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An antisymmetric effective Hall matrix

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Abstract

A periodic composite of four isotropic phases with an almost antisymmetric effective Hall matrix is constructed. This is done through the use of a microstructure, with high contrast conductivity, which acts to twist the direction of the induced Hall field. For an applied current along the y_3 axis and a magnetic field h in the y_1 - y_2 plane the Hall field will be essentially parallel, rather than perpendicular, to h . It is also shown how to obtain composites with an effective Hall matrix matching any given symmetric positive definite matrix.

Keywords: Hall effect, homogenization, Hall matrix

AMS Classification: 35B27, 74Q15

1 Introduction

To maintain a specific current-flow direction in an isotropic homogeneous conductor in the presence of a perpendicular magnetic field, it is necessary to have an electric field with a component in the direction transverse to both the current and magnetic field. This transverse electric field, in the classical physics interpretation, is necessary to balance the magnetic force acting on the moving charge carriers. The ratio between the component of the transverse electric field and the current defines the Hall coefficient (see, e.g., [4, 29]). For an anisotropic material it is natural to introduce, analogously, a Hall matrix [10]. The question naturally arises as to what properties this Hall matrix can have. In particular can it be an antisymmetric matrix?

To address this question we consider anisotropic periodic composite materials and ask what effective Hall matrices the composite can have, and in particular if they can be antisymmetric. At the microscale the composite will be assumed to be locally isotropic, characterized by a periodic isotropic local conductivity $\sigma(y) I_3$ and a periodic isotropic local Hall matrix $r(y) I_3$. Then, in the presence of a small magnetic field $h \in \mathbb{R}^3$, the (assumed periodic) local electric field $e(y)$ has the expansion

$$e(y) = \rho_h(y) j(y) = \sigma(y)^{-1} j(y) + \underbrace{r(y) j(y) \times h}_{\text{Hall field}} + o(h). \quad (1.1)$$

where $j(y)$ is the periodic local current (see, e.g., [21] for details) and $\rho_h(y) = \sigma_h(y)^{-1}$ is the perturbed local (non-symmetric matrix-valued) resistivity. We call the transverse field

$r(y)j(y) \times h$ the Hall field as it generates the Hall voltage. The average electric field in the composite derived from (1.1) has the expansion

$$\langle e \rangle = \rho_h^* \langle j \rangle = (\sigma^*)^{-1} \langle j \rangle + \underbrace{\langle j \rangle \times (R^* h)}_{\text{Macroscopic Hall field}} + o(h), \quad (1.2)$$

in which $\langle \cdot \rangle$ denotes the average value over the period cell, ρ_h^* is the perturbed effective resistivity tensor, σ^* is the (matrix-valued) effective conductivity tensor in the absence of the magnetic field, and R^* is the effective Hall matrix. We will call $\langle j \rangle \times (R^* h)$ the macroscopic Hall field.

It was discovered by Bergman [5] that the effective Hall coefficient in a composite can be obtained from the local Hall coefficient and the knowledge of several current fields that solve the conductivity equations in the absence of magnetic field. His arguments extend directly to the anisotropic case and provide a formula for the effective Hall matrix [10]. First note that since the relation between $e(y)$ and $\langle e \rangle$ is linear we can write, in the absence of any magnetic field,

$$e(y) = P(y) \langle e \rangle \quad (h = 0) \quad (1.3)$$

which following Tartar [32] defines the matrix-valued unperturbed electric field $P(y) = DU(y)$. Then, upon multiplying this equation by $\sigma(y)$ and averaging over the unit cell, one sees that the effective matrix conductivity is given by the classical formula

$$\sigma^* = \langle \sigma P \rangle. \quad (1.4)$$

By contrast the effective Hall matrix is given by the more intricate formula ([10] Theorem 3)

$$\text{Cof}(\sigma^*) R^* = \langle r \text{Cof}(\sigma P)^T \rangle, \quad (1.5)$$

where Cof denotes the cofactor matrix.

The relation between $\rho_h(y)$ and ρ_h^* is still poorly understood. In two dimensions, one can shift $\rho_h(y)$ (or $\rho_h(y)^{-1}$) by a constant antisymmetric matrix and then ρ_h^* (respectively $(\rho_h^*)^{-1}$) will shift in exactly the same way [15]. For a fixed magnetic field h this observation led to a complete characterization, in two dimensions, of the possible tensors ρ_h^* that are obtainable from mixtures of two given, possibly anisotropic, phases with prescribed orientations and mixed in fixed proportions [26]. In three dimensions one can add a periodic divergence free antisymmetric matrix field to $\rho_h(y)^{-1}$ and then $(\rho_h^*)^{-1}$ will shift by the average of this tensor field (see [31] and section 4.4 of [27]). For fibre-reinforced composites (with structure independent of one coordinate) an amazing plethora of microstructure independent exact relations and links between effective tensors have been derived [7, 8, 30, 9, 17, 18]. For strong magnetic fields h the tensor ρ_h^* can have a very sensitive dependence on the orientation of h relative to the microstructure [6].

It would be interesting to know the range of values the matrix pair (σ^*, R^*) , or at least the matrix R^* , can take given some information about the pair of functions $(\sigma(y), r(y))$, with $\sigma(y) > 0$. Even for isotropic composites, with $R^* = r^* I_3$, there are some surprises: contrary to the common belief that the sign of the Hall coefficient determines the sign of the charge carrier, r^* in a certain geometry of cubic chain mail is negative even though $r(y)$ is non-negative everywhere [10]. Physically such an unusual effect arises essentially because the inclusions with appreciable Hall coefficient (situated between the links of the chain mail) have an current field flowing through them which is in the opposite direction of the applied current field. Also the matrix elements of R^* can be orders of magnitude greater than the maximum value of $r(y)$ in

certain microstructures [11], unlike in the two-dimensional case where the Hall coefficient is bounded above by the maximum value of $r(y)$ [12].

The present paper provides a partial answer to the following question: What are the possible effective Hall matrices, R^* , when $\sigma(y)$ and $r(y)$ are allowed to be any pair of positive functions? Starting from a nonnegative isotropic Hall matrix, we first note (see the preliminary Proposition 2.1) that any positive definite symmetric matrix is an effective Hall matrix. More surprisingly, and this is the main result of the paper (see Theorem 2.2) which adds to the menagerie of pathologies arising in the homogenization of the Hall effect, there exists a four-phase composite the effective Hall matrix of which is (asymptotically) antisymmetric!

Physically this effect arises because for an applied current in the y_3 direction, a portion of this current flows through the cylinders with non-zero Hall coefficient and the induced macroscopic Hall field is essentially a 90° degree rotation of the Hall field across these inclusions (see figure 1). Thus if the magnetic field h is parallel to the y_1 - y_2 plane, then the direction of the macroscopic Hall field will be essentially parallel to h . To achieve this change of direction we use a highly conductivity phase. This is reminiscent of the use of high contrast materials for achieving the reversal of sign of Hall coefficient in the cubic chain mail of [10], but the geometry is completely different.

2 Statement and proof of the main results

In dimension three the expression of the effective the Hall matrix incorporates the cofactor matrix of the local electric field $DU(y)$ (see formula (1.5)). Therefore it should not be surprising that our analysis uses the quasi-affine property satisfied by the minors of the matrix-valued gradient $DU(y)$. We refer to [13] (Section 4.2) for a complete presentation of the quasi-affinity. In the present context we will specifically use the quasi-affine property of the quadratic minors:

- In dimension two the determinant is quasi-affine. This means that for any function $V : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that DV is a Y -periodic matrix-valued function in $L^2(Y)^{2 \times 2}$,

$$\int_Y \det(DV) dy = \det \left(\int_Y DV dy \right). \quad (2.1)$$

- In dimension three the cofactor matrix is quasi-affine. This means that for any function $V : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that DV is a Y -periodic matrix-valued function in $L^2(Y)^{3 \times 3}$,

$$\int_Y \text{Cof}(DV) dy = \text{Cof} \left(\int_Y DV dy \right). \quad (2.2)$$

We start with the preliminary result of Proposition 2.1. This proposition uses the fact that any symmetric positive definite matrix is the effective matrix of a periodic isotropic conductivity, at least asymptotically. Indeed, it suffices to consider composites of two isotropic phases and use the multiple rank lamination due to Maxwell [25] and its extension to dimension greater than three by Tartar [33] (see also [24]), or the coated ellipsoid assemblage (see chapter 7 of [27] and references therein, and its extension to higher dimension by Tartar [33]), or the periodic constructions of Vigdergauz [34] and Liu, James, and Leo [23]. Then, letting the free parameters in the microstructure vary, any symmetric positive definite matrix can be obtained in any dimension as the effective matrix of a composite of two isotropic phases with suitably extreme conductivities, one close to zero and one large. Even though multiple rank laminates

are not periodic, they can be approximated arbitrarily closely by periodic constructions (see, e.g., [2] Theorem 1.3.23). Also it is easily seen that the coated ellipsoid assemblage can be made periodic.

Proposition 2.1. *Any positive definite symmetric 3×3 matrix A is the effective Hall matrix of a suitable periodic structure.*

Proof. Consider a Y -periodic isotropic conductivity $\sigma(y) I_3$ the homogenized conductivity of which is $\sigma^* = A/\sqrt{\det A}$, and the Y -periodic Hall coefficient is $r := \sigma^{-2}$. By (1.5) the effective Hall matrix R^* associated with this periodic structure is given by the formula

$$\int_Y r \operatorname{Cof}(\sigma DU)^T dy = \operatorname{Cof}(\sigma^*) R^*, \quad (2.3)$$

where DU is the periodic local matrix-valued electric field solving

$$\operatorname{Div}(\sigma DU) = 0 \text{ in } \mathbb{R}^3, \quad \text{with} \quad \int_Y DU dy = I_3, \quad (2.4)$$

(the j -th column of the matrix DU is the gradient of the j -th coordinate u_j of U , and the j -th coordinate of the vector $\operatorname{Div}(\sigma DU)$ is the divergence of $\sigma \nabla u_j$) and σ^* is the homogenized conductivity given by the classical formula

$$\sigma^* = \int_Y \sigma DU dy. \quad (2.5)$$

Hence, by the quasi-affinity of the cofactor matrix (2.2) and the equality $\operatorname{Cof}(\sigma^*) = A^{-1}$, we obtain

$$I_3 = \int_Y \operatorname{Cof}(DU)^T dy = \int_Y r \operatorname{Cof}(\sigma DU)^T dy = A^{-1} R^*, \quad (2.6)$$

which implies that $R^* = A$. $\square$

On the contrary the derivation of an antisymmetric effective Hall matrix is much more delicate. To this end we have the following asymptotic result:

Theorem 2.2. *There exists a periodic conductivity associated with an isotropic Hall coefficient parametrized by a large number $\kappa \gg 1$, such that the homogenized Hall matrix $R^{*,\kappa}$ satisfies*

$$\lim_{\kappa \rightarrow \infty} R^{*,\kappa} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.7)$$

Proof. Let $Y := (-\frac{1}{2}, \frac{1}{2})^3$. We consider a columnar Y -periodic structure along the y_3 -axis, the period cell of which is represented in figure 1. On the one hand, the Y -periodic local conductivity σ^κ , for $\kappa > 0$, is defined by

$$\sigma^\kappa(y) := \begin{cases} \operatorname{diag}(\kappa, \kappa, 1) & \text{if } y \in Q_s = Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \\ I_3 & \text{if } y \in Y \setminus Q_s, \end{cases} \quad (2.8)$$

where Q_s is the grey region in figure 1. The conductivity $\sigma^\kappa(y)$ in the y_3 direction is chosen to be 1 in the region Q_s , rather than κ , to ensure that a significant fraction of the current flows

Figure 1: The cross section of the columnar period cell. The regions Q_1, Q_2, Q_3, Q_4 are highly conducting in the plane, and only the central square cylinder K_1 has a non-zero Hall coefficient.

through the material with non-zero Hall coefficient (which will be the square cylinder at the center) when the applied current is in the y_3 direction.

Although σ^κ is anisotropic it can be replaced, if desired, when $\kappa > 1$ by a laminate of two isotropic phases, finely layered in the y_3 -direction so its local effective conductivity is σ^κ . The conductivity matrix σ^κ is associated with the corrector DU^κ , where $U^\kappa = (u_1^\kappa, u_2^\kappa, u_3^\kappa)$ is the

unique function with zero Y -average which is the solution of

$$\operatorname{Div}(\sigma^\kappa DU^\kappa) = 0 \text{ in } \mathbb{R}^3, \quad y \mapsto U^\kappa(y) - y \text{ is } Y\text{-periodic}, \quad \int_Y DU^\kappa dy = I_3, \quad \nabla u_3^\kappa = e_3. \quad (2.9)$$

Note that the components u_1, u_2 of U^κ do not depend on the y_3 -coordinate due to the columnar geometry. Then, the cofactor matrix of DU^κ satisfies the equality

$$\operatorname{Cof}(DU^\kappa)^T = \begin{pmatrix} \frac{\partial u_2^\kappa}{\partial y_2} & -\frac{\partial u_2^\kappa}{\partial y_1} & 0 \\ -\frac{\partial u_1^\kappa}{\partial y_2} & \frac{\partial u_1^\kappa}{\partial y_1} & 0 \\ 0 & 0 & \frac{\partial u_1^\kappa}{\partial y_1} \frac{\partial u_2^\kappa}{\partial y_2} - \frac{\partial u_1^\kappa}{\partial y_2} \frac{\partial u_2^\kappa}{\partial y_1} \end{pmatrix}. \quad (2.10)$$

On the other hand, the Y -periodic local Hall matrix is defined by

$$R(y) := r_\kappa 1_{K_1}(y) I_3, \quad \text{for } y \in Y, \quad (2.11)$$

where $r_\kappa > 0$ will be chosen later and 1_{K_1} is the characteristic function of the central square K_1 of side $\ell < \frac{1}{3}$ in figure 1.

Due to the invariance of figure 1 by a rotation of 90° , the homogenized conductivity $\sigma^{*,\kappa}$ reads as

$$\sigma^{*,\kappa} = \begin{pmatrix} a_\kappa & 0 & 0 \\ 0 & a_\kappa & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{where } a_\kappa > 0. \quad (2.12)$$

By analogy with a two-phase $(1, \kappa)$ checkerboard structure we have the following result proved at the end of the paper:

Lemma 2.3. *The coefficient a_κ of the homogenized matrix $\sigma^{*,\kappa}$ satisfies the estimates*

$$\forall \kappa > 0, \quad c_1 \kappa^{\frac{1}{2}} \leq a_\kappa \leq c_2 \kappa^{\frac{1}{2}}, \quad (2.13)$$

where c_1, c_2 are two positive constants.

Remark 2.4. The factor of $\kappa^{\frac{1}{2}}$ in (2.13) arises because of the resistance of the corners where highly conducting regions meet.

From now on u_1, u_2 will be regarded as functions of the two variables y_1, y_2 , and the sets Y, Q_s, Q_i, K_i will be identified with their cross sections in the transverse plane y_1-y_2 as shown in figure 1.

According to the variational principle (3.1) we have

$$a_\kappa = \sigma_{11}^{*,\kappa} = \int_Y \sigma^\kappa \nabla u_1^\kappa \cdot \nabla u_1^\kappa dy \geq \kappa \int_{Q_s} |\nabla u_1^\kappa|^2 dy, \quad (2.14)$$

which combined with estimate (2.13) implies that ∇u_1^κ strongly converges to zero in $L^2(Q_s)$. That is expected because the electric field should be close to zero in the highly conducting phase, except near the corner contact points. Then, the Poincaré-Wirtinger inequality (see, e.g., [16] p. 164) in the regular connected open set Q_i , for $i = 1, \dots, 4$, yields

$$\overline{\int_{\partial K_1 \cap \partial Q_i} u_1^\kappa dy} - \overline{\int_{\partial Y \cap \partial Q_i} u_1^\kappa dy} = o(1), \quad \text{as } \kappa \rightarrow \infty, \quad (2.15)$$

where the integral with a bar through it denotes an average over the interval. In other words, the average electric potential along the boundary $\partial K_1 \cap \partial Q_i$ should be close to that along

the boundary $\partial Y \cap \partial Q_i$, as expected because the region Q_i is highly conducting in the plane. Moreover, since $u_1^\kappa(y) - y_1$ is Y -periodic, we have (see figure 1)

$$\int_{\partial Y \cap \partial Q_1} u_1^\kappa dy = \int_{\partial Y \cap \partial Q_3} u_1^\kappa dy \quad \text{and} \quad \int_{\partial Y \cap \partial Q_2} u_1^\kappa dy = \int_{\partial Y \cap \partial Q_4} u_1^\kappa dy + \ell. \quad (2.16)$$

Therefore, it follows from (2.15) and (2.16) that

$$\begin{cases} \int_{K_1} \frac{\partial u_1^\kappa}{\partial y_1} dy = \int_{\partial K_1 \cap \partial Q_1} u_1^\kappa dy - \int_{\partial K_1 \cap \partial Q_3} u_1^\kappa dy = o(1) \\ \int_{K_1} \frac{\partial u_1^\kappa}{\partial y_2} dy = \int_{\partial K_1 \cap \partial Q_4} u_1^\kappa dy - \int_{\partial K_1 \cap \partial Q_2} u_1^\kappa dy = -\ell + o(1), \end{cases} \quad \text{as } \kappa \rightarrow \infty. \quad (2.17)$$

Similarly, with a change of sign we get for the function u_2^κ

$$\begin{cases} \int_{K_1} \frac{\partial u_2^\kappa}{\partial y_1} dy = \int_{\partial K_1 \cap \partial Q_1} u_2^\kappa dy - \int_{\partial K_1 \cap \partial Q_3} u_2^\kappa dy = \ell + o(1) \\ \int_{K_1} \frac{\partial u_2^\kappa}{\partial y_2} dy = \int_{\partial K_1 \cap \partial Q_4} u_2^\kappa dy - \int_{\partial K_1 \cap \partial Q_2} u_2^\kappa dy = o(1). \end{cases} \quad \text{as } \kappa \rightarrow \infty. \quad (2.18)$$

Finally, taking into account (2.8), (2.11) the formula (1.5) for the homogenized Hall matrix reads as

$$R^{*,\kappa} = \frac{\sigma^{*,\kappa}}{\det \sigma^{*,\kappa}} \int_{K_1} r_\kappa \text{Cof} (DU^\kappa)^T dy. \quad (2.19)$$

This combined with (2.10), (2.12), (2.17), (2.18) yields

$$R^{*,\kappa} = \frac{r_\kappa}{a_\kappa^2} \begin{pmatrix} a_\kappa & 0 & 0 \\ 0 & a_\kappa & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} o(1) & -\ell + o(1) & 0 \\ \ell + o(1) & o(1) & 0 \\ 0 & 0 & c_\kappa \end{pmatrix}, \quad (2.20)$$

where

$$c_\kappa := \int_{K_1} \left(\frac{\partial u_1^\kappa}{\partial y_1} \frac{\partial u_2^\kappa}{\partial y_2} - \frac{\partial u_1^\kappa}{\partial y_2} \frac{\partial u_2^\kappa}{\partial y_1} \right) dy = \int_{K_1} \det (\nabla u_1^\kappa, \nabla u_2^\kappa) dy. \quad (2.21)$$

Moreover, by the Alessandrini, Nesi [1] positivity result we have $\det (\nabla u_1^\kappa, \nabla u_2^\kappa) > 0$ in Y . This combined with the quasi-affinity of the determinant (2.1) yields

$$0 < c_\kappa \leq \int_Y \det (\nabla u_1^\kappa, \nabla u_2^\kappa) dy = \det \left(\int_Y (\nabla u_1^\kappa, \nabla u_2^\kappa) dy \right) = 1. \quad (2.22)$$

Therefore, choosing

$$r_\kappa := \frac{a_\kappa}{\ell}, \quad (2.23)$$

the estimate from below of (2.13) and (2.22) give

$$\lim_{\kappa \rightarrow \infty} \frac{r_\kappa c_\kappa}{a_\kappa^2} = \lim_{\kappa \rightarrow \infty} \frac{c_\kappa}{\ell a_\kappa} = 0, \quad (2.24)$$

which implies the desired limit (2.7). $\square$

3 Proof of Lemma 2.3.

This lemma basically states that the effective conductivity in the y_1 - y_2 plane is dominated by the resistance of the corners where the highly conducting regions meet and has the asymptotic form expected from the analysis of Keller [20].

First of all, recall that for a given square Z of $\mathbb{R}^2$, the effective conductivity σ^* of a Z -periodic symmetric conductivity $\sigma(y)$ is given by the variational principal (see, e.g., [3])

$$\sigma^* \lambda \cdot \lambda = \min \left\{ \int_Z \sigma \nabla v \cdot \nabla v \, dy : (y \mapsto v(y) - \lambda \cdot y) \text{ } Z\text{-periodic} \right\} = \int_Z \sigma \nabla w^\lambda \cdot \nabla w^\lambda \, dy, \quad (3.1)$$

where the minimizer w^λ , for $\lambda \in \mathbb{R}^2$, is the unique solution (up to an additive constant) in $H_{\text{loc}}^1(\mathbb{R}^2)$ of the equation

$$\operatorname{div}(\sigma \nabla w^\lambda) = 0 \text{ in } \mathbb{R}^2, \quad \text{with} \quad (y \mapsto w^\lambda(y) - \lambda \cdot y) \text{ } Z\text{-periodic}. \quad (3.2)$$

Proof of the upper bound of (2.13): Let K_j , for $j = 1, 2, 3$, be the square centered at the origin and of side $j\ell$. On the one hand, consider for $\kappa > 0$ the potential w_κ associated with the two-phase $(1, \kappa)$ checkerboard of period K_2 (see figure 1) which is the solution of the problem

$$\operatorname{div}(\chi_\kappa \nabla w_\kappa) = 0 \text{ in } \mathbb{R}^2, \quad y \mapsto w_\kappa(y) - y_2 \text{ is } K_2\text{-periodic} \quad \text{and} \quad \int_{K_2} w_\kappa \, dy = 0, \quad (3.3)$$

where χ_κ is the K_2 -periodic conductivity of the checkerboard defined by

$$\chi_\kappa(y) := 1 + (\kappa - 1) 1_{Q_s}(y) \quad \text{for } y \in K_2. \quad (3.4)$$

Note that χ_κ and the first entry σ_{11}^κ of σ^κ (2.8) agree in the square K_3 as shown in figure 1. The formula (3.1) for the effective conductivity of a checkerboard (see, e.g., [14]) reads as

$$\int_{K_2} \chi_\kappa |\nabla w_\kappa|^2 \, dy = \kappa^{\frac{1}{2}}, \quad (3.5)$$

which implies by the Poincaré-Wirtinger inequality in K_2 ,

$$\int_{K_2} w_\kappa^2 \, dy + \int_{K_2} |\nabla w_\kappa|^2 \, dy = O(\kappa^{\frac{1}{2}}). \quad (3.6)$$

On the other hand, let v_κ be function defined by

$$v_\kappa := \frac{1}{2\ell} \int_{Q_i \cap \partial K_2} w_\kappa \, dy \quad \text{in } Q_i \cap (Y \setminus K_2), \quad \text{for } i = 1, \dots, 4, \quad (3.7)$$

and extended by interpolation in the whole set $Y \setminus K_2$. Since $y \mapsto w_\kappa(y) - y_2$ is K_2 -periodic, we have

$$\begin{cases} v_\kappa(\frac{1}{2}, y_2) = v_\kappa(-\frac{1}{2}, y_2) - 1 & \text{for } y_2 \in (-\frac{\ell}{2}, \frac{\ell}{2}) \\ v_\kappa(y_1, \frac{1}{2}) = v_\kappa(y_1, -\frac{1}{2}) & \text{for } y_1 \in (-\frac{\ell}{2}, \frac{\ell}{2}). \end{cases} \quad (3.8)$$

Moreover, by the trace inequality in K_2 (see, e.g., [22]):

$$\exists C > 0, \quad \forall v \in H^1(K_2), \quad \int_{\partial K_2} v^2 \, ds \leq C \int_{K_2} (v^2 + |\nabla v|^2) \, dx, \quad (3.9)$$

combined with the Cauchy-Schwarz inequality in $Q_i \cap \partial K_2$ and estimate (3.6), we have

$$\int_{Q_i \cap \partial K_2} w_\kappa dy = O(\kappa^{\frac{1}{4}}), \quad \text{for } i = 1, \dots, 4. \quad (3.10)$$

Then, we can construct v_κ in $Y \setminus K_2$ in such a way that $y \mapsto v_\kappa(y) + y_1$ is 1-periodic on ∂Y and

$$\int_{Y \setminus K_2} v_\kappa^2 dy + \int_{Y \setminus K_2} |\nabla v_\kappa|^2 dy = O(\kappa^{\frac{1}{2}}). \quad (3.11)$$

Let φ be a smooth Y -periodic function such that $\varphi = 1$ in $Y \setminus K_3$ and $\varphi = 0$ in K_2 . Define the function

$$u_\kappa := \varphi v_\kappa + (1 - \varphi) \frac{w_\kappa}{2\ell}. \quad (3.12)$$

Since $y \mapsto u_\kappa(y) + y_1$ is Y -periodic, the function $-u_\kappa$ can be used as a test function in the variational principle giving a_κ , hence

$$a_\kappa \leq \int_Y (1 + (\kappa - 1) 1_{Q_s}) |\nabla u_\kappa|^2 dy. \quad (3.13)$$

By (3.6) and (3.11) we have

$$\begin{aligned} & \int_{(Y \setminus K_3) \cup K_2} (1 + (\kappa - 1) 1_{Q_s}) |\nabla u_\kappa|^2 dy \\ &= \int_{Y \setminus K_3} |\nabla v_\kappa|^2 dy + \frac{1}{(2\ell)^2} \int_{K_2} \chi_\kappa |\nabla w_\kappa|^2 dy = O(\kappa^{\frac{1}{2}}). \end{aligned} \quad (3.14)$$

It remains to estimate the conduction energy of u_κ in the transition region $K_3 \setminus K_2$ where

$$\nabla u_\kappa := \nabla \varphi \left(v_\kappa - \frac{w_\kappa}{2\ell} \right) + \varphi \left(\nabla v_\kappa - \frac{\nabla w_\kappa}{2\ell} \right).$$

We easily deduce from (3.6), (3.11) and the K_2 -periodicity of ∇w_κ the estimate

$$\int_{K_3 \setminus K_2} (1 + (\kappa - 1) 1_{Q_s}) \varphi^2 \left| \nabla v_\kappa - \frac{\nabla w_\kappa}{2\ell} \right|^2 dy = O(\kappa^{\frac{1}{2}}), \quad (3.15)$$

and out of the region of high conductivity Q_s the estimate

$$\int_{(K_3 \setminus K_2) \setminus Q_s} |\nabla \varphi|^2 \left(v_\kappa - \frac{w_\kappa}{2\ell} \right)^2 dy = O(\kappa^{\frac{1}{2}}). \quad (3.16)$$

Finally, in each high-conductivity set $Q_i \cap (K_3 \setminus K_2)$, for $i = 1, \dots, 4$, the Poincaré-Wirtinger inequality yields

$$\begin{aligned} \int_{Q_i \cap (K_3 \setminus K_2)} |\nabla \varphi|^2 \left(v_\kappa - \frac{w_\kappa}{2\ell} \right)^2 dy &= \frac{1}{(2\ell)^2} \int_{Q_i \cap (K_3 \setminus K_2)} |\nabla \varphi|^2 \left(w_\kappa - \oint_{Q_i \cap \partial K_2} w_\kappa dy \right)^2 dy \\ &\leq c \int_{Q_i \cap (K_3 \setminus K_2)} |\nabla w_\kappa|^2 dy, \end{aligned} \quad (3.17)$$

which multiplying by κ gives

$$\int_{Q_i \cap (K_3 \setminus K_2)} \kappa |\nabla \varphi|^2 \left(v_\kappa - \frac{w_\kappa}{2\ell} \right)^2 dy \leq c \int_{K_3} \chi_\kappa |\nabla w_\kappa|^2 dy = O(\kappa^{\frac{1}{2}}). \quad (3.18)$$

The estimates (3.14)-(3.18) combined with (3.13) imply the upper bound of (2.13).

Proof of the lower bound of (2.13): The Keller [19] duality (see also [14]) and the upper bound proved above give the equalities $a_{\kappa}^{-1} = a_{\kappa^{-1}} = O(\kappa^{-\frac{1}{2}})$. This implies the lower bound of (2.13) and concludes the proof of Lemma 2.3. $\square$

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