Giant Hall Effect in Composites

Marc Briane, Graeme W Milton

GIANT HALL EFFECT IN COMPOSITES *

MARC BRIANE† AND GRAEME W. MILTON‡

Abstract. This paper deals with the homogenization of the Hall effect in three-dimensional composites. We prove that the homogenized Hall coefficients are bounded from above by means of prescribed bounds on the local conductivity and the local Hall coefficient. Then, three microstructures of different nature are presented in order to show that arbitrarily large effective Hall coefficients may be obtained in dimension three if the previous prescribed bounds do not hold.

Key words. Hall effect, homogenization, multi-scale, composites

AMS subject classifications. 35B27, 74Q15

1. Introduction. In electrodynamics a low magnetic field h modifies the conductivity of a conductor. The leading, first-order in h, effect is called the Hall effect (see e.g. [11] and [14]), and the second-order in h effect is called the magnetoresistance. Due to quantum effects, multilayer metal composites can have giant magnetoresistances and thus have a conductivity which is very sensitive to small magnetic fields. Using them has enabled the miniaturization of read-heads on magnetic recording devices. The importance of the discovery of such materials was recognized by the award of the 2007 Physics Nobel Prize (see e.g. [1]). This paper addresses the question of whether, within the framework of classical electrodynamics, composites can exhibit giant Hall effects, when the constituent materials do not.

At first-order in h the resistivity matrix is equal to the unperturbed resistivity in the absence of magnetic field plus an antisymmetric first-order term. In dimension two one coefficient arises in the first-order term, it is called the Hall coefficient. In dimension three the first-order term involves a whole 3×3 matrix which is called the Hall matrix. In the three-dimensional isotropic case the Hall matrix is proportional to the identity matrix and the coefficient of proportionality is again called the Hall coefficient.

Following the seminal work of Bergman [3] it is interesting to study the homogenized (or effective) Hall effect in composites. The situation is radically different in dimension two and in dimension three. Indeed, we proved in [9] that in dimension two the effective Hall coefficient satisfies the same bounds as the local Hall coefficient (i.e. the Hall coefficient in the original microstructure before homogenization). This result is similar to that of the homogenized conductivity which preserves the bounds of the local conductivity by virtue of the arithmetic-harmonic mean bounds. Curiously the bounds are not preserved for three-dimensional Hall coefficients. In particular, we built in [8] a cubic chain mail (of the Middle Age armor type) with perfectly conductive rings, and a suitable positive local Hall coefficient in such a way that the effective Hall matrix is isotropic but negative. Hence, the bound from below is not preserved by the homogenization process.

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It is then natural to see the influence of homogenization on the bound from above on the Hall coefficient. So, is it possible to obtain arbitrarily large effective Hall coefficients from composites with local Hall coefficients satisfying given bounds? In this paper we show that the answer is positive. Previously Bergman, Kantor, Stroud, and Webman [5] had shown numerically that discrete random resistor networks could exhibit enormous Hall coefficients near percolation, and Rohde and Micklitz [15] had obtained experimental evidence of huge Hall coefficient near percolation in granular mixtures. We mention, in passing, that there are also a number of interesting magnetotransport effects associated with composites in strong magnetic fields (see e.g. [6] and [4], and references therein).

We first prove (see Theorem 2.9) a sufficient condition of boundedness which claims that the homogenized Hall matrix is bounded by a constant times the upper bound on the local Hall coefficient provided that the local conductivity is bounded from above (at least in the region where the Hall coefficient is nonzero) and the homogenized conductivity matrix is bounded from below by prescribed constants. If one of the two previous assumptions is not satisfied we can obtain arbitrarily large homogenized Hall coefficients. Specifically, we present three microstructures, each of quite different nature, but which all can have arbitrarily large effective Hall coefficients:

- The first microstructure (see subsection 3.1) acts like n batteries in series. We find that one of the effective Hall coefficients is of order $n \gg 1$, while the local Hall coefficient is bounded above by 1.
- The second microstructure (see subsection 3.2) is an isotropic, two-phase, rank-three laminate (with 8 elementary layers) based on the Schulgasser construction [16]. One of the phases has a low conductivity $t \ll 1$ and occupies a high volume fraction $1-t$, and the other one has conductivity 1. We find that the effective conductivity is of order t, and that the effective Hall matrix is isotropic and of order $t^{-1} \gg 1$, while the local Hall coefficient is still bounded by 1.
- The third microstructure (see subsection 3.3) is an isotropic two-phase rank-five laminate (with 16 elementary layers) still based on the Schulgasser construction. But this time one of the two phases has a high conductivity t^{-1} occupying a small volume fraction t, and the other one has conductivity 1. We find that the effective conductivity is of order 1, while the effective Hall matrix is asymptotically isotropic and of order $t^{-1} \gg 1$. Surprisingly we obtain a composite with an effective conductivity of order 1 but inducing a giant Hall effect.

The paper is divided in two sections. In Section 2 we recall the Hall effect principle, a few results of the Murat-Tartar H-convergence and of the homogenization of the Hall effect, and we give a sufficient condition for preserving the bound from above of the three-dimensional homogenized Hall matrix. Section 3 is devoted to the study of the three composites which have unboundedly high effective Hall coefficients.

Notations.

- Y_d denotes the cube $(0,1)^d$;
- for any measurable set of $\mathbb{R}^d$, $|E|$ denotes the Lebesgue measure of E;
- I_d denotes the unit matrix of $\mathbb{R}^{d\times d}$;
- for any $A \in \mathbb{R}^{d\times d}$, A^T denotes the transpose of A, $\det(A)$ its determinant, $\mathrm{tr}(A)$ its trace, and $\mathrm{Cof}(A)$ its Cofactor matrix;

- $|\cdot|$ denotes the Euclidian norm in $\mathbb{R}^d$; for any $A \in \mathbb{R}^{d\times d}$,

$$|A| := \sup\left\{|Ax| \,:\, x \in \mathbb{R}^d \text{ with } |x| = 1\right\};$$

- for $\alpha, \beta > 0$, $\mathcal{M}(\alpha, \beta; \Omega)$ denotes the set of the matrix-valued functions $A : \Omega \longrightarrow \mathbb{R}^{d\times d}$ such that

$$\forall\, \xi \in \mathbb{R}^d, \quad A(x)\xi\cdot\xi \geq \alpha\,|\xi|^2 \text{ and } A^{-1}(x)\xi\cdot\xi \geq \beta^{-1}\,|\xi|^2, \quad \text{a.e. } x \in \Omega; \tag{1.1}$$

- $H^1_\sharp(Y_d)$ denotes the space of functions which are Y_d-periodic in $\mathbb{R}^d$ and belong to $H^1_{\text{loc}}(\mathbb{R}^d)$;
- for $u : \mathbb{R}^d \longrightarrow \mathbb{R}$, $\nabla u := \left(\dfrac{\partial u}{\partial x_i}\right)_{1\leq i\leq d}$;
- for $U : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $U = (u_1, \ldots, u_d)$, $DU := \left(\dfrac{\partial u_j}{\partial x_i}\right)_{1\leq i,j\leq d}$;
- for $\Sigma : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$,

$$\mathrm{Div}\,(\Sigma) := \left(\frac{\partial \Sigma_{ij}}{\partial x_i}\right)_{1\leq j\leq d}, \quad \mathrm{Curl}\,(\Sigma) := \left(\frac{\partial \Sigma_{ik}}{\partial x_j} - \frac{\partial \Sigma_{jk}}{\partial x_i}\right)_{1\leq i,j,k\leq d};$$

- for a sequence of functions $f_\varepsilon : O \longrightarrow H$, $\varepsilon > 0$, where O is a neighborhood of 0 in $\mathbb{R}^d$ and $(H, \|\cdot\|)$ a Banach space, we denote

$$f_\varepsilon(h) = o_H(h) \quad \Longleftrightarrow \quad \lim_{h\to 0}\left(\frac{1}{|h|}\sup_{\varepsilon>0}\|f_\varepsilon(h)\|\right) = 0, \tag{1.2}$$

  i.e. the $o_H(h)$ is uniform with respect to ε;
- $\mathcal{D}'(\Omega)$ denotes the space of the distributions on Ω.

2. The Hall effect and Homogenization.

2.1. The three-dimensional Hall effect in a microstructure. Let Ω be a bounded open subset of $\mathbb{R}^3$, and let $\alpha, \beta > 0$. Consider a sequence $\Sigma^\varepsilon(h)$, for $\varepsilon > 0$ and $h \in \mathbb{R}^3$, of matrix-valued functions in $\mathcal{M}(\alpha, \beta; \Omega)$, which represents the conductivity matrix of a heterogeneous conducting material in the presence of a constant magnetic field h, and the microstructure of which depends on the small parameter ε (which in the simplest case of a periodic composite may represent the length of the unit cell). We assume that $\Sigma^\varepsilon(h)$ belongs to $\mathcal{M}(\alpha, \beta; \Omega)$ and satisfies the uniform Lipschitz condition

$$\exists\, C > 0,\ \forall\, h, h' \in \mathbb{R}^3, \quad \|\Sigma^\varepsilon(h) - \Sigma^\varepsilon(h')\|_{L^\infty(\Omega)^{3\times 3}} \leq C\,|h - h'|. \tag{2.1}$$

From the physics of the problem it can be shown (see e.g. Section 21 of [11] pages 132-135) that the resistivity $\rho^\varepsilon(h) := \Sigma^\varepsilon(h)^{-1}$ satisfies the property

$$\forall\, h \in \mathbb{R}^3, \quad \rho^\varepsilon(h)^T = \rho^\varepsilon(-h), \tag{2.2}$$

which implies that the symmetric part of $\rho^\varepsilon(h)$ is even and the antisymmetric one is odd with respect to h. Then, in the presence of a low magnetic field h we assume that $\rho^\varepsilon(h)$ has a first-order expansion around $h = 0$. The zeroth-order term $\rho^\varepsilon := \rho^\varepsilon(0)$ of the expansion is the unperturbed resistivity which is a symmetric matrix. By contrast, the first-order term is an antisymmetric matrix and is linear with respect to h. Therefore, the first-order expansion of $\rho^\varepsilon(h)$ reads as (see [8] for more details)

$$\rho^\varepsilon(h) = \rho^\varepsilon + \mathcal{E}(R^\varepsilon h) + o_{L^\infty(\Omega)^{3\times 3}}(h), \tag{2.3}$$

where R^ε is the local Hall matrix which is bounded in $L^\infty(\Omega)^{3\times 3}$, and $\mathcal{E}$ is the Levi-Civita tensor which maps a vector to an antisymmetric matrix according to

$$\mathcal{E}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix} \qquad \text{for } a_1, a_2, a_3 \in \mathbb{R}. \tag{2.4}$$

Remark 2.1. In dimension two the first-order term in the expansion (2.3) reads as $r_\varepsilon h\, J$, where r_ε, h are scalar and J is the rotation matrix by an angle of $90°$. So, the Hall matrix reduces to the single Hall coefficient r_ε (see [9]).

2.2. Review of homogenization. Let Ω be a bounded open set of $\mathbb{R}^d$, and let $\alpha, \beta > 0$. We recall here the definition of the Murat-Tartar [13] H-convergence for a sequence of matrix-valued functions in $\mathcal{M}(\alpha, \beta; \Omega)$ (see the notation (1.1)), some properties of H-convergence, and the definition of a corrector:

Definition 2.2. **(Murat-Tartar [13])** A sequence A^ε in $\mathcal{M}(\alpha, \beta; \Omega)$ is said to H-converge to A^* in $\mathcal{M}(\alpha, \beta; \Omega)$ if, for any $f \in H^{-1}(\Omega)$, the solution u_ε of problem

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla u_\varepsilon) = f & \text{in } \mathcal{D}'(\Omega) \\ u_\varepsilon \in H^1_0(\Omega), \end{cases} \tag{2.5}$$

weakly converges in $H^1_0(\Omega)$ to the solution u of

$$\begin{cases} -\operatorname{div}(A^* \nabla u) = f & \text{in } \mathcal{D}'(\Omega) \\ u \in H^1_0(\Omega), \end{cases} \tag{2.6}$$

and the sequence $A^\varepsilon \nabla u_\varepsilon$ weakly converges to $A^* \nabla u$ in $L^2(\Omega)^d$. Then, the matrix-valued function A^* is called the homogenized matrix or the H-limit of A^ε.

Murat and Tartar proved that any sequence in $\mathcal{M}(\alpha, \beta; \Omega)$ admits a subsequence which H-converges in $\mathcal{M}(\alpha, \beta; \Omega)$. We will confine our attention to convergent subsequences in the sequel.

The periodic case provides a classical formula for the H-limit (see e.g. [2]): Let $A^\sharp$ be a Y_d-periodic matrix-valued function in $\mathcal{M}(\alpha, \beta; \mathbb{R}^d)$. Then, the oscillating sequence $A^\varepsilon(x) := A^\sharp(\frac{x}{\varepsilon})$ H-converges on every bounded open subset of $\mathbb{R}^d$ to the constant matrix A^* defined by

$$A^* := \int_{Y_d} A^\sharp DU\, dy, \tag{2.7}$$

where U is the unique (up to an additive constant) solution in $H^1_{\mathrm{loc}}(\mathbb{R}^d)^d$ of the periodic cell problem

$$\begin{cases} \operatorname{Div}\left(A^\sharp DU\right) = 0 & \text{in } \mathcal{D}'(\mathbb{R}^d) \\ U^\varepsilon(y) - y & \text{is } Y_d\text{-periodic.} \end{cases} \tag{2.8}$$

In Definition 2.2 the electric field ∇u^ε does not strongly converge to ∇u in $L^2(\Omega)^d$ due to the oscillations of the microstructure. Murat and Tartar introduced the notion of corrector in order to measure the oscillations of the electric field and to recover the H-limit:

DEFINITION 2.3. **(Murat-Tartar [13])** Let A^ε be a sequence in $\mathcal{M}(\alpha,\beta;\Omega)$ which H-converges to A^*. Any matrix-valued function P^ε in $L^2(\Omega)^3$ satisfying

$$\begin{cases} P^\varepsilon \rightharpoonup I_d & \text{weakly in } L^2(\Omega)^{d\times d} \\ \operatorname{Curl}(P^\varepsilon) \longrightarrow 0 & \text{strongly in } H^{-1}(\Omega)^{d\times d\times d} \\ \operatorname{Div}(A^\varepsilon P^\varepsilon) \longrightarrow \operatorname{Div}(A^*) & \text{strongly in } H^{-1}(\Omega)^{d\times d}, \end{cases} \tag{2.9}$$

is called a corrector associated with A^ε. The interest in correctors comes from the following result:

PROPOSITION 2.4. *Let A^ε be a sequence in $\mathcal{M}(\alpha,\beta;\Omega)$ which H-converges to A^*, and let P^ε be a corrector associated with A^ε. Then, the following convergence holds true*

$$A^\varepsilon P^\varepsilon \rightharpoonup A^* \quad \text{weakly in } L^2(\Omega)^{d\times d}. \tag{2.10}$$

Moreover, the potentials u_ε and u which are respectively solutions of (2.5) and (2.6), satisfy the strong convergence

$$\nabla u_\varepsilon - P^\varepsilon \nabla u \longrightarrow 0 \quad \text{in } L^1(\Omega)^d. \tag{2.11}$$

The following result allows us to build correctors:

PROPOSITION 2.5. **(Murat-Tartar [13])** *Let A^ε be a sequence in $\mathcal{M}(\alpha,\beta;\Omega)$ which H-converges to A^*. Let U^ε be the solution in $H^1(\Omega)^3$ of the problem*

$$\begin{cases} \operatorname{Div}(A^\varepsilon DU^\varepsilon) = \operatorname{Div}(A^*) & \text{in } \Omega \\ U^\varepsilon(x) = x & \text{on } \partial\Omega. \end{cases} \tag{2.12}$$

Then, DU^ε is a corrector associated with A^ε. In the periodic case there is a more explicit corrector:

EXAMPLE 2.6. Let $A^\sharp$ be a Y_d-periodic matrix-valued function in $\mathcal{M}(\alpha,\beta;\mathbb{R}^d)$. Then, the sequence $P^\varepsilon := DU(\frac{x}{\varepsilon})$, where the vector-valued function U solves (2.8), is a corrector associated with the oscillating sequence $A^\varepsilon(x) := A^\sharp(\frac{x}{\varepsilon})$.

2.3. The homogenized Hall matrix. Let Ω be a bounded open set of $\mathbb{R}^3$ and let $\alpha,\beta > 0$. We consider a conductivity matrix $\Sigma^\varepsilon(h)$ in $\mathcal{M}(\alpha,\beta;\Omega)$, with $h \in \mathbb{R}^3$, which satisfies the uniform Lipschitz condition (2.1) and which is such that the resistivity $\rho^\varepsilon(h) := \Sigma^\varepsilon(h)^{-1}$ satisfies the first-order expansion (2.3) at $h = 0$. We assume that $\Sigma^\varepsilon(h)$ H-converges to some conductivity $\Sigma^*(h)$ for any h (this is not a restrictive assumption if h belongs to a countable set).

Since $\Sigma^\varepsilon(h)^T = \Sigma^\varepsilon(-h)$ H-converges to $\Sigma^*(h)^T$ by a classical property of H-convergence, we also get that $\Sigma^*(h)^T = \Sigma^*(-h)$ for any h. Therefore, one can prove (see [9] and [8]) that there exists a matrix-valued S^* in $L^2(\Omega)^{3\times 3}$ such that the H-limit $\Sigma^*(h)$ satisfies the first-order expansion

$$\Sigma^*(h) = \Sigma^* + \mathcal{E}(S^*h) + o_{L^2(\Omega)^{3\times 3}}(h). \tag{2.13}$$

We also assume that S^* belongs to $L^\infty(\Omega)^{3\times 3}$. Then, (2.13) leads us to the first-order expansion satisfied by the homogenized resistivity $\rho^*(h) := \Sigma^*(h)^{-1}$:

$$\rho^*(h) = \rho^* + \mathcal{E}(R^* h) + o_{L^2(\Omega)^{3\times 3}}(h), \tag{2.14}$$

where $R^* \in L^\infty(\Omega)^{3\times 3}$ is the homogenized Hall matrix.

The homogenization problem is now to derive the pair (ρ^*, R^*) of homogenized matrices from the sequence $(\rho^\varepsilon, R^\varepsilon)$ of local matrices. As suggested by the work of Bergman [3] the result simply depends on the solutions for the current field without any magnetic field present. More precisely, in [8] we proved the following distributional convergence:

For any corrector P^ε associated with the unperturbed conductivity $\Sigma^\varepsilon := \Sigma^\varepsilon(0)$, the following convergence holds true

$$\mathrm{Cof}\,(\Sigma^\varepsilon P^\varepsilon)^T\, R^\varepsilon \;\rightharpoonup\; \mathrm{Cof}\,(\Sigma^*)^T\, R^* = \mathrm{Cof}\,(\Sigma^*)\, R^* \quad \text{in } \mathcal{D}'(\Omega)^{3\times 3}. \tag{2.15}$$

Remark 2.7. In the two-dimensional case of [9], the convergence (2.15) reduces to

$$\det\,(\Sigma^\varepsilon P^\varepsilon)\, r_\varepsilon \;\rightharpoonup\; \det\,(\Sigma^*)\, r_* \quad \text{in } \mathcal{D}'(\Omega)^{3\times 3}, \tag{2.16}$$

where r_ε is the local Hall coefficient and r_* is the homogenized one.

Remark 2.8. The norm of the homogenized Hall matrix R^* is invariant by an orthogonal change of variables. Indeed, let O be an orthogonal matrix of $\mathbb{R}^{3\times 3}$, i.e. $OO^T = I_3$. Making the change of variables $x' = Ox$, the homogenized resistivity ρ^*_O in the new coordinates x' is given by the formula $\rho^*_O(x') = O\,\rho^*(x)\,O^T$ (see e.g. Lemma 38 of [19]). This also yields for the perturbed resistivities $\rho^*_O(h')(x') = O\,\rho^*(h)(x)\,O^T$, where $h' := Oh$ is the magnetic field in the new coordinates. Then, by (2.14) the homogenized Hall matrix R^*_O and the magnetic field h' in the new coordinates satisfy

$$\mathcal{E}\big(R^*_O(x')h'\big) = O\,\mathcal{E}\big(R^*(x)h\big)\,O^T. \tag{2.17}$$

On the other hand, using Lemma 1 of [8] and the orthogonality of O the right-hand side of (2.17) is also equal to $\mathcal{E}\big(OR^*(x)h\big)$, which thus implies the equalities

$$R^*_O(x') = O\,R^*(x)\,O^T \quad \text{and} \quad \big|R^*_O(x')\big| = \big|R^*(x)\big|. \tag{2.18}$$

2.4. Sufficient conditions of boundedness of the homogenized Hall matrix. In dimension two we proved [9] that the homogenized Hall coefficient r_* (see Remark 2.7) preserves the bounds of the local Hall coefficient r_ε. This is not the case in dimension three. The next Section 3 is devoted to three counterexamples. However, if the local conductivity of the microstructure is uniformly bounded from above, and the homogenized conductivity is bounded, then the homogenized Hall matrix is controlled by the bounds on the local Hall matrix as in dimension two. More precisely, we have the following result:

Theorem 2.9. *Let Ω be a bounded open subset of $\mathbb{R}^3$, and let $\alpha, \beta, \beta_{\mathrm{H}}, r_{\mathrm{H}} > 0$. Consider a microstructure in Ω with a conductivity matrix $\Sigma^\varepsilon \in \mathcal{M}(a, \beta; \Omega)$, with*

$0 < a \le \alpha$, and an isotropic Hall matrix $R^\varepsilon := r_\varepsilon\, I_3$ in $L^\infty(\Omega)^{3\times 3}$. We assume that Σ^ε H-converges to some

$$\Sigma^* \in \mathcal{M}(\alpha, \beta; \Omega), \tag{2.19}$$

and that

$$|\Sigma^\varepsilon| \le \beta_{\rm H} \quad \text{a.e. in } \Omega \cap \{r_\varepsilon \neq 0\}, \tag{2.20}$$

and

$$|r_\varepsilon| \le r_{\rm H} \quad \text{a.e. in } \Omega. \tag{2.21}$$

Then, the homogenized Hall matrix R^* satisfies the bound

$$|R^*| \le 18\, \frac{\beta_{\rm H}}{\alpha}\, r_{\rm H} \quad \text{a.e. in } \Omega. \tag{2.22}$$

Remark 2.10. The homogenized Hall matrix is bounded, up to a multiplicative constant, by the product of the contrast $\frac{\beta_{\rm H}}{\alpha}$ of the local conductivity times the bound $r_{\rm H}$ of the local Hall coefficient r_ε. Note that this contrast is less or equal to the global contrast $\frac{\beta}{\alpha}$ of the conductivity, since by (2.20) $\beta_{\rm H}$ is the bound obtained in the region where the Hall coefficient is nonzero.

Proof. Let $P^\varepsilon := DU^\varepsilon$ be the corrector defined by Proposition 2.5, with $U^\varepsilon := (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$. By (2.15) the sequence $S^\varepsilon := \operatorname{Cof}(\Sigma^\varepsilon P^\varepsilon)^T R^\varepsilon$ converges in the distributions sense to the matrix-valued function $S^* := \operatorname{Cof}(\Sigma^*)^T R^*$ which belongs to $L^\infty(\Omega)^{3\times 3}$. Let us estimate the sequence S_{11}^ε.

First, the Cauchy-Schwarz inequality implies that for any $i, j = 1, 2, 3$,

$$\left|(\Sigma^\varepsilon P^\varepsilon)_{ij}\right| \le \left|\Sigma^\varepsilon P^\varepsilon e_j\right| \le |\Sigma^\varepsilon|^{\frac12} \left|(\Sigma^\varepsilon)^{\frac12} P^\varepsilon e_j\right| = |\Sigma^\varepsilon|^{\frac12} \left(\Sigma^\varepsilon \nabla u_j^\varepsilon \cdot \nabla u_j^\varepsilon\right)^{\frac12}. \tag{2.23}$$

Let $\varphi \in C_c^1(\Omega)$, $\varphi \ge 0$. Using successively (2.20), (2.23) and the Cauchy-Schwarz inequality for the integrals we have

$$\begin{aligned}
&\left|\int_\Omega S_{11}^\varepsilon\, \varphi\, dx\right| \\
&\le \int_\Omega |r_\varepsilon| \left|\operatorname{Cof}\left(\Sigma^\varepsilon P^\varepsilon\right)^T_{11}\right| \varphi\, dx \\
&= \int_\Omega |r_\varepsilon| \left|(\Sigma^\varepsilon P^\varepsilon)_{22}(\Sigma^\varepsilon P^\varepsilon)_{33} - (\Sigma^\varepsilon P^\varepsilon)_{23}(\Sigma^\varepsilon P^\varepsilon)_{32}\right| \varphi\, dx \\
&\le \int_\Omega |r_\varepsilon| \left|(\Sigma^\varepsilon P^\varepsilon)_{22}\right| \left|(\Sigma^\varepsilon P^\varepsilon)_{33}\right| \varphi\, dx + \int_\Omega |r_\varepsilon| \left|(\Sigma^\varepsilon P^\varepsilon)_{23}\right| \left|(\Sigma^\varepsilon P^\varepsilon)_{32}\right| \varphi\, dx \\
&\le 2\, \beta_{\rm H} r_{\rm H} \int_\Omega \left(\Sigma^\varepsilon \nabla u_2^\varepsilon \cdot \nabla u_2^\varepsilon\right)^{\frac12} \left(\Sigma^\varepsilon \nabla u_3^\varepsilon \cdot \nabla u_3^\varepsilon\right)^{\frac12} \varphi\, dx \\
&\le 2\, \beta_{\rm H} r_{\rm H} \left(\int_\Omega \Sigma^\varepsilon \nabla u_2^\varepsilon \cdot \nabla u_2^\varepsilon\, \varphi\, dx\right)^{\frac12} \left(\int_\Omega \Sigma^\varepsilon \nabla u_3^\varepsilon \cdot \nabla u_3^\varepsilon\, \varphi\, dx\right)^{\frac12}.
\end{aligned} \tag{2.24}$$

Similarly, by (2.23) we have for S^ε_{12}

$$\begin{aligned}
&\left|\int_\Omega S^\varepsilon_{12}\,\varphi\,dx\right| \\
&\leq \int_\Omega |r_\varepsilon|\left|\mathrm{Cof}\left(\Sigma^\varepsilon P^\varepsilon\right)^T_{12}\right|\varphi\,dx \\
&= \int_\Omega |r_\varepsilon|\left|(\Sigma^\varepsilon P^\varepsilon)_{12}(\Sigma^\varepsilon P^\varepsilon)_{33}-(\Sigma^\varepsilon P^\varepsilon)_{13}(\Sigma^\varepsilon P^\varepsilon)_{32}\right|\varphi\,dx \\
&\leq 2\,\beta_{\mathrm{H}}r_{\mathrm{H}}\left(\int_\Omega \Sigma^\varepsilon\nabla u^\varepsilon_2\cdot\nabla u^\varepsilon_2\,\varphi\,dx\right)^{\frac12}\left(\int_\Omega \Sigma^\varepsilon\nabla u^\varepsilon_3\cdot\nabla u^\varepsilon_3\,\varphi\,dx\right)^{\frac12}.
\end{aligned}\tag{2.25}$$

Moreover, the Murat-Tartar div-curl Lemma [18] combined with the Definition 2.2 implies the convergences

$$\int_\Omega \Sigma^\varepsilon\nabla u^\varepsilon_i\cdot\nabla u^\varepsilon_i\,\varphi\,dx \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega \Sigma^*_{ii}\,\varphi\,dx,\qquad \text{for } i=1,2,3.\tag{2.26}$$

Therefore, passing to the limit in estimate (2.24) we get

$$\left|\int_\Omega S^*_{11}\,\varphi\,dx\right| \leq 2\,\beta_{\mathrm{H}}r_{\mathrm{H}}\left(\int_\Omega \Sigma^*_{22}\,\varphi\,dx\right)^{\frac12}\left(\int_\Omega \Sigma^*_{33}\,\varphi\,dx\right)^{\frac12}.\tag{2.27}$$

Since the functions $S^*_{11},\Sigma^*_{22},\Sigma^*_{33}$ belong to $L^\infty(\Omega)$, by a density argument estimate (2.27) still holds true for any $\varphi\in L^1(\Omega)$.

Then consider a Lebesgue point x common to the matrix-valued functions S^* and Σ^*. By making an orthogonal change of variables which preserves the norms $|\Sigma^*|,|R^*|$ (see Remark 2.8 above), we can assume that $\Sigma^*(x)$ is a diagonal matrix without loss of generality. Using the test function $\varphi := 1_{B(x,\delta)}/|B(x,\delta)|$ in (2.27) (where $B(x,\delta)$ denotes the ball of center x and radius δ) and passing to the limit as $\delta\to 0$, we obtain the inequality

$$|S^*_{11}(x)| \leq 2\,\beta_{\mathrm{H}}r_{\mathrm{H}}\left(\Sigma^*_{22}(x)\,\Sigma^*_{33}(x)\right)^{\frac12}.\tag{2.28}$$

Similarly, we have for any $i,j=1,2,3$,

$$|S^*_{ij}(x)| \leq 2\,\beta_{\mathrm{H}}r_{\mathrm{H}}\left(\Sigma^*_{i_1i_1}(x)\,\Sigma^*_{i_2i_2}(x)\right)^{\frac12},\quad \text{where}\quad \{i,i_1,i_2\}=\{1,2,3\}.\tag{2.29}$$

Now, let us go back to the homogenized Hall matrix

$$R^* = \mathrm{Cof}\,(\Sigma^*)^{-T}\,S^* = \det\,(\Sigma^*)^{-1}\,\Sigma^*S^*.$$

Since $\Sigma^*(x)$ is a positive definite diagonal matrix, we have for any $i,j=1,2,3$,

$$|\Sigma^*_{ij}(x)| \leq \left(\Sigma^*_{ii}(x)\,\Sigma^*_{jj}(x)\right)^{\frac12}.$$

This combined with the estimates (2.29) yields for any $i,j=1,2,3$,

$$\begin{aligned}
|R^*_{ij}(x)| &= \frac{1}{\det\left(\Sigma^*(x)\right)}\left|\,\Sigma^*_{i1}(x)\,S^*_{1j}(x)+\Sigma^*_{i2}(x)\,S^*_{2j}(x)+\Sigma^*_{i3}(x)\,S^*_{3j}(x)\,\right| \\
&\leq \frac{6\,\beta_{\mathrm{H}}r_{\mathrm{H}}}{\det\left(\Sigma^*(x)\right)}\left(\Sigma^*_{ii}(x)\right)^{\frac12}\left(\Sigma^*_{11}(x)\,\Sigma^*_{22}(x)\,\Sigma^*_{33}(x)\right)^{\frac12} \\
&= 6\,\beta_{\mathrm{H}}r_{\mathrm{H}}\left(\frac{\Sigma^*_{ii}(x)}{\Sigma^*_{11}(x)\,\Sigma^*_{22}(x)\,\Sigma^*_{33}(x)}\right)^{\frac12} \leq \frac{6\,\beta_{\mathrm{H}}r_{\mathrm{H}}}{\alpha},\quad (\Sigma^*_{kk}(x)\geq\alpha),
\end{aligned}\tag{2.30}$$

which implies (2.22) since $|R^*(x)|\leq 3\,\max_{1\leq i,j\leq 3}|R^*_{ij}(x)|$. $\square$

3. Unbounded homogenized Hall coefficients. This section is devoted to three examples of three-dimensional microstructures which have unbounded effective Hall coefficients due to the presence of low or/and high conductivity regions.

3.1. A battery of Hall materials. In this Section, $\Omega = \omega \times (0,1)$ is an open bounded cylinder of height 1 and of cross section $\omega \subset \mathbb{R}^2$.

THEOREM 3.1. *There exists a microstructure in Ω depending on the positive integer n, with an εY_3-periodic conductivity $\Sigma^{\varepsilon,n}$, and a εY_3-periodic isotropic Hall matrix $R^{\varepsilon,n} := r_{\varepsilon,n}\, I_3$, such that*

$$\Sigma^{\varepsilon,n} = I_3 \quad \textit{a.e. in } \Omega \cap \{r_{\varepsilon,n} \neq 0\}, \tag{3.1}$$

$$r_{\varepsilon,n} \in \{0,1\} \quad \textit{a.e. in } \Omega, \tag{3.2}$$

Moreover, the homogenized conductivity matrix $\Sigma^{*,n}$ and the homogenized Hall matrix $R^{*,n}$ are constant and satisfy

$$\lim_{n\to\infty} |R^{*,n}| = \infty. \tag{3.3}$$

REMARK 3.2. The assumptions (2.20) and (2.21) of Theorem 2.9 hold in Theorem 3.1 with $\beta_{\rm H} = 1$ and $r_{\rm H} = 1$, thanks to conditions (3.1) and (3.2). However, the sequence $\Sigma^{*,n}$ is not bounded from below by $\alpha\, I_3$, where α is a constant independent of n. So, the arbitrarily large homogenized Hall coefficients, implied by (3.3), are not forbidden by Theorem 2.9. As a consequence, by virtue of (3.3) we can obtain arbitrarily large homogenized Hall coefficients.

3.1.1. Description of the microstructure. We consider in Ω a εY_3-periodic columnar microstructure aligned along the x_3-axis, the cross section of which is represented in figure 3.1. The period cell $Y_2 = (0,1)^2$ of the cross section of the microstructure is divided into three regions as shown in figure 3.2:

- The 1-conductivity (grey) region Q^n is composed of n rectangles ($n = 3$ in figure 3.2) $Q_1^n, \dots, Q_n^n$. Each rectangle Q_k^n has an x_1-length equal to $\frac{L_1}{n}$ and a x_2-length equal to L_2. The distance between two consecutive rectangles Q_k^n and Q_{k+1}^n is of order $\frac{1}{n}$.
- The high conductivity (black) region Q_s^n has n connected components in the two-dimensional torus which link successive pairs of the n rectangles $Q_1^n, \dots, Q_n^n$, and which link Q_n^n with Q_1^n across the boundary of the unit cell, as sketched in the figure.
- The low conductivity (white) region Q_w^n is the complementary of $Q^n \cup Q_s^n$ in Y_2. It contains the three rectangles R_0, R_1, R_2 the width of each of which is of order 1, as well as additional thin connecting regions, which serve to insulate the various components.

On the other hand, the conductivity of the microstructure is defined by the two parameters $\kappa \gg 1$ and $n \in \mathbb{N} \setminus \{0\}$ as follows. Let $\sigma_{\sharp,\kappa,n}$ be the Y_2-periodic function defined by its restriction on Y_2

$$\sigma_{\sharp,\kappa,n}(y) := \begin{cases} 1 & \text{if } y \in Q^n = Q_1^n \cup \cdots \cup Q_n^n \\ \kappa & \text{if } y \in Q_s^n \\ \dfrac{1}{n^3} & \text{if } y \in Q_w^n. \end{cases} \tag{3.4}$$

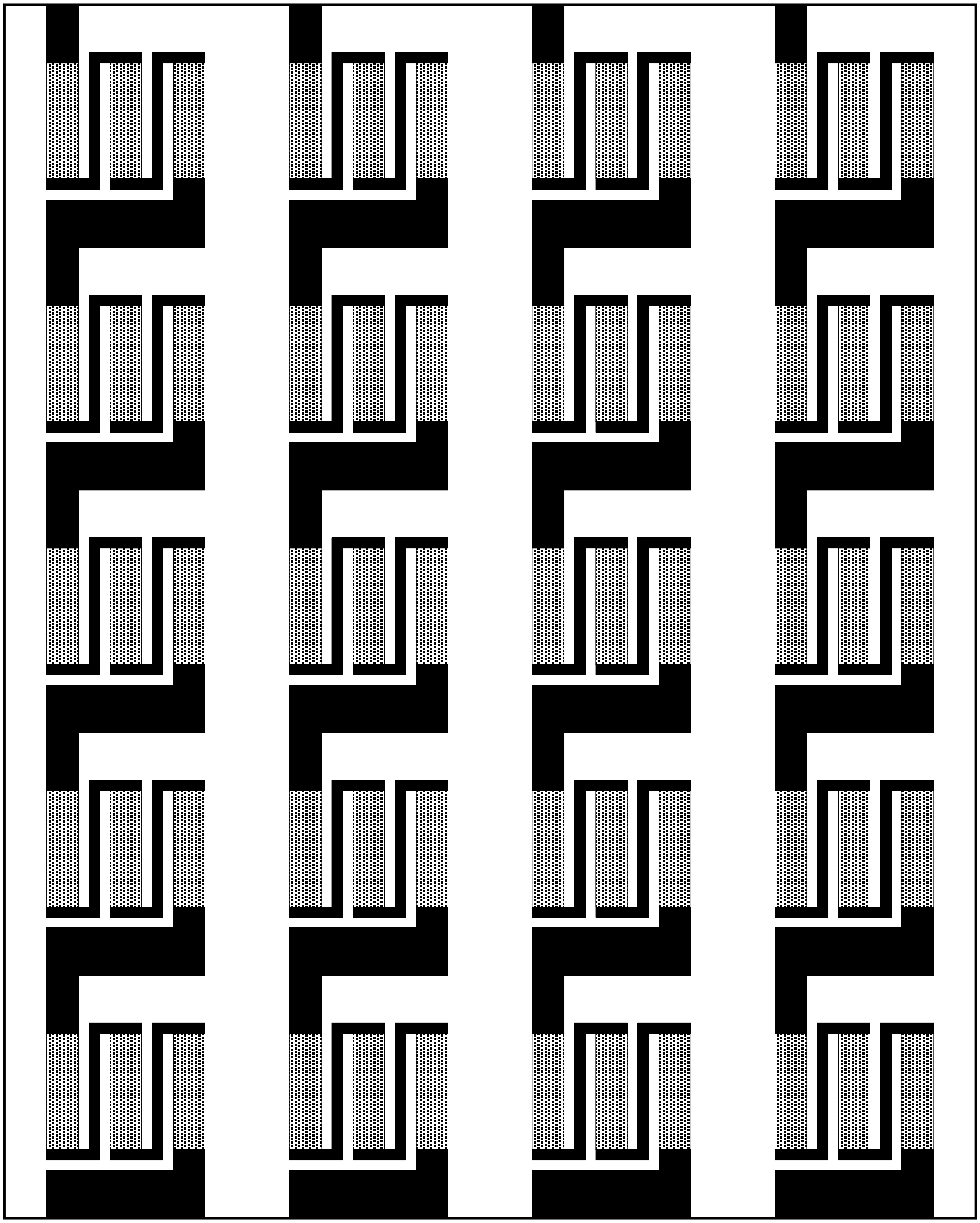

FIG. 3.1. *The cross section of the columnar microstructure*

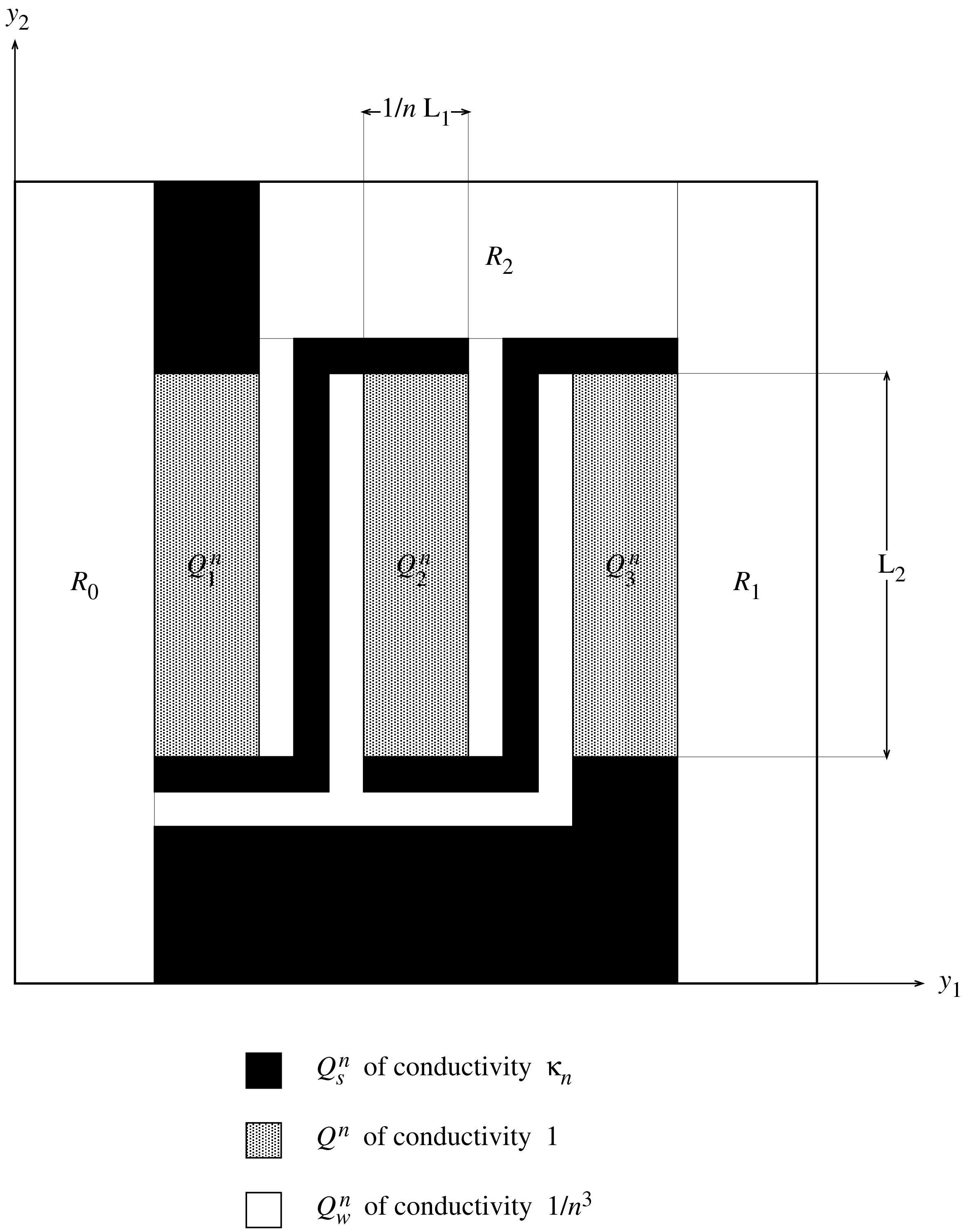

FIG. 3.2. *The cross section of the microstructure period cell*

Denote by $\chi : \mathbb{R} \to \mathbb{R}$ the 1-periodic function which agrees on the interval $(0,1)$ with the characteristic function of $(0, \frac{1}{2})$, and denote by $\chi_Q : \mathbb{R}^2 \to \mathbb{R}$ the Y_2-periodic function which agrees on Y_2 with the characteristic function of any set $Q \subset Y_2$. We consider the εY_3-periodic and anisotropic conductivity $\Sigma^{\varepsilon,\kappa,n}$ associated with the microstructure of figure 3.1 and defined for $x = (x', x_3) \in \Omega$, by

$$\Sigma^{\varepsilon,\kappa,n}(x) = \Sigma^{\sharp,\kappa,n}\left(\tfrac{x'}{\varepsilon}\right) := \begin{pmatrix} \sigma_{\sharp,\kappa,n} & 0 & 0 \\ 0 & \sigma_{\sharp,\kappa,n} & 0 \\ 0 & 0 & \chi_{Q_s^n}\, a + (1 - \chi_{Q_s^n})\, \sigma_{\sharp,\kappa,n} \end{pmatrix} \left(\tfrac{x'}{\varepsilon}\right). \tag{3.5}$$

where a is a constant which will be chosen later.

Remark 3.3. The conductivity $\Sigma^{\varepsilon,\kappa,n}$ is anisotropic only in the high conductivity (black) region to avoid a strong conductivity along the x_3-direction. We could have considered an isotropic conductivity by introducing in the black region a laminate along the x_3-direction at a smaller scale than ε. Using the reiterated homogenization for a multi-scale microstructure (see e.g. [2]), this two-scale isotropic conductivity does the same job as the anisotropic conductivity $\Sigma^{\varepsilon,\kappa,n}$. For the sake of clarity we directly start from the anisotropic one. By the periodic homogenization formula (2.7) and since $\Sigma^{\varepsilon,\kappa,n}$ is independent of x_3, the H-limit of the sequence $\Sigma^{\varepsilon,\kappa,n}$ is the constant matrix

$$\Sigma^{*,\kappa,n} := \begin{pmatrix} \Sigma_{11}^{*,\kappa,n} & \Sigma_{12}^{*,\kappa,n} & 0 \\ \Sigma_{21}^{*,\kappa,n} & \Sigma_{22}^{*,\kappa,n} & 0 \\ 0 & 0 & \Sigma_{33}^{*,\kappa,n} \end{pmatrix}, \tag{3.6}$$

where $\Sigma_{33}^{*,\kappa,n}$ is the average value of the Y_2-periodic function $\Sigma_{33}^{\sharp,\kappa,n}$, i.e.

$$\Sigma_{33}^{*,\kappa,n} = a\,|Q_s^n| + |Q^n| + \frac{1}{n^3}\,|Q_w^n|. \tag{3.7}$$

The entries $\Sigma_{ij}^{*,\kappa,n}$, $i,j = 1,2$, are given by

$$\begin{aligned} \Sigma_{ij}^{*,\kappa,n} := \int_{Y_2} \sigma_{\sharp,\kappa,n} \nabla u_i^{\kappa,n} \cdot \nabla u_j^{\kappa,n}\, dy \quad &= \int_{Y_2} \sigma_{\sharp,\kappa,n} \nabla u_i^{\kappa,n} \cdot e_j\, dy \\ &= \int_{Y_2} \sigma_{\sharp,\kappa,n}\, \frac{\partial u_i^{\kappa,n}}{\partial y_j}\, dy, \end{aligned} \tag{3.8}$$

where the functions $u_i^{\kappa,n}$, $i = 1,2$, solve the period cell problem

$$\begin{cases} \operatorname{div}\left(\sigma_{\sharp,\kappa,n} \nabla u_i^{\kappa,n}\right) = 0 & \text{in } \mathscr{D}'(\mathbb{R}^2) \\ u_i^{\kappa,n}(y) - y_i & \text{is } Y_2\text{-periodic.} \end{cases} \tag{3.9}$$

Finally, we consider in Ω the εY_3-periodic x_3-independent isotropic Hall matrix defined by

$$R^{\varepsilon,\kappa,n}(x) := r_{\varepsilon,\kappa,n}(x)\, I_3 \quad \text{where} \quad r_{\varepsilon,\kappa,n}(x) := \chi_{Q^n}\left(\tfrac{x'}{\varepsilon}\right), \quad \text{for } x = (x', x_3) \in \Omega. \tag{3.10}$$

Note that the local Hall coefficient $r_{\varepsilon,\kappa,n}$ is concentrated in the (grey) region of conductivity 1.

3.1.2. A physical approach. To simplify the physical understanding, let us slightly change the problem and suppose the isotropic white region has zero conductivity, and the black region has infinite transverse conductivity in the x_1-x_2 plane. Next suppose the average current is directed along the x_3 axis. Since the black and grey regions have finite conductivities a and 1 in this direction, a fixed portion of the total current will flow in the grey material in the direction x_3. If the magnetic field h is directed in the x_1 direction this current will induce a Hall voltage V_H (independent of n) in the x_2 direction across each grey region Q^n_k, for $k = 1, \ldots, n$. Thus, each of these grey regions will act in the transverse plane as a battery, and the black connecting regions serve to connect them in series. Since the black regions are perfectly conducting in the transverse direction there will be zero voltage drop in the transverse direction across each connected black segment. Therefore, the total voltage drop in the x_2 direction across the unit cell will be nV_H. This implies the Hall matrix coefficient R^*_{11} is proportional to n, which becomes unboundedly large as $n \to \infty$.

3.1.3. A mathematical proof. Taking into account the Example 2.6, the convergence (2.15) combined with the εY_3-periodicity of $\Sigma^{\varepsilon,\kappa,n}$ (3.5) and $R^{\varepsilon,\kappa,n}$ (3.10), imply that the homogenized Hall matrix $R^{*,\kappa,n}$ is given by

$$\operatorname{Cof}\left(\Sigma^{*,\kappa,n}\right) R^{*,\kappa,n} = \int_{Q^n} \operatorname{Cof}\left(\Sigma^{\sharp,\kappa,n} DU^{\kappa,n}\right)^T dy, \tag{3.11}$$

where $U^{\kappa,n} := (u_1^{\kappa,n}, u_2^{\kappa,n}, u_3^{\kappa,n})$ is defined by (3.9), and $u_3^{\kappa,n} = y_3$ (up to an additive constant) due to the y_3-independence of $\Sigma^{\sharp,\kappa,n}$ in (3.5). Since $\Sigma^{\sharp,\kappa,n} = I_3$ in Q^n and $U^{\kappa,n} := (u_1^{\kappa,n}, u_2^{\kappa,n}, y_3)$, we also have

$$R^{*,\kappa,n} = \frac{\Sigma^{*,\kappa,n}}{\det\left(\Sigma^{*,\kappa,n}\right)} \int_{Q^n} \operatorname{Cof}\left(DU^{\kappa,n}\right)^T dy, \tag{3.12}$$

where the transpose of the corrector Cofactor matrix reads as

$$\operatorname{Cof}\left(DU^{\kappa,n}\right)^T = \begin{pmatrix} \frac{\partial u_2^{\kappa,n}}{\partial y_2} & -\frac{\partial u_2^{\kappa,n}}{\partial y_1} & 0 \\ -\frac{\partial u_1^{\kappa,n}}{\partial y_2} & \frac{\partial u_1^{\kappa,n}}{\partial y_1} & 0 \\ 0 & 0 & \frac{\partial u_1^{\kappa,n}}{\partial y_1}\frac{\partial u_2^{\kappa,n}}{\partial y_2} - \frac{\partial u_1^{\kappa,n}}{\partial y_2}\frac{\partial u_2^{\kappa,n}}{\partial y_1} \end{pmatrix}. \tag{3.13}$$

From now on, we choose a in (3.7) in such a way that

$$\Sigma_{33}^{*,\kappa,n} = 1, \tag{3.14}$$

the coefficient a being then of order 1 as $n \to \infty$.

Let us focus on the Hall coefficient $R_{11}^{*,\kappa,n}$. The formula (3.12) combined with (3.14), (3.6) and (3.13), yields

$$R_{11}^{*,\kappa,n} = \frac{\Sigma_{11}^{*,\kappa,n} \displaystyle\int_{Q^n} \frac{\partial u_2^{\kappa,n}}{\partial y_2}\, dy - \Sigma_{12}^{*,\kappa,n} \int_{Q^n} \frac{\partial u_1^{\kappa,n}}{\partial y_2}\, dy}{\Sigma_{11}^{*,\kappa,n}\, \Sigma_{22}^{*,\kappa,n} - \left(\Sigma_{12}^{*,\kappa,n}\right)^2}. \tag{3.15}$$

We will estimate $R_{11}^{*,\kappa,n}$ passing successively to the limits $\kappa \to \infty$ and $n \to \infty$.

First step : Passage to the limit $\kappa \to \infty$.

For a fixed positive integer n and $i = 1, 2$, by equicoercivity of $\Sigma^{\kappa,n}$ the sequence $u_i^{\kappa,n}$ weakly converges, as $\kappa \to \infty$, in $H^1_{\mathrm{loc}}(\mathbb{R}^2)$ to the function u_i^n which solves the periodic cell problem

$$\left\{ \begin{aligned} \int_{Y_2} \sigma_{\sharp,n} \nabla u_i^n \cdot \nabla v \, dy = 0 \quad & \forall\, v \in H^1_\sharp(Y_2),\ \nabla v = 0 \text{ in } Q_s^n, \\ \nabla u_i^n = 0 \quad & \text{in } Q_s^n, \\ u_i^n(y) - y_i \quad & \text{is } Y_2\text{-periodic}, \end{aligned} \right. \tag{3.16}$$

where the conductivity $\sigma_{\sharp,n}$ is the Y_2-periodic function defined by

$$\sigma_{\sharp,n}(y) := \begin{cases} 1 & \text{if } y \in Q^n \\ \dfrac{1}{n^3} & \text{if } y \in Q_w^n. \end{cases} \tag{3.17}$$

By equation (3.9) and the weak convergence of $\nabla u_i^{\kappa,n}$ to ∇u_i^n in $L^2(Y_2)$, we also have

$$\int_{Y_2} \sigma_{\sharp,\kappa,n} \left|\nabla u_i^{\kappa,n} - \nabla u_i^n\right|^2 dy = \int_{Y_2} \sigma_{\sharp,n} \nabla u_i^n \cdot \nabla(u_i^n - u_i^{\kappa,n})\, dy \underset{\kappa\to\infty}{\longrightarrow} 0. \tag{3.18}$$

This combined with (3.6) yields for any $i, j = 1, 2$,

$$\Sigma_{ij}^{*,\kappa,n} = \int_{Y_2} \sigma_{\sharp,\kappa,n} \nabla u_i^{\kappa,n} \cdot \nabla u_j^{\kappa,n}\, dy \underset{\kappa\to\infty}{\longrightarrow} \Sigma_{ij}^{*,n} := \int_{Y_2} \sigma_{\sharp,n}\, \nabla u_i^n \cdot \nabla u_j^n\, dy, \tag{3.19}$$

hence we deduce that the coefficient $R^{*,\kappa,n}$ of (3.12) satisfies the following limit

$$\lim_{\kappa\to\infty} R_{11}^{*,\kappa,n} = R_{11}^{*,n} := \frac{\Sigma_{11}^{*,n} \displaystyle\int_{Q^n} \frac{\partial u_2^n}{\partial y_2}\, dy - \Sigma_{12}^{*,n} \int_{Q^n} \frac{\partial u_1^n}{\partial y_2}\, dy}{\Sigma_{11}^{*,n}\, \Sigma_{22}^{*,n} - \left(\Sigma_{12}^{*,n}\right)^2}. \tag{3.20}$$

Second step : Estimate of $\Sigma_{ij}^{*,n}$, for $i, j = 1, 2$.

Let us start by $\Sigma_{11}^{*,n}$. By (3.16) the function u_i^n, $i = 1, 2$, solves the minimization problem

$$\begin{aligned} \Sigma_{ii}^{*,n} &= \int_{Y_2} \sigma_{\sharp,n}\, |\nabla u_i^n|^2\, dy \\ &= \min\left\{ \int_{Y_2} \sigma_{\sharp,n}\, |\nabla v|^2\, dy \ :\ v(y) - y_i \in H^1_\sharp(Y_2),\ \nabla v = 0 \text{ in } Q_s^n \right\}. \end{aligned} \tag{3.21}$$

Consider a test function $v_1 = v_1(y_1)$ in (3.21) with $v_1(1) = 1$, such that v_1 is equal to zero in $Y_2 \setminus R_1$ (see figure 3.2). Then, by the definition (3.17) of $\sigma_{\sharp,n}$ we have

$$\Sigma_{11}^{*,n} \leq \frac{1}{n^3} \int_{R_1} |\nabla v_1|^2\, dy \leq \frac{c_1}{n^3}. \tag{3.22}$$

Moreover, by the Cauchy-Schwarz inequality and the 1-periodicity of $(y_1 \mapsto u_1(y) - y_1)$, we get

$$\Sigma_{11}^{*,n} \geq \frac{1}{n^3} \int_{Y_2} \left(\frac{\partial u_1^n}{\partial y_1}\right)^2 dy \geq \frac{1}{n^3} \left(\int_{Y_2} \frac{\partial u_1^n}{\partial y_1}\, dy\right)^2 = \frac{1}{n^3}. \tag{3.23}$$

Therefore, there exists a constant $c_1 > 0$ such that

$$\frac{1}{n^3} \le \Sigma_{11}^{*,n} \le \frac{c_1}{n^3}. \tag{3.24}$$

The estimate of $\Sigma_{22}^{*,n}$ is more delicate. On the one hand, we can build a test function v_2^n of (3.21) satisfying the following properties (see figure 3.2):

$$\begin{cases} v_2^n(y_1,0)=0,\ v_2^n(y_1,1)=1, \\ v_2^n = 1-\frac{k}{n} \text{ in each connected component of } Q_s^n \text{ joining } Q_k^n \text{ and } Q_{k+1}^n, \\ \nabla v_2^n = \frac{1}{nL_2}\, e_2 \text{ in } Q^n, \text{ i.e. } v_2^n \text{ is an affine function of } y_2 \text{ in each set } Q_k^n, \\ |\nabla v_2^n| \text{ is bounded in } Y_2 \text{ by a constant independent of } n. \end{cases} \tag{3.25}$$

To deduce the fourth property of (3.25) from the values of v_2^n given by the first three ones, we first make an interpolation in the rectangles between two consecutive sets Q_k^n and Q_{k+1}^n, the width of these rectangles being of order $\frac{1}{n}$. Then, we make a second interpolation in the (white) rectangles R_0, R_1, R_2 the width of which is of order 1 (see figure 3.2).

Putting the test function v_2^n in (3.21) we get by the third and fourth properties of (3.25),

$$\begin{aligned} \Sigma_{22}^{*,n} \le \int_{Y_2} \sigma_{\sharp,n}\, |\nabla v_2^n|^2\, dy &= \int_{Q^n} |\nabla v_2^n|^2\, dy + \frac{1}{n^3}\int_{Q_w^n} |\nabla v_2^n|^2\, dy \\ &= \frac{L_1}{n^2 L_2} + O\Big(\frac{1}{n^3}\Big). \end{aligned} \tag{3.26}$$

On the other hand, let $c_0^{n,i}$ be the value of u_i^n in the (black) upper connected component of Q_s^n in figure 3.2, let $c_n^{n,i}$ be the one in the lower connected component, and let $c_k^{n,i}$ be the value of the connected component of Q_s^n which joins two consecutive sets Q_k^n and Q_{k+1}^n, for $i = 1,2$ and $k = 1,\dots,n-1$. Then, by the Cauchy-Schwarz inequality we have

$$\Sigma_{22}^{*,n} \ge \int_{Q^n} |\nabla u_2^n|^2\, dy \ge \int_{Q_n} \left(\frac{\partial u_2^n}{\partial y_2}\right)^2 dy \ge \frac{1}{|Q^n|}\left(\int_{Q^n} \frac{\partial u_2^n}{\partial y_2}\, dy\right)^2. \tag{3.27}$$

Moreover, since $c_0^{n,2} - c_n^{n,2} = 1$ by the 1-periodicity of $(y_2 \mapsto u_2^n(y) - y_2)$, we obtain owing to an integration by parts

$$\frac{1}{|Q^n|}\left(\int_{Q^n} \frac{\partial u_2^n}{\partial y_2}\, dy\right)^2 = \frac{1}{L_1 L_2}\left(\frac{L_1}{n}\sum_{k=1}^{n}\big(c_{k-1}^{n,2} - c_k^{n,2}\big)\right)^2 = \frac{L_1}{n^2 L_2}. \tag{3.28}$$

This combined with (3.27) and (3.26) yields

$$\frac{L_1}{n^2 L_2} \le \int_{Q^n} |\nabla u_2^n|^2\, dy \le \Sigma_{22}^{*,n} \le \frac{L_1}{n^2 L_2} + O\Big(\frac{1}{n^3}\Big), \tag{3.29}$$

which implies that

$$\begin{aligned} \Sigma_{22}^{*,n} &= \frac{L_1}{n^2 L_2} + O\Big(\frac{1}{n^3}\Big), \\ \frac{1}{n^3}\int_{Q_w^n} |\nabla u_2^n|^2\, dy &= \Sigma_{22}^{*,n} - \int_{Q^n} |\nabla u_2^n|^2\, dy = O\Big(\frac{1}{n^3}\Big). \end{aligned} \tag{3.30}$$

Moreover, using successively (3.30), (3.25), the first property of (3.16) with $v := v_2^n - u_2^n$, and again (3.30) we get

$$\begin{aligned}
&\int_{Q^n} |\nabla u_2^n - \nabla v_2^n|^2\,dy \\
&= \int_{Q^n} |\nabla u_2^n|^2\,dy + \int_{Q^n} |\nabla v_2^n|^2\,dy - 2\int_{Q^n} \nabla u_2^n \cdot \nabla v_2^n\,dy \\
&= \frac{2\,L_1}{n^2 L_2} + O\Big(\frac{1}{n^3}\Big) - 2\int_{Y_2} \sigma_{\sharp,n}\nabla u_2^n \cdot \nabla v_2^n\,dy + O\Big(\frac{1}{n^3}\Big) \\
&= \frac{2\,L_1}{n^2 L_2} - 2\int_{Q^n} \sigma_{\sharp,n}\nabla u_2^n \cdot \nabla u_2^n\,dy + O\Big(\frac{1}{n^3}\Big) \\
&= \frac{2\,L_1}{n^2 L_2} - 2\,\Sigma_{22}^{*,n} + O\Big(\frac{1}{n^3}\Big) = O\Big(\frac{1}{n^3}\Big).
\end{aligned} \tag{3.31}$$

Finally, let us estimate $\Sigma_{12}^{*,n}$. By definition (3.19), (3.24) and the second estimate of (3.30) we have

$$\begin{aligned}
\Sigma_{12}^{*,n} &= \int_{Y_2} \sigma_{\sharp,n}\nabla u_1^n \cdot \nabla u_2^n\,dy \\
&= \int_{Q^n} \nabla u_1^n \cdot \nabla u_2^n\,dy + O\Big(\frac{1}{n^3}\Big) \\
&= \int_{Q^n} \nabla u_1^n \cdot \nabla v_2^n\,dy + \int_{Q^n} \nabla u_1^n \cdot (\nabla u_2^n - \nabla v_2^n)\,dy + O\Big(\frac{1}{n^3}\Big).
\end{aligned} \tag{3.32}$$

On the one side, by the definition of the constants $c_k^{1,n}$ (defined after (3.26)) and the 1-periodicity of $(y_2 \mapsto u_1^n(y))$, it follows that

$$\begin{aligned}
\int_{Q^n} \nabla u_1^n \cdot \nabla v_2^n\,dy = \frac{1}{nL_2}\sum_{k=1}^{n}\int_{Q_k^n}\frac{\partial u_1^n}{\partial y_2}\,dy &= \frac{L_1}{n^2 L_2}\sum_{k=1}^{n}\big(c_{k-1}^{n,1} - c_k^{n,1}\big) \\
&= \frac{L_1}{n^2 L_2}\big(c_0^{n,1} - c_n^{n,1}\big) = 0.
\end{aligned} \tag{3.33}$$

On the other side, the Cauchy-Schwarz inequality combined with estimates (3.24) and (3.31) yields

$$\begin{aligned}
&\left|\int_{Q^n} \nabla u_1^n \cdot (\nabla u_2^n - \nabla v_2^n)\,dy\right| \\
&\le \big(\Sigma_{11}^{*,n}\big)^{\frac{1}{2}} \left(\int_{Q^n} |\nabla u_2^n - \nabla v_2^n|^2\,dy\right)^{\frac{1}{2}} = O\Big(\frac{1}{n^3}\Big).
\end{aligned} \tag{3.34}$$

Therefore, from (3.32), (3.33) and (3.34) we deduce that

$$\Sigma_{12}^{*,n} = O\Big(\frac{1}{n^3}\Big). \tag{3.35}$$

Third step : Estimate of the homogenized Hall coefficient $R_{11}^{*,n}$.

First of all, by proceeding as in (3.27) and (3.24) we obtain

$$\int_{Q^n} \frac{\partial u_i^n}{\partial y_2}\, dy = \frac{L_1}{n} \sum_{k=1}^{n} \left(c_{k-1}^{n,i} - c_k^{n,i}\right) = \frac{L_1}{n} \left(c_0^{n,i} - c_n^{n,i}\right) = \begin{cases} 0 & \text{if } i = 1 \\ \dfrac{L_1}{n} & \text{if } i = 2. \end{cases} \tag{3.36}$$

This combined with estimates (3.24), (3.30) and (3.35) implies that the sequence $R_{11}^{*,n}$ defined by (3.20) satisfies

$$\begin{aligned} R_{11}^{*,n} &= \frac{\Sigma_{11}^{*,n}}{\Sigma_{11}^{*,n}\Sigma_{22}^{*,n} - \left(\Sigma_{12}^{*,n}\right)^2} \int_{Q^n} \frac{\partial u_2^n}{\partial y_2}\, dy \\ &\underset{n\to\infty}{\approx} \frac{1}{\Sigma_{22}^{*,n}} \int_{Q^n} \frac{\partial u_2^n}{\partial y_2}\, dy \underset{n\to\infty}{\approx} \frac{n^2 L_2}{L_1}\,\frac{L_1}{n} = n\, L_2. \end{aligned} \tag{3.37}$$

Fourth step : Proof of (3.3).

By virtue of the limit (3.20) combined with the estimate (3.37), there exists a sequence κ_n such that

$$R_{11}^{*,\kappa_n,n} \geq \frac{1}{2}\, n\, L_2, \quad \text{for any large enough } n. \tag{3.38}$$

Finally, we consider the microstructure of figure 3.1 associated with the local conductivity matrix $\Sigma^{\varepsilon,n} := \Sigma^{\varepsilon,\kappa_n,n}$ of (3.5) and the local Hall coefficient $r_{\varepsilon,n} := r_{\varepsilon,\kappa_n,n}$ of (3.10). Therefore, (3.38) implies that the homogenized Hall matrix $R^{*,n} := R^{*,\kappa_n,n}$ of (3.12) satisfies the desired result (3.3).

3.2. An isotropic laminate with a low effective conductivity. The microstructure we have just studied achieves a large Hall effect by combining transverse Hall voltages like batteries in series. By contrast, the laminate materials analyzed in this subsection and the subsequent subsection achieve large Hall voltages through large local electrical currents flowing through a small volume of material having a Hall coefficient of order 1, and conductivity much greater than the surrounding material.

We have the following result:

THEOREM 3.4. *There exists a laminate microstructure associated with an isotropic conductivity $\Sigma^{\varepsilon,t}$ depending on a small parameter $t > 0$, and an isotropic Hall matrix $R^\varepsilon = r_{\varepsilon,t}\, I_3$, with $r_{\varepsilon,t} \in \{0, 1\}$, such that the homogenized conductivity $\Sigma^{*,t} = \sigma_{*,t}\, I_3$ and the homogenized Hall matrix $R^{*,t} = r_{*,t}\, I_3$ are isotropic, with*

$$\lim_{t\to 0} \sigma_{*,t} = 0 \quad \textit{and} \quad \lim_{t\to 0} r_{*,t} = \infty. \tag{3.39}$$

Proof. The proof of Theorem 3.4 is based on a two-phase rank-one laminate in the direction $\xi \in \{e_1, e_2, e_3\}$, where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$. Let Σ_1, Σ_2 be the conductivity phases of the laminate with the respective volume fractions $\theta, 1-\theta$. By virtue of [12] and [7] there exists an associated corrector (electric field) which has the same laminate structure with components P_1, P_2 in the two phases satisfying the relation

$$P_1 = M P_2 \quad \text{where} \quad M := I_3 + \frac{1}{\Sigma_1 \xi \cdot \xi}\, (\xi \otimes \xi)\, (\Sigma_2 - \Sigma_1)\,. \tag{3.40}$$

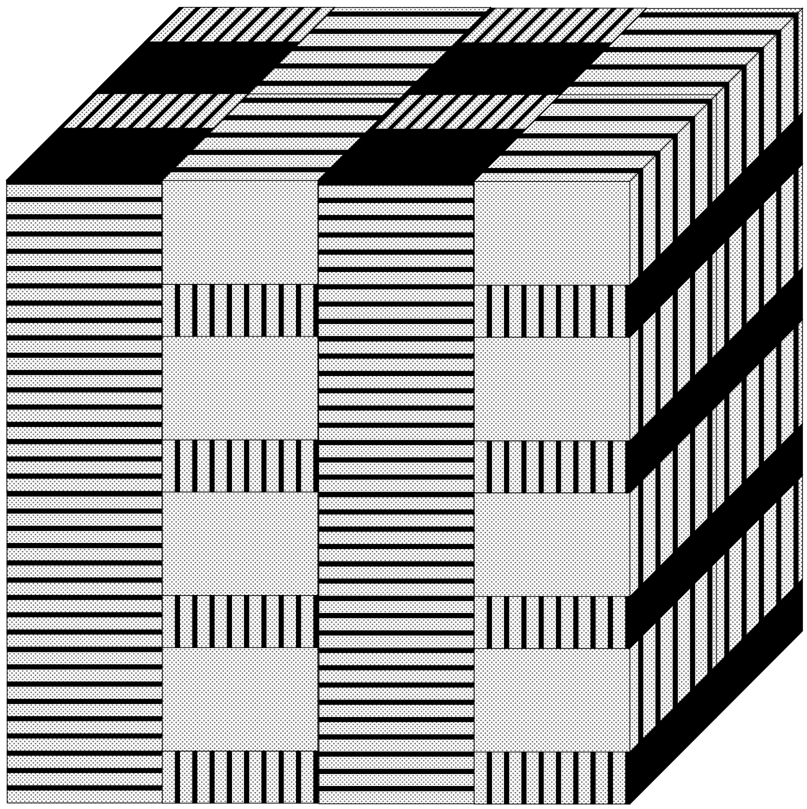

Fig. 3.3. *The Schulgasser laminate*

Moreover, the effective conductivity matrix of the laminate is given by

$$\Sigma^* := \Sigma_1 + (1-\theta)\,(\Sigma_2 - \Sigma_1)\,N^{-1} \quad \text{where} \quad N := I_3 + \frac{\theta}{\Sigma_1\xi\cdot\xi}\,(\xi\otimes\xi)\,(\Sigma_2-\Sigma_1)\,, \tag{3.41}$$

in such a way that

$$\theta\,\Sigma_1 P_1 + (1-\theta)\,\Sigma_2 P_2 = \Sigma^*\bar{P} \quad \text{where} \quad \bar{P} := \theta\,P_1 + (1-\theta)\,P_2. \tag{3.42}$$

To obtain an isotropic composite we will apply the Schulgasser construction [16] to a two-phase rank-one laminate of the previous type, considered as a single crystal with the effective conductivity matrix (3.41). The construction will provide a rank-three laminate based on 4 elementary two-phase rank-one laminates (that is to say 8 layers at the smallest scale) as shown in figure 3.3. Following the principle stated in [12] and proved in [7] there exists a corrector which is constant in each layer of the laminate. This combined with (2.15) will allow us to derive an explicit formula for the effective Hall matrix.

First step: construction of 4 elementary two-phase rank-one laminates.
We start from a two-phase rank-one laminate in the direction e_1 composed of the hard (high conducting) phase $\Sigma_1 := I_3$ and the soft (low conducting) phase $\Sigma_2 := t\, I_3$, with the respective volume fractions $\theta := t,\, 1-t$, where $t \ll 1$. So, the soft phase occupies the major part of the volume. The effective conductivity matrix of the laminate is given by

$$\Sigma_{11}^* := I_3 - (1-t)^2 \left[I_3 + t\,(t-1)\,(e_1 \otimes e_1) \right]^{-1} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}, \tag{3.43}$$

$$\text{where} \qquad \lambda := \frac{t}{1-t+t^2} \quad \text{and} \quad \mu := 2t - t^2. \tag{3.44}$$

Thanks to (3.40) the correctors P_1^h (h for the hard phase) and P_1^s (s for the soft phase) associated with this first rank-one laminate satisfy

$$P_1^h = M_{11}\, P_1^s \quad \text{where} \quad M_{11} := I_3 + (t-1)\,(e_1 \otimes e_1)\,. \tag{3.45}$$

The second rank-one laminate is similar but in the direction e_2, with the effective conductivity matrix

$$\Sigma_{12}^* := I_3 - (1-t)^2 \left[I_3 + t\,(t-1)\,(e_2 \otimes e_2) \right]^{-1} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}, \tag{3.46}$$

and the associated correctors P_2^h, P_2^s satisfying

$$P_2^h = M_{12}\, P_2^s \quad \text{where} \quad M_{12} := I_3 + (t-1)\,(e_2 \otimes e_2)\,. \tag{3.47}$$

The third rank-one laminate is a copy of the first one with the effective conductivity matrix $\Sigma_{13}^* := \Sigma_{11}^*$ and the associated correctors P_3^h, P_3^s satisfying

$$P_3^h = M_{11}\, P_3^s. \tag{3.48}$$

The fourth rank-one laminate is similar to the first one in the direction e_3, with the effective conductivity matrix

$$\Sigma_{14}^* := I_3 - (1-t)^2 \left[I_3 + t\,(t-1)\,(e_3 \otimes e_3) \right]^{-1} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \tag{3.49}$$

and the associated correctors P_4^h, P_4^s satisfying

$$P_4^h = M_{14}\, P_4^s \quad \text{where} \quad M_{14} := I_3 + (t-1)\,(e_3 \otimes e_3)\,. \tag{3.50}$$

Second step: construction of two rank-two laminates with the composite phases of the first step.

The first rank-two laminate is a rank-one laminate in the direction e_3 of the composite phases Σ^*_{11}, Σ^*_{12}, with the respective volume fractions $\frac{1}{3}$, $\frac{2}{3}$. By (3.41) the effective conductivity of this laminate is given by

$$\begin{aligned}\Sigma^*_1 \quad &:= \Sigma^*_{11} + \frac{2}{3}\,(\Sigma^*_{12} - \Sigma^*_{11})\left[I_3 + \frac{1}{3\,\Sigma^*_{11} e_3\cdot e_3}\,(e_3\otimes e_3)\,(\Sigma^*_{12} - \Sigma^*_{11})\right]^{-1} \\ &= \begin{pmatrix} \frac{\lambda+2\mu}{3} & 0 & 0 \\ 0 & \frac{2\lambda+\mu}{3} & 0 \\ 0 & 0 & \mu \end{pmatrix}.\end{aligned} \tag{3.51}$$

We set

$$\bar{P}_i := t\,P^h_i + (1-t)\,P^s_i, \quad \text{for } i = 1,\ldots,4. \tag{3.52}$$

By [12] and [7] this rank-two laminate can be regarded as a two-phase rank-one laminate the correctors of which are (averaging at the smallest scale of the rank-three laminate) $\bar{P}_1$, $\bar{P}_2$ satisfying

$$\bar{P}_1 = M_1\,\bar{P}_2 \quad \text{where} \quad M_1 := I_3 + \frac{1}{\Sigma^*_{11} e_3\cdot e_3}\,(e_3\otimes e_3)\,(\Sigma^*_{12} - \Sigma^*_{11}) = I_3. \tag{3.53}$$

The second rank-two laminate is a rank-one laminate in the direction e_2 of the composite phases Σ^*_{13}, Σ^*_{14}, with the respective volume fractions $\frac{1}{3}$, $\frac{2}{3}$. By (3.41) and by the equality $\Sigma^*_{13} = \Sigma^*_{11}$ its effective conductivity is given by

$$\begin{aligned}\Sigma^*_2 \quad &:= \Sigma^*_{11} + \frac{2}{3}\,(\Sigma^*_{14} - \Sigma^*_{11})\left[I_3 + \frac{1}{3\,\Sigma^*_{11} e_2\cdot e_2}\,(e_2\otimes e_2)\,(\Sigma^*_{14} - \Sigma^*_{11})\right]^{-1} \\ &= \begin{pmatrix} \frac{\lambda+2\mu}{3} & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \frac{2\lambda+\mu}{3} \end{pmatrix},\end{aligned} \tag{3.54}$$

and the associated correctors $\bar{P}_3$, $\bar{P}_4$ satisfy

$$\bar{P}_3 = M_2\,\bar{P}_4 \quad \text{where} \quad M_2 := I_3 + \frac{1}{\Sigma^*_{11} e_2\cdot e_2}\,(e_2\otimes e_2)\,(\Sigma^*_{14} - \Sigma^*_{11}) = I_3. \tag{3.55}$$

Third step: construction of the isotropic rank-three laminate.
The final composite is a two-phase rank-one laminate in the direction e_1 of the composite phases Σ^*_1, Σ^*_2 given by the second step, with the volume fractions $\frac{1}{2}$, $\frac{1}{2}$. By (3.41) its effective conductivity is the isotropic matrix

$$\begin{aligned}\Sigma^{*,t} = \sigma_{*,t}\,I_3 \quad &:= \Sigma^*_1 + \frac{1}{2}\,(\Sigma^*_2 - \Sigma^*_1)\left[I_3 + \frac{1}{2\,\Sigma^*_1 e_1\cdot e_1}\,(e_1\otimes e_1)\,(\Sigma^*_2 - \Sigma^*_1)\right]^{-1} \\ &= \left(\frac{\lambda+2\mu}{3}\right) I_3.\end{aligned} \tag{3.56}$$

We set with (3.52)

$$\bar{\bar{P}}_1 := \frac{1}{3}\,\bar{P}_1 + \frac{2}{3}\,\bar{P}_2 \quad \text{and} \quad \bar{\bar{P}}_2 := \frac{1}{3}\,\bar{P}_3 + \frac{2}{3}\,\bar{P}_4. \tag{3.57}$$

By [12] and [7] this isotropic composite can be considered as a two-phase rank-one laminate the correctors of which are (averaging at the meso-scale of the rank-three laminate) $\bar{\bar{P}}_1$, $\bar{\bar{P}}_2$ satisfying

$$\bar{\bar{P}}_1 = M\,\bar{\bar{P}}_2 \quad \text{where} \quad M := I_3 + \frac{1}{\Sigma_1^* e_1 \cdot e_1}\,(e_1 \otimes e_1)\,(\Sigma_2^* - \Sigma_1^*) = I_3. \tag{3.58}$$

Moreover, by the first convergence of (2.9) the whole average of the rank-three laminate correctors is equal to I_3, hence

$$\frac{1}{2}\,\bar{\bar{P}}_1 + \frac{1}{2}\,\bar{\bar{P}}_2 = I_3. \tag{3.59}$$

This combined with (3.58) yields

$$\bar{\bar{P}}_1 = \bar{\bar{P}}_2 = I_3. \tag{3.60}$$

Therefore, by combining (3.45), (3.47), (3.48), (3.50), (3.52), (3.53), (3.55), and (3.60) we obtain successively the following explicit formulas for the 8 elementary correctors P_i^h, P_i^s, $i = 1, \dots 4$:

$$\begin{cases} P_4^s = \left(t\,M_{14} + (1-t)\,I_3\right)^{-1}, & P_4^h = M_{14}\,P_4^s \\ P_3^s = \left(t\,M_{11} + (1-t)\,I_3\right)^{-1}, & P_3^h = M_{11}\,P_3^s \\ P_2^s = \left(t\,M_{12} + (1-t)\,I_3\right)^{-1}, & P_2^h = M_{12}\,P_2^s \\ P_1^s = \left(t\,M_{11} + (1-t)\,I_3\right)^{-1}, & P_1^h = M_{11}\,P_1^s. \end{cases} \tag{3.61}$$

Now, we consider the isotropic local Hall matrix $R^{\varepsilon,t} = r_{\varepsilon,t}\,I_3$ defined by

$$r_{\varepsilon,t} := \begin{cases} 1 & \text{in the layers of the corrector } P_1^h \text{ of volume fraction } \frac{t}{6} \\ 1 & \text{in the layers of the corrector } P_2^h \text{ of volume fraction } \frac{t}{3} \\ 1 & \text{in the layers of the corrector } P_3^h \text{ of volume fraction } \frac{t}{6} \\ 1 & \text{in the layers of the corrector } P_4^h \text{ of volume fraction } \frac{t}{3} \\ 0 & \text{elsewhere.} \end{cases} \tag{3.62}$$

Then, since the conductivity is equal to 1 in the layers of the correctors P_i^h, the formula (2.15) giving the effective Hall matrix $R^{*,t}$ reads as

$$\mathrm{Cof}\left(\Sigma^{*,t}\right) R^{*,t} = \frac{t}{6}\,\mathrm{Cof}\left(P_1^h\right)^T + \frac{t}{3}\,\mathrm{Cof}\left(P_2^h\right)^T + \frac{t}{6} \times \mathrm{Cof}\left(P_3^h\right)^T + \frac{t}{3}\,\mathrm{Cof}\left(P_4^h\right)^T. \tag{3.63}$$

Therefore, taking into account that the correctors are diagonal matrices we obtain the following effective Hall matrix

$$R^{*,t} = \frac{\Sigma^{*,t}}{\det\left(\Sigma^{*,t}\right)}\left[\frac{t}{6}\,\mathrm{Cof}\left(P_1^h\right) + \frac{t}{3}\,\mathrm{Cof}\left(P_2^h\right) + \frac{t}{6}\,\mathrm{Cof}\left(P_3^h\right) + \frac{t}{3}\,\mathrm{Cof}\left(P_4^h\right)\right]. \tag{3.64}$$

By (3.56) and (3.44) we have

$$\sigma_{*,t} = \frac{t\left(5 - 6t + 6t^2 - 2t^3\right)}{3\left(1 - t + t^2\right)} = \frac{5}{3}\,t + O\left(t^3\right). \tag{3.65}$$

Moreover, using Maple the formulas (3.61) and (3.64) combined with (3.56) imply that

$$R^{*,t} = r_{*,t}\, I_3 \quad \text{where} \quad r_{*,t} := \frac{3\,(1+t+t^2)\,(1-t+t^2)}{t\,(5-6t+6t^2-2t^3)^2} = \frac{3}{25}\, t^{-1}\big(1+O(t)\big). \tag{3.66}$$

The previous results show that the effective conductivity is isotropic and tends to zero as t, and that the effective Hall matrix is isotropic and blows up as t^{-1}. The proof of Theorem 3.4 is thus done. □

3.3. An isotropic laminate with an effective conductivity of order 1. In the two previous examples (subsection 3.1 and subsection 3.2) the effective conductivity is not bounded from below and only the assumption (2.19) of Theorem 2.9 does not hold. By contrast in the following example the condition (2.19) is satisfied but not the condition (2.20). We obtain a composite with an isotropic effective conductivity of order 1, while the effective Hall matrix is asymptotically isotropic and unbounded with respect to a small parameter.

Theorem 3.5. *There exists a laminate associated with an isotropic conductivity $\Sigma^{\varepsilon,t} \geq I_3$ depending on a small parameter $t > 0$, and an isotropic Hall matrix $R^{\varepsilon,t} = r_{\varepsilon,t}\, I_3$, with $r_{\varepsilon,t} \in \{0,1\}$, such that the homogenized conductivity $\Sigma^{*,t} = \sigma_{*,t}\, I_3$ is isotropic and the homogenized Hall matrix $R^{*,t} = r_{*,t}\big(I_3 + o(1)\big)$ is asymptotically isotropic, with*

$$\lim_{t\to 0} \sigma_{*,t} = \frac{4}{3} \quad \textit{and} \quad \lim_{t\to 0} r_{*,t} = \infty. \tag{3.67}$$

Proof. The microstructure is still based on the isotropic Schulgasser construction. However, the role of the single crystal is now played by a rank-three laminate defined as follows (this role is played by the rank-one laminate with the effective conductivity (3.43) in the Schulgasser construction of subsection 3.2):

- At the first level of lamination we construct a rank-one laminate in the direction $e_1 := (1,0,0)$ built from two phases with conductivities $t^{-1}I_3, I_3$ with the respective volume fractions $t, 1-t$. Its effective conductivity matrix is given by

$$\Sigma^*_{11} := \begin{pmatrix} \frac{1}{1-t+t^2} & 0 & 0 \\ 0 & 2-t & 0 \\ 0 & 0 & 2-t \end{pmatrix}. \tag{3.68}$$

- At the second level of lamination we construct a rank-one laminate in the direction $e_2 := (0,1,0)$ composed by the two phases Σ^*_{11}, I_3 with the respective volume fractions $\frac{4}{5}, \frac{1}{5}$. Its effective conductivity matrix is given by

$$\Sigma^*_{12} := \begin{pmatrix} \frac{5-t+t^2}{5\,(1-t+t^2)} & 0 & 0 \\ 0 & \frac{5\,(2-t)}{6-t} & 0 \\ 0 & 0 & \frac{9-4t}{5} \end{pmatrix}. \tag{3.69}$$

- At the third level of lamination we construct a rank-one laminate in the direction $e_3 := (0,0,1)$ composed by the two phases Σ^*_{12}, I_3 with the respective volume fractions $\frac{3}{4}, \frac{1}{4}$. Therefore, the effective conductivity of the rank-three laminate is given by

$$\Sigma_1^{*,t} := \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix} \quad \text{where} \quad \lambda := \frac{5-2t+2t^2}{5\,(1-t+t^2)} \quad \text{and} \quad \mu := \frac{9-4t}{6-t}, \tag{3.70}$$

Note that the effective matrix $\Sigma_1^{*,t}$ is similar to the one of (3.43). But this time $\Sigma_1^{*,t}$ is of order 1 with respect to t since the hard phase of conductivity t^{-1} is surrounded by the three soft phases of conductivity 1 in this rank-three laminate. Also note that the specific choice of the prescribed volume fractions $\frac{4}{5}, \frac{3}{4}$ allows us to obtain an effective matrix with two equal eigenvalues and so to simplify a lot the next step.

Now, we proceed to the Schulgasser construction from the previous four-phase rank-three laminate regarded as a three-dimensional single crystal with conductivity (3.70). On the one hand, this leads us to a rank-five laminate (i.e. with five ordered micro-scales) composed of $4 \times 4 = 16$ elementary layers (there are $2 \times 4 = 8$ elementary layers in the laminate of subsection 3.2 since the starting crystal is a two-phase rank-one laminate). Resting on the lamination formulas (3.41) and (3.42) we obtain 16 associated correctors. The computations are quite similar to the ones of subsection 3.2, hence we do not give the details. In particular, using Maple we get that the correctors P_1, P_2, P_3, P_4 associated with the 4 hard layers of conductivity t^{-1}, are given by

$$P_1 = P_3 := \begin{pmatrix} \nu_1 & 0 & 0 \\ 0 & \nu_2 & 0 \\ 0 & 0 & \nu_3 \end{pmatrix}, \quad P_2 := \begin{pmatrix} \nu_2 & 0 & 0 \\ 0 & \nu_1 & 0 \\ 0 & 0 & \nu_3 \end{pmatrix}, \quad P_4 := \begin{pmatrix} \nu_3 & 0 & 0 \\ 0 & \nu_2 & 0 \\ 0 & 0 & \nu_1 \end{pmatrix}, \tag{3.71}$$

$$\text{where} \quad \nu_1 := \frac{20t}{29-20t+20t^2}, \quad \nu_2 := \frac{20}{38-9t} \quad \text{and} \quad \nu_3 := \frac{20}{33-4t}. \tag{3.72}$$

Moreover, the homogenized conductivity of this rank-five laminate is the isotropic matrix

$$\Sigma^{*,t} = s_{*,t}\, I_3 \qquad \text{where} \qquad s_{*,t} := \frac{40-49t+48t^2-14t^3}{5\,(6-t)\,(1-t+t^2)} = \frac{4}{3} + O(t), \tag{3.73}$$

which establishes the first part of (3.67).

On the other hand, we consider the local Hall coefficient $r_{\varepsilon,t}$ defined by (compare to (3.62))

$$r_{\varepsilon,t} := \begin{cases} 1 & \text{in the layers of } P_1 \text{ of volume fraction } \frac{4}{5}\,\frac{3}{4}\,\frac{t}{6} = \frac{t}{10} \\ 1 & \text{in the layers of } P_2 \text{ of volume fraction } \frac{4}{5}\,\frac{3}{4}\,\frac{t}{3} = \frac{t}{5} \\ 1 & \text{in the layers of } P_3 \text{ of volume fraction } \frac{4}{5}\,\frac{3}{4}\,\frac{t}{6} = \frac{t}{10} \\ 1 & \text{in the layers of } P_4 \text{ of volume fraction } \frac{4}{5}\,\frac{3}{4}\,\frac{t}{3} = \frac{t}{5} \\ 0 & \text{elsewhere.} \end{cases} \tag{3.74}$$

Then, since the conductivity is equal to t^{-1} in the layers of the correctors P_i, the formula (2.15) giving the effective Hall matrix $R^{*,t}$ reads as

$$\begin{aligned} \operatorname{Cof}\left(\Sigma^{*,t}\right) R^{*,t} &= \frac{t}{10} \operatorname{Cof}\left(t^{-1}P_1\right)^T + \frac{t}{5} \operatorname{Cof}\left(t^{-1}P_2\right)^T \\ &+ \frac{t}{10} \operatorname{Cof}\left(t^{-1}P_3\right)^T + \frac{t}{5} \operatorname{Cof}\left(t^{-1}P_4\right)^T, \end{aligned} \tag{3.75}$$

or equivalently,

$$R^{*,t} = \frac{\Sigma^{*,t}}{\det\left(\Sigma^{*,t}\right)}\left[\frac{1}{10t}\mathrm{Cof}\big(P_1\big) + \frac{1}{5t}\mathrm{Cof}\big(P_2\big) + \frac{1}{10t}\mathrm{Cof}\big(P_3\big) + \frac{1}{5t}\mathrm{Cof}\big(P_4\big)\right]. \quad (3.76)$$

Therefore, the formula (3.76) combined with (3.73), (3.71) and (3.72) implies that $R^{*,t}$ is diagonal with

$$R^{*,t}_{11} = \frac{R^{*,t}_{22} + R^{*,t}_{33}}{2}, \quad (3.77)$$

and satisfies the asymptotic expansion

$$R^{*,t} = \frac{15}{418}\, t^{-1}\left(I_3 + O(t)\right). \quad (3.78)$$

This yields the second part of (3.67) and concludes the proof. $\square$

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