\right) + \left(\sum_{\alpha \in \Delta} n_\alpha (b) \alpha\right)$$

so that for any $\alpha \in \Delta$, we have $\delta_\alpha n'_\alpha \left(\frac{c}{2}\right) = \delta_\alpha n'_\alpha(a) + n_\alpha(b)$. Hence

$$\begin{aligned} l''_c = 2l''_{\frac{c}{2}} = \delta_{\frac{c}{2}} l''_{\frac{c}{2}} &= \delta_\theta l_{-\theta} + \sum_{\alpha \in \Delta} \delta_\alpha n'_\alpha \left(\frac{c}{2}\right) l_\alpha \\ &= \delta_\theta l_{-\theta} + \sum_{\alpha \in \Delta} \delta_\alpha n'_\alpha (a) l_\alpha + \sum_{\alpha \in \Delta} n_\alpha(b) l_\alpha \\ &= \delta_a l''_a + l'_b = l''_a + l'_b \end{aligned}$$

We have $l''_a \in \Gamma_a$ by (1) since a is non-divisible and we have $l'_b \in \Gamma_b$ by Proposition 4.2.6(1) since $b \in \Phi^+$. Thus, by Proposition 4.1.3, for any $u \in U_{c,l''_c}$, there exist elements $v \in U_{a,l''_a}$ and $v' \in U_{b,l'_b}$ such that $u = [v, v']$. Hence $U_{c,l''_c} \subset [H, H]$.

• **Case of a non-divisible long root when Φ is non-reduced:** Assume that Φ is irreducible non-reduced and that $c \setminus \{-\theta\}$ is a long root in Φ_{nd}. Then c is a short root in Φ_{nm} and, by Lemma 4.2.7 applied in Φ_{nm}, there is a positive short root $b \in \Phi_{\mathrm{nm}}^+$ and a root $a \in \Phi_{\mathrm{nm}}$ such that $c = a + b$. Then b is a long root in Φ_{nd}^+ so that $\delta_b = 1$. Thus $l''_a \in \Gamma_a$ by (1) and $l'_b \in \Gamma_b$ by Proposition 4.2.6(1). Note that $\Phi(a, b)$ is a rank 2 root system contained in Φ_{nm}, therefore of type A_2 or B_2.

First subcase: $\Phi(a, b)$ is of type A_2. We have $(a, b) = \{a, b, a+b\}$ and we have shown in (2) that $l''_c = l''_a + l'_b$. By Proposition 4.1.3, for any $u \in U_{c,l''_c}$, there exist elements $v \in U_{a,l''_a}$ and $v' \in U_{b,l'_b}$ such that $u = [v, v']$. Hence $U_{c,l''_c} \subset [H, H]$ because $l''_a \geq l_a$ and $l'_b \geq l_b$.

Second subcase: $\Phi(a, b)$ is of type B_2. Then a is divisible in Φ since a has to be long in Φ_{nm} and we have $(a, b) = \{a, b, a + b, a + 2b\}$. Let $a' = \frac{1}{2}a$ so that $l''_a = 2l''_{a'}$ and $\delta_{a'} = 2$ by definition. We have

$$a + b = \delta_{a'} a' + b = \left(-\delta_\theta \theta + \sum_{\alpha \in \Delta} n'_\alpha(a') \delta_\alpha \alpha\right) + \left(\sum_{\alpha \in \Delta} n_\alpha(b) \alpha\right).$$

Moreover, since $\delta_c = 1$, we have

$$c = -\delta_\theta \theta + \sum_{\alpha \in \Delta} n'_\alpha(c) \delta_\alpha \alpha$$

and, since $\delta_{a'+b} = 2$, we have

$$a + 2b = \delta_{a'+b}(a' + b) = -\delta_\theta \theta + \sum_{\alpha\in\Delta} n'_\alpha(a' + b)\delta_\alpha \alpha$$
$$= \delta_{a'} a' + 2b = \left(-\delta_\theta\theta + \sum_{\alpha\in\Delta} n'_\alpha(a')\delta_\alpha\alpha\right) + 2\left(\sum_{\alpha\in\Delta} n_\alpha(b)\alpha\right).$$

Thus we have $\delta_\alpha n'_\alpha(c) = \delta_\alpha n'_\alpha(a') + n_\alpha(b)$ and $\delta_\alpha n'_\alpha(a' + b) = \delta_\alpha n'_\alpha(a') + 2n_\alpha(b)$ so that $l''_c = l''_{a'} + l'_b$ and $l''_{a+2b} = 2l''_{a'+b} = \delta_{a'+b} l''_{a'+b} = \delta_{a'} l''_{a'} + 2l'_b = l''_a + 2l'_b$. By Proposition 4.1.3, for any $u \in U_{c,l''_c}$, there exist elements $v \in U_{a,l''_a}$ and $v' \in U_{b,l'_b}$ and $v'' \in U_{a+2b,l''_a+2l'_b}$ such that $u = [v, v']v''$. Moreover, the divisible root satisfies $a + 2b \neq -2\theta$ since b is a positive root and 2θ is the highest root of Φ. We have already shown, because $a+2b \neq -2\theta$ is a divisible root, that the group $U_{a+2b,l''_a+2l'_b} = U_{a+2b,l''_{a+2b}}$ is a subgroup of $[H, H]$. Hence $U_{c,l''_c} \subset [H, H]$.

- **Case of a long root when Φ is reduced:** Let c be a long root of Φ_{nd}. Then $\delta_c = 1$ by definition. Suppose that $c = c^\delta \notin \{-\theta, -h\}$, where h denotes the highest root of Φ_{nd}. By Lemma 4.2.1 applied to Φ_{nd}, there exist non-collinear roots $a, b \in \Phi_{\mathrm{nd}}$ such that $b \in \Phi^+_{\mathrm{nd}}$ and $c = a + b$. Thus $l''_a \in \Gamma_a$ by (1) and $l'_b \in \Gamma_b$ by Proposition 4.2.6(1).

First subcase: $\Phi(a, b)$ is of type A_2. We have $(a, b) = \{a, b, a+b\}$ and we have shown in (2) that $l''_c = l''_a + l'_b$. By Proposition 4.1.3, for any $u \in U_{c,l''_c}$, there exist elements $v \in U_{a,l''_a}$ and $v' \in U_{b,l'_b}$ such that $u = [v, v']$. Hence $U_{c,l''_c} \subset [H, H]$ because $l''_a \geq l_a$ and $l'_b \geq l_b$.

Second subcase: $\Phi(a, b)$ is of type B_2 or G_2. In this case, necessarily, the long root c is the sum of two short roots. Indeed, c has to be the sum of root of same length, and if a, b were long in the case G_2, then $\Phi(a, b) = (\mathbb{Z}a+\mathbb{Z}b)\cap\Phi$ would only contain long roots which would contradict the condition on the type of $\Phi(a, b)$. Thus, we have $(a, b) = \{a, b, a + b\}$ and $\delta_a = \delta_b = \delta_\theta$ because a, b are short. Hence we have $a = -\theta + \sum_\alpha \frac{\delta_\alpha}{\delta_\theta} n'_\alpha(a)\alpha$ and $c = -\delta_\theta\theta + \sum_{\alpha\in\Delta} \delta_\alpha n'_\alpha(c)\alpha$. We have:

$$0 = a + b - c = (\delta_\theta - 1)\theta + \sum_\alpha \Big(\frac{\delta_\alpha}{\delta_\theta} n'_\alpha(a) + n_\alpha(b) - n'_\alpha(c)\delta_\alpha\Big)\alpha.$$

By uniqueness of coefficients, for any $\alpha \in \Delta$, we have

$$-(\delta_\theta - 1)n_\alpha(\theta) = \frac{\delta_\alpha}{\delta_\theta} n'_\alpha(a) + n_\alpha(b) - n'_\alpha(c)\delta_\alpha.$$

Hence

$$l''_c - l''_a - l'_b = (\delta_\theta - 1)l_{-\theta} + \sum_\alpha (\delta_\theta - 1)n_\alpha(\theta)l_\alpha = (\delta_\theta - 1)(l_{-\theta} + l'_\theta).$$

Because $l_{-\theta} + l'_\theta \geq l_{-\theta} + l_\theta \geq 0$ by axiom (C2), we obtain $l''_c \geq l''_a + l'_b$. By Proposition 4.1.3, for any $u \in U_{c,l''_c} \subset U_{c,l''_a+l'_b}$, there exist elements $v \in U_{a,l''_a}$ and $v' \in U_{b,l'_b}$ such that $u = [v, v']$. Hence $U_{c,l''_c} \subset [H, H]$.

• **The lowest root when it is not $-\theta$ and Φ is reduced:** We treat the case, when it appears, of the non-divisible root $c = -h \neq -\theta$ where h is the highest root of Φ_{nd}. In that case we necessarily have that $\Phi^\delta_{\mathrm{nd}} = \Phi^D_{\mathrm{nd}} \neq \Phi_{\mathrm{nd}}$ (this appears only for G of type ${}^2A_{2l+1}$, ${}^2D_{l+1}$, ${}^2E_6$, ${}^3D_4$ or ${}^6D_4$ with a ramified extension L'/K). In this case, we have $\delta_\theta > 1$ and h is a long root. In particular, the integer $(\delta_\theta - 2)$ is non-negative. We write c as a sum $h = c = a + b$ of two short roots a and b, so that $\delta_a = \delta_b = \delta_\theta$ and $\delta_c = 1$. Moreover $(a, b) = \{a, b, a + b\}$. We have:

$$\begin{aligned} c = a + b =& \Big(-\theta + \sum_\alpha \frac{\delta_\alpha}{\delta_a} n'_\alpha(a)\alpha \Big) + \Big(-\theta + \sum_\alpha \frac{\delta_\alpha}{\delta_b} n'_\alpha(b)\alpha \Big) \\ =& -2\theta + \sum_\alpha \Big(\frac{\delta_\alpha}{\delta_\theta} n'_\alpha(a) + \frac{\delta_\alpha}{\delta_\theta} n'_\alpha(b) \Big)\alpha \\ =& -\delta_\theta\theta + \sum_\alpha \Big(\frac{\delta_\alpha}{\delta_\theta} n'_\alpha(a) + \frac{\delta_\alpha}{\delta_\theta} n'_\alpha(b) + (\delta_\theta - 2) n_\alpha(\theta) \Big)\alpha \end{aligned}$$

Hence we obtain:

$$\begin{aligned} l''_c =& \delta_\theta l_{-\theta} + \frac{1}{\delta_\theta}(\delta_a l''_a - \delta_\theta l_{-\theta}) + \frac{1}{\delta_\theta}(\delta_b l''_b - \delta_\theta l_{-\theta}) + (\delta_\theta - 2) l'_\theta \\ =& l''_a + l''_b + (\delta_\theta - 2)(l_{-\theta} + l'_\theta) \\ \geq& l''_a + l''_b \end{aligned}$$

By (1), we know that $l''_a \in \Gamma_a$ and $l''_b \in \Gamma_b$. Hence, by Proposition 4.1.3, for any $u \in U_{c,l''_c} \subset U_{c,l''_a+l''_b}$, there exist elements $v \in U_{a,l''_a}$ and $v' \in U_{b,l''_b}$ such that $u = [v, v']$.

• **Case of a short root:** Let $c \in \Phi_{\mathrm{nd}}$ be a short root. Then $\delta_c = \delta_\theta$ by definition. Suppose that $c \neq -\theta$ and that $-c^D$ is not the highest root of Φ^D_{nd}. Denote by h the highest root of Φ_{nd}. By Lemma 4.2.7 applied to Φ_{nd}, there exist non-collinear roots $a, b \in \Phi_{\mathrm{nd}}$ such that $b \in \Phi^+_{\mathrm{nd}}$, the root a is short and $c = a + b$.

First subcase: case of two short roots a and b. We have $\delta_a = \delta_b = \delta_c = \delta_\theta$ and we have shown in (2) that $l''_c = l''_a + l'_b$. The rank 2 root subsystem $\Phi(a, b)$ is of type A_2 or G_2.

If $\Phi(a, b)$ is of type A_2, $(a, b) = \{a, b, a+b\}$. By Proposition 4.1.3, for any $u \in U_{c,l''_c}$, there exist elements $v \in U_{a,l''_a}$ and $v' \in U_{b,l'_b}$ such that $u = [v, v']$.

If $\Phi(a, b)$ is of type G_2, we have $(a, b) = \{a, b, a + b, 2a + b, a + 2b\}$. Because b is a positive root, we have that $a + 2b \neq -h$ since it is higher that $a + b$.

► Suppose that $\delta_\theta > 1$ or that $2a + b \neq -h$. If $\delta_\theta = 1$, then $\theta = h$ is a long root and by assumption $2a + b, a + 2b \in \Phi_{\mathrm{nd}} \setminus \{-h\} = \Phi_{\mathrm{nd}} \setminus \{-\theta\}$. If

$\delta_\theta > 1$, then θ is a short root. In both cases, the roots $2a+b, a+2b$ are long and different from $-\theta$.

We have:

$$\begin{aligned}2a+b =&2\Big(-\theta+\sum_\alpha \frac{\delta_\alpha}{\delta_a} n'_\alpha(a)\alpha\Big)+\sum_\alpha n_\alpha(b)\alpha\\ =&-\delta_\theta\theta+\sum_\alpha\Big(2\frac{\delta_\alpha}{\delta_a}n'_\alpha(a)+n_\alpha(b)+(\delta_\theta-2)n_\alpha(\theta)\Big)\alpha,\end{aligned}$$

Because $2a+b$ is long, we have $\delta_{2a+b}=1$, so that:

$$\begin{aligned}l''_{2a+b} =&\delta_\theta l_{-\theta}+\sum_\alpha\Big(2\frac{\delta_\alpha}{\delta_a}n'_\alpha(a)+n_\alpha(b)+(\delta_\theta-2)n_\alpha(\theta)\Big)l_\alpha\\ =&\delta_\theta l_{-\theta}+\frac{2}{\delta_a}(\delta_a l''_a-\delta_\theta l_{-\theta})+l'_b+(\delta_\theta-2)l'_\theta\\ =&2l''_a+l'_b+(\delta_\theta-2)(l_{-\theta}+l'_\theta)\end{aligned}$$

In the same way, one can show that $l''_{a+2b}=l''_a+2l'_b+(\delta_\theta-1)(l'_\theta+l_{-\theta})$.

If $\delta_\theta=1$, because $l_{-\theta}+l'_\theta\geq 0$, we get $l''_{2a+b}\leq 2l''_a+l'_b$ and $l''_{a+2b}=l''_a+2l'_b$. Hence, we get $U_{2a+b,l''_{2a+b}}\supset U_{2a+b,2l''_a+l'_b}$ and $U_{a+2b,l''_{a+2b}}=U_{a+2b,l''_a+2l'_b}$.

Otherwise, $\delta_\theta=3$ and G is a trialitarian D_4 with a ramified extension L'/K. In that case, we assumed that $l_{-\theta}+l'_\theta\leq\omega(\varpi_{L'})=0^+\in\Gamma_{L'}$. Because $l''_{a+2b}, l''_{2a+b}\in\Gamma_K=3\Gamma_{L'}$, we obtain that $0\leq(\delta_\theta-1)(l'_\theta+l_{-\theta})<3\omega(\varpi_{L'})=0^+\in\Gamma_K$. The same is true for $(\delta_\theta-2)(l'_\theta+l_{-\theta})$. Hence, we have the equalities of root groups: $U_{a+2b,l''_a+2l'_b}=U_{a+2b,l''_{a+2b}-(\delta_\theta-1)(l'_\theta+l_{-\theta})}=U_{a+2b,l''_{a+2b}}$ and $U_{2a+b,2l''_a+l'_b}=U_{2a+b,l''_{2a+b}-(\delta_\theta-2)(l'_\theta+l_{-\theta})}=U_{2a+b,l''_{2a+b}}$.

Since $l''_c=l''_a+l'_b$ has been proven in (2) (case where $\delta_a=\delta_b=\delta_c=\delta_\theta$), $l''_a\in\Gamma_a$ and $l'_b\in\Gamma_b$, then by Proposition 4.1.3, for any $u\in U_{c,l''_c}$, there exist elements $v\in U_{a,l''_a}$ and $v'\in U_{b,l'_b}$ and $v''\in U_{2a+b,2l''_a+l'_b}U_{a+2b,l''_a+2l'_b}$ such that $u=[v,v']v''$. In both cases, because $2a+b$ and $a+2b$ are long and different from $-\theta$, we have shown in a previous case that the root groups $U_{2a+b,l''_{2a+b}}$ and $U_{a+2b,l''_{a+2b}}$ are contained in $[H,H]$. Thus, $v''\in[H,H]$. Hence $U_{c,l''_c}\subset[H,H]$.

► Suppose that $\delta_\theta=1$ and that $2a+b=-h=-\theta$. Then a is a negative root since b is a positive root and $2a+b$ is a negative root. Set $a'=2a+b=-\theta$ and $b'=-a\in\Phi^+_{\mathrm{nd}}$. Then $c=a+b=(2a+b)+(-a)=a'+b'$ and $(a',b')=\{a',b',a'+b',a'+2b',a'+3b',2a'+3b'\}$. Because $\delta_\theta=1$, we have $\delta_\gamma=1$ for any root $\gamma\in\Phi$. We have $a'=-\theta$ and $b'=\sum_\alpha n_\alpha(b')\alpha$. Hence the equality $c=a'+b'$ gives $n'_\alpha(c)=n_\alpha(b')$ for any $\alpha\in\Delta$, so that $l''_c=l_{-\theta}+l'_{b'}=l''_{a'}+l'_{b'}$. By the same way, we have $l''_{a'+2b'}=l''_{a'}+2l'_{b'}$ and $l''_{a'+3b'}=l''_{a'}+3l'_{b'}$. Moreover,

$$2a'+3b'=-2\theta+3b'=-\theta+\sum_{\alpha\in\Delta}\Big(3n_\alpha(b')-n_\alpha(\theta)\Big)\alpha$$

so that

$$\begin{aligned}
l''_{2a'+3b'} =& l_{-\theta} + \sum_{\alpha\in\Delta} \Big(3n_\alpha(b') - n_\alpha(\theta)\Big) l_\alpha \\
=& 2l_{-\theta} + 3\left(\sum_{\alpha\in\Delta} n_\alpha(b') l_\alpha\right) - \left(l_{-\theta} + \sum_{\alpha\in\Delta} n_\alpha(\theta) l_\alpha\right) \\
=& 2l''_{a'} + 3l'_{b'} - \big(l_{-\theta} + l'_\theta\big) \\
\leq& 2l''_{a'} + 3l'_{b'}
\end{aligned}$$

By Proposition 4.1.3, for any $u \in U_{c,l''_c}$, there exist elements $v \in U_{a',l''_{a'}}$ and $v' \in U_{b',l'_{b'}}$ and $v'' \in U_{a'+2b',l''_{a'}+2l'_{b'}} U_{a'+3b',l''_{a'}+3l'_{b'}} U_{2a'+3b',2l''_{a'}+3l'_{b'}}$ such that $u = [v, v']v''$. Because $a'+2b' = (a'+b')+(b')$ is a sum of two short roots with b' a positive root and $2a'+3b' \neq -h = a'$, it is not the lowest dual root and, we have seen in the previous case that $U_{a'+2b',l''_{a'}+2l'_{b'}} = U_{a'+2b',l''_{a'+2b'}} \subset [H,H]$. Because $a'+3b' \neq -\theta = a'$ is a long root, we have seen that $U_{a'+3b',l''_{a'}+3l'_{b'}} = U_{a'+3b',l''_{a'+3b'}} \subset [H,H]$. Because $2a'+3b' \neq -\theta = a'$ is a long root, we have seen that $U_{2a'+3b',2l''_{a'}+3l'_{b'}} \subset U_{2a'+3b',l''_{2a'+3b'}} \subset [H,H]$. Thus, $v'' \in [H,H]$. Hence $U_{c,l''_c} \subset [H,H]$.
Second subcase: a is short and b is long. If $\Phi(a,b)$ is of type G_2, then c also can be written as a sum of two short roots a', b' with b' a positive root and this case has been treated, so that we can assume that $\Phi(a,b)$ is not of type G_2. Because a, b do not have the same length, the rank 2 root subsystem $\Phi(a,b)$ is of type B_2 or BC_2. We have $\delta_a = \delta_c = \delta_\theta$ and $\delta_b = 1$. Precisely, we have $(a,b) = \{a, b, a+b, 2a+b\}$ if Φ is a reduced root system and $(a,b) = \{a, b, a+b, 2a, 2a+b, 2a+2b\}$ otherwise. Then $2a+b$ is a long root of Φ_{nd} and we have $\delta_{2a+b} = 1$.

▶ Suppose that $2a+b \neq -h$ or that $\delta_\theta > 1$. We have

$$\delta_c c = \delta_\theta\Big(-\theta + \sum_\alpha \big(\frac{\delta_\alpha}{\delta_a} n'_\alpha(a) + n_\alpha(b)\big)\alpha\Big) = -\delta_\theta\theta + \sum_\alpha \big(\delta_\alpha n'_\alpha(a) + \delta_\theta n_\alpha(b)\big)\alpha.$$

Hence $\delta_c l''_c = \delta_a l''_a + \delta_\theta l'_b$. Thus $l''_c = l''_a + l'_b$. By Proposition 4.1.3, for any $u \in U_{c,l''_c}$, there exist elements $v \in U_{a,l''_a}$ and $v' \in U_{b,l'_b}$ and $v'' \in U_{2a+b,2l''_a+l'_b}$ such that $u = [v, v']v''$. It remains to check that $v'' \in [H,H]$. We have:

$$\begin{aligned}
\delta_{2a+b}(2a+b) = 2a+b =& 2\Big(-\theta + \sum_\alpha \frac{\delta_\alpha}{\delta_a} n'_\alpha(a)\alpha\Big) + \sum_\alpha n_\alpha(b)\alpha \\
=& -\delta_\theta\theta + \sum_\alpha \Big(\delta_\alpha \frac{2}{\delta_a} n'_\alpha(a) + n_\alpha(b) + (\delta_\theta - 2)n_\alpha(\theta)\Big)\alpha
\end{aligned}$$

Hence:

$$\begin{aligned}l''_{2a+b} =& \delta_\theta l_{-\theta} + \frac{2}{\delta_a}(\delta_a l''_a - \delta_\theta l_{-\theta}) + l'_b + (\delta_\theta - 2)l'_\theta \\ =& \delta_\theta l_{-\theta} + 2l''_a - 2l_{-\theta} + l'_b + (\delta_\theta - 2)l'_\theta \\ =& 2l''_a + l'_b + (\delta_\theta - 2)(l_{-\theta} + l'_\theta)\end{aligned}$$

Because $\delta_\theta \in \{1,2\}$ and $l_{-\theta}+l'_\theta \geq 0$, we obtain the inequality $l''_{2a+b} \leq 2l''_a+l'_b$. Moreover, if $\delta_\theta \neq 1$ so that $\theta \neq h$, then $-\theta$ is short whereas $2a+b$ is long. Otherwise $-\theta = -h \neq 2a+b$ by assumption. Hence $2a+b$ is a long root in $\Phi_{\mathrm{nd}} \setminus \{-\theta\}$, and we have already shown that $U_{2a+b,2l''_a+l'_b} \subset U_{2a+b,l''_{2a+b}} \subset [H,H]$. Finally, if Φ is non-reduced and $2a+2b$ is a root, we get that $l''_{2c} = l''_{2a+2b} = 2l''_{a+b} = 2l''_c = 2l''_a + 2l'_b$. Since $c \neq -\theta$, we have $2a+2b = 2c \neq -2\theta$ and we know that $U_{2a+2b,2l''_a+2l'_b} = U_{2a+2b,l''_{2a+2b}} \subset [H,H]$ since we have already treated the case of such a divisible root. Hence $v'' \in [H,H]$ and it follows that $U_{c,l''_c} \subset [H,H]$.

▶ Suppose that $2a+b = -h$ and that $\delta_\theta = 1$, so that Φ is reduced and $h = \theta$. Then a is a negative root since b is a positive root and $2a+b$ is a negative root. Set $a' = 2a+b = -\theta$ and $b' = -a \in \Phi^+$. Then $c = a+b = (2a+b)+(-a) = a'+b'$ and $(a',b') = \{a',b',a'+b',a'+2b'\}$.

Because $\delta_\theta = 1$, we have $\delta_\gamma = 1$ for any root $\gamma \in \Phi$. We have $a' = -\theta$ and $b' = \sum_\alpha n_\alpha(b')\alpha$. Hence the equality $c = a'+b'$ gives $n'_\alpha(c) = n_\alpha(b')$ for any $\alpha \in \Delta$, so that $l''_c = l_{-\theta} + l'_{b'} = l''_{a'} + l'_{b'}$. Moreover,

$$a' + 2b' = -\theta + 2b' = -\theta + \sum_{\alpha \in \Delta} 2n_\alpha(b')\alpha$$

so that

$$l''_{a'+2b'} = l_{-\theta} + 2l'_{b'} = l''_{a'} + 2l'_{b'}.$$

By Proposition 4.1.3, for any $u \in U_{c,l''_c}$, there exist elements $v \in U_{a',l''_{a'}}$ and $v' \in U_{b',l'_{b'}}$ and $v'' \in U_{a'+2b',l''_{a'}+2l'_{b'}}$ such that $u = [v,v']v''$. Because $a'+2b' \neq -\theta = a'$ is a long root, we have seen that $U_{2a'+b',2l''_{a'}+l'_{b'}} = U_{a'+2b',l''_{a'+2b'}} \subset [H,H]$. Thus, $v'' \in [H,H]$. Hence $U_{c,l''_c} \subset [H,H]$.

Now, it only remain the case of a short negative root c such that $-c^D$ is the highest root of Φ^D when $h = \theta$.

• **The lowest dual root:** Assume that c is the negative root of Φ_{nd} such that $-c^D$ is the highest root of Φ^D_{nd} and $h = \theta \neq -c$ so that $\Phi^D_{\mathrm{nd}} \neq \Phi^\delta_{\mathrm{nd}} = \Phi_{\mathrm{nd}}$. This case appears only if L'/K is unramified and Φ is a reduced non simply laced root system: indeed, if Φ were non-reduced, then h would be a long root in Φ_{nd} and since θ is multipliable, it is a short root in Φ_{nd}; if Φ were reduced and simply-laced, then all roots would have same length. In this case, we have $\delta_\gamma = 1$ for any root $\gamma \in \Phi$ and the rank 2 root subsystem $\Phi(a,b)$ is reduced for any $a,b \in \Phi$. By Lemma 4.2.1(2), there exists $a \in \Phi^-_{\mathrm{nd}}$

and $b \in \Phi^+_{\mathrm{nd}}$ such that $c = a + b$. If a is short, we can proceed as before. Hence we assume that a is a long root, b and c are short roots.

▶ If $\Phi(a, b)$ is of type B_2, then $(a, b) = \{a, b, a+b, a+2b\}$ and we have the equalities $l''_{a+b} = l''_a + l'_b$ and $l''_{a+2b} = l''_a + 2l'_b$. By Proposition 4.1.3, for any $u \in U_{c,l''_c}$, there exist elements $v \in U_{a,l''_a}$ and $v' \in U_{b,l'_b}$ and $v'' \in U_{a+2b,l''_a+2l'_b}$ such that $u = [v, v']v''$. Since b is a positive root, we have $a + 2b \neq -h = -\theta$. Since $a + 2b$ is a long root of $\Phi_{\mathrm{nd}} = \Phi$, we have already shown that $U_{a+2b,l''_a+2l'_b} = U_{a+2b,l''_{a+2b}} \subset [H, H]$. Hence $U_{c,l''_c} \subset [H, H]$.

▶ If $\Phi(a, b)$ is of type G_2, then $(a, b) = \{a, b, a+b, a+2b, a+3b, 2a+3b\}$ If $2a + 3b = -h = -\theta$, then, up to replace a by $-2a - 3b = -\theta = -h$ which is a long root and b by $-a - 2b = h - c$ which is a positive short root, we can assume that $2a + 3b \neq -h = -\theta$. Since b is a positive root, we have $a + 3b \neq -h = -\theta$ and $a + 2b \neq -h = -\theta$.

We have the equalities $l''_{a+b} = l''_a + l'_b$, $l''_{a+2b} = l''_a + 2l'_b$ and $l''_{a+3b} = l''_a + 3l'_b$. Moreover, we have $l''_{2a+3b} = 2l''_a + 3l'_b - (l_{-\theta} + l'_\theta) \leq 2l''_a + 3l'_b$. By Proposition 4.1.3, for any $u \in U_{c,l''_c}$, there exist elements $v \in U_{a,l''_a}$ and $v' \in U_{b,l'_b}$ and $v'' \in U_{a+2b,l''_a+2l'_b} U_{a+3b,l''_a+3l'_b} U_{2a+3b,2l''_a+3l'_b}$ such that $u = [v, v']v''$. Since $a+3b$ and $2a+3b$ are long roots of $\Phi_{\mathrm{nd}} = \Phi$, we have already shown that $U_{a+3b,l''_a+3l'_b} = U_{a+3b,l''_{a+3b}} \subset [H, H]$ and that $U_{2a+3b,2l''_a+3l'_b} \subset U_{2a+3b,l''_{2a+3b}} \subset [H, H]$. Since $a + 2b \neq -\theta$ can be written as the sum of the two short roots b and $a + b$, we have shown that $U_{a+2b,l''_a+2l'_b} = U_{a+2b,l''_{a+2b}} \subset [H, H]$. Hence $U_{c,l''_c} \subset [H, H]$. This finishes the proof. □

4.2.10 Remark. Proposition 4.2.6 and Proposition 4.2.9 do not restrict the choice of the basis Δ but only the choice of values l_a. In fact, the conditions $l_a \in \Gamma_a$ for any $a \in \Delta$ and $l_{-\theta} \in \Gamma_{-\theta}$ limit the available choices for the basis Δ.

4.2.11 Lemma. *Let Φ be an irreducible non-reduced root system and Δ be a basis of Φ. Let $a \in \Delta$ be the multipliable simple root. Let θ be the half highest root of Φ relatively to the basis Δ. Then $\Delta' = (\Delta \cup \{-\theta\}) \setminus \{a\}$ is another basis of Φ; and $-a$ is the half highest root of Φ relatively to the basis Δ'.*

Proof. We consider the following Euclidean geometric realisation of the root system $\Phi = \{\pm e_i,\ 1 \leq i \leq l\} \cup \{\pm e_i \pm e_j,\ 1 \leq i < j \leq l\} \cup \{\pm 2e_i,\ 1 \leq i \leq l\}$ where (e_i) denotes the canonical basis of the Euclidean space $\mathbb{R}^l$. We denote by $a_i = e_i - e_{i+1}$ for any $1 \leq i \leq l-1$ and by $a_l = e_l$. The set $\Delta = \{a_1, \ldots, a_l\}$ is a basis of Φ and $\theta = e_1 = a_1 + \cdots + a_l$ is the half highest root of Φ.

Let $w \in \mathrm{GL}_l(\mathbb{R})$ be the element of the Weyl group $W(\Phi)$ defined by $w(e_i) = -e_{l-i+1}$. We observe that w stabilises $\Delta \setminus \{a_l\}$, that $w(-\theta) = a_l$ and that $w(a_l) = -\theta$.

If D is a half-space of $\mathbb{R}^l$ defining the basis Δ, then $w(D)$ is also a half-space of $\mathbb{R}^l$ and it defines the basis $\Delta' = (\Delta \setminus \{a_l\}) \cup \{-\theta\}$. The half highest root of Φ relatively to Δ' is then $-a_l$. □

4.2.3 Lower bounds for valued root groups of the Frattini subgroup

We want to apply Propositions 4.2.6 and 4.2.9 to the maximal pro-p subgroup P corresponding to the fundamental alcove $\mathbf{c}_{\mathrm{af}}$ described in Section 3.1.

4.2.12 Theorem. *Assume that the irreducible relative root system Φ is of rank $l \geq 2$ and that the residue characteristic p of K satisfies Hypothesis 4.1.2 and $p \neq 2$. Let P be a maximal pro-p subgroup of $G(K)$ and let $\mathbf{c}$ be the (unique) alcove fixed by P. Assume that $\mathbf{c} \subset \mathbb{A}$. For any non-divisible root $a \in \Phi_{\mathrm{nd}}$, if the wall $\mathcal{H}_{a,f'_{\mathbf{c}}(a)}$ (this notation has been defined in Section 3.1.1 and Notation 3.1.1) contains a panel of $\mathbf{c}$, then we have $[P,P] \supset U_{a,f'_{\mathbf{c}}(a)^+}$ and, if $2a \in \Phi$, $[P,P] \supset U_{2a,f'_{\mathbf{c}}(2a)^+}$; otherwise, we have $[P,P] \supset U_{a,f'_{\mathbf{c}}(a)}$ and, if $2a \in \Phi$, $[P,P] \supset U_{2a,f'_{\mathbf{c}}(2a)}$.*

Proof. We normalize $\Gamma_{L'} = \mathbb{Z}$. Up to conjugation, we can assume that $\mathbf{c} = \mathbf{c}_{\mathrm{af}}$ is the fundamental alcove, defined in Section 3.1.2, and bounded by the following walls:

- $\mathcal{H}_{a,0}$ for all simple roots $a \in \Delta$;
- $\mathcal{H}_{-\theta,1}$ if Φ is reduced;
- $\mathcal{H}_{-\theta,\frac{1}{2}}$ if Φ is non-reduced.

If the root system Φ is reduced, according to section 3.1.2, the alcove $\mathbf{c}$ is the intersection of half-apartments $D(a,0)$ for $a \in \Phi^+$ and $D(b,0^+)$ for $b \in \Phi^-$. This provides $f'_{\mathbf{c}}(a) = 0$ for $a \in \Phi^+_{\mathrm{nd}}$ and $f'_{\mathbf{c}}(b) = 0^+ \in \Gamma_b$ for $b \in \Phi^-_{\mathrm{nd}}$. But $\Gamma_b = \Gamma_{L_b} = \frac{1}{\delta_b}\Gamma_K = \frac{1}{\delta_b}\delta_\theta\Gamma_{L'}$ so that $0^+ = \frac{\delta_\theta}{\delta_b}$ in the group Γ_b.

If the root system Φ is non-reduced, according to section 3.1.2, the alcove $\mathbf{c}$ is the intersection of half-apartments $D(a,0)$ for $a \in \Phi^+_{\mathrm{nd}}$ and $D(b,1)$ for $b \in \Phi^-_{\mathrm{nd}}$ non-multipliable and $D(c,\frac{1}{2})$ for $c \in \Phi^-_{\mathrm{nd}}$ multipliable. But for any non-divisible root, $\delta_a = 2$ if a is multipliable and $\delta_a = 1$ if a is non-multipliable (Proposition 3.1.2) so that $f'_{\mathbf{c}}(a) = \frac{1}{\delta_a}$ for any $a \in \Phi^-_{\mathrm{nd}}$.

Thus, we have the following values:

- $f'_{\mathbf{c}}(a) = 0$ if $a \in \Phi^+$;
- $f'_{\mathbf{c}}(a) = \frac{\delta_\theta}{\delta_a} \in \{1, d'\}$ if $a \in \Phi^-$ and Φ is reduced;
- $f'_{\mathbf{c}}(a) = \frac{1}{\delta_a} \in \{\frac{1}{2}, 1\}$ if $a \in \Phi^-_{\mathrm{nd}}$ and Φ is non-reduced.

We set $l_a = f_{\mathbf{c}}(a) \in \mathbb{R}$ for any $a \in \Phi$. We know, by [BT84, 4.6.26], that the maps $a \mapsto l_a$ is concave. The wall bounding the alcove $\mathbf{c}$ are directed by the relative roots $\Delta \cup \{-\theta\}$. Hence, for any $a \in \Delta \cup \{-\theta\}$, we get $f_{\mathbf{c}}(a) = f'_{\mathbf{c}}(a) = l_a \in \Gamma_a$. Moreover, $f_{\mathbf{c}}(-\theta) \in \{1, \frac{1}{2}\}$ and $l'_\theta = 0$ so that the sum satisfies $f_{\mathbf{c}}(-\theta) + l'_\theta \leq 1 = \omega(\varpi_{L'})$. Finally, if Φ is non-reduced so that

θ is multipliable, we have that $2l_{-\theta} = 2f_{\mathbf{c}}(-\theta) = \frac{2}{\delta_\theta} = 1 \in \Gamma_{2\theta}$ according to Lemma 2.1.10. As a consequence, we can apply Propositions 4.2.6 and 4.2.9 to the group P and the values $l_a = f_{\mathbf{c}}(a)$ where $a \in \Phi$.

For any non-divisible non-simple positive root $b \in \Phi^+_{\mathrm{nd}} \setminus \Delta$, by Proposition 4.2.6, because $l'_b = 0$, we have $[P,P] \supset U_{b,0} = U_{b,f'_{\mathbf{c}}(b)}$.

For any non-divisible root $c \in \Phi^-_{\mathrm{nd}} \setminus \{-\theta\}$, by definition, we have $\delta_c l''_c = \delta_\theta f'_{\mathbf{c}}(-\theta)$. If Φ is reduced, then we have $l''_c = \frac{\delta_\theta}{\delta_c} = f'_{\mathbf{c}}(c)$. If Φ is non-reduced, then we have $l''_c = \frac{1}{\delta_c} = f'_{\mathbf{c}}(c)$ because $\delta_{-\theta} l_{-\theta} = 1$. Hence, for any non-divisible root $c \in \Phi^-_{\mathrm{nd}} \setminus \{-\theta\}$, by Proposition 4.2.9, we have $[P,P] \supset U_{c,f'_{\mathbf{c}}(c)}$.

We suppose that Φ is reduced. Let $a \in \Delta \cup \{-\theta\}$. Then, by Proposition 2.2.3, we know that $[P,P] \supset U_{a,l_a^+} = U_{a,f'_{\mathbf{c}}(a)^+}$.

We suppose that Φ is non-reduced. Let $a \in \Delta$. By definition, we get $\delta_a l''_a = \delta_\theta f'_{\mathbf{c}}(-\theta) = 1$. We have $l''_a = \frac{1}{\delta_a} = 0^+ = f'_{\mathbf{c}}(a)^+$. Indeed, if a is multipliable, $l''_a = \frac{1}{2}$; otherwise $l''_a = 1$ is the smallest positive value of Γ_a. By Proposition 4.2.9, we have $[P,P] \supset U_{a,f'_{\mathbf{c}}(a)^+}$.

For any divisible root $c \in \Phi$, according to remarks below Notation 3.1.1, we have that $U_{c,f'_{\mathbf{c}}(c)} = U_{c,f_{\mathbf{c}}(c)} \subset U_{\frac{c}{2},\frac{1}{2}f_{\mathbf{c}}(c)} = U_{\frac{c}{2},f_{\mathbf{c}}(\frac{c}{2})} = U_{\frac{c}{2},f'_{\mathbf{c}}(\frac{c}{2})}$. Thus we have, $U_{c,f'_{\mathbf{c}}(c)^+} \subset U_{\frac{c}{2},f'_{\mathbf{c}}(\frac{c}{2})^+} \subset [P,P]$. If the wall $\mathcal{H}_{\frac{c}{2},f'_{\mathbf{c}}(\frac{c}{2})}$ does not contains a panel of $\mathbf{c}$, then we have $U_{c,f'_{\mathbf{c}}(c)} \subset U_{\frac{c}{2},f'_{\mathbf{c}}(\frac{c}{2})} \subset [P,P]$ in that case.

Finally, when Φ is non-reduced, we can apply Lemma 4.2.11 to exchange the roles of the multipliable simple root $a \in \Delta$ and the opposite of the half highest root $-\theta$. We write $\theta = \sum_{b\in\Delta} n_b b$ where $n_b \in \mathbb{N}^*$. In fact, we have $n_b = 1$ for any $b \in \Delta$ (one can use description of roots given in [Bou81, VI.4.14]) so that $-\theta = \theta + (-2\theta) = a + \sum_{b\in\Delta\setminus\{a\}} b + 2(-\theta)$. With respect to the basis Δ', we get $\delta_{-\theta} l''_{-\theta} = \delta_a l_a + \sum_{b\in\Delta\setminus\{a\}} \delta_b l_b + 2\delta_{-\theta} l_{-\theta}$. Thus $l''_{-\theta} = 2l_{-\theta} = 1 = l_\theta^+$. By applying Proposition 4.2.9 to the basis $\Delta' = (\Delta \setminus \{a\}) \cup \{-\theta\}$, we obtain $[P,P] \supset U_{-\theta,f'_{\mathbf{c}}(-\theta)^+}$. □

4.2.13 Remark. As an immediate consequence, the derived group $[P,P]$ contains $U_{c,f'_{B(\mathbf{c},1)\cap\mathbb{A}}(c)}$ for any root $c \in \Phi$.

In the rank 1 case, we have a lack of rigidity that could make $[P,P]$ smaller than expected. Typically, Propositions 4.2.6 and 4.2.9 cannot be applied.

In Proposition 3.1.7, given a basis Δ_{f} corresponding to the choice of a positive set of roots Φ^+, we defined a fundamental alcove $\mathbf{c}_{\mathrm{af}}$ of $\mathbb{A}$. Let $\mathbf{c}$ be any alcove of $\mathbb{A}$ and $n \in \mathcal{N}_G(S)(K)$ be such that $\mathbf{c} = n \cdot \mathbf{c}_{\mathrm{af}}$. We say that the basis $\Delta = n \cdot \Delta_{\mathrm{f}}$ corresponds to $\mathbf{c}$. Note that Δ does not depend on the choice of n: by simply-connectedness assumption, the setwise stabilizer of $\mathbf{c}_{\mathrm{af}}$ fixes it pointwise [BT84, 4.6.32] (i.e. the action is type-preserving) and therefore any two such n differ from an element of $\ker \nu \subset T(K)$, which fixes Δ_{f} [BT72, 6.1.11(ii)].

4.2.14 Corollary. *We assume that p satisfies assumption 4.1.2 and that Φ is of rank at least 2. Let $P = P_{\mathbf{c}}^+$ be a maximal pro-p subgroup of $G(K)$ fixing an alcove $\mathbf{c} \subset \mathbb{A}$. Let Δ be a basis of Φ corresponding to $\mathbf{c}$. For any non-divisible root $a \in \Phi_{\mathrm{nd}}$, we write $P \cap U_a(K) = U_{a,l_a}$ where $l_a \in \Gamma_a$. Let $a \in \Delta \cup \{-\theta\}$,*

- *if a is multipliable, if the extension L_a/L_{2a} is unramified, and if $l_a \in \Gamma'_a$, then we have the inclusions $U_{a,l_a^+} \subset [P,P] \cap U_a(K) \subset U_{a,l_a^+} U_{2a,2l_a}$;*
- *otherwise, we have the equality $[P,P] \cap U_a(K) = U_{a,l_a^+}$.*

If $a \in \Phi \setminus (\Delta \cup \{-\theta\})$, then we have the equality $[P,P] \cap U_a(K) = U_{a,l_a}$.

Proof. This results immediately from Theorem 4.2.12 and Proposition 3.2.2. ◻

5 Generating set of a maximal pro-p subgroup

As before, G is an absolutely simple quasi-split simply-connected K-group and P is a maximal pro-p subgroup of $G(K)$. In Corollary 5.2.2, we obtain the minimal number of topological generators of the pro-p Sylow P in the various cases.

5.1 The Frattini subgroup

In order to compute a minimal generating set of the maximal pro-p subgroup $P = P_{\mathbf{c}}^+$ for some $\mathbf{c} \subset \mathbb{A}$, we know by [DdSMS99, 1.9] that it suffices to compute a minimal generating set of the p-elementary commutative group $P/\mathrm{Frat}(P)$, where $\mathrm{Frat}(P)$ denotes the Frattini subgroup of P. Up to conjugation, we can assume that $\mathbf{c} = \mathbf{c}_{\mathrm{af}}$ is the fundamental alcove of $\mathbb{A}$ defined in Proposition 3.1.7. Let Δ be a basis corresponding to $\mathbf{c}$ and Φ^+ the positive roots in Φ with respect to this basis. According to [Loi16, 3.2.9], we know that P can be written as a product $P = \left(\prod_{a\in\Phi_{\mathrm{nd}}^-} U_{a,\mathbf{c}}\right) T(K)_b^+ \left(\prod_{a\in\Phi_{\mathrm{nd}}^+} U_{a,\mathbf{c}}\right)$.

We want to describe the Frattini subgroup $\mathrm{Frat}(P)$, in the same way, in terms of valued root groups $U_{a,\widehat{l_a}}$, with suitable values $\widehat{l_a} \in \mathbb{R}$, and a subgroup of $T(K)_b^+$ that we have to determinate. Since P is a pro-p group, by [DdSMS99, 1.13], we have $\mathrm{Frat}(P) = \overline{P^p[P,P]}$. Hence $P/\mathrm{Frat}(P)$ is a $\mathbb{Z}/p\mathbb{Z}$ vector space of dimension $d(P)$ that we want to compute explicitly.

5.1.1 Theorem (Descriptions of the Frattini subgroup of a maximal pro-p subgroup: the reduced case)**.** *We suppose that the relative root system Φ is reduced and that $p \neq 2$. If Φ is of type G_2, we require that $p \geq 5$. Then:*

Profinite description: *The pro-p group P is topologically of finite type and* $\mathrm{Frat}(P) = [P,P]$.

Description by the valued root groups datum: *For any $a \in \Phi$, we set:*

$$V_{a,\mathbf{c}} = \begin{cases} U_{a,f_{\mathbf{c}}(a)^+} & \text{if } a \in \Delta \cup \{-\theta\} \\ U_{a,\mathbf{c}} & \text{otherwise} \end{cases}$$

where θ is the root defined in Notation 3.1.6. This group depends only on the root $a \in \Phi$ and the alcove $\mathbf{c} \subset \mathbb{A}$, not on the chosen basis Δ.

We have the following writing, as product:

$$\mathrm{Frat}(P) = \left(\prod_{a\in\Phi^+} V_{-a,\mathbf{c}}\right) T(K)_b^+ \left(\prod_{a\in\Phi^+} V_{a,\mathbf{c}}\right)$$

Geometrical description: *The Frattini subgroup* $\mathrm{Frat}(P)$ *is the maximal pro-p subgroup of the pointwise stabilizer in $G(K)$ of the combinatorial ball centered at $\mathbf{c}$ of radius* 1.

Proof. For any $a \in \Phi$, we let $l_a = f_{\mathbf{c}}(a)$, so that $l_a \in \Gamma_a$ for any $a \in \Delta \cup \{-\theta\}$ and the map $a \mapsto l_a$ is concave. We define $\widehat{l_a} = \begin{cases} l_a^+ & \text{if } a \in \Delta \cup \{-\theta\} \\ l_a & \text{otherwise} \end{cases}$.
We define $Q = \prod_{a\in\Phi^-} U_{a,\widehat{l_a}} \cdot T(K)_b^+ \cdot \prod_{a\in\Phi^+} U_{a,\widehat{l_a}}$. We prove the chain of inclusions $Q \subset [P,P] \subset \mathrm{Frat}(P) \subset Q$.

The inclusion $[P,P] \subset \overline{P^p[P,P]} = \mathrm{Frat}(P)$ is immediate.

By Corollary 3.2.10, we have $\mathrm{Frat}(P) \subset Q$.

If the reduced irreducible root system Φ is of rank $l \geq 2$, by Theorem 4.2.12, we have $\forall a \in \Phi,\ [P,P] \supset U_{a,\widehat{l_a}}$. If Φ is of rank 1, by Proposition 2.2.3, we have $\forall a \in \Phi,\ [P,P] \supset U_{a,\widehat{l_a}}$. Moreover, by Proposition 2.2.3, we also have $T^a(K)_b^+ \subset [P,P]$ for any $a \in \Phi$. Because G is a simply-connected semi-simple group, $T(K)_b^+$ is generated by the groups $T^a(K)_b^+$, hence $T(K)_b^+ \subset [P,P]$. As a consequence, $Q \subset [P,P]$.

Hence, we obtain that $Q = \mathrm{Frat}(P) = [P,P]$.

By Proposition 3.2.11, we know that $\mathrm{Frat}(P) = Q$ is the maximal pro-p subgroup of the pointwise stabilizer of the combinatorial closure of the combinatorial unit ball centered in $\mathbf{c}$. □

In the case of a non-reduced root system Φ, we have seen that computation of $[P,P]$ is different from the reduced case because of non-commutativity of root groups. We have to study this case separately.

5.1.2 Theorem (Descriptions of the Frattini subgroup of a maximal pro-p subgroup: the non-reduced case)**.** *We suppose that Φ is a non-reduced root system of rank $l \geq 2$, and that $p \geq 5$. Then:*

Profinite description: *The pro-p group P is topologically of finite type and* $\mathrm{Frat}(P) = [P,P]$.

Description by the valued root groups datum: *Let $a \in \Phi_{\mathrm{nd}}$ be a non-divisible root. If $a \notin \Delta \cup \{-\theta\}$, we set $V_{a,\mathbf{c}} = U_{a,\mathbf{c}}$.*

If $a \in \Delta \cup \{-\theta\}$, we set:

$$V_{a,\mathbf{c}} = \begin{cases} U_{a,f'_{\mathbf{c}}(a)^+} U_{2a,2f'_{\mathbf{c}}(a)} & \text{if } a \text{ is multipliable, } L'/K \text{ is unramified and } f'_{\mathbf{c}}(a) \in \Gamma'_a, \\ U_{a,f_{\mathbf{c}}(a)^+} & \text{otherwise.} \end{cases}$$

Then $\mathrm{Frat}(P) = \left(\prod_{a \in \Phi^-_{\mathrm{nd}}} V_{a,\mathbf{c}} \right) T(K)^+_b \left(\prod_{a \in \Phi^+_{\mathrm{nd}}} V_{a,\mathbf{c}} \right)$.

Proof. Let $Q = \left(\prod_{a \in \Phi^-_{\mathrm{nd}}} V_{a,\mathbf{c}} \right) T(K)^+_b \left(\prod_{a \in \Phi^+_{\mathrm{nd}}} V_{a,\mathbf{c}} \right)$. By Corollary 3.2.10, we have $\mathrm{Frat}(P) \subset Q$.

Let $a \in \Phi_{\mathrm{nd}}$. If $a \notin \Delta \cup \{-\theta\}$, we have $U_{a,\mathbf{c}} = V_{a,\mathbf{c}} \subset [P,P]$ by Theorem 4.2.12.

If $a \in \Delta \cup \{-\theta\}$, we have $U_{a,f'_{\mathbf{c}}(a)^+} \subset [P,P]$ by Theorem 4.2.12. Assume that a is multipliable and that L'/K is unramified. Normalize the valuation so that $\Gamma_{L'} = \mathbb{Z}$. If $f'_{\mathbf{c}}(a) \in \Gamma'_a$, since $\Gamma'_a = \mathbb{Z}$ by Lemma 2.1.10, we have $U_{2a,2f'_{\mathbf{c}}(a)} = [U_{a,f'_{\mathbf{c}}(a)}, U_{a,f'_{\mathbf{c}}(a)}]$ by Lemma 2.3.12 so that $V_{a,\mathbf{c}} = U_{a,f'_{\mathbf{c}}(a)^+} U_{2a,2f'_{\mathbf{c}}(a)} \subset [P,P]$ in this case. In other cases, we have $V_{a,\mathbf{c}} = U_{a,f'_{\mathbf{c}}(a)^+} \subset [P,P]$.

As before, we normalize the valuation by $\Gamma_{L'} = \mathbb{Z}$. For the multipliable simple root a, if $l_a = -f'_{\mathbf{c}}(-a) = -\frac{1}{2} + f'_{\mathbf{c}}(a)$, then we have $U_{a,l_a+1}, U_{-a,l_a+\frac{1}{2}} \subset [P,P]$. Thus, we can apply Proposition 2.3.1 and Proposition 2.3.11. For a positive multipliable root $a \in \Phi^+$, because $f_{\mathbf{c}}(a) = 0$, we have $\varepsilon = 0$, and so $T^a(K)^+_b \subset [P,P]$. For any non-multipliable root $a \in \Phi$, by Proposition 2.2.3, we have $T^a(K)^+_b \subset [P,P]$. Hence, $T(K)^+_b$ is a subgroup of $\mathrm{Frat}(P)$. As a consequence, we have $Q \subset \mathrm{Frat}(P)$.

Moreover, because Q is an open subgroup of P (of finite index), the Frattini subgroup $\mathrm{Frat}(P) = Q$ is open in P. By [DdSMS99, 1.14], we know that P is topologically of finite type. By [DdSMS99, 1.20], we deduce $\mathrm{Frat}(P) = [P,P]$. □

5.2 Minimal number of generators

5.2.1 Corollary (of Theorems 5.1.1 and 5.1.2)**.** *We assume $p \neq 2$.*

If the root system Φ is reduced, we assume that $p \neq 3$ or Φ is not of type G_2. If the root system Φ is non-reduced, we assume that $p \geq 5$ and that Φ is not of rank 1.

Then $P/\mathrm{Frat}(P)$ is isomorphic to the following direct product of p-elementary commutative groups: $\prod_{a \in \Phi_{\mathrm{nd}}} U_{a,\mathbf{c}}/V_{a,\mathbf{c}}$, where the groups $V_{a,\mathbf{c}}$ for $a \in \Phi_{\mathrm{nd}}$ are defined in Theorems 5.1.1 and 5.1.2.

Proof. Let $A = \prod_{a \in \Phi_{\mathrm{nd}}} U_{a,\mathbf{c}}/V_{a,\mathbf{c}}$ be the considered direct product of quotient groups. Let $B = \left(\prod_{a \in \Phi^-_{\mathrm{nd}}} U_{a,\mathbf{c}}\right) \times T(K)^+_b \times \left(\prod_{a \in \Phi^+_{\mathrm{nd}}} U_{a,\mathbf{c}}\right)$ be the direct product of the valued root groups with respect to $\mathbf{c} = \mathbf{c}_{\mathrm{af}}$, and of the maximal pro-p subgroup of the bounded torus. Let $C = \left(\prod_{a \in \Phi^-_{\mathrm{nd}}} V_{a,\mathbf{c}}\right) \times T(K)^+_b \times \left(\prod_{a \in \Phi^+_{\mathrm{nd}}} V_{a,\mathbf{c}}\right)$ be the direct product of the valued root groups provided by Theorems 5.1.1 and 5.1.2.

We want to define a surjective group homomorphism $B \to P/\mathrm{Frat}(P)$. Let $\pi : P \to P/\mathrm{Frat}(P)$ be the quotient homomorphism. For any inclusion $j_a : U_{a,\mathbf{c}} \to P$ (resp. $j_0 : T(K)^+_b \to P$), we define a group homomorphism $\phi_a = \pi \circ j_a : U_{a,\mathbf{c}} \to P/\mathrm{Frat}(P)$ (resp. $\phi_0 = \pi \circ j_0$). Since $P/\mathrm{Frat}(P)$ is commutative, the multiplication map induces a group homomorphism $\mu : B \to P/\mathrm{Frat}(P)$. Applying [Loi16, 3.2.9] to P, we deduce that the homomorphism μ is surjective.

By Theorems 5.1.1(2) and 5.1.2(2), we get $\ker \mu = C$. Passing to the quotient, we deduce a group isomorphism $B/C \simeq P/\mathrm{Frat}(P)$. Furthermore, there is a canonical group isomorphism $A \simeq B/C$. Hence $P/\mathrm{Frat}(P)$ is isomorphic to A. □

Since $P/\mathrm{Frat}(P)$ is a p-elementary commutative group, we deduce that so are the quotient groups $U_{a,\mathbf{c}}/V_{a,\mathbf{c}}$. Hence, we can compute their dimension as $\mathbb{F}_p$-vector space. According to [DdSMS99, 1.9], we know that the minimal number of elements in a generating set of a pro-p group is $d(P) = \dim_{\mathbb{F}_p}\big(P/\mathrm{Frat}(P)\big)$. It can also be computed by $d(P) = \dim_{\mathbb{Z}/p\mathbb{Z}}\big(H^1(P, \mathbb{Z}/p\mathbb{Z})\big)$ according to [Ser94, 4.2 Corollaire 5]. We apply this to our maximal pro-p subgroup P of $G(K)$.

In order to give explicit formulas for these numbers, we introduce the following integers. We denote by e' the ramification index of L'/K and by f' its residue degree; we let $m = \log_p(\mathrm{Card}(\kappa_K))$ so that $\kappa_K \simeq \mathbb{F}_{p^m}$.

5.2.2 Corollary. *As above we assume that K is a non-Archimedean local field of residue characteristic p and residue field $\kappa \simeq \mathbb{F}_{p^m}$. We assume that G is an absolutely simple simply-connected quasi-split K-group and that $p \neq 2$. We keep notations of 2.1.3. Let n be the rank of the irreducible absolute root system $\widetilde{\Phi}(G_{\widetilde{K}}, \widetilde{K})$ and l be the rank of the irreducible relative root system $\Phi(G, K)$.*

(1) If Φ is of type G_2 or if Φ is non-reduced of rank $l \geq 2$, suppose that $p \geq 5$. If L'/K is ramified, then $d(P) = m(l+1)$; if L'/K is unramified, then $d(P) = m(n+1)$.

(2) Suppose that Φ is of type BC_1 and that $p \geq 5$. If L'/K is ramified, then $2m \leq d(P) \leq 6m$; if L'/K is unramified, then $3m \leq d(P) \leq 9m$.

5.2.3 Remark (Summary in terms of quasi-split groups classification). We recall that f' denotes the residue degree of L'/K and that there are, case by case, identities between d, l and n. In Corollary 5.2.2, if the quasi-split group is of type ${}^dX_{n,l}$ (with notations of [Tit66]; Tits indices are not necessary in this study because of quasi-splitness assumption), we have $d(P) = m\xi$ where:

Type	(in)equality	Assumption
${}^1X_l,\ l \geq 1,\ X \neq G$	$\xi = l+1$	$p \geq 3$
${}^1G_2$	$\xi = 3$	$p \geq 5$
${}^2A_{2l-1},\ l \geq 2$	$\xi = f'(l-1)+2$	$p \geq 3$
${}^2D_{l+1},\ l \geq 3$	$\xi = l+f'$	$p \geq 3$
${}^2E_6$	$\xi = 3+2f'$	$p \geq 3$
${}^3D_4$ and ${}^6D_4$	$\xi = 2+f'$	$p \geq 5$
${}^2A_{2l},\ l \geq 2$	$\xi = f'l+1$	$p \geq 5$
${}^2A_2$	$f'+1 \leq \xi \leq 3f'+3$	$p \geq 5$

Proof. **(1) Suppose that Φ is reduced.** By definition of the groups $V_{a,\mathbf{c}}$ 5.1.1(2), we have $U_{a,\mathbf{c}}/V_{a,\mathbf{c}} \simeq \begin{cases} X_{a,f_{\mathbf{c}}(a)} & \text{if } a \in \Delta \cup \{-\theta\} \\ 0 & \text{otherwise} \end{cases}$, where the quotient groups $X_{a,f_{\mathbf{c}}(a)}$ are defined as in Proposition 3.1.12. Applying Corollary 5.2.1, we write $P/\mathrm{Frat}(P) \simeq \prod_{a\in\Delta\cup\{-\theta\}} X_{a,f_{\mathbf{c}}(a)}$. We know by Proposition 3.1.12 that the group $X_{a,f_{\mathbf{c}}(a)}$ is a κ_{L_a}-vector space of dimension 1. The finite field κ_{L_a} is of order p^{mf_a} where f_a denotes the residue degree of the extension L_a/K. Thus, we obtain $\dim_{\mathbb{F}_p}(P/\mathrm{Frat}(P)) = \sum_{a\in\Delta\cup\{-\theta\}} mf_a$. It remains to compute $\xi = \sum_{a\in\Delta\cup\{-\theta\}} f_a$. Let $a \in \Delta\cup\{-\theta\}$. If a is a long root, then $L_a = K$ and $f_a = 1$. Otherwise $L_a = L'$ and $f_a = f'$.

Suppose that L'/K is ramified. Then $f_a = f' = 1$ for any root so that $\xi = \mathrm{Card}(\Delta\cup\{-\theta\}) = l+1$.

Suppose that L'/K is unramified. We know that θ is the highest root of Φ with respect to Δ. Hence, $-\theta$ is a long root and $L_{-\theta} = K$, so that $f_{-\theta} = 1$. We have $f_a = \mathrm{Card}(a)$ where any simple root $a \in \Delta$ is seen as an orbit of absolute simple roots $\alpha \in \widetilde{\Delta}$. Thus $\xi = f_{-\theta} + \sum_{a\in\Delta} f_a = 1 + \mathrm{Card}(\widetilde{\Delta}) = 1+n$.

(2) Suppose that Φ is non-reduced of rank $l \geq 2$.

Denote by $a \in \Delta$ the multipliable simple root. We have a group isomorphism $P/\mathrm{Frat(P)} \simeq \prod_{b\in\Delta\cup\{-\theta\}} U_{b,l_b}/V_{b,\mathbf{c}}$. We can express each $U_{b,l_b}/V_{b,\mathbf{c}}$ in terms of $X_{b,l}$ (and of $X_{2b,2l}$ if $b \in \{a,-\theta\}$ is a multipliable root).

First case: b is non-multipliable. In this case, we have $V_{b,\mathbf{c}} = U_{b,f_{\mathbf{c}}(b)^+}$. By 3.1.12, we know that $U_{b,f_{\mathbf{c}}(b)}/U_{b,f_{\mathbf{c}}(b)^+} = X_{b,f_{\mathbf{c}}(b)}$ is a κ_{L_b}-vector space of dimension 1, hence a $\mathbf{F}_p$-vector space of dimension $f'm$.

Second case: b is multipliable and L_b/L_{2b} is ramified. By Lemmas 3.1.14 and 2.1.10, we know that $U_{b,f_{\mathbf{c}}(b)}/V_{b,\mathbf{c}} = U_{b,f_{\mathbf{c}}(b)}/U_{b,f_{\mathbf{c}}(b)^+} = X_{b,f_{\mathbf{c}}(b)}$ is a $\kappa_{L_a} \simeq \kappa_K$-vector space of dimension 1, hence a $\mathbf{F}_p$-vector space of dimension $m = f'm$.

Third case: b is multipliable, L_b/L_{2b} is unramified and $f'_{\mathbf{c}}(b) \notin \Gamma'_a$. By Proposition 3.1.12 and Lemma 3.1.14, we know that $U_{b,f_{\mathbf{c}}(b)}/V_{b,\mathbf{c}} = U_{b,f_{\mathbf{c}}(b)}/U_{b,f_{\mathbf{c}}(b)^+} = X_{2b,2f_{\mathbf{c}}(b)}$ is a $\kappa_{L_{2b}}$-vector space of dimension 1, hence a $\mathbf{F}_p$-vector space of dimension m.

Fourth case: b is multipliable, L_b/L_{2b} is unramified and $f'_{\mathbf{c}}(b) \in$

$\Gamma'_b = \Gamma'_a$. By Proposition 3.1.12, we know that $U_{b,f_{\mathbf{c}}(b)}/V_{b,\mathbf{c}} = U_{b,f_{\mathbf{c}}(b)}/\Big(U_{b,f_{\mathbf{c}}(b)^+}U_{2b,2f_{\mathbf{c}}(b)}\Big) = X_{b,f_{\mathbf{c}}(b)}/X_{2b,2f_{\mathbf{c}}(b)}$ is a κ_{L_b}-vector space of dimension 1, hence a $\mathbf{F}_p$-vector space of dimension $2m = f'm$.

We know that in the unramified situation, with the normalization of the valuation $\Gamma_{L'} = \mathbb{Z}$, we have $\Gamma_c = \frac{1}{2}\mathbb{Z}$ and $\Gamma'_c = \mathbb{Z}$ for any multipliable root $c \in \Phi$ according to Lemma 2.1.10. Since $a, -\theta$ are multipliable and $f_{\mathbf{c}}(-\theta) + f_{\mathbf{c}}(a) = 0^+ = \frac{1}{2}$, we note that we have the alternative: either $f_{\mathbf{c}}(a) \in \Gamma'_a$ and $f_{\mathbf{c}}(-\theta) \notin \Gamma'_{-\theta}$, or $f_{\mathbf{c}}(a) \notin \Gamma'_a$ and $f_{\mathbf{c}}(-\theta) \in \Gamma'_{-\theta}$.

Hence, the sum of dimensions over $\mathbb{F}_p$ of $U_{a,f_{\mathbf{c}}(a)}/V_{a,\mathbf{c}}$ and $U_{-\theta,f_{\mathbf{c}}(-\theta)}/V_{-\theta}$ is always equal to $(f'+1)m$.

Since there are $l-1$ non-multipliable simple roots, we get

$$d(P) = mf'(l-1) + (1+f')m = m(lf'+1).$$

Let ξ be such that $d(P) = m\xi$. If L'/K is unramified, then $f' = 2$ and $\xi = n+1$. If L'/K is ramified, then $f' = 1$ and $\xi = l+1$.

(3) Suppose that Φ is non-reduced of rank 1. In this case, we cannot apply Theorem 5.1.2 and its corollary. Let $H = U_{-a,\frac{1}{2}}T(K)_b^+U_{a,0}$ be a maximal pro-p subgroup of $G(K) \simeq \mathrm{SU}(h)(K)$, so that $\varepsilon = 0$. Let $l'' = \max(1,3) = 3$.

Suppose that L/K is unramified. By Lemma 2.3.12, by Lemma 2.3.4 and by Proposition 2.3.1, we have:

$$U_{-2a,2}U_{-a,\frac{3}{2}}T(K)_b^{l''}U_{a,1}U_{2a,0} \subset [H,H]H^p \subset U_{-2a,2}U_{-a,1}T(K)_b^+U_{a,\frac{1}{2}}U_{2a,0}$$

On the one hand, thanks to computation with the quotient groups $X_{a,l}$, we get the κ-vector spaces $U_{a,0}/U_{a,\frac{1}{2}}U_{2a,0} \simeq X_{a,0}/X_{2a,0}$ of dimension $d(a,0) = 2$ and $U_{-a,\frac{1}{2}}/U_{-2a,2}, U_{-a,1} \simeq X_{-a,\frac{1}{2}}$ of dimension $d\left(-a,\frac{1}{2}\right) + d(-2a,1) = 0+1 = 1$. Hence $d(H) \geq 3m$. On the other hand, $U_{a,0}/U_{a,1}U_{2a,0}$ have to be isomorphic to a subgroup of $X_{a,0}/X_{2a,0} \oplus X_{a,\frac{1}{2}}/X_{2a,1}$, of dimension $d(a,0)+d(a,\frac{1}{2}) = 2$ as κ-vector space. In the same way, $U_{-a,\frac{1}{2}}/U_{-2a,2}U_{-a,\frac{3}{2}}$ is isomorphic to a subgroup of $X_{-a,\frac{1}{2}} \oplus X_{-a,1}/X_{-2a,-2}$, of dimension $d(-a,\frac{1}{2}) + d(-2a,1)+d(-a,1) = 0+1+2 = 3$. Finally, $T(K)_b^+/T(K)_b^{l''}$ is of dimension $2(l''-1) = 4$. Thus $d(H) \leq m(5+4) = 9m$.

Suppose that L/K is ramified. By Lemma 2.3.12, by Lemma 2.3.4 and by Proposition 2.3.1, we have:

$$U_{-2a,3}U_{-a,2}T(K)_b^{l''}U_{a,\frac{3}{2}}U_{2a,1} \subset [H,H]H^p \subset U_{-2a,3}, U_{-a,1}T(K)_b^+U_{a,\frac{1}{2}}U_{2a,1}$$

On the one hand, thanks to computation with the quotient groups $X_{a,l}$, we get the κ-vector spaces $U_{a,0}/U_{a,\frac{1}{2}}U_{2a,1} \simeq X_{a,0}$ of dimension $d(a,0) + d(2a,0) = 1+0$ and $U_{-a,\frac{1}{2}}/U_{-2a,3}, U_{-a,1} \simeq X_{-a,\frac{1}{2}}$ of dimension $d(-a,\frac{1}{2} + d(-2a,1) = 0+1 = 1$. Hence $d(H) \geq 2m$. On the other hand, $U_{a,0}/U_{a,\frac{3}{2}}U_{2a,1}$

have to be isomorphic to a subgroup of $X_{a,0} \oplus X_{a,\frac{1}{2}}/X_{2a,1} \oplus X_{a,1}/X_{2a,2}$, of dimension $d(a,0)+d(2a,0)+d(a,\frac{1}{2})+d(a,1) = 1+0+0+1 = 2$ as κ-vector space. In the same way, $U_{-a,\frac{1}{2}}/U_{-2a,3}U_{-a,2}$ is isomorphic to a subgroup of $X_{-a,\frac{1}{2}} \oplus X_{-a,1} \oplus X_{-a,\frac{3}{2}}/X_{2a,3}$, of dimension $d(-a,\frac{1}{2})+d(-2a,1)+d(-a,1)+d(-2a,2)+d\left(-a,\frac{3}{2}\right) = 0+1+1+0+0 = 2$. Finally, $T(K)_b^+/T(K)_b^{l''}$ is of dimension $(l''-1) = 2$. Thus $d(H) \leq m(4+2) = 6m$. □

5.2.4 Remark (Generating set in terms of root groups). A generating set of $P/\mathrm{Frat}(P)$ always come from a topologically generating set of P. Hence, for instance, when the relative root system Φ is reduced and L'/K is ramified, a system of generators of P is given by:

$$\left\{x_a(\lambda_i),\ 1 \leq i \leq m \text{ and } a \in \Delta\right\} \cup \left\{\{x_{-\theta}(\lambda_i \varpi_{L'}),\ 1 \leq i \leq m\right\}$$

where $(\lambda_i)_{1 \leq i \leq m}$ is a family of elements of $\mathcal{O}_K$ such that $(\lambda_i \mathcal{O}_K/\mathfrak{m}_K)_{1 \leq i \leq m}$ is a basis of κ; the root θ is chosen as in Section 3.1; and $\varpi_{L'} \in \mathcal{O}_{L'}$ is a uniformizer.

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