A posteriori error estimation and adaptivity for XFEM in 3D linear elastic fracture mechanics: from theory to industrial applications

Marc Duflot, Stéphane P A Bordas

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Marc Duflot[1] **and Stéphane P. A. Bordas**[2]

[1] CENAERO
Bâtiment ÉOLE
Rue des Frères Wright, 29
B-6041 Gosselies, Belgium marc.duflot@cenaero.be

[2] University of Glasgow
Civil Engineering
Rankine Building
Glasgow, G12 8LT, Écosse (Scotland) UK
stephane.bordas@alumni.northwestern.edu

Résumé — This paper describes two recovery techniques for XFEM and presents adaptivity results based on those smoothing operators.

Mots clés — XFEM, error estimation, adaptivity.

1 Basic features of the error estimates

This paper shows how two *recovery based error estimation* techniques for *extended finite element methods*, *XFEM*, or, more generally, methods based on *extrinsic partition of unity enrichment* can be used to drive $h-$ and $e-$ adaptivity [1, 2, 3].

The first estimator employs derivative recovery with intrinsically enriched eXtended Moving Least Squares (XMLS) approximants and diffraction to account for the discontinuity through the crack. MLS derivative recovery in finite elements was first proposed in [4] of which this work is a generalization. The smoothness of the recovered derivatives is identical to that of the MLS weight function, in the examples proposed, they are C_1.

The second is a generalization to enriched approximations of the simple concept of *global derivative recovery* introduced in [5, 6] for the finite element method. The starting point of global derivative recovery is the remark that when only C_0 continuity of functions in the trial space is assumed in finite element methods, the strain and stress fields are discontinuous through element boundaries. The principle presented in [5, 6] is to construct an *enhanced* stress field interpolated with the same ansatzt functions as the displacements, and such that the L_2 norm *over the whole domain* of the difference between the enhanced and original finite element strains (stresses) is minimized. Through global minimization, we obtain an enhanced strain field, which is a better aproximant to the exact solution, itself unknown.

In both techniques, comparing the orginal (raw) XFEM strains to the enhanced strains, as in standard recovery-based error estimation [7], we define a local (element-wise) error which can be used to drive adaptive strategies. The conclusions of the studies, reported in detail in [1], [2] [3] are that : (i) both XMLS and XGR methods yield error estimates which are valid, i.e. their effectivity tends to zero as the mesh is refined, (ii) the XMLS method yields smoother recovered fields than XGR, (iii) XGR is cheaper computationally than XMLS, at least in its initial formulation, (iv) XGR is more easily implemented in existing codes, and is well-suited to engineering analysis.

The adaptive studies show that the proposed methods are able to improve the quality of the stresses as well as the convergence rates, both for academic and industrial examples.

2 Essential results

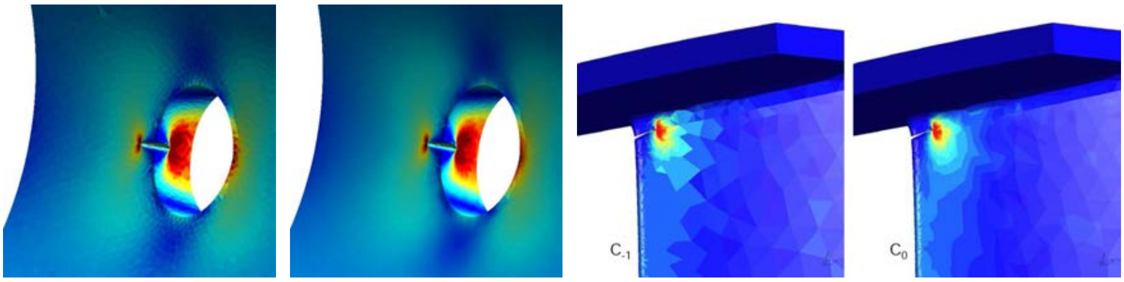
For the sake of conciseness, we will only recall the key results associated with both estimators, and show how they are applied to three dimensional fracture problems. The readers are referred to [1, 2, 3] for details on the formulation and more numerical illustrations.

2.1 Extended moving least squares (XMLS) recovery

- The recovered strain/stress fields are C_1.
- In [1, 2], we show the necessity for the addition of the near-tip fields to the MLS basis, if these functions are not added, the effectivity index of the proposed error indicator does not tend to unity as the mesh is refined.
- For problems where the exact solution is not known, and thus where the effectivity index cannot be computed, we check that the L_2 norm of the difference between the raw XFEM strain field and the XMLS recovered strain field converges to zero with a rate close to the optimal rate of 1.0 as the mesh size tends to zero, as long as a fixed area is enriched around the crack tip during mesh refinement. If only the crack tip element is enriched, the convergence rate is close to $1/2$, which is the strength of the crack tip stress singularity. This corroborates the findings of References [8, 9]. Note that the higher the enrichment radius, the lower the error, the straighter the convergence line, and the closer to optimal the convergence rate is.
- Larger XMLS recovery smoothing lengths lead to higher effectivity indices, but we believe that the increase is not significant enough to justify the additional computational cost.
- In [1, 2], we show the superiority of the XMLS recovered solution with the now standard Superconvergent Patch Recovery (SPR) of [7], for fracture problems.

2.2 eXtended Global Recovery (XGR)

- The recovered strain/stress fields are C_0.
- The L_2 norm of the difference between the XGR strain and the raw XFEM strain vanishes upon mesh refinement.
- More importantly, we show that the effectivity index of the error indicator converges toward unity upon mesh refinement. This proves that the *approximate error* converges to the *exact error*, and, therefore, that the error indicator is indeed a correct measure of the error.
- The larger the XFEM enrichment radius, the closer the convergence rate is to 1. This corroborates earlier findings in the context of the XMLS recovery technique [1, 2] and is explained by the fact that larger enrichment radii lead to more accurate solutions, thus more accurate recovered solutions, and therefore an approximate error which is close to the exact error.
- Comparing the converged values of the effectivity index for XGR to that obtained for XMLS and published in [1, 2], we note that the XGR effectivities converge between 93 and 96%, whereas the XMLS effectivities are in the vicinity of 99%. For the whole range of mesh sizes, the XMLS effectivities are better than the XGR effectivities, this is due to the fact that the XMLS approximation is C_2 where as the XGR approximation is only C_0. We also



(a) raw XFEM strain field

(b) XGR strain field

(c) Turbine blade

Figure 1 – XGR : quarter-circular crack emanating from a hole in a cylindrical shell under internal pressure ; the enrichment radius r_{enr} is equal to the radius of the quarter-penny crack. Application to a turbine blade.

notice that the XMLS results are less sensitive to the value of the enrichment radius than the XGR results. This is not surprising, since the XMLS recovery is built with a global intrinsic enrichment of the MLS approximation, whereas the enrichment used for the strain recovery in XGR is only active in a small[1] ball (tube) around the crack tip (front).

3 Three-dimensional validation of the error measures

In this section, we summarize the three-dimensional example of a quarter-circular crack emanating from a hole in a cylindrical shell subjected to a uniform internal pressure. The elements are linear tetrahedral elements. In this problem, elements in a tube centered on the crack front and with a variable radius r_{enr} are enriched with near-tip fields. The results are very interesting. They show that increasing the enrichment radius from $r_{crack}/5$ (roughly one element size) to r_{crack} decreases the error, and reduces the size of the *peak error zone*. For $r_{enr} = r_{crack}$, the estimated error around the crack front is approximately the same as the error on the other, uncracked side of the hole. This corroborates our findings in two dimensions, as well as the conclusions drawn in References [8, 9]. Results are shown in Figure 1.

4 Adaptivity

This section discusses various adaptivity techniques driven by the aforementioned error measures. Usual adaptive techniques rely on $h-$ or $p-$ adaptivity. As discussed in [1, 2, 3], XFEM offers one more possibility which we named $e-$ adaptivity and relies on the adaptation of (i) the enrichment functions ; (ii) the zone over which they are active. This is summarized in Figure **??**.

A simple example of a near-tip crack problem is given in Figure **??**, for a target error of 0.01, which shows that the $h-$ adaptive strategy increases, as expected, the convergence rate. This figure also shows that the mesh converges towards a pattern involving heavy mesh refinement near the tip, despite the near-tip enrichment. A sharp mesh density transition is also observed at the boundary between the enriched and non-enriched domain, which we attribute to blending effects, which are not treated in the current implementation. Note that the refinement strategy used here aims at distributing the error uniformly across the domain.

[1]with respect to the domain size

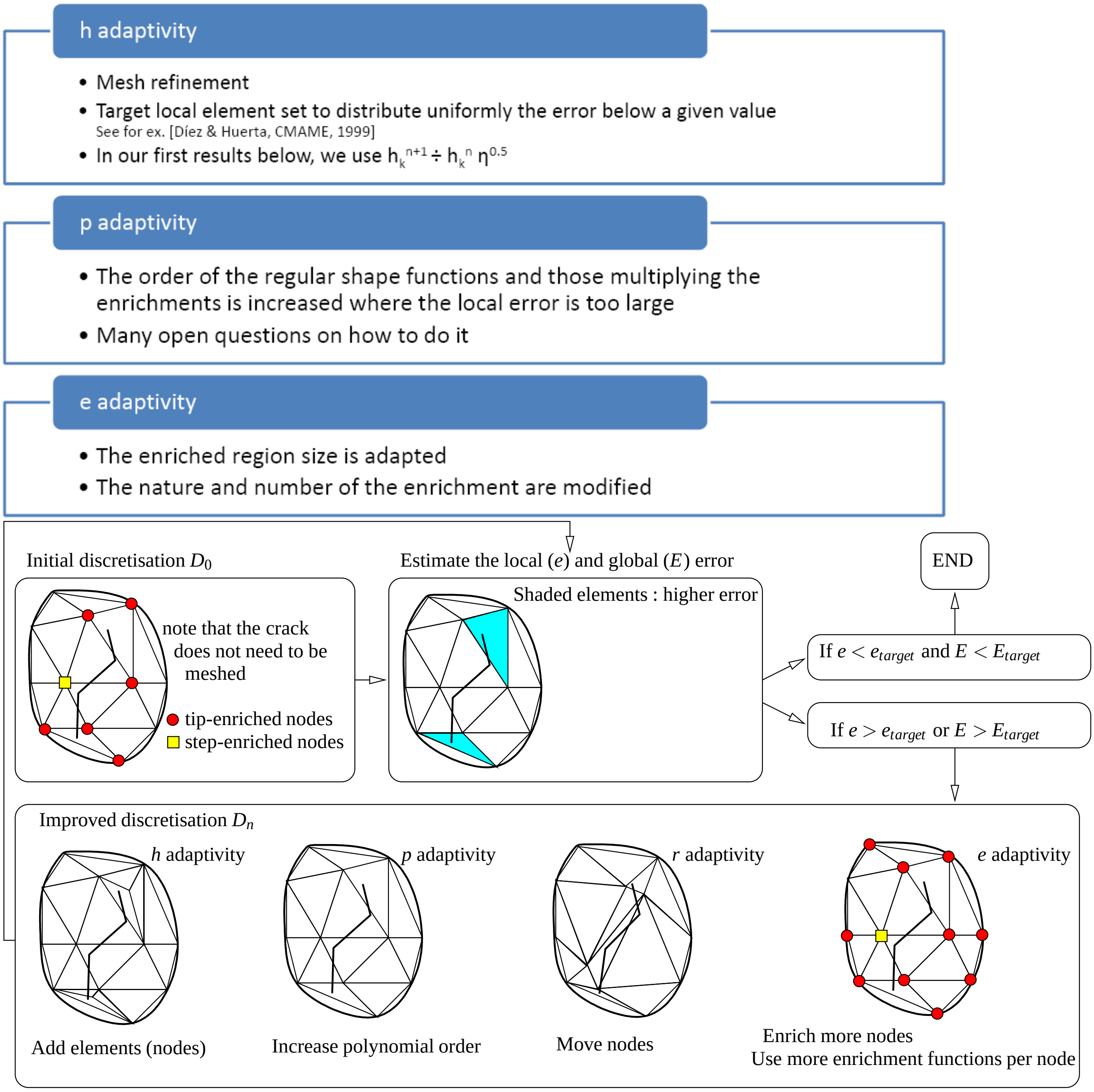


Figure 2 – Possibilities for adaptivity within XFEM.

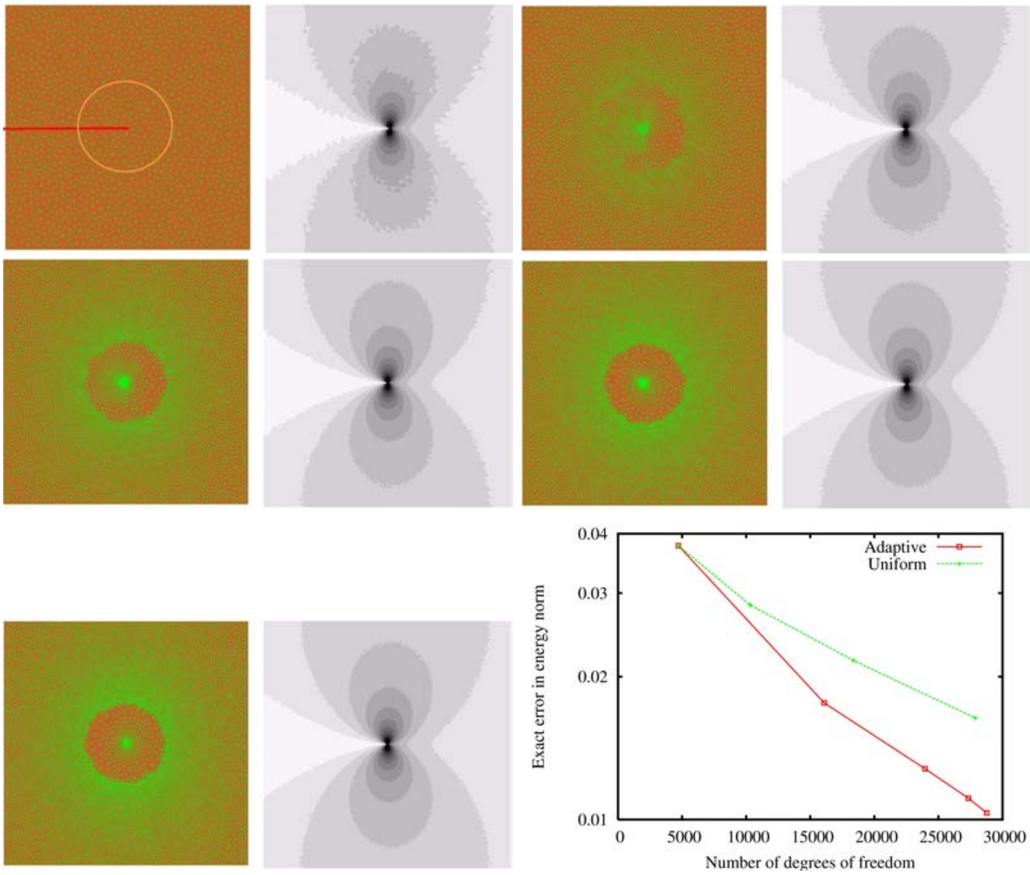


Figure 3 – Adaptivity for near-tip mode I problem.

5 Conclusions

We presented the basic results of our study of two a posteriori error estimates for the extended finite element method (XFEM). They suggest a strategy for $h-$ adaptivity in enriched finite element methods, and hint at a new approximation adaptation scheme specifically tailored to enriched approximations.

These two techniques were exercised in the context of academic and industrial fracture mechanics problems and showed to be suitable drivers for $h-$ adaptivity.

Our experience indicates that an effective error minimization technique consists in *evaluating the optimal XFEM enrichment radius* r_{enr}[2] and, second, if the overall or/and local errors are still above the tolerance specified, proceed to $h-$ or/and $p-$ refinement, while keeping the enrichment radius constant. This procedure of estimating the optimal enrichment radius can be seen as a generalization of $h-$ and $p-$ adaptivity to encompass the non-polynomial functions present in the XFEM approximation. This new adaptivity could be coined *enrichment-adaptivity*, or *e-adaptivity*, and is subject to on-going research.

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[2]this optimal enrichment radius is problem dependent. In our experience, it is situated in the vicinity of the length of the crack.