On the History of Differentiable Manifolds

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Sommario

We discuss central aspects of the history of the concept of an affine differentiable manifold, as a proposal confirming the need for using some quantitative methods (drawn from elementary Model Theory) in Mathematical Historiography. In particular, we prove that this geometric structure is a syntactic rigid designator in the sense of Kripke-Putnam.

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1 Introduction

It is well-known (see, for instance, [39], [59]) that the sources of the modern concept of an affine differentiable manifold should be searched in the Weyl's work [65], where he gave an axiomatic description, in terms of neighborhoods (following Hilbert's work on the foundations of geometry), of a Riemann surface (that is to say, in modern terms, a real two-dimensional analytic differentiable manifold).

Moreover, the well-known geometrical works of Gauss and Riemann[1] are considered as prolegomena respectively of the topological and metric aspects

[1]Nevertheless, following what has been said in the *Introduction* to [42], we may say that *«for a modern reader, it is very tempting to regard his* [that is, of Riemann] *efforts as an endeavor to define a "manifold", and it is precisely the clarification of Riemann's ideas, as understood by his successors, which led gradually to the notions of manifold and Riemannian space as we know them today».*

of the structure of a differentiable manifold respectively in $\mathbb{R}^3$ and $\mathbb{R}^n$, $n \geq 3$ (see [6]).

All these common claims are well-established in the History of Mathematics, as witnessed, for example, by the crucial work of E. Scholz (see [59]).

Nevertheless, in this paper we would like to propose other possible viewpoints, about the same historical question, which are corroborated by some elementary methods of Model Theory applied to Mathematical Historiography. To be precise, we wish to show that Dini's work on implicit function theorems provides an essential syntactic tool, which was at the foundation of the modern theory of differentiable manifolds (see Examples 5 and 6, Section 1.1 of [26]).
We may think the Dini's theory on implicit functions as a theory, in a certain sense, deductively equivalent (from the syntactical point of view) to the modern abstract theory of differentiable manifolds, via the fundamental works of H. Whitney. For a modern treatment of the theory of differentiable manifolds strictly related to Dini's and Whitney's theorems (and for other interesting imbedding results), see [43].

Furthermore, in this perspective, we wish to relate (logically) Dini's work with some arguments and statements of Lagrange's Analytical Mechanics, in such a way that the latter may be seen as necessary physical (hence, semantical) and formal motivations to the birth of the structure of differentiable manifold (as we know it nowadays). At last (but not least), we prove that the geometric structure "differentiable manifold" is a mathematical entity that should be understood as a syntactic rigid designator in the sense of Kripke-Putnam.

2 The papers of Hassler Whitney

With the papers [67], [68] and [69], Hassler Whitney begins a detailed study of the structure of a differentiable manifold, mainly starting[2] from the works of O. Veblen and J.H.C. Whitehead (see next § 7).
Subsequently, he improves and extends part of these results: for instance, his celebrated imbedding theorem is first stated in [68] for compact manifold, and extended to every paracompact manifold in [70].

In the *Introduction* to [68], he says that

> «*A differentiable manifold may be defined in two ways: as a point set with neighborhoods homeomorphic with Euclidean spaces* $\mathbb{R}^n$ (hence, according to Weyl), *or as a subset of* $\mathbb{R}^n$ *defined near*

[2]See footnote [2] of page 645 of [68].

> *each point by expressing some of the coordinates in terms of others by differentiable functions* (hence, according to Dini, as we will see).
>
> *The first fundamental theorem is that the first definition is no more general than the second; any differentiable manifold may be imbedded in Euclidean space. In fact, it may be made into an analytic manifold in some* $\mathbb{R}^n$*»*.

In [68], Whitney uses many results of [67] and, especially, he uses some approximation theorems of the Weierstrass type (see I.6. of [68]).

In II.8. of [68], he proves (a first version of) the following, celebrated *imbedding theorem* (of *Whitney*)

> *«Any* C^r*- manifold of dimension* m *(with* $r \geq 1$ *finite or infinite) is* C^r*-homeomorphic with an analytic manifold in Euclidean space* $\mathbb{R}^{m+1}$*»*.

There is another fundamental theorem proved by Whitney in [68], namely the Theorem 2 (expounded in II.8., after the above mentioned Theorem 1), that plays a crucial role in the proof of the various Lemmas to Theorem 1. In the proof of Theorem 2 of [68], many results of the theory of real analytic functions and their approximations, are used.

Finally, we recall what he says in I.1. of [69]

> *«Let* $f_1, ..., f_{n-m}$ *be differentiable functions defined in an open subset of* $\mathbb{R}^n$*. At each point* p *at which all* f_i *vanish, let the gradients* $\nabla f_1, ..., \nabla f_{n-m}$ *be independent. Then the vanishing of the* f_i *determines a differentiable manifold* M *of dimension* m*. Any such manifold we shall say is in "regular position" in* $\mathbb{R}^n$*. Only certain manifolds are in regular position [...]. The purpose of the paper is to show that any* m*-manifold* M *in regular position in* $\mathbb{R}^n$ *may be imbedded in a* $(n-m)$*-parameter family of homeomorphic analytic manifold; these fill out a neighborhood of* M *in* $\mathbb{R}^n$*.*
>
> *We may extend the above definition as follows:* M *is in regular position if, roughly, there exist* $n-m$ *continuous vector functions in* M *which, at each point* p *of* M*, are independent and independent of vectors determined by pairs of points of* M *near* p*. If* M *is differentiable, the two definitions agree; the* ∇f_i *are the required vectors. The theorem holds also for this more general class of manifolds»*.

Clearly, the recalls to the Dini's work are evident.

Moreover, as has been made subsequently to the works of Whitney (see, for instance, § 1.1 and Theorem 3.2. of [27]; see also [43]), the Theorem 2 of [68], nowadays called *regular value theorem,* may be re-expressed and simplified through the implicit function theorem, starting from the original Whitney's proof, with a few modifications.

Furthermore, the implicit function theorems are at the basis of the important notion of transversality, a modern differential topology tool (see [27]) that specifies the intuitive concept of "generic position" (drawn from algebraic geometry) of a manifold.

However, we are mainly interested in the above fundamental Theorem 1, for the following reason: we shall use this imbedding theorem for proving a certain logical (syntactical) equivalence between the theory of differentiable manifolds according to Weyl (that is to say, the modern one) and that deducible by the work of U. Dini.

3 The implicit function theorem: a brief history

The most complete work on the history of implicit function theorem, is [31], to which we refer for a deepening of the subject; but, for the chief aspects of this history, see also [41].

The germs of the idea for the implicit function theorem, can be retraced both in the works of I. Newton, G.W. Leibniz, J. Bernoulli and L. Euler on Infinitesimal Analysis, and in the works of R. Descartes on algebraic geometry. Later on, in the context of analytic functions, J.L. Lagrange finds a theorem that may be seen as a first version of the present-day inverse function theorem (see also, [32], § 2, for the limitations of this theorem). We shall return on this question in regard to the influences of the Theory of Analytic Functions and Algebraic Geometry, in the birth of the modern notion of differentiable manifold.

Subsequently, A.L. Cauchy gives a rigorous formulation of the previous semi-theories of implicit functions assuming that they are expressible as power series, a restriction removed by Dini (see [8], p. 431).
Indeed, from here on, the implicit function theorem evolves until up the definitive Dini's generalized real-variable version (see [16], [17]), related to functions of any number of real variables.

But, only with Dini's works, we have a first complete, general and organic theory of implicit functions (at least, from the syntactic viewpoint).

4 The work of Ulisse Dini

Ulisse Dini (1845-1918) was a pupil of Ottavio Fabrizio Mossotti (1791-1863) and Enrico Betti (1823-1892). The former was a physicist and a mathematician, deeply influenced by the works of J.L. Lagrange[3], who taught Geodesy at the University of Pisa when Dini was a student. The latter was professor of Mathematical Physics at the University of Pisa and supervisor of the Dini's thesis.

In 1864, Dini publishes a paper on an argument of his thesis suggested by Betti. This first paper is followed by many other works on differential geometry and geodesy. In that period, Dini is in a scientific correspondence with E. Beltrami who took Geodesy chair left by the late Mossotti. At the same time, Dini is into touch with B. Riemann, at the time visiting professor at Pisa under Betti's interests.

In 1865, Dini spends one year of specialization in Paris under the supervision of J. Bertrand, where he continues his thesis arguments, with further researches in differential geometry, geodesy, algebra and analysis.

In 1866, Dini comes back to Pisa, where he starts his academic teaching career, as a professor of Geodesy and Advanced Analysis.

Nearly seventy, Dini settles an important work on a rigorous revision of the mathematical foundations of Analysis, with his celebrated *Lezioni di Analisi Infinitesimale* (see [16], [17]) and the *Fondamenti per la teorica delle funzioni di variabili reali* (see [18]). In these works, many original results and contributions of the Author are inserted: among these, the (so-called Dini's) theory of implicit functions, in [16], [17].

We are interested in the *Lezioni di Analisi Infinitesimale.*

These are the lessons given by the Author in the Academic Year 1876-1877 at the University of Pisa, and there exist two contemporaneous autographed (or lithographed) editions: the edition published by the printing-work Bertini, and the edition published by the printing-work Gozani. Both editions are in a unique volume, but divided into two parts: the first devoted to the Differential Calculus (with Chapters *I-XXXII*), the second devoted to the Integral Calculus (with Chapters *I-XXIII*).

Dini's theory on implicit functions is expounded in the following Chapter (of [16])

> *XIII. Derivate e differenziali dei vari ordini di funzioni implicite di una o più variabili indipendenti,*

[3] On the other hand, O.F. Mossotti was a close colleague and collaborator of G.A.A. Plana at Torino, and the latter was a pupil of J.L. Lagrange at the Paris École Polytechnique.

whereas, in the Chapter (of [16])

XV. Cangiamento delle variabili indipendenti,

Dini deals with certain forms of the so-called inverse function theorem. Finally, in the following Chapters (of [16]), Dini exposes some geometrical and analytical[4] applications of some theorems of the previous Chapters *XIII* and *XV.*

At the beginning of the 20^{th} century, Dini publishes a new revised and enlarged edition of the previous lessons [16], into two volumes (and each volume, into two parts). Nevertheless, as specified in the Preface to each volume, the new edition is different from the first only in notations and terminology, but not in the contents: indeed, he notices that the Editorial publication of these lessons is motivated by the will to giving a historical evidence to his teaching of 1876-1877, with a lesser provisional publication.

For our purposes, we are interested in part 1^a and part 2^a of the vol. I of [17]; the part 1^a, with total pages 372, contains the Chapters *I-XVII,* where the last Chapter has the following title

XVII. Massimi e minimi delle funzioni di una o più variabili indipendenti.

The part 2^a, with total pages 345, starts with the following headline

– APPLICAZIONI GEOMETRICHE DEL CALCOLO DIFFERENZIALE –

and contains the Chapters *XVIII-XXXVI.* It is completely devoted to the geometrical applications of the tools and methods developed in part 1^a: indeed, it is a very, organic treatise on differential geometry, fully based on the previous lessons [16]. Above all, in the Chapters *XIX-XXXVI* he uses extensively the theory of implicit functions (of the previous Chapters *XIII* and *XV* of part 1^a): for a modern (only) terminological reformulation of these Dini's (geometric) applications, see for example Cap. 2 of [2].

5 The paper of Henry Poincaré

Following E. Scholz ([59]; see also [40]), in the paper [54] may be found another possible source of the concept of a manifold.

[4]Where, among other things, the Author introduces the famous *Dini's numbers* of Mathematical Analysis.

In fact, H. Poincaré, in § 1 and § 3 of [54], gives a constructive definition of (unilateral/bilateral[5]) manifold as follows.
If $x_1, ..., x_n$ are generic variables of $\mathbb{R}^n$ ($n \geq 2$), then he considers the following system of p equalities and q inequalities

$$(1) \qquad \begin{cases} F_1(x_1, ..., x_n) = 0 \\ \quad .... \\ F_p(x_1, ..., x_n) = 0 \\ \varphi_1(x_1, ..., x_n) > 0 \\ \quad .... \\ \varphi_q(x_1, ..., x_n) > 0, \end{cases}$$

with F_i, φ_j continuous and uniform functions, with continuous derivatives in such a way that $J = \left\| \dfrac{\partial F_i}{\partial x_k} \right\| \neq 0$ in each point of the common definition domain of the F_i. If $p = 0$, we have a *domain.*

The system (1) defines a manifold of dimension $m = n - p$, that, when[6] $q = 0$, it is possible to prove (see § 3 of [54]) to be equivalent to a manifold defined by a system of equations of the following type

$$(2) \qquad \begin{cases} x_1 = \theta_1(y_1, ..., y_m) \\ \quad .... \\ x_n = \theta_n(y_1, ..., y_m). \end{cases}$$

Again, the (syntactic) recalls to the implicit function theory, are evident.

However, the main historical interest of the paper [54] is known to be related to the origins of Algebraic Topology, and not to the (possible) concept of differentiable manifold (see [58]).

6 The work of Hermann Weyl

The first definition of a complex two-dimensional topological manifold, as we know it nowadays, is exposed in § 4 of [65], while in § 6 of [65], the Author gives the notion of a differentiable structure on such a manifold type.

Weyl's analysis starts from a geometrical representation of an analytic form (according to K. Weierstrass and Riemann), and attaining to a particular structure of (Riemann) surface[7], through the new topological developments achieved

[5]The distinction between unilateral and bilateral manifolds is given in § 8 of [54]. We refer to the bilateral case.

[6]Henceforth, if not otherwise specified, when we shall consider the equivalence between (1) and (2), it is understood that $q = 0$.

[7]This is not a surface, in the sense of Analysis Situs.

by D. Hilbert and others. In particular, the local Hausdorff's concept of "neighborhood" of a point, has played a crucial role in the Weyl's construction of a topological manifold.

Moreover, some geometrical aspects of Complex Analysis of that time, have also played a fundamental (syntactic) role in the Weyl's work (as we shall see later).

The central Weyl's idea is that of local homeomorphism of a manifold with $\mathbb{R}^n$.

Subsequently, Weyl introduces a differentiable structure on a topological manifold by means of such a local homeomorphism of this manifold with $\mathbb{R}^n$, taking into account some previous works of F. Klein.

For our purposes, it is necessary to examine such little known works of F. Klein on Riemann surfaces.

Klein wrote a fundamental monograph[8] on the concept of a Riemann surface, more general than the formulation used by Riemann in his studies on the theory of analytic functions.

Klein based his work on previous Riemann's studies on Abelian functions, on the fundamental 1870 paper of H.A. Schwarz on the integration of the bidimensional Laplace equation $\Delta u = 0$, and on a 1877 paper of R. Dedekind. In all these works, there are some first results relative to a particular class of $\mathbb{R}^n$-imbedded surfaces, generated by analytic functions.

Klein also known other works on $\mathbb{R}^n$-imbedded surfaces as, for instance, those of A. Tonelli (*Atti della R. Accademia Reale dei Lincei, ser. II, v. 2, 1875*), W.K. Clifford (1876), F. Prym (1874) and P. Koebe.

As said by Weyl himself, these works of Klein seem to assume an important role in the (Weyl's) definition of a differentiable structure on a manifold.

Furthermore, Klein's *Erlangen Program* viewpoint seems to be at the base of Weyl's definition of compatibility relations among local coordinate systems of a generic point of the manifold, since he introduces a group of local coordinate transformations Γ, that leaves fixed the origin of $\mathbb{R}^2$; such a group characterizes the manifold, and Weyl speaks of a *surface of type* Γ.

Later on, in [66] the author makes use of what is said in [65], with applications to General Relativity.

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[8]Entitled *Über Riemann's Theorie der algebraischen Funktionen und ihrer Integrale,* Leipzig, 1882. See also F. Klein, "Neue Beiträge zur Riemannschen Funktionentheorie", *Mathematische Annalen*, 21 (1883).

7 The works of O. Veblen and J.H.C. Whitehead

O. Veblen and J.H.C. Whitehead, in the paper [63] (and, more extensively, in [64]), introduce two definitions of a n-dimensional (regular) affine manifold through three groups of axioms.

In the *Introduction,* the Authors define

> *«a manifold as a class of elements, called points, having a structure which is characterized by means of coordinate systems»,*

where the notion of (local) *coordinate system* is the same of the Weyl's one. Next, they introduce the notion of *regular transformation* by means of Dini's implicit function theorem (see p. 552 of [63]). This notion is put at the foundation of the definition of *regular manifold,* through the further notion of *pseudo-group* of transformations (see [44], [30] or [11]), via three groups of axioms that, on the whole, characterize the concept of manifold (see § 5).

The next sections of [63] are devoted to the consistency and independence of the previous groups of axioms, to some topological considerations and to a few analytic applications.

Also in this case, Dini's implicit function theorems play a crucial role in the definition of manifold, since this is characterizable as an abstract entity locally diffeomorphic to $\mathbb{R}^n$, via allowable – through regular transformations – local coordinate systems[9] (see Examples 5. and 6., Section 1.1 of [26], and also the next paragraph).

8 The role of Dini's theory on implicit functions in differential geometry

In this paragraph, we want to put in evidence the existence of important logical (and historical) links between the theory of implicit functions, as settled by Ulisse Dini, and the construction of the abstract theory of a (topological) affine manifolds.

It is possible to build up a theory of affine manifolds in $\mathbb{R}^n$, by means of the Dini's implicit function theorem and the inverse function theorem: see, for instance, [50], [51], [52], [53] – in particular [52], parte I, Capitolo 2, § 2 and parte II, Capitolo 7, § 3 – and [15], secondo volume, Cap. V.

Implicit function theorem and inverse function theorem, characterize the local structure of any manifolds (see the "parametrization" technique of [60],

[9]Besides, the Authors devote § 2 of Chapter III of [63], to explain the Implicit Function Theorem, as a fundamental tool that will be used in the remaining text.

Cap. 5; see also Chapter 5 of [25]): to this end, see, for instance, Theorems 3 and 4, Chapter 5, of [62].

Moreover, a manifold (in $\mathbb{R}^n$) may be thought, in a certain sense, as the zero values of a given system of functions of the type (1) (equivalent to (2)), already discussed in the previous § 5.

Here, we do not develop the detailed calculations connected with these claims, since we have other interests and aims. Nevertheless, it is necessary to recall the main definitions and theorems related to such a question, following, respectively, the expositions given by [22] in Chapters VII and VIII, and by [55] in Chapter 4.

According to the exposition given by [22], the local character of the Dini's implicit function theorems led to important applications even having local character: among these, there are the inverse function theorems (or local invertibility theorems).

Roughly speaking, a differentiable manifold is a subset $\Gamma \subseteq \mathbb{R}^n$ that may be locally represented as a set of zeros of a many-variables function whose Jacobian matrix has maximum rank. For example, we may consider a surface $\Gamma \subseteq \mathbb{R}^3$ given by $g(x_1, x_2, x_3) = 0$ with $\nabla_x g \neq 0$ for each $x = (x_1, x_2, x_3) \in \Gamma$, or the geometric entity Γ given by the non-degenerate intersection of p (≥ 2) hyperplanes $\Gamma_1, ..., \Gamma_p$ of $\mathbb{R}^n$. In this last case, if Γ_i is represented by the linear function $g_i(x) = \sum_{j=1}^n a_{ij} x_j, \quad x = (x_1, ..., x_n), \ i = 1, ..., p$, then Γ is represented by the zeros of the linear function $g(x) = (g_1(x), ..., g_p(x))$, so that, if $\Gamma = Ker \ g$, then $dim \ \Gamma = dim \ Ker \ g = n - dim \ Im \ g = n - rank \ A$ where $A = \|a_{ij}\|$; moreover, we suppose that $det \ A \neq 0$. Finally, if we wish that such a Γ have dimension $m \in \mathbb{N}$ with $m < n$, then we must impose that either $p = n - m$ and $rank \ A = n - m$, or $p = rank \ A = n - m$. Therefore, if we extend these last examples to the case in which g is non-linear, then we should impose that its Jacobian matrix have maximum rank, and since this is variable with the variation of the points of Γ, it follows that the representation of Γ as sets of zeros can only have a local nature.

Generalizing this, we have that Dini's theorem implies that a manifold may be locally thought either as non-degenerate intersection of diagrams of regular functions (definition 1) and as images of regular functions (definition 2), in both these cases the Jacobian matrices having maximum rank; furthermore, from the pointwise variability of the Jacobian matrix, it follows the possibility of introducing local coordinate systems.

Hence, a first definition of manifold arises when we consider this latter as the result of gluing together many pieces each of which is a curved (due to the non-linearity of the various functions g) subset of $\mathbb{R}^n$ obtained intersecting a subspace (of $\mathbb{R}^n$) with an open set (of $\mathbb{R}^n$). Precisely, we have the following

Definition 1. Let $\Gamma \subseteq \mathbb{R}^n$, $m \in \mathbb{N}$ with $m < n$, and $k \in \mathbb{N}$ or $k = \infty$. Then we say that Γ is a *C^k-manifold* of $\mathbb{R}^n$ with dimension m, when, for each $x_0 \in \Gamma$, there exists an open neighborhood I of x_0 and a function $g \in C^k(I, \mathbb{R}^{n-m})$, such that $\Gamma \cap I = \{x; x \in I, g(x) = 0\}$ and $rank \ J(g)(x) = n - m$ for each $x \in \Gamma \cap I$.

Here, $J(g)(x)$ is the Jacobian matrix of g computed in x.

The following definition of a manifold arises when we consider it locally identified as image of regular functions. Exactly, we have the following

Definition 2. Let $\Gamma \subseteq \mathbb{R}^n$, $m \in \mathbb{N}$ with $m < n$, and $k \in \mathbb{N}$ or $k = \infty$. Then we say that Γ is *locally the diagram* of a m-variables C^k-function when, for each $x_0 = (x_{10}, ..., x_{n0}) \in \Gamma$, there exists two open neighborhood I' and I'' respectively of the points $(x_{10}, ..., x_{m0})$ and $(x_{(m+1)0}, ..., x_{n0})$, and there is a C^k-function $h : I' \to I''$, such that, setting $I = I' \times I''$, we have

$$\Gamma \cap I = \{(x_1, ..., x_n); (x_1, ..., x_n) \in I, (x_{m+1}, ..., x_n) = h(x_1, ..., x_m)\},$$

unless unessential permutations of $x_1, ..., x_n$. In such a case, we say that Γ has a structure of a C^k*-manifold* with dimension m.

The latter is the definition of a manifold via *parametrizations,* which are the result of a formalization of the geographical mapping that put into bijective correspondence a geographical chart C with a certain zone C' of Earth's surface; in such a way, it is evident that the tools and methods of the Geodesy have played a fundamental role in developing the intuitive idea of what a manifold can be[10]. Indeed, from this last point of view, we reach the following

Definition 3. Let $\Gamma \subseteq \mathbb{R}^n$, $m \in \mathbb{N}$ with $m < n$, and $k \in \mathbb{N}$ or $k = \infty$. If $x_0 \in \Gamma$ and Ω is an open set of $\mathbb{R}^n$, then a *local* m*-chart* of class C^k of Γ at x_0 is an injective C^k-function $r : \Omega \to \mathbb{R}^n$ such that there exists an open neighborhood I of x_0 in such a way that $\Gamma \cap I = r(\Omega)$ and *rank* $J(r)(t) = m$ for any $t \in \Omega$. An m*-atlas* of class C^k of Γ is a family $\{r_i\}_{i \in \Xi}$ (Ξ is a set of indices) of local m-charts of class C^k such that the union of the related image sets is Γ. Finally, Γ is a C^k*-manifold* of dimension m if it has a m-atlas of class C^k. The parametric map $r^{-1} : \Gamma \cap I \to \Omega$ provides a *local coordinate system* in such a way that, if $x \in \Gamma \cap I$, then the Cartesian coordinates $t_1, ..., t_m$ of $r^{-1}(x)$ are said to be the *local coordinates* of x with respect to the given local coordinate system.

Now, we consider a particular, simple situation that allows us to put into evidence that certain conditions, imposed on the rank of the various Jacobian matrices, are necessary in order that be possible to prove the equivalence among the above mentioned definitions of manifold.

Let $n = 3, m = 2$ and Γ be a plane of $\mathbb{R}^3$ containing the origin of $\mathbb{R}^3$. Such a plane may be considered as the set of zeros of a suitable linear operator with rank $n - m = 1$; let $a_1x_1 + a_2x_2 + a_3x_3 = 0$ such an operator with $(a_1, a_2, a_3 =) \neq 0$, and, for instance, let $a_3 \neq 0$. Hence we have $x_3 = px_1 + qx_2$, so that such a plane is also the diagram of a linear operator from $\mathbb{R}^2$ to $\mathbb{R}$ (that is to say, from $\mathbb{R}^m$ to $\mathbb{R}^{n-m}$); finally, the same plane has the following parametric equations $x_1 = t_1, x_2 = t_2$ and $x_3 = pt_1 + qt_2$, so that it is the image of the linear operator $(t_1, t_2) \to (t_1, t_2, pt_1 + qt_2)$, operating from $\mathbb{R}^2$ to $\mathbb{R}^3$ (that is to say, from $\mathbb{R}^m$ to $\mathbb{R}^n$), with rank 2 (that is to say, m) whatever be p, q. From here, in the more general case in which $\Gamma \subseteq \mathbb{R}^n$ be a subspace of dimension $m(< n)$, it is possible to prove that Γ may be represented as the set of zeros of the linear map associated to a certain $(n-m, m)$-matrix with maximum rank, as the diagram of a certain linear operator from $\mathbb{R}^m$ to $\mathbb{R}^{n-m}$, and as the image of a certain linear operator from $\mathbb{R}^m$ to $\mathbb{R}^n$ with rank m.

Finally, if Γ is a manifold, then instead having to do with a linear operator (like in the previous examples, in which such a linear operator globally represents Γ), we shall have to do with regular non-linear operators providing (in general, only) a local representation of such a manifold. The fundamental tools which allow us to get such a local representation are the Dini's theorem and the inverse function theorem. Indeed, as seen in the above mentioned example concerning a plane of $\mathbb{R}^3$, it has been necessary to solve an implicit equation with respect to one of its three variables, so that, in the general case, it will be necessary to solve a system of the type $g(x) = 0$ with respect to $n - m$ out of its n variables, and the

[10]Till to the first middle of the 20-th century, Geodesy was a common subject-matter of mathematical academic studies.

implicit function theorem is the most natural tool for solving such a problem. This theorem, nevertheless, may only provide local representations, also in the case in which the manifold is globally given as the set of zeros of a unique function.

In Theorem 1.11 of Chapter VIII of [22], it is proved the following, fundamental result:

Theorem 1. *Let $\Gamma \subseteq \mathbb{R}^n$, $m \in \mathbb{N}$ with $m < n$, and $k \in \mathbb{N}$ or $k = \infty$. Then, the following conditions are equivalent:*

1. *Γ is a C^k-manifold of dimension m, according to the Definition 1;*
2. *Γ is locally the diagram of a m-variables function of class C^k, according to the Definition 2;*
3. *Γ is a manifold having a m-atlas of class C^k, according to the Definition 3.*

In the proof of implication *1.* $\Rightarrow$ *2.* it is used the Dini's implicit function theorem, whereas in the proof of the implication *3.* $\Rightarrow$ *1.* it is used the inverse function theorem. Among other things, the above theorem 1. provides a useful criterion to verify whether a certain subset of $\mathbb{R}^n$ is a manifold or not.

Lastly, we observe that Dini's implicit function theorem and the inverse function theorem are strictly correlated of each other. The above exposition, drew from [22], starts from Dini's theorems toward inverse function theorems. Instead, according to the exposition of [55], it is possible to start from inverse function theorems toward Dini's theorems.

For instance, Chapter 4 of [55] begins with problems concerning possible inversions of differentiable functions between $\mathbb{R}^n$-type spaces, hence with problems of local inversion of functions of many variables. The first historical methods related to this problem type concerns the class of differentiable functions, as the differential of a function is the first, natural linear approximation tool for these functions, and we have a large class of results for linear applications (like the differential map) suitable to answer to the above mentioned inversion problems. Therefore, the principle of the method consist in a generalization of what is known about linear maps to the more extended class of differentiable maps. Because of the local nature of the differential map, it is clear that the so obtained results from this generalization, have a local character as well.

In § 2 of [55], the author deals with some problems concerning the local inversion of maps. If $\Omega \subseteq \mathbb{R}^m$ and $\Lambda \subseteq \mathbb{R}^n$ are non-void open sets, then let $f : \Omega \to \Lambda$ be a continuous map; if $\bar{x} \in \Omega$, then let $\bar{y} = f(\bar{x})$. We say that f is *locally injective* on $\bar{x}$ if there exists a neighborhood U of $\bar{x}$ such that $f_{|U}$ is an injective map. We say that f is *locally surjective* on $\bar{x}$ if, for any neighborhood U of $\bar{x}$, $f(U)$ is a neighborhood of $\bar{y}$.

As regard the local inversion problems of maps, let us consider the following examples.

Given two open sets $\Omega, \Lambda \subseteq \mathbb{R}^n$, for a[11] C^1-map $f : \Omega \to \Lambda$ to be invertible in a point $x \in \Omega$, it is necessary and sufficient that its Jacobian matrix in x, say $J(f)(x)$, is not singular when $n = m$); so, we obtain a characterization of the local invertibility of a C^1-function in the case $n = m$.

The general case of arbitrary $n, m \in \mathbb{N}$, is as follows.

Let $f : \Omega \to \mathbb{R}^n$ be a C^1-map defined on an open set $\Omega \subseteq \mathbb{R}^m$; if we want to study locally f in a neighborhood of a point $x \in \Omega$, then we have to consider the rank of the Jacobian matrix $J(f)(x)$ of f at x (that represents the differential of f at x).
If $r_x = rank\ J(f)(x)$, then $r_x \leq \min\{n, m\}$ for every $n, m \in \mathbb{N}$ and $x \in \Omega$, the local invertibility of f in x being possible only when r_x is the highest.

[11] The C^1-regularity hypothesis is a fundamental one.

Therefore, we first consider the case $r_x = \min\{n, m\}$, in such a way that it remains maximum in a neighborhood of x, since $f \in C^1(\Omega, \mathbb{R}^n)$.

Let $r_x = m < n$. In such a case, we have the following inverse function theorem:

Theorem 2. *Let $f : \Omega \to \mathbb{R}^n$ be a C^1-map defined on an open set $\Omega \subseteq \mathbb{R}^m$, and let $r_x = rank\ J(f)(x) = m$. Then, f is locally injective in x, and, moreover, the image, through f, of an open neighborhood of x, is a regular*[12] *Cartesian graph with base an open subset of $\mathbb{R}^m$.*

For a proof, see Theorem 4.3 of [55].

Instead, if $r_x = n < m$, then f is locally surjective, so that it follows the problem of studying the inverse image of every point $y \in \mathbb{R}^n$ that lies into a neighborhood of $f(x)$. To this end, Dini's theorems are fundamental tools for the resolution of such a problem. For instance, if we consider the case-study $m = 2$ and $n = r_x = 1$, then it holds the following Dini's implicit function theorem

Theorem 3. *Let $\mathbb{R}^2 = \mathbb{R}' \times \mathbb{R}''$ with $\mathbb{R}' \cong \mathbb{R}'' \cong \mathbb{R}$. Let $f : \Omega \to \mathbb{R}$ be a continuous function defined on an open set $\Omega \subseteq \mathbb{R}^2$ with f_y continuous on it; let $P_0 = (x_0, y_0) \in \Omega$ be a point such that $f(x_0, y_0) = 0$ and $f_y(x_0, y_0) \neq 0$. Then, f is locally surjective in P_0. Moreover, there exists a neighborhood U of x_0 on $\mathbb{R}'$ and a neighborhood V of y_0 on $\mathbb{R}''$, such that the set of zeros of f on $U \times V$ is a regular Cartesian diagram with base U, that is to say, there exists a neighborhood W of 0 on $\mathbb{R}$, such that, for each $z \in W$, the set $\{(x, y); (x, y) \in \Omega, f(x, y) = z\} \cap (U \times V)$ is a regular Cartesian diagram with base U.*

Such a theorem is applied to the study of the set of zeros of a real function f of two variables: for example, if f is a function verifying the same hypotheses of the previous theorem, $\Gamma_f = \{(x, y); (x, y) \in \Omega, f(x, y) = 0\}$ and there is a point $(\bar{x}, \bar{y}) \in \Gamma_f$ such that $(f_x(\bar{x}, \bar{y}), f_y(\bar{x}, \bar{y})) \neq (0, 0)$, then Γ_f, in a neighborhood of $(\bar{x}, \bar{y})$, is of the form $y = \varphi(x)$ or $x = \psi(y)$, for certain C^1-functions φ or ψ.
This result fails into the degenerate case $f_x(\bar{x}, \bar{y}) = f_y(\bar{x}, \bar{y}) = 0$, that is to say, on the singular points of Γ_f.

In the general case, we have the following Dini's theorem

Theorem 4. *Let $f : \Omega \to \mathbb{R}^n$ be a C^1-function defined on an open set $\Omega \subseteq \mathbb{R}^m$, with $n < m$. If $r_{\bar{x}} = rank\ J(f)(\bar{x}) = n$ in a point $\bar{x} \in \Omega$, then f is locally surjective on $\bar{x}$. Moreover, there exists a neighborhood V of $\bar{y} = f(\bar{x})$, a neighborhood U of $\bar{x}$ and a $(m - n)$-dimensional open set V'' of $\mathbb{R}^m$ such that, for every $\bar{y} \in V$, $f^{-1}(\{\bar{y}\}) \cap U$ is a regular Cartesian diagram with base V''.*

For a proof (making use of the above mentioned theorem 3.), see Theorem 4.8 of [55].

Finally, we have functional dependence in the case in which $r_x < \min\{n, m\}$ and, in general, it is no longer true that such a value r_x remains constant in a neighborhood of x. Nevertheless, in such a case, if we suppose that such a value r_x remains constant in, at least, one neighborhood of x, then we have the following

Theorem 5. *Let $f : \Omega \to \mathbb{R}^n$ be a C^1-map defined on an open set $\Omega \subseteq \mathbb{R}^m$. Given a point $\bar{x} \in \Omega$, we suppose that $r_{\bar{x}} = rank\ J(f)(\bar{x}) < \min\{n, m\}$ is constant in a neighborhood of $\bar{x}$. Then, locally, the image of f is a regular Cartesian diagram, say Γ_f, with base an open subset of a coordinated r-dimensional subspace of $\mathbb{R}^n$. Moreover, the inverse image of an arbitrary point of Γ_f, is a Cartesian diagram with base an open subset of a $(m - r)$-dimensional coordinated space of $\mathbb{R}^m$.*

At this point, the author introduces the notion of a differentiable manifold on $\mathbb{R}^m$.

[12] That is to say, a diagram of class C^1.

Precisely, if we wish to introduce particular subsets of $\mathbb{R}^m$ that are locally like some affine numerical space $\mathbb{R}^n$, with $n \leq m$, then the above mentioned theorems are fundamental tools for this problematic context.

This problem has, in general, only solutions of local nature: for example, it is well-known that a circle of $\mathbb{R}^2$ and a line, are locally homeomorphic but not globally; on the other hand, the intersection point of two distinct lines is not even locally homeomorphic to a point of a line.

Thus, a (topological) n-dimensional *manifold* of $\mathbb{R}^m$ (with $n \leq m$) is a subset $\Gamma \subseteq \mathbb{R}^m$ such that every point of it, has a neighborhood homeomorphic to some open subset of $\mathbb{R}^n$, namely, for each $x \in \Gamma$, there exists an open neighborhood U of x on $\mathbb{R}^n$, an open set V of $\mathbb{R}^n$ and a bijective continuous map $r : V \to U \cap \Gamma$ with continuous inverse; in such a case, we say that r is a *local coordinate system* (or a *local chart*) of x.
In general, further properties are required to holding for such a map r: among these, we mainly require that it is continuously differentiable (or of class C^k, with $k \in \mathbb{N}$ or $k = \infty$), and, in such a case, we speak of a *differentiable chart* of class C^1 (or of class C^k).

If every point $x \in \Gamma$ has a differentiable chart of class C^k, then we say that Γ has the structure of a n-dimensional *differentiable manifold* of class C^k.

We have the following[13]

Theorem 6. *For a subset $\Gamma \subseteq \mathbb{R}^m$ to be a n-dimensional differentiable manifold of class C^k, it is necessary and sufficient that, for every $x \in \Gamma$, there exists an open neighborhood U of x such that $\Gamma \cap U$ is a Cartesian diagram of class C^k, with base an open subset B of a n-dimensional coordinated space.*

For a proof (making use of the above mentioned Theorem 2.), see Theorem 6.4 of [55].

From the previous Theorems 2. and 6., it follows that any inverse local chart $r^{-1} : \Gamma \cap U \to V$ can be factorized into $r^{-1} = \Delta \circ p$ where p is the canonical projection of the given Cartesian diagram (of Theorem 6.) over the base B, whereas $\Delta : B \to V$ is a C^k-bijective map with continuous inverse.

At this point, a natural question is to treat the case in which a same point $x \in \Gamma$ is into two distinct local charts, say r_1 and r_2. Exactly, let $r_i : V_i \to \Gamma \cap U, \quad i = 1, 2$ be two local charts on the same open neighborhood U with $x \in U$; then, it is possible to prove (see Theorem 6.6 of [55]) that $r_2^{-1} \circ r_1$ and $r_1^{-1} \circ r_2$ are real homeomorphisms of class C^k: the proof follows from the decomposition $r^{-1} = \Delta \circ p$.
The differentiability properties of a manifold lies just in the differentiability of its transition maps among allowable coordinate systems, and it is clear that these last properties do not subsist in the abstract case, that is to say, these must be explicitly postulated: from here, it follows the abstract (Weyl's) definition of a differentiable manifold. Nevertheless, the author himself (see Remark 6.7 of [55]) says that the degree of (syntactic) logical generality of the abstract theory of differentiable manifolds is no higher than that of the real differentiable manifold theory, because of the works of Whitney. However, the axiomatic approach has methodological and pragmatic advantages since, for instance, we may define such a structure over arbitrary mathematical objects (with a some predefined topology).

Finally, we may define a differentiable manifold by means of Dini's Theorem 4. other than through the inverse function theorem (see the above Theorem 2) as done in Theorem 6., as follows

Theorem 7. *For a subset $\Gamma \subseteq \mathbb{R}^m$ to be a n-dimensional differentiable manifold of class C^k (with $n \leq m$) it is necessary and sufficiency that, for each $\bar{x} \in \Gamma$, there exists an*

[13] The Theorem 6., among other, is a useful criterion for determining whether a subset Γ is a manifold or not.

open neighborhood (of $\mathbb{R}^n$) and a C^k-function $\psi : U \to \mathbb{R}^{m-n}$ with maximum rank on U such that $\Gamma \cap U = \{x; x \in U, \psi(x) = 0\}$.

In other words, the latter says that there are $m - n$ real C^k-functions $\psi_1, ..., \psi_{m-n}$, defined on U and whose Jacobian matrix has rank $m-n$, such that[14] $\Gamma = \bigcap_{i=1}^{m-n} \Gamma_i$, having put $\Gamma_i = \{x; x \in U, \psi_i(x) = 0\}$ for $i = 1, ..., m-n$.

For a proof (making use of the above mentioned Dini's Theorem 4.), see Theorem 6.8 of [55].

This last theorem, assures us that a n-dimensional differentiable manifold of class C^k is locally representable as the set of zeros of a certain multivalued function.

In conclusion, from the viewpoint of the treatment given by [55], Chapter 4, the inverse function theorem is related to the problem of local injectivity of a regular function, whereas Dini's theorem is related to the problem of local surjectivity of a regular function. From both these points of view, we may get a definition of a differentiable manifold (respectively, like in Theorem 6. as regard the problem of local injectivity, and like in Theorem 7. as regard the problem of local surjectivity) in $\mathbb{R}^n$, so that it is evident the historical importance played by Dini's works on implicit function theorem regarding the foundations of modern differential geometry.

However, it would be an historical mistake to think that Ulisse Dini had in mind such a manifold theory (although in $\mathbb{R}^n$): in fact, he only settled the fundamental *syntactic tools* need for the next modern constructions of an abstract affine manifold, although it might be probable that some problems of $\mathbb{R}^n$-imbedded surfaces (as seen in the previous § 6) had been (more or less unconsciously) at the base of his work[15].

As we shall see later, there is no (explicit) semantic link between Dini's work on implicit functions and the theory of manifolds; there holds, instead, only strong links of syntactic nature (that, despite all, have a proper historical importance, as we shall see in the next paragraphes, as regard the notion of syntactic rigid designator).

We have already mentioned the possible role played by Algebraic Geometry (see, for instance, [31]) and Complex Analysis in regard to the mind-setting of modern concept of a differentiable manifold. We wish to outline some a few words about these last aspects.

The work of H. Weyl, as seen in § 6, is centered around the study of the geometrical representation of certain analytic functions.
On the other hand, we also remember that, for instance, Salvatore Pincherle, in Chapter XI of [49], exposes the implicit functions theory in the complex context, following Dini's work in the real case. In Chapter XII, he applies what has been said in the previous one, to the algebraic functions theory, whereas, in Chapter XIII, he resumes Lagrange's work on inverse function theorem in

[14] The maximum rank condition assures that such an intersection is non-degenerate.

[15] This last consideration has to be considered valid only in an semi-intuitive context, or like a kind of insight, with respect to the Dini's work on implicit function theorems.

view of its analytical applications. This plan is common to all major treatises on Analytic Function Theory of that time.

From all that, it is possible to guess (as, for instance, made by [40]) some not negligible influences of the 19-th century Algebraic Geometry, in the developments of some aspects of the Theory of Differentiable Manifolds, because many algebraic geometry tools and methods are applied to the study of the so-called Riemannian surfaces of an algebraic function[16].

A posteriori, these conjectures find some partial (syntactic) confirmations by the so-called *Nash-Tognoli imbedding theorems* of Algebraic Geometry (see [5], Chapter 14), a sort of algebraic geometry analogous of the Whitney's theorems, proving that any compact smooth manifold is diffeomorphics to a well-defined nonsingular real algebraic manifold.

Hence, the works of 19-th century algebraic geometers, it should be also considered have had some influences on the possible sources of the modern theory of differentiable manifolds. Nevertheless, just limited to the aim of the present paper, the comparison with the Nash-Tognoli theorems mentioned above, seems do not have had a great historical importance within the question related to the born of modern theory of differentiable manifolds, differently by the case of Dini's and Whitney's works (see next § 11).

At this point, it is necessary formally to introduce the minimal Model Theory notions, which will be essential for the following critical remarks: indeed, we want to introduce these basic quantitative tools to precise better, in a rigorous manner, the previous historical remarks, as well as to establish rigorously the historical relevance of the possible syntactic links among theories, in our case, between theory of functions and differential geometry.

9 Some notions of Model Theory

According to [10], roughly speaking, Model theory is Universal Algebra plus Logic. In this section, we recall some notions of Model Theory, need for the following. Our main references are [12], [13], [14], [36], [21], [38].

9.1 Syntactic and semantic models

Every scientific axiomatic theory has both a syntactic component and a semantic one, and, often, these two components are mixed of each other into a concrete (that is, non-axiomatic or intuitive) scientific theory.

[16] This is an historical remark which should be thought back in regard to André Weil work on algebraic manifolds.

Therefore, in general, the formalization process of a scientific theory is an axiomatization process working out over an initial structure of intuitive theory, towards an abstract (axiomatic) structure, called *model.*

The Model Theory deals with problems and methods of such a construction. In this problematic context, syntactic and semantic questions arise: for instance, the works of K. Gödel and A. Tarski show the possible existence of a non-contradictory syntactically closed theories, and the non-existence of a non-contradictory semantically closed theories. Hence, there exist limitative theorems on the syntactic and semantic capacities of an axiomatic theory.

Nevertheless, from these limitations, it also follows the reciprocal inseparability of the syntactic and semantic components.

9.1.1 Syntactic models

The formalization process leads to the so-called notion of *formal system.* It is composed both by syntactic components and semantic ones. In this section, we expose the syntactic aspects.

An *elementary* (*syntactic*) *formal system* (or a *syntactic theory*) $\mathfrak{F}$ is a tuple $\mathfrak{F} = \langle\langle \mathcal{L}, \mathcal{D} \rangle\rangle$ with *language scheme* $\mathcal{L} = \langle Al, Te, Wr, E \rangle$ and *deductive scheme* $\mathcal{D} = \langle Ax, Ru \rangle$, where

- there exists disjoint sets Co, Qu, Fu, Pr, Va, Au, in such a way that $Al = \bigcup\{Co, Qu, Fu, Pr, Va, Au\}$ is the *alphabet* (or the *set of symbols*) of $\mathfrak{F}$, with Co the set of *logic connectives,* Qu the set of *logic quantifiers,* Fu the set of *functors,* Pr the set of *predicates,* Va the set of *individual variables* and Au the set of *auxiliary symbols* (with $Co \cup Qu$ the set of *logic constants* and $Pr \cup Fu$ the set of *descriptive constants* or *vocabulary*);

- Wr is the set of *words,* Te ($\subseteq Wr$) is the set of *atomic terms,* E ($\subseteq Wr$) is the set of *atomic expressions* (with $Te \cup E$ the set of *well-formed words,* and $Prop$ the set of *propositions* defined as a subset of E whose elements have no free variables);

- Ax ($\subseteq E$) is the set of (*logic* and *specific*) *axioms,* whereas Ru is the set of *logic deduction rules* (with respect to a given Logic).

Note. In this section, from now on, we only speak of a formal system (or theory), without specifies the term "syntactic".

$\mathcal{L}$ determines the set of (*explicit* and *implicit*) *definitions* (say De) of $\mathfrak{F}$, whereas $\mathcal{D}$ determines the set of *proofs* (say Pf), and the set of *theorems* (say Th), of $\mathfrak{F}$.

Therefore, a formal system (a theory) is a tuple of the type

$$\mathfrak{F} = \langle\langle \mathcal{L}, \mathcal{D} \rangle\rangle = \langle\langle\ \langle Al, Te, Wr, E, De \rangle, \langle Ax, Ru, Pf, Th \rangle\ \rangle\rangle. \tag{1}$$

We may think to $\mathcal{D}$ as the predicative, or propositional, or enunciative calculus of a theory $\mathfrak{F}$.

If α is a theorem of $\mathfrak{F}$, we write $\vdash_{\mathfrak{F}} \alpha$. If an expression α of $\mathfrak{F}$ is a logical derivation by a set of expressions M of $\mathfrak{F}$, then we write $M \vdash_{\mathfrak{F}} \alpha$.

If the set of axioms Ax is decidable, then $\mathfrak{F}$ is said to be *axiomatizable,* whereas, if the set of specific axioms is finite, then $\mathfrak{F}$ is said to be *finitely axiomatizable.*

If $\mathcal{L}$ is a formal [not formal (or intuitive)] language, then we say that $\mathfrak{F}$ is a *formal* [*not formal*] theory.

We should make some clarifications about the elements of Fu and Pr. Fu is the class of all n-functor $Fu^n = \{f_i^n\}_{0\leq i<j}$ with $0 \leq j \leq \omega$ and $0 \leq n < \omega$, where Fu^0 is the set of *individual constants,* with $Fu^n = \emptyset$ if $j = 0$. Pr is the class of all n-predicate $Pr^n = \{P_i^n\}_{0\leq i<j}$ with $0 \leq j \leq \omega$ and $0 < n < \omega$; Pr^2 contains, at least, the element P_0^2 said to be the *identity* predicate, with $Pr^n = \emptyset$ if $j = 0$.

Let $\mathfrak{F}_1, \mathfrak{F}_2$ be two theories of the type (1); we say that

- $\mathfrak{F}_2$ is a *predicative linguistic extension* of $\mathfrak{F}_1$ when $Pr_1 \subseteq Pr_2$;
- $\mathfrak{F}_2$ is a *functorial linguistic extension* of $\mathfrak{F}_1$ when $Fu_1 \subseteq Fu_2$;
- $\mathfrak{F}_2$ is a *linguistic extension* of $\mathfrak{F}_1$ (and we write $\mathcal{L}_1 \subseteq \mathcal{L}_2$) when $\mathfrak{F}_2$ is a predicative and functorial linguistic extension of $\mathfrak{F}_1$;
- $\mathfrak{F}_2$ is a *deductive extension* of $\mathfrak{F}_1$ when $Ax_1 \subseteq Th_2$;
- $\mathfrak{F}_2$ is a *theoretical extension* of $\mathfrak{F}_1$ (or that $\mathfrak{F}_1$ is a *sub-theory* of $\mathfrak{F}_2$) when $\mathfrak{F}_2$ is a deductive and linguistic extension of $\mathfrak{F}_1$; in such a case, we write $\mathfrak{F}_1 \preccurlyeq \mathfrak{F}_2$, and we say that $\preccurlyeq$ is the *theoretical inclusion relation*;
- a theoretical extension $\mathfrak{F}_2$ of $\mathfrak{F}_1$ is a *linguistically invariant* extension when $\mathcal{L}_1 = \mathcal{L}_2$, that is to say, when $\mathfrak{F}_2$ is an improper linguistic extension of $\mathfrak{F}_1$;
- a theoretical extension $\mathfrak{F}_2$ of $\mathfrak{F}_1$ is an *inessential*[17] extension when $Th_1 = Th_2 \cap E_1$.

[17] See next *Remark 1.*

If $\mathfrak{F}_1 \preccurlyeq \mathfrak{F}_2$ and $\mathfrak{F}_2 \preccurlyeq \mathfrak{F}_1$, then we say that $\mathfrak{F}_1$ is *equivalent* to $\mathfrak{F}_2$, and we write $\mathfrak{F}_1 \approx \mathfrak{F}_2$; we say that $\approx$ is the *theoretical equivalence relation.*

We refer to [13], Capitolo 1, § 3, Definizione 7, for the definition of the elements of De (the set of predicative and functorial definitions of a theory $\mathfrak{F}$).

We say that $\mathfrak{F}_2$ is a *simple definitional extension* of $\mathfrak{F}_1$ if and only if there exists a predicative [functorial] definition $\delta^{P_i^n}$ [$\delta^{f_i^n}$] in $\mathfrak{F}_2$, such that

1. $P_i^n \notin Pr_1$ [$f_i^n \notin Fu_1$];
2. $Pr_2 = Pr_1 \cup \{P_i^n\}$, $Fu_2 = Fu_1$ [$Fu_2 = Fu_1 \cup \{f_i^n\}$, $Pr_2 = Pr_1$];
3. $Ax_2 = Ax_1 \cup \{\delta^{P_i^n}\}$ [$Ax_2 = Ax_1 \cup \{\delta^{f_i^n}\}$].

We say that $\mathfrak{F}_2$ is a *definitional extension* of $\mathfrak{F}_1$ when there exists a sequence of theories $\mathfrak{F}_{k_1}, ..., \mathfrak{F}_{k_p}$ $(1 < p \leq \omega)$, such that:

1. $\mathfrak{F}_1 = \mathfrak{F}_{k_1}$ and $\mathfrak{F}_2 = \mathfrak{F}_{k_p}$;
2. for each $1 \leq i \leq \omega$, $\mathfrak{F}_{i+1}$ is a simple definitional extension of $\mathfrak{F}_i$.

In other words,

$$\mathfrak{F}_1 = \mathfrak{F}_{k_1} \to ... \to \mathfrak{F}_{k_i} \to ... \to \mathfrak{F}_{k_p} = \mathfrak{F}_2 \qquad 1 < i < p,$$

is a chain of simple definitional extensions.

Every simple definitional extension is a (proper) deductive and linguistic extension as well. Moreover, we have the following

Theorem 1. *If $\mathfrak{F}_2$ is a [simple] definitional extension of $\mathfrak{F}_1$, then $\mathfrak{F}_2$ is an inessential extension of $\mathfrak{F}_1$.*
For a proof, see [13], Capitolo 1, § 3, Teoremi 5, 6.

Remark 1. The Theorem 1. is the final result of a part of the works of Giuseppe Peano, Alessandro Padoa and Mario Pieri on the logical analysis of formal systems; a consequence of the so-called (*Peano-Padoa-Pieri*) *non-creativity principle*[18] of the logical definitions, is that the definitions (elements of De) of a formal theory $\mathfrak{F}$, should not determine deductive novelties[19] but only expressive novelties. From here, it follows why a [simple] definitional extension is proved to be "inessential".

If $\mathcal{L}$ is a pure syntactic [or not syntactic] language, then we say that $\mathfrak{F}$ is a *pure syntactic* [*not pure syntactic*] theory. This last classification leads us to an extra-syntactic area, as we will see later, when we shall introduce the notion of semantic model.

[18] See [38], Cap. III, § 5, and Cap. VI, § 2, or [4], Cap. I.

[19] That is, the definitions should not involve the demonstrability of new theorems, or rather, it should not widen the deductive capacity of a theory.

We now introduce the notions of theoretical homomorphisms (for details, see [13], Capitolo 2).

Let $\mathfrak{F}_1, \mathfrak{F}_2$ be two theories of the type (1).

A *theoretical representation* of $\mathfrak{F}_1$ into $\mathfrak{F}_2$ is a map $\rho : Wr_1 \to Wr_2$; hence, we write $\rho : \mathfrak{F}_1 \to \mathfrak{F}_2$.

Remembering that $E, Th \subseteq Wr$, we can say that a theoretical representation $\rho : \mathfrak{F}_1 \to \mathfrak{F}_2$ is

- an *expressive homomorphism* if $\rho(E_1) \subseteq E_2$;
- a *theorematical homomorphism* if ρ is an expressive homomorphism and $\rho(Th_1) \subseteq Th_2$;
- a *deductive homomorphism* if $\rho(Pr_1) \subseteq Pr_2$.

A deductive homomorphism is a theorematical homomorphism as well. This last classification defines the so-called class of *theoretical homomorphisms.*

A theoretical representation $\rho : \mathfrak{F}_1 \to \mathfrak{F}_2$ is said to be

- a *version* of $\mathfrak{F}_1$ into $\mathfrak{F}_2$, if there exists a map (called the *base* of this version) $\psi : Fu_1 \cup Pr_1 \to Th_2 \cup E_2$, satisfying a certain set of compatibility properties (see [13], Cap. 2, § 1, Def. 3, a));
- a *quasi-relativization* of $\mathfrak{F}_1$ into $\mathfrak{F}_2$, if there exists an expression $\alpha(v) \in E_2$ (v is a free variable) and a map $\psi : Fu_1 \cup Pr_1 \to Th_2 \cup E_2$, verifying a set of compatibility properties (see [13], Cap. 2, § 1, Def. 3, b)); we say $B_\rho =< \alpha(v), \psi >$ to be the *base* of this quasi-relativization;
- a *relativization* of $\mathfrak{F}_1$ into $\mathfrak{F}_2$, if there exists a quasi-relativization ρ' of $\mathfrak{F}_1$ into $\mathfrak{F}_2$, with base $B_{\rho'} =< \alpha(v), \psi >$, in such a way that $\rho(\beta) \Rightarrow \rho'(\beta)$ for each $\beta \in E_1$, and $\rho(\beta) = \rho'(\beta)$ for each $\beta \in Pr_1$.

Versions, quasi-relativizations and relativizations, are expressive homomorphisms.

A theorematical homomorphism ρ of $\mathfrak{F}_1$ into $\mathfrak{F}_2$ is said to be

- a *translation,* if $\rho(\neg\beta) = \neg\rho(\beta)$ for each $\beta \in Pr_1$;
- an *interpretation,* if ρ is a version of $\mathfrak{F}_1$ into $\mathfrak{F}_2$;
- a *relative interpretation,* if ρ is a relativization of $\mathfrak{F}_1$ into $\mathfrak{F}_2$;
- an *isomorphism,* if $\rho : Wr_1 \to Wr_2$ is a bijection such that $\rho(Ax_1) = Ax_2$, and there exists a map $\psi : Al_1 \to Al_2$, commuting with ρ, such that $\psi(Fu_1) \subseteq Fu_2, \psi(Pr_1) \subseteq Pr_2, \psi(Va_1) \subseteq Va_2, \psi(Au_1) \subseteq Au_2, \psi(Co_1 \cup Qu_1) \subseteq Co_2 \cup Qu_2$.

Therefore, we say that $\mathfrak{F}_1$ is *translatable, interpretable,* and *relatively interpretable* into $\mathfrak{F}_2$ if, respectively, there exists a translation, an interpretation, and a relative interpretation of $\mathfrak{F}_1$ into $\mathfrak{F}_2$. We say that $\mathfrak{F}_1$ is *isomorphic* to $\mathfrak{F}_2$ if there exists an isomorphism between $\mathfrak{F}_1$ and $\mathfrak{F}_2$, and we write $\mathfrak{F}_1 \sim \mathfrak{F}_2$.

An isomorphism is a deductive homomorphism as well, but not conversely, in general (see [13], Capitolo 2, § 1, Teorema 5).

It is possible to prove (see [13], Cap. 2, § 2, Teorema 1) the following

Theorem 2. *If $\mathfrak{F}_1 \preccurlyeq \mathfrak{F}_2$, then $\mathfrak{F}_1$ is translatable, relatively interpretable and interpretable into $\mathfrak{F}_2$; moreover, if it is also $\mathcal{L}_1 = \mathcal{L}_2$ and $\mathfrak{F}_1$ is isomorphic to $\mathfrak{F}_2$, then $\mathfrak{F}_1 \approx \mathfrak{F}_2$, the converse being not true, in general.*

The relations of translatability, relative interpretability and interpretability, are pre-orders.

We have the following chain of implications (see [13], Capitolo 2, § 1, Teoremi 6, 7, 8, 9)

$$\text{Isomorphism} \Rightarrow \text{Interpretation} \Rightarrow$$

$$\Rightarrow \text{Relative Interpretation} \Rightarrow \text{Traducibility}.$$

If a representation ρ, inducing a certain theoretical homomorphism [isomorphism], is computable, then we speak of an *effective* theoretical homomorphism [isomorphism]. If $\mathfrak{F}_1$ is relatively interpretable into $\mathfrak{F}_2$, then we say that $\mathfrak{F}_1$ has a *syntactic model* into $\mathfrak{F}_2$, and we write $\mathfrak{F}_1 \precsim \mathfrak{F}_2$.

It is important the following

Theorem 3. *If $\mathfrak{F}_1$ is [relatively] interpretable in $\mathfrak{F}_2$, then $\mathfrak{F}_2$ has a definitional extension $\mathfrak{F}_2'$ containing a sub-theory $\mathfrak{F}_1'$ isomorphic to $\mathfrak{F}_1$.*

For a proof, see [13], Capitolo 2, § 2, Teoremi 10, 11.

Among the theoretical homomorphisms defined above, for our historiographical purposes, we are interested in the interpretable and relatively interpretable ones. The adjective "interpretable" leads us towards the semantic context. To each formal theory $\mathfrak{F} = \langle\langle \mathcal{L}, \mathcal{D} \rangle\rangle$ of the type (1), it is associable a particular universe U, that is, the set of truth values of its statements (propositions, theorems, expressions, and so on); its choice is independent[20] by the syntactic structure of $\mathfrak{F}$.

Therefore, the interpretability of $\mathfrak{F}_1$ into $\mathfrak{F}_2$, means that it is always possible to give an interpretation of the concepts of $\mathfrak{F}_1$ in the terms of the concepts of $\mathfrak{F}_2$, in such a way that what $\mathfrak{F}_1$ says to be true with respect to its universe U_1, is also true − by means of such an interpretation − with respect to the universe U_2 of $\mathfrak{F}_2$.

[20] We shall take again this argument in the semantic context.

Instead, the relative interpretation of $\mathfrak{F}_1$ into $\mathfrak{F}_2$, means that it is always possible to give an interpretation of the concepts of $\mathfrak{F}_1$ in terms of the concepts of $\mathfrak{F}_2$, but in such a way that what $\mathfrak{F}_1$ says to be true with respect to its universe U_1, is also true with respect to a particular sub-universe U_α of U_2, determined by the relativization condition $\alpha(v)$ (of the base $B_\rho =< \alpha(v), \psi >$ of the given representation $\rho : \mathfrak{F}_1 \to \mathfrak{F}_2$).

At this point, it is possible to apply these considerations to the historiographical context, as follows. Indeed, a central problem in the Historiography of Exact Sciences, is the determination of the possible relations among different theories, as, for instance, those that hold among a concrete (or intuitive) theory and its formalizations[21].

A first rational (or quantitative) comparison of this last type, it is possible, for instance, when one takes into account the possible existence of a theoretical representation among the theories under comparison: for example, if there exists an interpretation, or a relative interpretation, of a theory $\mathfrak{F}_1$ into a theory $\mathfrak{F}_2$, then we can say that $\mathfrak{F}_1$ is, in a certain sense, «included» into $\mathfrak{F}_2$.

Analogously, the possible determination of a syntactic model (and the possible theoretical connections that it may provide) gives a useful criterion for the «reducibility» of a theory into another. These types of (syntactic) connections, provide "natural" interpretations of certain theories into others, also in the case in which their (historical) sources are very far off.

Nevertheless, for methodological motivations, we should consider such syntactic comparison criteria, with the suitable cautions.

However, at this point, we may do a simple historical application of what has been said above. If $\mathfrak{F}_1^{Dini}$ is the theory of differentiable manifolds in the Dini's sense, while $\mathfrak{F}_2^{Weyl}$ is the theory of differentiable manifolds in the Weyl's sense (that is, the modern one), then it is obvious that $\mathfrak{F}_1^{Dini}$ is interpretable into $\mathfrak{F}_2^{Weyl}$.
On the other hand, by means of Whitney's theorems, we can say too that $\mathfrak{F}_2^{Weyl}$ is interpretable into $\mathfrak{F}_1^{Dini}$. By Theorem 3, it follows that $\mathfrak{F}_1^{Dini} \sim \mathfrak{F}'^{Weyl}_2 \preccurlyeq \tilde{\mathfrak{F}}_2^{Weyl}$ and $\mathfrak{F}_2^{Weyl} \sim \mathfrak{F}'^{Dini}_1 \preccurlyeq \tilde{\mathfrak{F}}_1^{Dini}$, for certain definitional extensions $\tilde{\mathfrak{F}}_i$ of $\mathfrak{F}_i$ $i = 1, 2$. Moreover, we may suppose the equality[22] between the languages of $\mathfrak{F}_1^{Dini}$ and $\mathfrak{F}'^{Weyl}_2$, and of $\mathfrak{F}_2^{Dini}$ and $\mathfrak{F}'^{Weyl}_1$, so that, by Theorem 2., we have $\mathfrak{F}_1^{Dini} \approx \mathfrak{F}'^{Weyl}_2$ and $\mathfrak{F}_2^{Weyl} \approx \mathfrak{F}'^{Dini}_1$. From here, it does not follow the (syntactic) equivalence $\mathfrak{F}_1^{Dini} \approx \mathfrak{F}_2^{Weyl}$, but a "minor" equivalence, as follows. If we take

[21] Although, it would be more correct to consider such a type of logical comparison only among theories having almost the same syntactic degree of formalization.

[22] In fact, even by Whitney's works, it is no restrictive to think any abstract smooth n-manifold as a closed subset of some $\mathbb{R}^N$ (with $N = N(n) > n$), locally representable (according to Dini) as an intersection of the diagrams of a system of differentiable functions defined on some common open subset of $\mathbb{R}^n$, with values into $\mathbb{R}^s$, $\quad s = N - n$.

into account the notion of deductive equivalence, then we may say that $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are *deductively equivalents,* and we write $\mathfrak{F}_1 \simeq \mathfrak{F}_2$, when $\mathfrak{F}_2$ is a deductive extension of $\mathfrak{F}_1$, and vice versa. Therefore, if we take into account what has been said in *Remark 1,* about the inessentiality of the definitional extensions, then we may set $\mathfrak{F}_i \simeq \tilde{\mathfrak{F}}_i$ $i = 1, 2$. Thus, the relations $\mathfrak{F}_1^{Dini} \preccurlyeq \tilde{\mathfrak{F}}_2^{Weyl} \simeq \mathfrak{F}_2^{Weyl}$ and $\mathfrak{F}_2^{Weyl} \preccurlyeq \tilde{\mathfrak{F}}_1^{Dini} \simeq \mathfrak{F}_1^{Dini}$, implies the following deductive equivalence $\mathfrak{F}_1^{Dini} \simeq \mathfrak{F}_2^{Weyl}$.
On the other hand, it is clear that this equivalence cannot be extended to the theoretical syntactic equivalence $\approx$, because there is no linguistic equivalence between $\mathfrak{F}_1^{Dini}$ and $\mathfrak{F}_2^{Weyl}$: indeed, in $\mathfrak{F}_1^{Dini}$, there exists neither the explicit nor the implicit definition of manifold. In conclusion, $\mathfrak{F}_2^{Weyl}$ is a proper linguistic and inessential extension of $\mathfrak{F}_1^{Dini}$.

Another, almost equivalent way leading to the same conclusions (about the relations between $\mathfrak{F}_1^{Dini}$ and $\mathfrak{F}_2^{Weyl}$), is centered around the (logic) immersion theorems (see [36], Cap. 2, § 2.3), through which we have $\mathfrak{F}_1^{Dini} \sim \mathfrak{F}_2^{Weyl}$.

Let $\mathcal{T}$ be the class of all possible elementary theories, and $\mathfrak{T} = \mathcal{T}/\approx$ the set of equivalence classes of $\mathcal{T}$, with respect to the equivalence relation $\approx$. If $\preccurlyeq_{ri}$ is the relation of relative interpretability, then $(\mathfrak{T}, \preccurlyeq_{ri}^*)$ is a pre-ordered set, putting $[\mathfrak{F}_1] \preccurlyeq_{ri}^* [\mathfrak{F}_2]$ if and only if $\mathfrak{F}_1 \preccurlyeq_{ri} \mathfrak{F}_2$ (this is a well-posed definition).

We call *rational power* of a theory $\mathfrak{F}$, its equivalence class $[\mathfrak{F}] \in (\mathfrak{T}, \preccurlyeq_{ri}^*)$: intuitively, $[\mathfrak{F}]$ is the class of all theories $\mathfrak{F}'$ containing a sub-theory 'which says the same things said' by $\mathfrak{F}$, whereas, on its turn, $\mathfrak{F}$ contains a sub-theory 'which says the same things said' by $\mathfrak{F}'$.

Analogously, if $\preccurlyeq_{eri}$ denotes the effective relative interpretation relation, we have that $(\mathfrak{T}, \preccurlyeq_{eri}^*)$ is a pre-ordered set; $[\mathfrak{F}] \in (\mathfrak{T}, \preccurlyeq_{eri}^*)$ is said to be the *rational content* of $\mathfrak{F}$, and, intuitively, it 'contains everything said by $\mathfrak{F}$ and, also, everything said' by the weaker theories of $\mathfrak{F}$.
Since it is possible to prove the existence of a (syntactic) isomorphism between $(\mathfrak{T}, \preccurlyeq_{ri}^*)$ and $(\mathfrak{T}, \preccurlyeq_{eri}^*)$, the unique formal entity they determined, is called a *theoretical pre-order.*

Therefore, it is possible to consider this theoretical pre-order, as a tool to determine a certain "axiological scale of importance" among theories; further, it may turn out to be also useful in certain historical classifications of the 'importance' of a theory identified by its rational content. Moreover, such a pre-ordering may correspond to the historical development of the theories, so that it is evident the usefulness of the syntactic tools here exposed, in the possible historic-critical comparison between theories.

9.1.2 Semantic models

In this section, we discuss the elementary semantic aspects of a (syntactic) formal system.

The emergence of the semantic context, has the following motivations. The above exposed syntactic methods, may turn out to be useful when we are mainly interested in the syntactic comparison of theories: for instance, with these methods, it is possible a comparison of theories with different languages.

Nevertheless, the historical comparison is often oriented towards a language comparison, and the syntax shows its own limits[23] with respect to this framework. A method to avoid these limits, consists in the introduction of the so-called Metamathematical Semantics.

Roughly speaking, the Semantics studies the sets of possible meanings (or interpretations) associable to syntactic symbols.
In [13], Capitolo 4, it is possible to find a purely abstract formalization of the Semantics; instead, we are interested in a more extended setting, suitable to historical questions. To this end, we refer to [12], [14], [21] and [38].

We follow the algebraic viewpoint of the Semantics, as developed by the Polish school. One of the central concepts of Algebraic Semantics is that of (Peirce-Schröder) *logical matrix,* built up on a syntactic system $\mathfrak{F} = \langle\langle \mathcal{L}, \mathcal{D} \rangle\rangle$. Such a logical matrix is a tuple of the type $\mathcal{M} = \langle\langle \mathfrak{F}, \mathfrak{D} \rangle\rangle$, where $\mathfrak{D}$ is the set of the so-called *appointed* (or *designated*) *values,* defined as follows. If $\mathcal{C} = Fu \cup Pr$ is the set of descriptive constants[24] of $\mathfrak{F}$, $\mathcal{U}$ is a *possible world* (or a *universe of discourse*) and $v : \mathcal{C} \to \mathcal{U}$ is a *valuation,* then $\mathcal{R} = (\mathcal{U}, v)$ is said to be a (Frege) *extensional interpretation* of $\mathfrak{F}$. Therefore, we may define (extensively) $\mathfrak{D}$ as follows: for each formula $\mathcal{F}$ of $\mathfrak{F}$, it is $v(\mathcal{F}) \in \mathfrak{D}$ if and only if $\mathcal{F}$ is true. $\mathcal{F}$ is a tautology if and only if $v(\mathcal{F}) \in \mathfrak{D}$ for every valuation v. If $E_v(\mathcal{M})$ is the set of all formulas true under v (that is to say, such that $v(\mathcal{F}) \in \mathfrak{D}$), then we set $E(\mathcal{M}) = \bigcap_v E_v(\mathcal{M})$. In such a way, the logical matrix generalizes the concept of (Tarski-Huntington-Bernstein) *deductive system* (or *deductive theory*); in general, $\mathfrak{F}$ is a Boolean algebra and $\mathfrak{D}$ is a filter on $\mathfrak{F}$ (instead, the set of not true formulas, is an ideal of this algebra). We say that $\mathcal{R}$ is a *semantic interpretation* of the language $\mathcal{L}$ of $\mathfrak{F}$. We may also write $\mathcal{M} = \langle\langle \mathfrak{F}, \mathcal{R} \rangle\rangle = \langle\langle \mathfrak{F}, (\mathcal{U}, v) \rangle\rangle$, instead of $\mathcal{M} = \langle\langle \mathfrak{F}, \mathfrak{D} \rangle\rangle$.

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[23]There are further problematic limits of the syntactic context: for instance, there exists finiteness problems, connected with the attempts to avoid the impossible identification between mathematical truth and demonstrability, that led to the failure of the Hilbert's formalistic program.

[24]Descriptive constants (or atomic propositions) and specific axioms, characterize (syntactically) a formal theory.

We can now introduce the fundamental notion of Lindenbaum-Tarski algebra.

If $\mathcal{M}$ is a deductive theory (according to Tarski), we define the following pre-order

$$\phi \leq_{\mathcal{M}} \psi \quad \overset{def.}{\Leftrightarrow} \quad \mathfrak{F} \vdash_{\mathcal{M}} \phi \Rightarrow \psi.$$

Its symmetrization gives the following equivalence relation[25]

$$\phi \equiv_{\mathcal{M}} \psi \quad \overset{def.}{\Leftrightarrow} \quad \big(\mathcal{M} \vdash_{\mathcal{M}} \phi \Rightarrow \psi\big) \wedge \big(\mathcal{M} \vdash_{\mathcal{M}} \psi \Rightarrow \phi\big),$$

and it is immediate to prove that $\mathcal{A}_{\mathcal{M}} = \mathcal{M}/\equiv_{\mathcal{M}}$ is a Boolean algebra with

$$[\phi] \cup [\psi] = [\phi \vee \psi], \ [\phi] \cap [\psi] = [\phi \wedge \psi],$$

$$\neg[\phi] = [\neg\phi], \ 0 = [(\forall x)(x \neq x)], \ 1 = [(\forall x)(x = x)].$$

Frequently, the above Lindenbaum-Tarski construction is made on $\mathfrak{F}$ instead of the whole $\mathcal{M}$, so that we obtain the following (syntactic) Lindenbaum-Tarski algebra $\mathcal{A}_{\mathfrak{F}} = \mathfrak{F}/\equiv_{\mathfrak{F}}$. It is possible to prove that $\mathcal{A}_{\mathcal{M}}$ is a free algebra generated by $\mathcal{C}$.

By means of the Lindenbaum-Tarski algebra, it is possible to set up a bijective correspondence between valuations and some particular homomorphisms of Boolean algebras, as follows.

Let $\mathcal{F}(\mathcal{L})$ be the set of all formulas of $\mathcal{L}$ (in $\mathfrak{F}$) (as defined in [9], Appendice B, B.1.), and let $\mathcal{A}_{\mathcal{F}(\mathcal{L})} = \mathcal{F}(\mathcal{L})/\equiv_{\mathcal{F}(\mathcal{L})}$ be the Lindenbaum-Tarski algebra of $\mathcal{F}(\mathcal{L})$; then, it is possible to prove that any valuation of $\mathcal{M}$, bijectively correspond to a well-determined homomorphism (of Boolean algebras) from $\mathcal{A}_{\mathcal{M}}$ to $\mathfrak{F}$, defined on the set of generators $\mathcal{C}$.

Moreover, if M is an arbitrary set of formulas of $\mathfrak{F}$ ($\subseteq \mathcal{F}(\mathcal{L})$) and $T(\mathcal{L}, M)$ is the set of all theorems of the formal system having language $\mathcal{L}$, and M the set of specific axioms (see [9], l.c.), then $T(\mathcal{L}, M)$ is a sub-theory of $\mathfrak{F}$, while $T(\mathcal{L}, M)/\equiv_{T(\mathcal{L},M)}$ is a filter of $\mathcal{A}_{\mathcal{M}}$. Thus, a Theory has a unique filter (on $\mathcal{A}_{\mathcal{M}}$) as an algebraic counterpart [precisely, a maximal filter for a (syntactically) complete Theory]: it is generated by the equivalence classes of the specific axioms M of $T(\mathcal{L}, M)$.

On the other hand, following [36], if it is given a language $\mathcal{L}$, a consistent set T of $\mathcal{L}$-sentences is, roughly speaking, a Theory (see the above $T(\mathcal{L}, M)$), while a *model* of T (or a T-*model*) is a $\mathcal{L}$-structure (see [36], Cap. 1, §§ 1.1, 1.2 and 1.3), say $\mathcal{S}$, such that every sentence in T is true into $\mathcal{S}$. We say that a

[25] There exists other equivalence relations leading to the so-called (Halmos) *polyadic algebras,* or to the so-called (Tarski) *cylindric algebras.* For simplicity, we restrict ourself to consider Lindenbaum-Tarski algebras.

theory T proves a $\mathcal{L}$-sentence ψ if $T \vdash_{\mathcal{S}} \psi$ for every model $\mathcal{S}$. Sometimes, the elements of T are called axioms, whereas the theorems (of T) are the sentences proved in T, that is, the deductive closure of T (see also [60]).

If we write, for simplicity's sake, $\mathcal{S} \vdash \psi$ instead of $T \vdash_{\mathcal{S}} \psi$, then $\vdash$ sets up a Galois connection between the class of models of T and the set of all $\mathcal{L}$-sentences of the deductive closure of T (see [12], Chapter 5, § 4).

Precisely, to each class C of T-models corresponds the set C^* of all $\mathcal{L}$-sentences true into every model of C, while, to each class S of $\mathcal{L}$-sentences of T, corresponds the class S^* of T-models with respect to which any sentence of S is true. Then, we have the following bijective correspondences $C \stackrel{\xi}{\to} C^*$ and $S \stackrel{\xi^{-1}}{\to} S^*$, induced by the above Galois connection.

On the other hand, if we consider the Lindenbaum-Tarski algebra associated (to the formal system corresponding) to the deductive closure of T, say $\mathcal{A}_T$, then the Galois connection, ξ, induces a Galois connection between $\mathcal{A}_M$ and the space of models of T, say $\mathfrak{M}_T$. Hence, we may write $\mathcal{A}_T \stackrel{\xi}{\cong} \mathfrak{M}_T$. The (logical) closure operators defines (following Kuratowski) a well-determined topology on the model space $\mathfrak{M}_T$, and the corresponding topological space is called the *Boole space* of T (see [12], Chapter 5, § 6); it is a Stone space.

If we want to apply these last considerations to the case related to the History of Differentiable Manifolds, then we may deduce, via Whitney's theorems[26], the existence of a Galois connection between $\mathfrak{M}_{\mathfrak{F}_1^{Dini}}$ and $\mathfrak{M}_{\mathfrak{F}_2^{Weyl}}$, hence between their corresponding Lindenbaum-Tarski algebras (computed with respect to the syntactic context, or with respect to the extensional semantic context).

At this point, it is necessary to specify some of the above expounded semantic concepts.

If $\mathcal{B} = (B, \vee', \wedge', \neg', 0, 1)$ is any Boolean algebra, then a *realization* (or *representation*) of the language $\mathcal{L}$ into $\mathcal{B}$, is a map $\rho : \mathcal{F}(\mathcal{L}) \to B$, such that

1. $\rho(\neg\alpha) = \neg'\rho(\alpha)$,
2. $\rho(\alpha \wedge \beta) = \rho(\alpha) \wedge' \rho(\beta)$,
3. $\rho(\alpha \vee \beta) = \rho(\alpha) \vee' \rho(\beta)$,
4. $\rho(\alpha \Rightarrow \beta) = \neg'\rho(\alpha) \vee' \rho'(\beta)$.

[26] This correspondence is bijective since, by a fundamental theorem due to H. Grauert (see [23], and [43]), any abstract manifold corresponds to a unique real manifold, via Whitney's imbedding. Hence, it follows the existence of a unique (Whitney) imbedded structure, for each assigned abstract manifold.

We say that ρ is a *model* of α, or that α is *true* with respect to ρ, if and only if $\rho(\alpha) = 1$. α is said to be *valid* into $\mathcal{B}$ if and only if it is true with respect to any realization ρ into $\mathcal{B}$; α is said to be *valid* if and only if it is valid into any Boole algebra $\mathcal{B}$.

A *semantic meaning* may be defined with respect to the (Frege) extensional context (*extensional semantic*) or with respect to the intensional context (*intensional semantics*).

We have seen that a possible extensional interpretation is given by $\mathcal{R} = (\mathcal{U}, v)$, where $\mathcal{U}$ is a possible world (or a *universe of discourse*), while v is a map that assigns a meaning, into $\mathcal{U}$, to the descriptive constants ($\in \mathcal{C}$) of $\mathcal{L}$. Then, according to G. Frege, v should satisfy the following conditions: 1) $v(a) \in \mathcal{U}$ for each $a \in \mathcal{C}$; 2) $v(P_i^n) \subseteq \mathcal{U}^n \quad \forall n \in \omega, \forall P_i^n \in Pr$. We say that $\mathcal{R}$ is an (extensional semantic) *interpretation* of $\mathcal{L}$, or a (extensional) *semantic structure* associated to $\mathcal{L}$.

Then, we say that a proposition $\psi \in Pr$ is *true* with respect to the interpretation $\mathcal{R} = (\mathcal{U}, v)$ if and only if $v(\psi) \subseteq \mathcal{U}^n$, where $n \in \omega$ is the arity of ψ. In general, for an arbitrary $\mathcal{L}$-sentence ψ, we say that ψ is *true* with respect to $\mathcal{R}$, and we write $\models_{\mathcal{R}} \psi$, if a set of (Frege) conditions are fulfilled (see [14], Capitolo 2, § 2.2.). These are the basic elements of the (Frege) extensional semantics in the modern formulation given by A. Tarski.

Nevertheless, especially in the historical context, it is more important to consider an intensional semantic context, as, for example, that given by Kripke's Semantics (of the general class of Modal Logics).

The main limit of Tarski Semantics is due to the fact that it comprehends only two possible cases: indeed, such a Semantics considers either one universe of discourse $\mathcal{U}$ or all possible universes of discourse.

Instead, S. Kripke (see [32]) considers a suitable system of possible universes of discourse in dependence on the uses and purposes of the given formal system. So, we speak of a *Kripke realization* with respect to a particular set of universes of discourse, those *accessible.* These universes of discourse are connected among them by the so-called *accessibility relations.* In such a way, we go towards the realm of Modal Logics (Temporal, Epistemic, etc.) and the intensional theories of meaning (as, for example, the Carnap's one). The Modal Logics may play a very important role in some historical interpretation, as we will see in the next section.

Finally, we may consider a *Kripke deductive system* as a tuple of the type $\mathcal{M}_{Kripke} = \langle\langle\mathfrak{F}, (\mathcal{U}_i, v_i)_{i\in J}\rangle\rangle$, where $\mathcal{R}_i = (\mathcal{U}_i, v_i)$, $i \in J$, is the Kripke's set of realizations of $\mathcal{M}_{Kripke}$ (if J is a singleton, or an infinite set, then we obtain a Tarskian deductive system). Mutatis mutandis, what has been said above about the Lindenbaum-Tarski methods, may be applied to $\mathcal{M}_{Kripke}$ as well.

Analogously to what has been said in section 9.1.1., the critical comparison between the Lindenbaum-Tarski algebras built up on a [Kripkian] deductive theory, may turn out to be useful for possible historical comparisons between the relative theories (see next § 11).

9.2 The work of Saul Kripke

Saul Kripke is considered one of the most important founders of the so-called Semantic Modal Logic, which gives a more extended semantic context than that of Tarski's one (for the Classical Logic) and of Gödel's one (for the Intuitionist Logic).

The book [32] is a philosophical continuation of the first sixty Kripke's researches on the semantic analysis of Modal Logic. This work has, among other things, a prominent role in the Historiography of Sciences, as we will see.

In [32], among other, is discussed the historical role of the Factuals, Counterfactuals and of the so-called *Historical Chains,* in the framework of the so-called *Possible Worlds*; there is a deep critical analysis of the Aristotele's distinction between Essential and Accidental properties, and of some related metaphysical Kantian conceptions (as the "a priori", the "analytical" and "necessity" truth Categories, and so on).

The kripkian logical and philosophical analysis, starts from a critical study of the already known (philosophical) concepts and notions of Name, Necessity, Possibility, Essence, Analytical Truth, Referent, Meaning, Reference, Description, rigid and not rigid Designators, Cluster Concept, and so forth.

He examines the modalities of the relations which hold between Names and Things; besides, in his first January 20, 1970 lesson, the author discusses the role of the concept of Possible World in the mathematical definitions, as regard the importance of the Identity Criterion along time (hence, from the historical viewpoint).

From a critical re-examination of the previous Name Theories (as, for example, the Name Reference Theory of G. Frege and B. Russell), Kripke reaches his semantic theory of Possible Worlds, with some possible its applications; among these, we recall those having usefulness in some epistemological questions: precisely, the author says that his theory is an essential tool to establish the existence, or not, of correct historical connections among historical facts. On the other hand, this is just what is necessary, for instance, for the historical comparison of the mathematical theories treated in this paper.

Saul Krikpe, with Hilary Putnam (see [56]), are the founders of the modern new reference theory.

10 The role of the principle of virtual works in differential geometry

In this section, we wish briefly to recall the important role played by the principle of virtual works of Analytical Mechanics. [39] is the main reference for the History of Mechanics up to 1920.

This principle has played a fundamental role in Lagrange's work (see [33]): in fact, it is at the base of the analytical mechanics arguments[27]. There are many modern texts on Analytical Mechanics whose first chapters, devoted to the formulations of the celebrated Lagrange's equations, just begin with the exposition of D'Alembert-Lagrange principle of virtual works. For instance, a modern historical exposition very similar to the original Lagrange's formulation, is given by [1], vol. I, Capitolo I: here, once again, the reference to Dini's work on implicit function theorems is evident and this proves the essential syntactic necessity of these methods for the formal setting of Analytical Mechanics and, hence, for the subsequent formulation of Differential and Riemannian Geometry. For a brief, but rigorous, exposition of these arguments, see [46], [47] and the more complete treatment of [20].

We briefly recall the main results of [1], vol. I, Capitolo I.

In § 2 of Chapter I, it is expounded the so-called *D'Alembert principle* $m_i\vec{a}_i = \vec{F}_i + \vec{R}_i \quad i = 1, ..., N$, for a system of N point particles, each of which has mass m_i, acceleration $\vec{a}_i$, and subjected to both the total active forces $\vec{F}_i$ and the total constraint forces $\vec{R}_i$. This principle reduces every dynamical problem to a statical one; it provides a well-defined equilibrium condition. For a smooth[28] systems with holonomic constraints, this last equilibrium condition being equivalent to the so-called *principle of virtual works,* whose analytical formulation is based on the invertibility of the virtual displacements δP_i of the point particle P_i, and it is $\sum_{i=1}^{N}(\vec{F}_i - m_i\vec{a}_i) \times \delta P_i = 0$, said to be the *general* (or *symbolic*) *equation* of Dynamics.

In § 3 and § 4, respectively, the [angular] momentum conservation theorems and the Lagrange's equations[29], are deduced from this symbolic equation.

In the following sections, many possible formal expressions of Lagrange's equations are deduced: the δ-d Lagrange's formalism is the main analytical

[27]In passing, this principle has also been used on some questions related to the constrained motion of a quantum particle (see, for instance, [28]).

[28]Here, the term 'smooth' means constraints without friction.

[29]There exist various forms of Lagrange's equations exposed in [1], vol. I, Capitolo I. In particular, in the subsection 2 of § 4., the authors expose a first form of Lagrange's equations using the Lagrange's multipliers rule, that plays a fundamental role in the extremum theory with side conditions (so that, again, we go back to Dini's works).

tool for deducing many formal dynamical properties of a holonomic smooth constrained systems, given in a (Hertz) form similar to (1) of our § 5, with $q = 0$ (equivalent to (2), where the y_i are replaced by the lagrangian coordinates q_i); these properties have both metrical nature (assuming it is assigned a certain metric given by the kinetic energy, according to Jacobi) and affine nature, and it is much probable that they have played a fundamental role in the subsequent development of Differential Geometry.

For instance, to this purpose, it is important to remember that the first differential topology tool explaining the basic differential geometry local concepts, is that of tangent space in a point of a manifold: historically, the first definitions of tangent space have been the result of a generalization of the main basic concepts and methods of Analytical Mechanics concerning the constrained motion of a particle over a smooth holonomic system (see the so-called *physicist's* definition and the *geometer's* definition, of a tangent space, equivalent between them – and to another, called the *algebraic* definition – given in Chap. 2 of [7]; see also [39]).

However, for our purposes, we follow the modern exposition given by V.I. Arnold in [3].

In Chapter IV, he gives a first modern definition of smooth holonomic constraint suggested[30] by M.A. Leontovic (see § 17, A.), with a second definition (see § 17, B.) where it is defined, substantially, a manifold in the Dini's sense (see (1) of our § 5); he returns on the definition of smooth holonomic constraint in B., Example 10 of § 18, where it is introduced the modern (Weyl's) definition of differentiable manifold.

In § 21, Arnold introduces D'Alembert principle, and, at the point B. of the same section, he proves the (syntactic) equivalence between D'Alembert-Lagrange principle (of virtual works) and the definition of smooth holonomic constraint given at the point B. of § 17, by means of the use (see the point C. of § 21) of a variational calculus arguments already known to Lagrange (see the point C. of § 21, where the author also exposes the original Lagrange's static formulation). A similar exposition may be found in [1], vol. I, Capitolo I.

The holonomy of such constraints has physical motivations, and, therefore, it is evident the mathematical physics sources of the concept of a smooth holonomic constraint, hence of the differentiable manifold: indeed, the principle of virtual works provides the local characterization of a manifold, locally like to $\mathbb{R}^n$, likewise to Dini's implicit function theorems.

[30]For a deduction of Lagrange's equation from this Leontovic's point of view, see [19], and reference therein.

On the other hand, it is well-known that the sources of Lagrange's inverse function theorem (already mentioned), should be traced in the Lagrange's works on some static problems, where, among other, he introduced the nowadays well-known "Lagrange's multipliers" (see [15], Capitolo V., § 5., footnote [5] of page 382). The latter, in turn, results to be related to the principle of virtual works as well, hence to the local structure of a differentiable manifold (via the connection of the Lagrange's multipliers with the inverse function theorem). To this end, we briefly recall the problem.

Almost every extremum problem with side conditions, historically started from questions of mechanics of constrained systems.

If Γ is a smooth constraint, hence a manifold described by a set of zeros of functions, then, by means of Lagrange's multipliers, an extremum problem on Γ is reduced to a local extremum problem related to the functions (locally) describing such a manifold Γ. To this end, we remember that, if Γ is a manifold of $\mathbb{R}^n$, $f : \Gamma \to \mathbb{R}$, $x_0 \in \Gamma$ and r is a chart of Γ (see § 8) containing x_0, then we say that x_0 is a relative maximum/minimum extremum for f if and only if $r^{-1}(x_0)$ is a relative maximum/minimum extremum for $f \circ r$.

Since, in general, it is a difficult task to determine the charts of a manifold, because of the local nature of the question, for such an extremum problem it is enough to consider the same extremum problem related to a restriction of f on $\Gamma' = \Gamma \cap I$ where I is a neighborhood of x_0.

Hence, we have the following

Lagrange's Multipliers Theorem. *Let $\Gamma \subseteq \mathbb{R}^n$ be a manifold of dimension $m(< n)$, and $x_0 \in \Gamma$; let I be an open neighborhood of x_0 and $g(x_0) = 0$ the local equation of Γ in x_0 with $g \in C^1(I)$ and rank $J(g)(x) = n - m$ for each $x \in \Gamma \cap I$. Let $f : \Gamma \to \mathbb{R}$ with $f \in C^1(I)$. If x_0 is a relative extremum for $f_{|\Gamma\cap I}$ then $\nabla f(x_0) \in N_{x_0}\Gamma$ (normal space to Γ at x_0), that is, there exist $n - m$ real numbers $\lambda_1, ..., \lambda_{n-m}$ such that $\nabla f(x_0) = \sum_{i=1}^{n-m} \lambda_i \nabla g_i(x_0)$ with $\lambda_1, ..., \lambda_{n-m}$ uniquely determined by x_0.*

If, for each $x_0 \in \Gamma$, we have $\nabla f(x_0) \in N_{x_0}\Gamma$, then we say that x_0 is a *stationary* (or a *critical*) point of f; such points are in bijective correspondence with the solutions of the system of equations $g(x) = 0$ and $\nabla f(x) = \sum_{i=1}^{n-m} \lambda_i \nabla g_i(x)$, whose solutions are of the type $(x_1, ..., x_n, \lambda_1, ..., \lambda_{n-m}) \in \mathbb{R}^n \times \mathbb{R}^{n-m}$, with $\lambda_1, ..., \lambda_{n-m}$ said to be *Lagrange's multipliers.*

For instance, in the case $n = 3$, if $\mathbb{R}^3$ is a model of the physical space and $\Gamma(\subseteq \mathbb{R}^3)$ represents a bilateral smooth holonomic constraint for the material point x_0 subjected to the force field ∇f, then the above Theorem says that the force acting over a critical point x_0 is orthogonal to the constraint Γ, whereas the values of the Lagrange's multipliers are connected with the intensity of the constraint reactions. From here, it arises evident links with the principle of virtual works.

In short, it is evident the existence of syntactic links between these analytical mechanics arguments and the basic formulations of the theory of differentiable manifolds, although it is a very difficult task to do sure historical claims about these suppositions, but, at most, having only probability nature.

The only certainty concerns the syntactic comparison among the previous arguments, whereas their possible semantic comparison may be conducted within the Kripkian context (or, more generally, into the Modal Logics context), if we choose the suitable Kripke's set of realizations upon which to interpret the syntactic contents of the previous statements. From this point of view, it is perhaps possible to think that the work of Lagrange (and others, as C.G.J. Jacobi, L. Euler, and so on) on Analytical Mechanics, were intuitively oriented towards a study of the (local) geometry of configuration space of a moving particle, subsequently formalized by both D'Alembert-Lagrange principle and a mathematical structure described by a system of the type (1) of our § 5 (with $q = 0$), by means of a large use of the so-called δ-d symbolism (typical of Classical Analytical Mechanics). The just mentioned historical connections are, however, rather probably (see next § 11).

We conclude this section, with an unusual remark on the work of Tullio Levi-Civita on his parallel displacement[31], from which it is possible to infer another prove of the importance played by the principle of virtual works in the foundations of Differential Geometry.

In fact, it is almost always affirmed (in the current relative literature) that Levi-Civita parallel displacement was motivated by the attempt of giving a geometrical interpretation to the so-called "covariant derivative" of Absolute Differential Calculus. Indeed, if one carefully reads the paper [35], it is clear that the historical verity is quite different. Levi-Civita was motivated by the attempts to simplify the computation of the curvature of a manifold through the Riemann symbols, as he says in the *Introduction* of his paper.

Then, once introduced a generic metric structure on a manifold defined by a system of the type (2) of our § 5, the author establishes a fundamental equation, the (I) of § 2.; the latter, is nothing but the principle of virtual works applied to such a manifold, thought as a smooth holonomic system subjected to (invertible) virtual displacements. From it, the author deduces an equivalent equation, the (8) of the same paragraph, hence another equivalent form, the (I_a) of § 3, from which he deduces the analytical conditions characterizing his celebrated notion of parallel displacement. In the remaining paragraphs, the author does not make any explicit mention to the covariant derivative, except a secondary application relative to Ricci's rotation coefficients (see § 13 of [35]).

11 Conclusions

Albeit it is surely erroneous to think that the concept of a differentiable manifold (as we know it nowadays) is already present in the works of Dini on

[31] As regard the historical importance of Levi-Civita parallel displacement in Physics (as, for example, in Gauge Theories), see [6].

implicit function, or in the foundations of Analytical Mechanics, nevertheless we may state, out of doubt, what follows.

Any mathematical theory does not born from nothing, but, instead, it starts from some previous ones[32]: precisely, it begins from those having, at least, a some syntactic link with it[33]. Hence, from here, it is evident the importance of the notion of syntactic model in searching such syntactic links, in such a way that it is possible to determine a chain of syntactic models which may remember the Kripkian "historical chains" of § 9.2.

So. we have exposed a case-study of such an historiographical methodology, precisely, that relative to the origins of the concept of a differentiable manifold.

Beyond such a first comparison among theories, further researches are possible concerning the semantic context, for instance, in the Modal Logic framework. Through this last perspective, it is subsequently possible to make suitable "interpretations" (on the basis of the previous syntactic comparison), which are more proper for a historical setting.

For instance, since we have seen that certain filters algebraically correspond to theories, then it is possible to compare two theories comparing their filters, and so on, to be then historically interpreted.

Analogously, it may be compared the corresponding (syntactic or semantic) Lindenbaum-Tarski algebras between them. In these last two cases, the resulting chains of filters, or algebras, may be considered as an "algebraic formalization" of the so-called "historical chains" of Historiography (already mentioned when we have discussed on Kripke's work).

We have therefore sketched such a line of historiographical methodology in relation to a particular case related to the History of Differential Geometry. Thus, in this context, it is very likely that Dini's works on implicit functions and the bases of Analytical Mechanics, have played a considerable role in the formulation of the modern theory of Differentiable Manifolds, both from syntactic and semantic viewpoint.

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[32] On the other hand, this statement finds a further confirmation on a certain, not casual epistemological "evolution" of a mathematical structure along the historical time (see [48]). As a concrete example of that, we recall the work of G. Peano on the axiomatization of Natural Numbers, that started from the previous works of R. Dedekind on the same argument (such a question, besides, it is treated, from the Modal Logic viewpoint, in the January 22, 1970 lesson of S. Kripke – see [32]).

[33] The further, not trivial question concerning the awareness, or not, of the existence of these theories on behalf of the author under historical examination, may be analyzed from suitable philosophical viewpoints. However, certain contemporaneously but independent (between them) mathematical discoveries/construction (like those mentioned above in the footnote [12]) prove that the previous syntactical capacity of a certain theoretical context reaches a certain degree in allowing a subsequent discovery/construction.

Furthermore, from what has been said so far, it is clear too that the geometric structure, called *differentiable manifold,* is a syntactic[34] rigid designator in the sense of the new reference theory of S. Kripke and H. Putnam; in fact, the same syntactic structure (or mathematical entity), i.e., that of differentiable manifold, has been identified in, at least, two different semantical contexts (or in two discourse worlds): that of the Theory of Implicit Functions and that of Lagrange's Analytical Mechanics. Moreover, following H. Putnam, we could say that it may exists a collective (not individual) historical chain (see above), so external to every individual, or else a series of "reference rings" transmitted through the time, in which it is possible to identify a certain constancy of the discourse's terms (rigidity of the reference) leading to a given entity (rigid designator): in our case, it deals with the syntactic structure "differentiable manifold". Hence, the quantitative methods used in this paper for such a particular case-study, may turn out to be of some usefulness also in regard to the nature of other mathematical entities (in the context of Mathematical Philosophy).

[34]Examples of semantic rigid designators are the physical entities (as, for instance, an elementary particle, a physical field, etc.).

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