HARMONIC COCYCLES, VON NEUMANN ALGEBRAS, AND IRREDUCIBLE AFFINE ISOMETRIC ACTIONS

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Abstract. Let G be a compactly generated locally compact group and $(\pi, \mathcal{H})$ a unitary representation of G. The 1-cocycles with coefficients in π which are harmonic (with respect to a suitable probability measure on G) represent classes in the first reduced cohomology $\overline{H}^1(G, \pi)$. We show that harmonic 1-cocycles are characterized inside their reduced cohomology class by the fact that they span a minimal closed subspace of $\mathcal{H}$. In particular, the affine isometric action given by a harmonic cocycle b is irreducible (in the sense that $\mathcal{H}$ contains no non-empty, proper closed invariant affine subspace) if the linear span of $b(G)$ is dense in $\mathcal{H}$. The converse statement is true, if π moreover has no almost invariant vectors. Our approach exploits the natural structure of the space of harmonic 1-cocycles with coefficients in π as a Hilbert module over the von Neumann algebra $\pi(G)'$, which is the commutant of $\pi(G)$. Using operator algebras techniques, such as the von Neumann dimension, we give a necessary and sufficient condition for a factorial representation π without almost invariant vectors to admit an irreducible affine action with π as linear part.

1. Introduction

Let G be a locally compact group and $(\pi, \mathcal{H})$ a continuous unitary (or orthogonal) representation of G on a complex (or real) Hilbert space $\mathcal{H}$. Recall that a 1-*cocycle* with coefficients in π is a continuous map $b : G \to \mathcal{H}$ such that $b(gh) = b(g) + \pi(g)b(h)$ for all $g, h \in \mathcal{H}$ and that a 1-cocycle is a *coboundary* if it is of the form ∂_v for some $v \in \mathcal{H}$, where $\partial_v(g) = \pi(g)v - v$ for $g \in G$. The space $Z^1(G, \pi)$ of 1-cocycles with coefficients in π is a vector space containing the space $B^1(G, \pi)$ of coboundaries as linear subspace. The 1-*cohomology* $H^1(G, \pi)$ is the quotient $Z^1(G, \pi)/B^1(G, \pi)$.

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The space $B^1(G,\pi)$ is not necessarily closed in $Z^1(G,\pi)$ (see Proposition 1) and the *reduced* 1-*cohomology* with coefficients in π is defined as $\overline{H}^1(G,\pi) = Z^1(G,\pi)/\overline{B^1(G,\pi)}$.

Assume now that G is compactly generated, that is, $G = \cup_{n\in\mathbb{Z}} Q^n$ for a compact subset Q, which we can assume to be a neighbourhood of the identity $e \in G$ and to be symmetric ($Q^{-1} = Q$).

Harmonic 1-cocycles in $Z^1(G,\pi)$, with respect to an appropriate probability measure on G, form a set of representatives for the classes in the reduced cohomology $\overline{H}^1(G,\pi)$, as we will shortly explain. Such cocycles appear in [BeV] in the case where π is the regular representation of a discrete group G, in relation with the first ℓ^2-Betti number of G; they play an important role in Ozawa's recent proof of Gromov's polynomial growth theorem ([Oza]) as well as in the work [ErO] and [GoJ].

Harmonic 1-cocycles were implicitly introduced in [Gui, Theorem 2]; it was observed there that $Z^1(G,\pi)$ can be identified with a closed subspace of the Hilbert space $L^2(Q,\mathcal{H},m_G)$, where m_G is a (left) Haar measure on G and so $\overline{H}^1(G,\pi)$ corresponds to the orthogonal complement $B^1(G,\pi)^\perp$ of $B^1(G,\pi)$ in $Z^1(G,\pi)$. Following [ErO], we prefer to embed $Z^1(G,\pi)$ in a more general Hilbert space, defined by a class of appropriate probability measures similar to those appearing there. For this, we consider the word length on G associated to Q, that is, the map $g \mapsto |g|_Q$, where

$$|g|_Q = \min\{n \in \mathbb{N} : g \in Q^n\}.$$

Definition 1. A probability measure μ on G is *cohomologically adapted* (or, more precisely, 1-cohomologically adapted) if it has the following properties:

- μ is symmetric;
- μ is absolutely continuous with respect to the Haar measure m_G;
- μ is adapted: the support of μ is a generating set for G;
- μ has a second moment: $\int_G |x|_Q^2 d\mu(x) < \infty$.

Observe that the class of cohomologically adapted measures is independent of the generating compact set Q, since the length functions associated to two compact generating sets are bi-Lipschitz equivalent.

We consider the Hilbert space $L^2(G,\mathcal{H},\mu)$ of measurable square-integrable maps $F : G \to \mathcal{H}$. Then $Z^1(G,\pi)$ is a subset of $L^2(G,\mathcal{H},\mu)$ (see Section 2). Moreover, the linear operator

$$\partial : \mathcal{H} \to Z^1(G,\pi), v \mapsto \partial_v$$

is bounded, has $B^1(G,\pi)$ as range, and it is straightforward to check that its adjoint is $-\frac{1}{2}M_\mu$, where

$$M_\mu : Z^1(G,\pi) \to \mathcal{H},\ b \mapsto \int_G b(x)d\mu(x).$$

So, the orthogonal complement $B^1(G,\pi)^\perp$ of $B^1(G,\pi)$ in $Z^1(G,\pi)$ can be identified with the space of harmonic cocycles in the sense of the following definition. In particular, the reduced cohomology $\overline{H}^1(G,\pi)$ can be identified with $\mathrm{H}ar_\mu(G,\pi)$.

Definition 2. A cocycle $b \in Z^1(G,\pi)$ is *μ-harmonic* if $M_\mu(b) = 0$, that is, $\int_G b(x)d\mu(x) = 0$. We denote by $\mathrm{H}ar_\mu(G,\pi)$ the space of μ-harmonic cocyles in $Z^1(G,\pi)$ and by

$$P_{\mathrm{H}ar} : L^2(G,\mathcal{H},\mu) \to \mathrm{H}ar_\mu(G,\pi)$$

the orthogonal projection on $\mathrm{H}ar_\mu(G,\pi)$.

Observe that, by the cocycle relation, $b \in Z^1(G,\pi)$ is μ-harmonic if and only if it has the mean value property

$$b(g) = \int_G b(gx)d\mu(x) \qquad \text{for all} \quad g \in G.$$

In our opinion, the Hilbert space structure of $\overline{H}^1(G,\pi)$ given by its realization as a space of harmonic cocycles, together with its module structure over the von Neumann algebra $\pi(G)'$ (see below), deserves more attention than it has received so far in the literature. Our aim in this paper is to use this structure in relation with a natural notion of irreducibility for affine isometric actions (see Definition 3).

Our first result shows that harmonic 1-cocycles b are characterized by a remarkable minimality property of the space $\overline{\mathrm{span}(b(G))}$, the closure of the linear span of $b(G)$.

Theorem 1. *Let G be a compactly generated group. Let $(\pi,\mathcal{H})$ be an orthogonal or unitary representation of G and μ a cohomologically adapted probability measure on G. Let $b \in \mathrm{H}ar_\mu(G,\pi)$ be a μ-harmonic cocycle. We have*

$$\overline{\mathrm{span}\,(b(G))} = \bigcap_{b'} \overline{\mathrm{span}\,(b'(G))},$$

where b' runs over the 1-cocycles in the cohomology class of b in $\overline{H}^1(G,\pi)$.

In particular, Theorem 1 shows that, for a μ-harmonic cocycle b, the closed linear subspace spanned by $b(G)$ only depends on the reduced cohomology class of b and not on the choice of μ.

Recall that, given a cocycle $b \in Z^1(G, \pi)$, a continuous action $\alpha_{\pi,b}$ of G on $\mathcal{H}$ by affine isometries is defined by the formula

$$\alpha_{\pi,b}(g)v = \pi(g)v + b(g) \qquad \text{for all} \quad g \in G, v \in \mathcal{H}.$$

Conversely, let α be a continuous action of G on $\mathcal{H}$ by affine isometries. Denote by $\pi(g)$ and $b(g)$ the linear part and the translation part of $\alpha(g)$ for $g \in G$. Then π is a unitary (or orthogonal) representation of G on $\mathcal{H}$, b is a 1-cocycle in $Z^1(G, \pi)$, and $\alpha = \alpha_{\pi,b}$. For all this, see Chapter 2 in [BHV].

The following notion of irreducibility of affine actions was introduced in [Ner] and further studied in [BPV].

Definition 3. An affine isometric action α of G on the complex or real Hilbert space $\mathcal{H}$ is *irreducible* if $\mathcal{H}$ has no non-empty, closed and proper $\alpha(G)$-invariant affine subspace.

First examples of irreducible affine isometric actions arise as actions $\alpha_{\pi,b}$, where π is an irreducible unitary representation of G with non trivial 1-cohomology and $b \in Z^1(G, \pi)$ a cocycle which is not a coboundary. By [Sha1, Theorem 0.2], such a pair (π, b) always exists, provided G does not have Kazhdan's Property (T). A remarkable feature of irreducible affine isometric actions of a locally compact group G is that they remain irreducible under restriction to "most" lattices in G (see [Ner, 3.6], [BHV, Theorem 4.2]), whereas this is not true in general for irreducible unitary representations.

Let $b \in Z^1(G, \pi)$. Observe that $\mathrm{span}(b(G))$ is $\alpha_{\pi,b}(G)$-invariant. So, for $\alpha_{\pi,b}$ to be irreducible, it is necessary that $\mathrm{span}(b(G))$ is dense in $\mathcal{H}$. This condition is not sufficient (see [BPV, Example 2.4]; however, see also Proposition 3 below). The following corollary of Theorem 1 relates harmonic cocycles to this question.

Corollary 1. *Let $G, (\pi, \mathcal{H})$, and μ be as in Theorem 1. Let $b \in Z^1(G, \pi)$ and $P_{\mathrm{H}ar} b$ its projection on* $\mathrm{H}ar_\mu(G, \pi)$.
(i) If $\mathrm{span}(P_{\mathrm{H}ar} b(G))$ *is dense in $\mathcal{H}$, then the affine action $\alpha_{\pi,b}$ is irreducible.*
(ii) Assume that $B^1(G, \pi)$ is closed; if the affine action $\alpha_{\pi,b}$ is irreducible, then $\mathrm{span}(P_{\mathrm{H}ar} b(G))$ *is dense in $\mathcal{H}$.*

Remark 1. (i) Point (ii) in Corollary 1 does not hold in general when $B^1(G, \pi)$ is not closed; indeed, let $G = \mathbb{F}_2$ denote the free group on 2. generators. Then $H^1(G, \pi) \neq 0$ for every unitary representation π of G (see [Gui, §9, Example 1]). On the other hand, there exists an irreducible unitary representation π of G with $\overline{H}^1(G, \pi) = 0$ (see [MaV, Theorem 1.1]), so that $\mathrm{H}ar_\mu(G, \pi) = 0$ for any cohomologically adapted

probability measure μ on G. Now, let b be a 1-cocycle in $Z^1(G,\pi)$ which is not a coboundary. Then the affine action $\alpha_{\pi,b}$ is irreducible.
(ii) Although we will not need it, we will give an explicit formula for the projection $P_{\mathrm{H}ar} : Z^1(G,\pi) \to \mathrm{H}ar_\mu(G,\mu)$ in the case where $B^1(G,\pi)$ is closed (see Proposition 4 below).

In view of Corollary 1, it is of interest to know when $B^1(G,\pi)$ is closed. Write $\mathcal{H} = \mathcal{H}^G \oplus \mathcal{H}^0$, where $\mathcal{H}^G$ is the space of $\pi(G)$-invariant vectors in $\mathcal{H}$ and $\mathcal{H}^0$ its orthogonal complement. Let π^0 denote the restriction of π to $\mathcal{H}^0$. Observe that $B^1(G,\pi^0) = B^1(G,\pi)$ and that $Z^1(G,\pi^0)$ is closed in $Z^1(G,\pi)$; so, the following result is both a (slight) strengthening and a consequence of Théorème 1 in [Gui].

Proposition 1. *(*[Gui]*) Let $(\pi,\mathcal{H})$ be an orthogonal or unitary representation of the σ-compact group G. Then $B^1(G,\pi)$ is closed in $Z^1(G,\pi)$ if and only if $(\pi^0,\mathcal{H}^0)$ does not weakly contain the trivial representation 1_G.*

Our approach to the proof of Theorem 1 uses the fact, observed in [BPV, §3.1] and [BeV] that $\overline{H}^1(G,\pi)$ is, in a natural way, a module over the (real or complex) von Neumann algebra $\pi(G)'$, which is the commutant of $\pi(G)$ in $\mathcal{B}(\mathcal{H})$; see Section 2. Viewing, as we do, $\overline{H}^1(G,\pi)$ as the Hilbert space $\mathrm{H}ar_\mu(G,\pi)$, one is lead to the study of $\mathrm{H}ar_\mu(G,\pi)$ as a Hilbert module over $\pi(G)'$.

For instance, if $\mathcal{M} := \pi(G)'$ is a finite von Neumann algebra (that is, if there exists a faithful finite trace on $\mathcal{M}$) then, we can define (as in [GHJ, Definition p.138] or [Bek, p. 327]) the *von Neumann dimension* of $\overline{H}^1(G,\pi)$ as

$$\dim_{\mathcal{M}} \overline{H}^1(G,\pi) := \dim_{\mathcal{M}} \mathrm{H}ar_\mu(G,\pi) \in [0,+\infty) \cup \{+\infty\};$$

for more details, see Section 2. It is worth mentioning that in case π is the regular representation of a discrete group G, $\dim_{\mathcal{M}} \overline{H}^1(G,\pi)$ coincides with $\beta_2^1(G)$, the L^2-Betti number of G (see [BeV, Proposition 2]).

We now give some applications of von Neumann techniques to the problem of the existence of an irreducible affine isometric action of G with a given linear part π. First, using Corollary 1, we can reformulate Corollary 3.7 from [BPV] in our setting. Recall that a vector v in a Hilbert module over a von Neumann algebra $\mathcal{M}$ is a *separating vector* for $\mathcal{M}$ if $Tv = 0$ for $T \in \mathcal{M}$ implies $T = 0$.

Proposition 2. *(*[BPV]*)*
(i) Assume that $\mathcal{M} = \pi(G)'$ has a separating vector b in $\mathrm{H}ar_\mu(G,\pi)$. Then $\alpha_{\pi,b}$ is irreducible.

(ii) Assume $B^1(G,\pi)$ is closed and that $\alpha_{\pi,b}$ is irreducible for some $b \in \mathrm{H}ar_\mu(G,\pi)$. Then b is a separating vector for $\mathcal{M}$.

For an application of the previous criterion in the case where G is a discrete finitely generated group and π a subrepresentation of a multiple of the regular representation of G, see [BPV, Theorem 4.25]. We extend this result to arbitrary factor representations, that is, to unitary representations $(\pi, \mathcal{H})$ such that the von Neumann subalgebra $\pi(G)''$ of $\mathcal{B}(\mathcal{H})$ generated by $\pi(G)$ is a factor (equivalently, such that $\pi(G)'$ is a factor). Concerning general facts about factors, such as their type classification, see [Dix1].

Theorem 2. *Let $(\pi, \mathcal{H})$ be a factor representation of the compactly generated locally compact group G on the separable complex Hilbert space $\mathcal{H}$. Assume that $B^1(G,\pi)$ is closed in $Z^1(G,\pi)$. Set $\mathcal{M} := \pi(G)'$ and let μ be a cohomologically adapted probability measure on G. Depending on the type of $\mathcal{M}$, there exists $b \in Z^1(G,\pi)$ such that $\alpha_{\pi,b}$ is irreducible if and only if:*

(i) *the factor $\mathcal{M}$ is of type I_∞ or of type II_∞ and its commutant in $\mathcal{B}(\mathrm{H}ar_\mu(G,\pi))$ is of infinite type (that is, of type I_∞ or II_∞, respectively);*
(ii) *the factor $\mathcal{M}$ is of finite type (that is, of type I_n for $n \in \mathbb{N}$ or of type II_1) and* $\dim_{\mathcal{M}} \mathrm{H}ar_\mu(G,\pi) \geq 1$;
(iii) *the factor $\mathcal{M}$ is of type III and* $\mathrm{H}ar_\mu(G,\pi) \neq \{0\}$.

Remark 2. Let $(\pi, \mathcal{H})$ be a unitary representation of G such that $B^1(G,\pi)$ is closed in $Z^1(G,\pi)$; let

$$\pi = \int_\Omega^\oplus \pi_\omega d\nu(\omega)$$

be the central integral decomposition of π, so that the π_ω's are mutually disjoint factor representations of G (see [Dix2, Theorem 8.4.2]). One checks that one has a corresponding decomposition of $\mathrm{H}ar_\mu(G,\pi)$ as a direct integral of Hilbert spaces:

$$\mathrm{H}ar_\mu(G,\pi) = \int_\Omega^\oplus \mathrm{H}ar_\mu(G,\pi_\omega) d\nu(\omega).$$

Moreover, $B^1(G,\pi_\omega)$ is closed in $Z^1(G,\pi_\omega)$ and there exists a separating vector for $\pi(G)'$ in $\mathrm{H}ar_\mu(G,\pi)$ if and only if there exists a separating vector for $\pi_\omega(G)'$ in $\mathrm{H}ar_\mu(G,\pi_\omega)$ for ν-almost every ω. So, Theorem 2 can be used to check the existence of an irreducible affine with *any* unitary representation π as linear part (provided $B^1(G,\pi)$ is closed in $Z^1(G,\pi)$).

As an illustration of the use of Theorem 2, we will treat the example of a wreath product of the form $\Gamma = G \wr \mathbb{Z}$ and a unitary representation π of Γ which factorizes through a representation of G; the reduced cohomology of such groups was considered in [Sha2, §5.4].

Theorem 3. *Let G be a finitely generated group, and let $(\pi, \mathcal{H})$ be a unitary representation of the wreath product $\Gamma = G \wr \mathbb{Z}$ in the separable Hilbert space $\mathcal{H}$. Assume that π factorizes through G and that $H^1(G, \pi) = 0$.*
(i) For a suitable cohomologically adapted probability measure μ on Γ, the space $\mathrm{H}ar_\mu(\Gamma, \mu)$ can be identified, as a module over $\pi(\Gamma)'$, with the Hilbert space $\mathcal{H}$.
(ii) There exists an irreducible affine action of Γ with linear part π if and only if the representation $(\pi, \mathcal{H})$ is cyclic.
(iii) Assume that G is not virtually abelian (that is, G does not have an abelian normal subgroup of finite index). Then G has a factorial representation π for which $\pi(G)'$ is of any possible type.

Remark 3. (i) When π is a factor representation, a necessary and sufficient condition for the existence of a cyclic vector for $\pi(G)$ (equivalently, a separating vector for $\pi(G)'$) in $\mathcal{H}$ is given in Theorem 2, with $\mathcal{H}$ replacing $\mathrm{H}ar_\mu(G, \mu)$ there.
(ii) By the Delorme-Guichardet theorem ([BHV, Theorem 2.12.4]), the condition $H^1(G, \pi) = 0$ is satisfied for every unitary representation π of G if (and only if) G has Kazhdan's property (T).

2. The space of harmonic cocycles as a von Neumann algebra module

Let G be a locally compact group which is generated by a compact subset Q, which we assume to be a symmetric neighbourhood of the identity $e \in G$. Let $(\pi, \mathcal{H})$ be an orthogonal or unitary representation of G. The map

$$b \mapsto \|b\|_Q = \sup_{x \in Q} \|b(x)\|$$

is a norm which generates the topology of uniform convergence on compact subsets and for which $Z^1(G, \pi)$ is a Banach space.

Let $\mathcal{M} := \pi(G)'$ be the commutant of $\pi(G)$ in $\mathcal{B}(\mathcal{H})$, that is,

$$\mathcal{M} = \{T \in B(\mathcal{H}) : T\pi(g) = \pi(g)T \quad \text{for all} \quad g \in G\};$$

$\mathcal{M}$ is a (real or complex) von Neumann algebra, that is, $\mathcal{M}$ is a unital self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed for the weak (or strong) operator topology.

As observed in [BPV, §3.1]), $H^1(G,\pi)$ is a module over $\mathcal{M}$; indeed, if $b \in Z^1(G,\pi)$ and $T \in \pi(G)'$, then $Tb \in Z^1(G,\pi)$, where Tb is defined by

$$Tb(g) = T(b(g)) \qquad \text{for all} \quad g \in G;$$

moreover, $T\partial_v = \partial_{Tv}$ for every vector $v \in \mathcal{H}$.

Let μ be a cohomologically adapted probability measure on G (Definition 1). We consider the Hilbert space $L^2(G,\mathcal{H},\mu)$ of measurable mappings $F : G \to \mathcal{H}$ such that

$$\|F\|_2^2 := \int_G \|F(x)\|^2 d\mu(x) < \infty.$$

Then every $b \in Z^1(G,\pi)$ belongs to $L^2(G,\mathcal{H},\mu)$; indeed, the cocycle relation shows that

$$\|b(x)\| \leq |x|_Q \|b\|_Q \qquad \text{for all} \quad x \in G,$$

and hence

$$\|b\|_2^2 \leq \|b\|_Q^2 \int_G |x|_Q^2 d\mu(x) < \infty.$$

In fact, the norms $\|\cdot\|_2$ and $\|\cdot\|_Q$ on $Z^1(G,\pi)$ are equivalent (see [ErO, Lemma 2.1]). So, we can (and will) identify $Z^1(G,\pi)$ with a closed subspace of the Hilbert space $L^2(G,\mathcal{H},\mu)$.

The von Neumann algebra $\mathcal{M}$ acts on $\mathcal{H}$ in the tautological way and on $L^2(G,\mathcal{H},\mu)$ by

$$TF(g) = T(F(g)) \qquad \text{for all} \quad T \in \pi(G)', F \in L^2(G,\mathcal{H},\mu), g \in G,$$

preserving $Z^1(G,\pi)$ and $B^1(G,\pi)$. Since the operator $M_\mu : Z^1(G,\mu) \to \mathcal{H}$ is equivariant for these actions, $\mathrm{H}ar_\mu(G,\pi) = \ker M_\mu$ as well as its orthogonal complement $\overline{B^1(G,\pi)}$ are modules over $\mathcal{M}$.

The image of $\mathcal{M}$ in $\mathcal{B}(L^2(G,\mathcal{H},\mu)) = \mathcal{B}(L^2(G,\mu)) \otimes \mathcal{H}$ is

$$\widetilde{\mathcal{M}} = I \otimes \pi(G)',$$

which is a von Neumann algebra isomorphic to $\mathcal{M}$. The orthogonal projection $P_{\mathrm{H}ar} : L^2(G,\mathcal{H},\mu) \to \mathrm{H}ar_\mu(G,\pi)$ belongs to the commutant

$$\widetilde{\mathcal{M}}' = \mathcal{B}(L^2(G,\mu)) \otimes \pi(G)''$$

of $\mathcal{M}$ in $\mathcal{B}(L^2(G,\mathcal{H},\mu))$, where $\pi(G)''$ is the von Neumann algebra generated by $\pi(G)$ in $\mathcal{B}(\mathcal{H})$. The commutant of $\mathcal{M}$ in $\mathrm{H}ar_\mu(G,\pi))$ is then the reduced von Neumann algebra (see Chap.1, §, Proposition 1 in [Dix1])

$$P_{\mathrm{H}ar}\widetilde{\mathcal{M}}'P_{\mathrm{H}ar} = P_{\mathrm{H}ar}(\mathcal{B}(L^2(G,\mu)) \otimes \pi(G)'')P_{\mathrm{H}ar}.$$

Assume now that $\mathcal{M}$ is a finite von Neumann algebra, with faithful normalized trace τ. Let $L^2(\mathcal{M})$ be the Hilbert space obtained from

τ by the GNS construction. We identify $\mathcal{M}$ with the subalgebra of $\mathcal{B}(L^2(\mathcal{M}))$ of operators given by left multiplication with elements from $\mathcal{M}$. The commutant of $\mathcal{M}$ in $\mathcal{B}(L^2(\mathcal{M}))$ is $\mathcal{M}' = J\mathcal{M}J$, where $J : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ is the conjugate linear isometry which extends the map $\mathcal{M} \to \mathcal{M}, x \mapsto x^*$. The trace on $\mathcal{M}'$, again denoted by τ, is defined by $\tau(JxJ) = \tau(x)$ for $x \in \mathcal{M}$.

The $\mathcal{M}$-module $L^2(G, \mathcal{H}, \mu)$ can be identified with an $\mathcal{M}$-submodule of $L^2(\mathcal{M}) \otimes \ell^2$, with $\mathcal{M}$ acting on $L^2(\mathcal{M}) \otimes \ell^2$ by $T \mapsto T \otimes I$. The orthogonal projection $Q : L^2(\mathcal{M}) \otimes \ell^2 \to L^2(G, \mathcal{H}, \mu)$ belongs to the commutant of $\mathcal{M}$ in $\mathcal{B}(L^2(\mathcal{M}) \otimes \ell^2)$, which is $\mathcal{M}' \otimes \mathcal{B}(\ell^2)$. The projection $P = P_{\mathrm{H}ar} \circ Q$ belongs therefore to the commutant of $\mathcal{M}$ in $\mathcal{B}(L^2(\mathcal{M}) \otimes \ell^2)$.

Let $\{e_n\}_n$ be a basis of ℓ^2 and let $(P_{ij})_{i,j}$ be the matrix of P with respect to the decomposition $L^2(\mathcal{M}) \otimes \ell^2 = \oplus_i (L^2(\mathcal{M}) \otimes e_i)$. Then every P_{ij} belongs to $\mathcal{M}'$ and the von Neumann dimension of the $\mathcal{M}$-module $\mathrm{H}ar_\mu(G, \pi)$ is

$$\dim_{\mathcal{M}} \mathcal{H} = \sum_i \tau(P_{ii}).$$

3. Proofs of the main results

3.1. **Proof of Theorem 1.** Let $b_0 \in \mathrm{H}ar_\mu(G, \pi)$. Let $b_1 \in \overline{B^1(G, \pi)}$ and set $b := b_0 + b_1$. We claim that $b_0(G)$ is contained in the closure of $\mathrm{span}(b(G))$.

Indeed, let $\mathcal{K}$ denote the closure of $\mathrm{span}(b(G))$ and $P_{\mathcal{K}} : \mathcal{H} \to \mathcal{K}$ the corresponding orthogonal projection. Since $\mathcal{K}$ is $\pi(G)$-invariant, $P_{\mathcal{K}}$ belongs to the commutant $\pi(G)'$ of $\pi(G)$. Therefore (see Section 2), $\mathcal{P}_{\mathcal{K}} b_0$ is contained in $\mathrm{H}ar_\mu(G, \pi)$ and $\mathcal{P}_{\mathcal{K}} b_1$ is contained in $\overline{B^1(G, \pi)}$. On the other hand, since b take its values in $\mathcal{K}$, we have that

$$P_{\mathcal{K}} b = b = b_0 + b_1.$$

It follows that $\mathcal{P}_{\mathcal{K}} b_0 = b_0$ and $\mathcal{P}_{\mathcal{K}} b_1 = b_1$. Therefore,

$$b_0(G) \subset \mathcal{K} = \overline{\mathrm{span}(b(G))},$$

as claimed. ■

3.2. **A characterization of irreducible affine isometric actions.** We will need for the proof of Corollary 1 one of the several characterizations of irreducible affine actions from Proposition 2.1 in [BPV]; for the convenience of the reader, we give a direct and short argument.

Proposition 3. *(*[BPV]*) For $b \in Z^1(G, \pi)$, the following properties are equivalent:*
(i) the action $\alpha = \alpha_{\pi,b}$ is irreducible;

(ii) the linear span of $(b+\partial_v)(G))$ *is dense in* $\mathcal{H}$ *for every* $v \in \mathcal{H}$.

Proof Observe that

$$\alpha_{\pi,b+\partial_v}(g) = t_{-v} \circ \alpha_{\pi,b}(g) \circ t_v \qquad \text{for all} \quad g \in G, \ v \in \mathcal{H},$$

where t_v is the translation by v. So, $\alpha_{\pi,b}$ is irreducible if and only if $\alpha_{\pi,b+\partial_v}$ is irreducible. This shows that (i) implies (ii).

To show the converse implication, let F be a non empty closed $\alpha_{\pi,b}(G)$-invariant affine subspace of $\mathcal{H}$. Then $F = v + \mathcal{K}$ for a vector $v \in \mathcal{H}$ and a closed linear subspace $\mathcal{K}$ of $\mathcal{H}$. Set $b_0 := b + \partial_v$. Then

$$v + b_0(g) = \alpha_{\pi,b}(g)v \in F \qquad \text{for all} \quad g \in G,$$

and $b_0(G)$ is hence contained in $\mathcal{K}$. Therefore, $\mathcal{K} = \mathcal{H}$, since $\mathrm{span}(b_0(G))$ is dense in $\mathcal{H}$.■

3.3. **Proof of Corollary 1.** Let $b \in Z^1(G,\pi)$ and set $b_0 := P_{\mathrm{H}ar} b \in \mathrm{H}ar_\mu(G,\pi)$.
(i) Assume that $\mathrm{span}(b_0(G))$ is dense in $\mathcal{H}$. By Theorem 1, the linear span of $(b+\partial_v)(G)$ is dense for every $v \in \mathcal{H}$, and Proposition 3 shows that $\alpha_{\pi,b}$ is irreducible.
(ii) Assume now that $B^1(G,\pi)$ is closed in $Z^1(G,\pi)$ and that $\alpha_{\pi,b}$ is irreducible. Write $b = b_0 + \partial_{v_0}$ for $b_0 = P_{\mathrm{H}ar} b$ and $v_0 \in \mathcal{H}$. Then $\alpha_{\pi,b_0} = \alpha_{\pi,b-\partial_{v_0}}$ is also irreducible, by Proposition 3; hence, $\mathrm{span}(b_0(G))$ is dense.■

3.4. **Proof of Theorem 2.** Let $(\pi,\mathcal{H})$ be a unitary representation of G; we assume that $B^1(G,\pi)$ is closed in $Z^1(G,\pi)$. Let μ be a cohomologically adapted probability measure on G.

In view of Proposition 2, we have to investigate under which conditions $\mathcal{M} = \pi(G)'$ has a separating vector in $\mathrm{H}ar_\mu(G,\pi)$. We may assume that $\mathrm{H}ar_\mu(G,\pi) \neq \{0\}$.

Observe that a vector in $\mathrm{H}ar_\mu(G,\pi)$ is separating for $\mathcal{M}$ if and only if it is cyclic for the commutant $\mathcal{N}$ of $\mathcal{M}$ in $\mathcal{B}(\mathrm{H}ar_\mu(G,\pi))$. Three cases cases can occur.

• *First case:* $\mathcal{N}$ is an infinite factor. Then $\mathcal{M}$ always has a separating vector (see Corollaire 11 in Chap. III, §8 of [Dix1]).

• *Second case:* $\mathcal{N}$ is a finite factor and $\mathcal{M}$ is an infinite factor. Then $\mathcal{N}$ has a cyclic vector in $\mathrm{H}ar_\mu(G,\pi)$ if and only if $\dim_{\mathcal{N}} \mathrm{H}ar_\mu(G,\pi) \leq 1$ (see [Bek, Corollary 1]). For this to happens a necessary condition is that $\mathcal{M}$ is a finite factor. So, $\mathcal{M}$ has no separating vector.

• *Third case:* $\mathcal{N}$ and $\mathcal{M}$ are finite factors. In this case, we have (see [GHJ, Prop. 3.2.5])

$$\dim_{\mathcal{M}} \mathrm{H}ar_\mu(G,\pi) \dim_{\mathcal{N}} \mathrm{H}ar_\mu(G,\pi) = 1;$$

hence, $\mathcal{M}$ has a separating vector in $\mathrm{H}ar_\mu(G,\pi)$ if and only if

$$\dim_{\mathcal{M}} \mathrm{H}ar_\mu(G,\pi) \geq 1.$$

Claims (i), (ii), and (iii) follow from this discussion.■

3.5. **Proof of Theorem 3.** We first consider the general case of the wreath product $\Gamma = G \wr H$ of two finitely generated groups G and H. Recall that $\Gamma = G \ltimes H^{(G)}$, for $H^{(G)} = \bigoplus_{g\in G} H$ and G acts on $H^{(G)}$ by shifting the copies of H. We view H as a subgroup of Γ, by identifying it with the copy of H inside $H^{(G)}$ indexed by e.

Let S_1 and S_2 finite symmetric generating sets for G and H, respectively. Then $S_1 \cup S_2$ is a finite symmetric generating set for Γ. Let μ_1 and μ_2 be cohomologically adapted probability measures on G and H respectively. Then $\mu = \dfrac{1}{2}(\mu_1 + \mu_2)$ is a cohomologically adapted probability measure on Γ.

Let $(\pi, \mathcal{H})$ be a unitary representation of G, viewed as representation of Γ. We have orthogonal $\pi(\Gamma)$-invariant decompositions

$$\ell^2(\Gamma, \mathcal{H}, \mu) = \ell^2(G, \mathcal{H}, \mu_1) \oplus \ell^2(H, \mathcal{H}, \mu_2)$$

and

$$\mathrm{H}ar_\mu(\Gamma,\pi) = \mathrm{H}ar_{\mu_1}(G,\pi) \oplus \mathrm{H}ar_{\mu_2}(H,\pi).$$

Since π is trivial on H, the space $Z^1(H,\pi)$ coincides with the set $\mathrm{H}om(H,\mathcal{H})$ of homomorphisms $H \to \mathcal{H}$. Observe that every $b \in \mathrm{H}om(H,\mathcal{H})$ is μ_2-harmonic, since

$$\sum_{h\in H} b(h)\mu_2(h) = \sum_{h\in H} b(-h)\mu_2(h) = -\sum_{h\in H} b(h)\mu_2(h).$$

Hence, $\mathrm{H}ar_{\mu_2}(H,\pi) = \mathrm{H}om(H,\mathcal{H})$ (alternatively, this follows from the fact that $B^1(H,\pi) = B^1(H, 1_H)$ is trivial); therefore, we have

$$\mathrm{H}ar_\mu(\Gamma,\pi) = \mathrm{H}ar_{\mu_1}(G,\pi) \oplus \mathrm{H}om(H,\mathcal{H}).$$

We specialize by taking $H = \mathbb{Z}$; then $\mathrm{H}om(H,\mathcal{H})$ can be identified with $\mathcal{H}$ and we have

$$\mathrm{H}ar_\mu(\Gamma,\pi) = \mathrm{H}ar_{\mu_1}(G,\pi) \oplus \mathcal{H};$$

moreover, the action of the von Neumann algebra $\pi(\Gamma)' = \pi(G)'$ on $\mathrm{H}ar_\mu(G,\mu)$ corresponds to the direct sum of the actions of $\pi(G)'$ on $\mathrm{H}ar_{\mu_1}(G,\mu_1)$ and on $\mathcal{H}$.

In particular, when the 1-cohomology $H^1(G,\pi)$ is trivial, we have

$$\mathrm{H}ar_\mu(\Gamma,\pi)=\mathcal{H},$$

so that Claim (i) is proved. Claim (ii) follows from Proposition 2.

To show Claim (iii), assume that G is not virtually abelian. Then G is not of type I, by Thoma's theorem ([Tho, Satz 6]).

First, observe that G has an irreducible unitary representation σ of infinite dimension; indeed, otherwise, G would be a liminal (or CCR) group and hence of type I, by [Dix2, 13.9.7]. Set $\pi=n\sigma$, a multiple of σ for $n\in\mathbb{N}$ or $n=\infty$; then $\pi(G)'$ is of type I_n.

Next, since G is not of type I, G has a factorial representation π such that both $\pi(G)''$ and $\pi(G)'$ are of type II_1, by [Tho, Lemma 19]. Then $\rho:=\infty\pi$ is factorial and $\rho(G)'$ is of type II_∞.

Finally, G has a factor representation such that $\pi(G)''$ (and hence $\pi(G)'$) is of type III, by Glimm's theorem [Gli, Theorem 1]). ■

4. An explicit formula for the projection on harmonic cocycles

We give an explicit formula for the orthogonal projection $P_{\mathrm{H}ar}$ in terms of an averaging (or Markov) operator associated to μ, in the case where $B^1(G,\pi)$ is closed.

Consider the operator $\pi^0(\mu)\in\mathcal{B}(\mathcal{H}^0)$ defined by

$$\pi^0(\mu)v=\int_G\pi(x)v\,d\mu(x)\qquad\text{for all}\quad v\in\mathcal{H}^0.$$

The operator $\pi^0(\mu)-I:\mathcal{H}^0\to\mathcal{H}^0$ is invertible if and only if π^0 does not weakly contain the trivial representation 1_G (see Proposition G.4.2 in [BHV]); in view of Proposition 1, this is the case if and only if $B^1(G,\pi)$ is closed.

Proposition 4. *Assume that $B^1(G,\pi)$ is closed. For $b\in Z^1(G,\pi)$, we have $P_{\mathrm{H}ar}b=b-\partial_v$, where*

$$v=(\pi^0(\mu)-I)^{-1}(M_\mu(b)).$$

Proof Indeed, observe first that $M_\mu(b)\in\mathcal{H}^0$; indeed, for every $w\in\mathcal{H}^G$, we have

$$\begin{aligned}\langle M_\mu(b),w\rangle&=\int_G\langle b(x),w\rangle d\mu(x)=\int_G\langle b(x),\pi(x)w\rangle d\mu(x)\\&=\int_G\langle\pi(x^{-1})b(x),w\rangle d\mu(x)=-\int_G\langle b(x^{-1}),w\rangle d\mu(x)\\&=-\int_G\langle b(x),w\rangle d\mu(x)=-\langle M_\mu(b),w\rangle.\end{aligned}$$

Moreover, for $v = (\pi^0(\mu) - I)^{-1}(M_\mu(b))$, we have

$$M_\mu(\partial_v) = \int_G (\pi(x)v - v)d\mu(x) = (\pi^0(\mu) - I)v = M_\mu(b).\blacksquare$$

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