# Mixed Integer-Real Mathematical Analysis, and The Lattice Refinement Approximation and Computation Paradigm

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**Abstract**

This draft is the third version of a preliminary document following a work presented at the $14^{th}$ International Conference on $p-$adic Analysis, in Aurillac, France, July 2016. This version is a minor update fixing a few gaps and improving the algebraic notions. The final version of this draft is to be submitted afterwards, at a date which is difficult to estimate due to the lack of time and support. See also:

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# 1 Algebraic Background

This section is devoted to presenting the rings and algebras which have enough properties to develop the subsequent theory. All the rings and algebras considered throughout the paper are assumed to be abelian, even when not specified.

## 1.1 Complete Archimedean Totally Ordered Abelian Rings

**Definition 1.1** *We call an ordered abelian ring* $(\mathcal{R}, +, ., \preceq)$ *any abelian ring on which is defined an order* $\preceq$*, such that*

1. *for* $r, s, t \in \mathcal{R}$ *with* $s \preceq t$ *then* $r + s \preceq r + t$ *(translation invariance).*

2. *for* $r, s, t \in \mathcal{R}$ *with* $0_{\mathcal{R}} \preceq r$ *and* $s \preceq t$ *then* $r.s \preceq r.t$ *(compatibility with the product).*

*Such a ring is called Dedekind-complete (or Complete for short) when any subset of* $\mathcal{R}$ *with an upper bound has a supremum and any subset of* $\mathcal{R}$ *with a lower bound has an infemum.*

**Definition 1.2** *A complete abelian ordered ring* $(\mathcal{R}, \preceq_{\mathcal{R}})$ *is called Archimedean if and only if for any positive and nonzero* $l \in \mathcal{R}^*$*, we have:*

$$\mathcal{R} = \bigcup_{n\in\mathbb{N}} \{r \in \mathcal{R} \ / \ r \preceq_{\mathcal{R}} n.l\} \text{ and } \mathcal{R} = \bigcup_{n\in\mathbb{N}} \{r \in \mathcal{R} \ / \ -(n.l) \preceq_{\mathcal{R}} r\}$$

*where* $n.l = (\sum_{i=1}^{n} l)$.

**Proposition 1.1** *A Dedekind-complete Archimedean totally ordered abelian ring* $\mathcal{R}$ *which is unitary (that is, the multiplication in* $\mathcal{R}$ *has a neutral element) is isomorphic (as an ordered ring) either to the usual ordered ring* $\mathbb{Z}$ *or to the usual ordered ring* $\mathbb{R}$*.*

*Proof.* Using a classical result on ordered rings, we see that $\mathcal{R}$ is isomorphic (as an ordered ring) to an induced sub-ring of the field $\mathbb{R}$ provided with the usual order. Up to this isomorphism, we may suppose now that $\mathcal{R}$ is an induced ordered sub-ring of $\mathbb{R}$. We consider two cases:

*First, assume* that $\inf(\mathcal{R}_+^*) = 1$. Let $r$ in $\mathcal{R}$ be greater than 1. Let $n_0 = \inf\{n \in \mathbb{N} \ / \ n > r\}$. Then we have $0 \leq r - (n_0 - 1) < 1$ so that $0 = r - (n_0 - 1)$. This shows that $\mathcal{R}_+^* = \mathbb{N}$ so that $\mathcal{R} = \mathbb{Z}$.

*Second, assume* that $\mathcal{R} \cap ]0; 1[ \neq \emptyset$. Then there exist in $\mathcal{R}$ some element $s$ such that $0 < s < \frac{1}{2}$. Let $x$ be a positive real number. Consider now $X = \{r \in \mathcal{R}; 0 \leq r \leq x\}$. It is obviously bounded in $\mathcal{R}$ so that it has a supremum $r_0 \in \mathcal{R}$. Now we prove that $x = r_0$ which, the number $x$ being arbitrary, will prove $\mathcal{R} = \mathbb{R}$.

The suppremum of $X$ must be less than $x$, which is an upper bound for $X$, which means $r_0 \leq x$. Now, if $x > r_0$, then for $n$ large enough we have $s^n < x - r_0$, so that $r_0 < r_0 + s^n < x$, yielding a contradiction. □

## 1.2 Complete Archimedean Totally Ordered Algebras

In all this section, $\mathcal{R}$ is a unitary Dedekind-complete Archimedean totally ordered abelian ring.

**Definition 1.3** *An ordered algebra on* $\mathcal{R}$ *is a tuple* $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$*, where* $\mathcal{R}$ *is a Dedekind-complete Archimedean totally ordered abelian ring,* $\mathcal{A}$ *is an* $\mathcal{R}-$*algebra (i.e. provided with an* $\mathcal{R}-$*module structure given by a product by scalars of* $\mathcal{R}$*, and an abelian addition operation denoted by* $+$*, and also provided with an internal product operation, denoted by ., with distributivity with respect to* $+$*, and which is here assumed to be abelian) and* $\preceq$ *is a complete partial order, which is compatible with the order in the ring* $\mathcal{A}$ *(Definition 1.1), and is also compatible with the order in* $\mathcal{R}$*, that is:*
*if* $a \in \mathcal{R}$ *and* $x, y \in \mathcal{A}$ *with* $x \preceq_{\mathcal{A}} y$*, if* $0_{\mathcal{R}} \preceq_{\mathcal{R}} a$ *then* $ax \preceq_{\mathcal{A}} ay$*, and if* $a \preceq_{\mathcal{R}} 0_{\mathcal{R}}$*, then* $ay \preceq_{\mathcal{A}} ax$*.*

**Definition 1.4** *An ordered algebra* $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$ *on* $\mathcal{R}$ *is called complete if and only if the order* $\preceq_{\mathcal{A}}$ *is a Dedekind-complete.*

**Definition 1.5** *A complete ordered algebra* $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$ *is called Archimedean if and only if it is Archimedean as a ring (Definition 1.2)*

**Remark 1.1** *The lexicographic order, which is of frequent use in computer sciences, does not define a complete ordered Archimedean algebra on the product of complete algebras.*

For instance, let us consider the lexicographic order $\preceq_{lex}$ on the cartesian product $\mathcal{A} = \mathbb{R}\times\mathbb{R}$. Let us denote $X = \left\{(1-\frac{1}{n}, n) \ / \ n \in \mathbb{N}^*\right\} \subset \mathcal{A}$. Then, $(1, b)$ is an upper bound for $X$, for any $b \in \mathbb{R}$, but no such couple is a supremum. Besides, no couple $(a, b)$ with $a \in \mathbb{R}$, $a < 1$ and $b \in \mathbb{R}$ is a supremum. Therefore, the subset $X$ of $\mathbb{R}\times\mathbb{R}$ has no supremum. The ordered algebra $\mathcal{A}$ is not Archimedean either, since $\mathbb{R}\times\mathbb{R}$ strictly contains

$$\bigcup_{n\in\mathbb{Z}} \{(u, v) \in \mathbb{R}\times\mathbb{R} \ / \ (u, v) \preceq_{lex} n(0, 1)\} = \mathbb{R}_- \times \mathbb{R}.$$

**Example 1.1** *Let $(\mathcal{A}_1, \preceq_1)$ and $(\mathcal{A}_2, \preceq_2)$ be two totally ordered Dedekind complete sets. If $\mathcal{A}_2$ has a minimum element and a maximum element, then $\mathcal{A}_1 \times \mathcal{A}_2$ is a totally ordered Dedekind complete set for the lexicographic order.*

Indeed, let $X \subseteq \mathcal{A}_1 \times \mathcal{A}_2$ be a non empty subset of $\mathcal{A}_1 \times \mathcal{A}_2$. Let $b_m$ be the minimum element of $\mathcal{A}_2$ and let $(a, b)$ be an upper bound for $X$. Let us denote by $a_M$ the supremum

$$a_M = \sup\{u_1 \in \mathcal{A}_1 \ / \ \exists u_2 \in \mathcal{A}_2 : (u_1, u_2) \in X\}.$$

- If $a_M \in \{u_1 \in \mathcal{A}_1 \ / \ \exists u_2 \in \mathcal{A}_2 : (u_1, u_2) \in X\}$, then let us denote $b_m$ the upper bound of $\{u_2 \in A_1 \ / \ (a_M, u_2) \in X\}$. In that case, $(a_M, b_m)$ is a supremum for $X$.
- If $a_M \notin \{u_1 \in \mathcal{A}_1 \ / \ \exists u_2 \in \mathcal{A}_2 : (u_1, u_2) \in X\}$. In that case, $(a_M, b_m)$ is a supremum for $X$.

The proof for infema is similar.

In order to enlarge the category of considered algebras, we now weaken the hypothesis on the considered orders, by introducing so called multi-Archimedean partial orders.

## 1.3 Multi-Archimedean Partially Ordered Algebra

Throughout this section, the ring $\mathcal{R}$ is a unitary Dedekind-complete abelian ring, which is partially ordered, but not necessarily totally ordered. It is important to note that the algebras involved are not necessarily unitary.

**Definition 1.6** *A partially ordered algebra on $\mathcal{R}$ is a tuple $(\mathcal{A}, \mathcal{R}, \preceq_\mathcal{A})$, where $\mathcal{R}$ is a Dedekind-complete ordered abelian ring, $\mathcal{A}$ is an $\mathcal{R}-$algebra (with operations also denoted by $+_\mathcal{A}$ and $\times_\mathcal{A}$) and $\preceq_\mathcal{A}$ is a partial order, compatible with the order in $\mathcal{R}$, that is:*
*if $r \in \mathcal{R}$ and $x, y \in \mathcal{A}$ with $x \preceq_\mathcal{A} y$, if $0_\mathcal{R} \preceq r$ then $rx \preceq_\mathcal{A} ry$, and if $r \preceq_\mathcal{R} 0_\mathcal{R}$, then $ry \preceq_\mathcal{A} rx$.*

The following definitions only need $\preceq_\mathcal{A}$ to be a partial order on a set $\mathcal{A}$.

**Definition 1.7 (Tight Comparability)** *Let $x \in \mathcal{A}$ and $y \in \mathcal{A}$.*

1. *An element $y \in \mathcal{A}$ is called a tight lower bound of $x$ if and only if the order induced by $\preceq_\mathcal{A}$ on the set $[y, x] = \{z \in \mathcal{A} \ / \ y \preceq_\mathcal{A} z \preceq_\mathcal{A} x\}$ is a total order.*
2. *We say that $y$ is a tight upper bound of $x$ if $x$ is a tight lower bound of $y$.*

3. *We say that $x$ and $y$ are called tightly comparable if $y$ is either a tight upper bound of $x$ or a tight lower bound of $x$.*

**Remark 1.2** *On a cartesian product $\mathcal{A} = \Pi_{a\in A}\mathcal{B}_a$ of totally ordered sets, define $x \preceq y$ if and only if each coordinate $x_a$ of $x$ is less than the corresponding coordinate $y_a$ of $y$. This order is called coordinate by coordinate order. Then, thightly comparable elements in $\mathcal{A}$ differ by at most one of their coordinates.*

**Definition 1.8** *Let $x$ and $y$ be elements of a partially ordered set $\mathcal{A}$. An element $z$ in $\mathcal{A}$ is said to be tightly between $x$ en $y$ if and only if:*

$$\begin{cases} \textit{either } \ x \preceq_{\mathcal{A}} z \preceq_{\mathcal{A}} y \textit{ or } y \preceq_{\mathcal{A}} z \preceq_{\mathcal{A}} x \\ \textit{and} \\ z \textit{ is tightly comparable to both } x \textit{ and } y \end{cases}$$

**Definition 1.9 (Tight Strict Order)** *Let $x$ and $y$ be two elements of a partially ordered set $\mathcal{A}$.*

1. *We say that $y$ is a strict upper bound of $x$ (or simply that $y$ is greater than $x$), and we denote $x \prec_{\mathcal{A}} y$, if and only if $y \neq x$ and $x \preceq y$.*

2. *We say that $y$ is a tight strict upper bound (or $y$ is thightly greater than $x$) of $x$ if and only if $y \neq x$ and $y$ is a tight upper bound of $x$.*

**Remark 1.3** *Suppose that $\mathcal{A} = \Pi_{a\in A}\mathcal{B}_a$ is a cartesian product of totally ordered sets. Let $x, y \in \mathcal{A}$. Besides the notions of strict upper bound and tight upper bound from Definition 1.9 above, another notion can be defined, of a coordinate by coordinate strict ordering relation:*

*We say that $y$ is a broad strict upper bound (or $y$ is broadly greater than $x$), and we denote $x <_{\mathcal{A}} y$ of $x$ if and only if $y_a$ is a strict upper bound of $x_a$ for all $a \in A$ .*

The notion of a broad strict upper bound, defined above in carthesian product, makes sense in an arbitrary partially ordered algebra:

**Definition 1.10 (Broad Strict Order)** *Let $x_1$ and $x_2$ be two elements in a partially ordered algebra $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$. We say that $x_1$ is broadly strictly greater than $x_2$ if and only if for any strict lower bound $y$ of $x_1$, there exists an element in $z \in \mathcal{A}$, with $x_2 \prec_{\mathcal{A}} z$, such that $z$ is tightly between $y$ and $x_1$.*

*We say that $x_1$ is broadly strictly less than $x_2$ if the element $x_2$ is broadly strictly greater than $x_1$. We say that $x_1$ is broadly strictly positive [respectively negative] if $x_1$ is broadly strictly greater than [respectively less than] $0_{\mathcal{A}}$.*

**Remark 1.4** *Note that $x_1 \in \mathcal{A}$ is broadly strictly greater than $x_2 \in \mathcal{A}$ if and only if $-x_1$ is broadly strictly less than $-x_2$.*

The last Definition 1.10 agrees with the notion introduced in Remark 1.3 because of Remark 1.2.

**Definition 1.11 (Rough Order and Rough Equality)** *Let $\lambda_1$ and $\lambda_2$ be two broadly stricly positive elements in a Dedekind Complete partially ordered algebra $\mathcal{A}$.*

1. *We say that* $\lambda_1$ *is roughly less than* $\lambda_2$*, and we denote* $\lambda_1 \ll \lambda_2$*, if there exists* $n \in \mathbb{N}$ *such that* $\lambda_1 \preceq_{\mathcal{A}} n.\lambda_2$.
2. *We say that* $\lambda_1$ *is roughly equivalent to* $\lambda_2$*, and we denote* $\lambda_1 \cong \lambda_2$*, if we have both* $\lambda_1 \ll \lambda_2$ *and* $\lambda_2 \ll \lambda_1$.

**Definition 1.12** *Let* $x \in \mathcal{A}$ *and* $\lambda$ *be a broadly strictly positive element of* $\mathcal{A}$*. We say that* $x$ *is upper* $\lambda-$*bounded [respectively lower* $\lambda-$*bounded] if there exists* $n \in \mathbb{N}$ *such that* $x \preceq_{\mathcal{A}} n.\lambda$ *[respectively* $-(n.\lambda) \preceq_{\mathcal{A}} x$*]. We say that* $x$ *is* $\lambda-$*bounded if it is both upper* $\lambda-$*bounded and lower* $\lambda-$*bounded.*

**Definition 1.13 (Multi-Archimedean partially ordered algebra)** *A Dedekind complete partially ordered algebra* $(\mathcal{A}, \preceq_{\mathcal{A}})$ *is called* $\lambda-$*multi-Archimedean if and only if any elemnt in* $\mathcal{A}$ *is* $\lambda-$*bounded. In other words,*

$$\mathcal{A} = \bigcup_{n\in\mathbb{N}} \{x \in \mathcal{A} \ / \ x \preceq_{\mathcal{A}} n.\lambda\} \text{ and } \mathcal{A} = \bigcup_{n\in\mathbb{N}} \{x \in \mathcal{A} \ / \ -(n.\lambda) \preceq_{\mathcal{A}} x\}$$

*where* $n.\lambda = (\sum_{i=1}^{n} \lambda)$.

*We say that* $\mathcal{A}$ *is a multi-Archimedean algebra if it is* $\lambda-$*multi-Archimedean for some broadly strictly positive element* $\lambda \in \mathcal{A}$.

**Definition 1.14** *A strongly multi-Archimedean algebra is a Dedekind complete partially ordered algebra which contains at least one broadly strictly positive element, and which is* $\lambda-$*multi-Archimedean for any broadly strictly positive element* $\lambda \in \mathcal{A}$.

**Definition 1.15** *A standard multi-Archimedean algebra is a unitary Dedekind complete partially ordered algebra which is* $1_{\mathcal{A}}-$*multi-Archimedean (which implies that* $1_{\mathcal{A}}$ *is broadly strictly positive).*

The following shows a typical example:

**Proposition 1.2** *Let* $U$ *be any set and let* $\mathcal{R}$ *be a Dedekind-complete Archimedean totally ordered abelian ring. Let us consider the set* $\mathcal{A} = \mathcal{R}^U$ *of maps with domain* $U$ *and range* $\mathcal{R}$*, provided with a partially ordered algebra structure by setting* $f \preceq_{\mathcal{R}^U} g$ *if and only if for all* $x$ *in* $U$*, we have* $f(x) \preceq_{\mathcal{R}} g(x)$*. Then,*

1. *the ordered set* $\left(\mathcal{R}^U, \preceq_{\mathcal{R}^U}\right)$ *is complete.*
2. *It is not strongly multi-Archimedean in general, but it is strongly multi-Archimedean when* $U$ *is finite.*
3. *For any broadly strictly positive element* $\lambda$ *in* $\left(\mathcal{R}^U, \preceq_{\mathcal{R}^U}\right)$*, the subset of all* $\lambda-$*bounded maps in* $\mathcal{R}^U$ *is* $\lambda-$*multi-Archimedean.*

*Proof.* *1.* We prove that any upper bounded subset $B$ of $\mathcal{A} = \mathcal{R}^U$ has a supremum. By definition, such a set $B$ has an upper bound $f_0 \in \mathcal{A}$ for the order $\preceq_{\mathcal{R}^U}$, that is, the map $f_0$ is such that for all $f$ in $B$ and $x$ in $U$, we have $f(x) \prec_{\mathcal{R}} f_0(x)$. Hence the sets $\{f(x) \ / \ f \in B\}$ are all bounded, and therefore have a supremum $g(x)$. As for all $f$ in $B$ and for all $x$ in $U$, we clearly have $f(x) \preceq_{\mathcal{R}} g(x)$, the map $g$ thus defined is a supremum for $B$. The proof for infema is similar.

*2* We first notice that a constant function on $U$ identically equal to a strictly positive element of $\mathcal{R}$ is broadly strictly positive in $\mathcal{A}$. Now assume that $U$ is finite. Let $\lambda \in \mathcal{R}^U$ be a broadly strictly positive element. This means, as $\mathcal{R}^U$ is naturally a Cartesian product, that $0_{\mathcal{R}} \prec_{\mathcal{R}} \lambda(x)$. As $\mathcal{R}$ is totally ordered and $U$ is finite, this implies that there exists some $\varepsilon \in \mathcal{R}$ such that $0_{\mathcal{R}} \prec_{\mathcal{R}} \varepsilon \prec_{\mathcal{R}} l(x)$ for all $x \in U$.

Now let $f \in \mathcal{R}^U$. Since the ring $\mathcal{R}$ is Archimedean, this implies that there exists $n \in \mathbb{N}$ with $f(x) \preceq_{\mathcal{R}} n.\varepsilon$ for all $x \in U$. Then we have $f \preceq_{\mathcal{R}^U} n.l$. By reasoning similarly to prove that $-n'.l \preceq_{\mathcal{R}^U} f$ for some $n' \in \mathbb{N}$, we conclude that $\mathcal{A} = \mathcal{R}^U$ is multi-Archimedean.

*3* follows directly from Point *1* and the definitions. □

**Definition 1.16 (Multi-Archimedean Ring)** *A unitary Dedekind-complete partially ordered ring is called multi-Archimedean if it is multi-Archimedean as an algebra (Definition 1.13) over the ring $\mathbb{Z}$, where the external product $n.x$, for $n \in \mathbb{Z}$ and $x \in \mathcal{R}$, is defined in a natural way by:*

$$n.x = \left(\sum_{i=1}^{n} x\right) \text{ if } 0 \leq n \text{ and } n.x = -((-n).x) \text{ if } n < 0$$

**Definition 1.17 (General and Simple Complete Multi-Archimedean Algebra)** *We distinguish between the following kinds of algebras:*

- *A General [strongly,$\lambda-$] Multi-Archimedean Algebra is a Dedekind-complete [strongly,$\lambda-$] multi-Archimedean Algebra over a multi-Archimedean Dedekind-complete partially ordered abelian ring.*
- *A Simple [strongly,$\lambda-$] Multi-Archimedean Algebra is a Dedekind-complete [strongly,$\lambda-$] multi-Archimedean Algebra over an Archimedean Dedekind-complete totally ordered abelian ring (i.e. the ring can be only the usual oredered ring structures on either $\mathbb{Z}$ or $\mathbb{R}$).*

**Remark 1.5** *A General Complete Multi-Archimedean Algebra can be naturally provided with a structure of a Simple Complete Multi-Archimedean Algebra over the ring $\mathbb{Z}$, using the naturally defined external multiplication as in Definition 1.16.*

## 1.4 Multi-Archimedean Algebra and Cartesian Product

We proved with Proposition 1.2 that a cartesian product of a finite numbers of copies of an Archimedean Dedekind-complete totally ordered ring is a Simple Complete Multi-Archimedean Algebra over that same ring. In general, we consider the following construction:

**Definition 1.18 (Ordered Space of Maps)** *Let $(\mathcal{A}, \preceq_{\mathcal{A}})$ be a General Complete Multi-Archimedean Algebra over a multi-Archimedean Dedekind-complete partially ordered ring $(\mathcal{R}, \preceq_{\mathcal{R}})$. Then the set $\mathcal{A}^U$ of maps with domain $U$ and range $\mathcal{A}$ is a partially ordered algebra by considering the value by value addition, multiplication and order between maps (i.e. setting $(f+g)(x) = f(x) + g(x)$ and $(f.g)(x) = f(x).g(x)$ and, for the order $f \preceq_{\mathcal{A}^U} g$ if and only if for all $x$ in $U$, we have $f(x) \preceq_{\mathcal{A}} g(x)$). This provides $\mathcal{A}^U$ with an ordered algebra structure called the canonical ordered algebra structure, or the canonical ordered algebra, or the value by value ordered algebra, or the product ordered algebra structure structure over $\mathcal{A}^U$.*

**Theorem 1.1** *Under the notations of Definition 1.18, then, for a finite set $U$, the value by value ordered algebra structure over $\mathcal{A}^U$ is a General Complete Strongly Multi-Archimedean Algebra over $\mathcal{R}$, as well as over $\mathcal{A}$, as well as over $\mathbb{Z}$, considered as a ring acting by coordinate by coordinate addition and multiplication over $\mathcal{A}^U$.*

*Proof.* We follow the lines of the proof used of Proposition 1.2. Our ordered algebra $\mathcal{A}^U$ is complete for the same reason: the suppremum and infemum can be constructed coordinate by coordinate. We also notice that a constant function on $U$ identically equal to a broadly strictly positive element of $\mathcal{A}$ is broadly strictly positive in $\mathcal{A}^U$.

Now, let $l \in \mathcal{A}^U$ be a broadly strictly positive element. This implies that $l(x)$ is a broadly strictly positive element of $\mathcal{A}$ for each $x \in U$. Then, the set $U$ being finite and $\mathcal{A}$ multi-archimedean, this implies that there exists $n_1 \in \mathbb{N}$ such that $1_\mathcal{A} \leq n_1.l(x)$ for all $x \in U$.

Now, let $f$ be any element in $\mathcal{A}^U$. As $U$ is finite and $\mathcal{A}$ is multi-archimedean, this implies that there exists an $n_2 \in \mathbb{N}$ such that $-(n_2.1_\mathcal{A}) \prec_\mathcal{A} f(x) \prec_\mathcal{A} (n_2.1_\mathcal{A})$ for all $x \in U$.

For $x \in U$, we have $-(n_2.n_1.l(x)) \prec_\mathcal{A} -(n_2.1_\mathcal{A}) \prec_\mathcal{A} f(x) \prec_\mathcal{A} (n_2.1_\mathcal{A}) \prec_\mathcal{A} (n_2.n_1).l(x)$. This means that $-(n_2.n_1).l \preceq_{\mathcal{A}^U} f \preceq_{\mathcal{A}^U} (n_2.n_1)l$, which proves that $\mathcal{A}^U$ is multi-archimedian (over both rings $\mathcal{R}$ and $\mathcal{A}$). □

**Theorem 1.2** *Under the notations of Definition 1.18, then, for any broadly strictly positive element $\lambda \in \mathcal{A}$, the set all $\lambda-$bounded elements of $\mathcal{A}$, provided with the value by value ordered algebra structure over $\mathcal{A}^U$ is a General Complete $\lambda-$Multi-Archimedean Algebra over $\mathcal{R}$, as well as over $\mathcal{A}$, as well as over $\mathbb{Z}$, considered as a ring acting by coordinate by coordinate addition and multiplication over $\mathcal{A}^U$.*

The proof is similar to that of Therorem 1.1.

**Definition 1.19** *A general multi-Archimedean Dedekind-complete partially ordered algebra $\mathcal{A}$ is called discrete if the infemum*

$$i = \inf(\{x \in \mathcal{A} \ / \ x \textit{ is broadly greater that } 0_\mathcal{A}\})$$

*is itself broadly greater than $0_\mathcal{A}$. This notion holds in particular for a multi-Archimedean Dedekind-complete partially ordered ring, seen as an algebra over itself.*

Due to Proposition 2.2, discrete **unitary** multi-Archimedean algebras can be characterized as follows:

**Remark 1.6** *A unitary general multi-Archimedean Dedekind-complete partially ordered algebra $\mathcal{A}$ is discrete if any element in $\mathcal{A}$ which is broadly greater than $0_\mathcal{A}$ is greater than or equal to $1_\mathcal{A}$.*

**Theorem 1.3** *Under the notations of Definition 1.18, let us also assume that the algebra $\mathcal{A}$ is discrete. We consider the induced ordered sub-algebra of the value by value ordered algebra structure over $\mathcal{A}^U$ on the subset $\mathcal{A}^U_{1_\mathcal{A}}$ of bounded maps (i.e. maps $f$ such that the set $\{f(x) \ / \ x \in U\}$ has an upper bound in $\mathcal{A}$) Then, for any set $U$, this provides $\mathcal{A}^U_b$ with a General Complete Multi-Archimedean Algebra structure over $\mathcal{R}$.*

At last, we have:

**Theorem 1.4** *Let $\mathcal{A} = \prod_{a=1}^{d} \mathcal{B}_a$ be a cartesian product of a finite number of General Complete Multi-Archimedean Algebras $(\mathcal{B}_a)_{a\in\{1,\dots,d\}}$. Then, $\mathcal{A}$ is naturally provided with a General Complete Multi-Archimedean Algebra structure by considering the coordinate by coordinate sum, product and order on $\mathcal{A}$.*

The proof is similar to that of Therorem 1.1.

**Definition 1.20 (Product Multi-Archimedean Algebra)** *Under the hypothesis and notations of either Therorem 1.1, Theorem 1.2, Therorem 1.3 or Therorem 1.4, the resulting multi-Archimedean algebra is called the product multi-Archimedean algebra.*

# 2 Separability, Classification, Integration

## 2.1 Suprema of Tightly Strictly Positive Elements

**Lemma 2.1** *Let $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$ be a multi-Archimedean partially ordered algebra. If $x \prec_{\mathcal{A}} y$, then there exists an element $z$ which is tightly strictly greater than $x$ and less than $y$.*

*Proof.* Let $\lambda$ be a broadly strictly positive element of $\mathcal{A}$ such that $\mathcal{A}$ is $\lambda-$multi-Archimedean. The element $x + \lambda$ is broadly strictly greater than $x$, so, for the element $y$ which is strictly greater than $x$, there exists $z_0 \in \mathcal{A}$ such that $z_0 \prec_{\mathcal{A}} y$ and $z_0$ is tightly between $x$ and $y$. Then, we conclude by taking $z = z_0$ of $x \prec_{\mathcal{A}} z_0$, and by taking $z = y$ if $z_0 = x$. □

**Proposition 2.1** *Let $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$ be a $\lambda-$multi-Archimedean partially ordered algebra. For any interval $[x, y] \subset \mathcal{A}$, there exists a totally ordered subset $M \subset [x, y]$, such that $x = \inf(M)$ and $y = \sup(M)$, and which is maximal in the sense that any element of $[x, y]$ which is comparable to all elements of $M$ belong to $M$.*

*Moreover, we may assume that any $z, z' \in M$, are tightly comparable.*

*Proof.* We plan to use the Zorn Lemma. We define $C = \{z \in \mathcal{A} \ / \ x \preceq_{\mathcal{A}} z \preceq_{\mathcal{A}} y\}$, and

$$P = \left\{ X \subset C \ \middle/ \ \begin{array}{c} \forall z, z' \in X \text{ we have } z \preceq_{\mathcal{A}} z' \text{ or } z' \preceq_{\mathcal{A}} z \\ \text{and } z \text{ and } z' \text{ are tightly comparable} \end{array} \right\}$$

Let us prove that the ordered set $(P, \subset)$ is inductive. For this purpose, we consider a chain $T \subset P$, that is, for any, $X, X' \in T$, we have $X \subset X'$ or $X' \subset X$. Then the set $X_M = \bigcup_{X \in T} X$ belongs to $P$. Indeed, for $z$ and $z'$ in $X_M$, then there exists $X \in T \subset P$ which contains both $z$ and $z'$. From the definition of $P$, we see that $z$ and $z'$ are comparable in $\mathcal{A}$, so that we can conclude that $X_M \in P$. From the Zorn lemma, there exists a maximal element $M \in P$ for inclusion, that is, there exists a totally ordered set $M \subset C \subset \mathcal{A}$ such that no other element $z$ of $C$ can be comparable to all elements of $M$. Clearly, $x = \inf(M)$ and $y = \sup(M)$. □

**Remark 2.1** *Let $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$ be a $\lambda-$multi-Archimedean partially ordered algebra. Let $z \in \mathcal{A}$ and $X \in \mathcal{A}$ such that for all $x \in X$ the interval $[z, x]$ is totally ordered. Then, the interval $[z, \sup(X)]$ is totally ordered.*

**Proposition 2.2** *Let $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$ be a $\lambda-$multi-Archimedean partially ordered algebra, and let $x \prec_{\mathcal{A}} y$ be two distinct ans comparable elements of $\mathcal{A}$. Then we have $y = \sup(Z)$ where*

$$Z = \left\{ z \in \mathcal{A} \ \middle/ \ \begin{array}{c} x \preceq_{\mathcal{A}} z \preceq_{\mathcal{A}} y \textit{ and} \\ z \textit{ is tightly greater than } x \end{array} \right\}$$

*Proof.* Let $s = \sup(Z)$. Due to Lemma 2.1, if $s \neq y$, there exists $z_0 \in \mathcal{A}$ such that $s \prec_{\mathcal{A}} z_0$ and $z_0$ is tightly strictly greater than $s$.

By applying Lemma 2.1, there is a maximal totally ordered set $M \subset [x, z_0]$ such that $\inf(M) = x$ and $\sup(M) = z_0$, and any $z, z' \in M$ are tightly comparable. For any $z' \in M$ with $z' \prec_{\mathcal{A}} z_0$, we have, by considering (Remark 2.1) the infemum on $z$, that $[x, z']$ is totally ordered. We conclude, by conidering $z_0$ as the supremum on $z'$, that $z_0$ is less than the supremum $s$ of a larger set. □

Now, we can define the absolute value of a non zero element as follows.

**Definition 2.1 (Absolute Value)** *Let $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$ be a multi-Archimedean partially ordered algebra. Let $x \in \mathcal{A}$.*

1. *If $x$ is tightly comparable to $0_{\mathcal{A}}$, then the absolute value $|x|$ of $x$ is defined by*

$$|x| = \begin{cases} x & \text{if} \quad 0_{\mathcal{A}} \preceq_{\mathcal{A}} x \\ -x & \text{if} \quad x \preceq_{\mathcal{A}} 0_{\mathcal{A}} \end{cases}$$

2. *We define the absolute value $|x|$ of $x$ by:*

$$|x| = \sup\left(\left\{ |y| \quad \Big/ \quad \begin{array}{c} 0_{\mathcal{A}} \preceq_{\mathcal{A}} y \preceq_{\mathcal{A}} x \text{ or } x \preceq_{\mathcal{A}} y \preceq_{\mathcal{A}} 0_{\mathcal{A}} \\ \text{and } y \text{ is tightly comparable to } 0_{\mathcal{A}} \end{array} \right\}\right)$$

**Proposition 2.3** *Let $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$ be a multi-Archimedean partially ordered algebra and $x \in \mathcal{A}$. Then,*

1. *We have $x \preceq_{\mathcal{A}} |x|$.*

2. *We have $x = |x|$ if and only if $0_{\mathcal{A}} \preceq_{\mathcal{A}} x$.*

The only non-trivial point is that if $0_{\mathcal{A}} \preceq_{\mathcal{A}} x$ we have $x = |x|$, but this follows from Proposition 2.2.

**Proposition 2.4** *Let $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$ be a $\lambda-$multi-Archimedean partially ordered algebra. Then, there exists a maximal set $Z \subset \mathcal{A}$ such that for any two distinct elements $z, z' \in Z$ the elements $z$ and $z'$ are not comparable and all elements in $Z$ are tightly strictly positive.*

*Moreover, any set $Z_0$ in which all elements are tightly strictly positive any two distinct elements are not comparable can be extended to such a maximal set $Z$.*

*Proof.* Let us consider

$$P = \left\{ Z_0 \subset X \subset \mathcal{A} \quad \Big/ \quad \begin{array}{c} \forall z, z' \in X \text{ the elements } z \text{ and } z' \text{ are not comparable} \\ \text{all elements of } X \text{ are tightly strictly positive} \end{array} \right\}$$

Let us prove that the ordered set $(P, \subset)$ is inductive. To that aim, we consider a chain $T \subset P$. Then the set $X_M = \bigcup_{X \in T} X$ belongs to $P$. Indeed, for $z, z' \in X_M$, there exists $X \in P$ which contains both $z$ and $z'$. This implies that $z$ is tightly stricly positive and $z, z'$ are not comparable. We conclude by the Zorn Lemma that there exists a maximal element $Z$ in $P$. □

**Proposition 2.5** *Under the hypothesis and notations of Proposition 2.4, we consider the element $b = \sup(Z)$. Then the element $b$ is broadly strictly positive. Moreover, we may choose $Z$ in such a way that $\sup(Z) = \lambda$.*

*Proof.* If we consider a tight strict lower bound $y$ of $b$, then, from the definition of $b$, the element $y$ must be smaller than some element $z \in Z$, which is tightly stricly greater than $0_{\mathcal{A}}$. We have $0_{\mathcal{A}} \prec_{\mathcal{A}} z$ and $z$ is tightly between $y$ and $b$.

Now, for $z \in Z$, let us set

$$g(z) = \sup\left(\left\{ u \in \mathcal{A} \quad \Big/ \quad \begin{array}{c} u \preceq_{\mathcal{A}} \lambda \text{ and} \\ u \text{ is tightly comparable to } 0_{\mathcal{A}} \text{ and to } z \end{array} \right\}\right)$$

Then, by Remark 2.1, $g(z)$ is tightly comparable to $0_{\mathcal{A}}$. Moreover, the set $Z' = \{g(z) \ / \ z \in Z\}$ is maximal in the sense of Proposition 2.4. At last, we have $\sup(Z') = \lambda$ by maximality of $Z'$. □

**Proposition 2.6** *Under the hypothesis and notations of Proposition 2.5, given an element* $0_{\mathcal{A}} \preceq_{\mathcal{A}} b_0$ *in* $\mathcal{A}$*, then there exists a positive element* $0_{\mathcal{A}} \preceq_{\mathcal{A}} b_1$ *in* $\mathcal{A}$ *such that* $b_0 + b_1$ *is broadly strictly positive, and such that no nonzero element which is tightly strictly greater than zero is comparable to both* $b_0$ *and* $b_1$*.*

*Proof.* In a similar way to the proof of Proposition 2.4, we show that there exists a maximal set $Z_0$ of tightly strictly positive elements of $\mathcal{A}$ which are comparable to $b_0$, and any two distinct elements of $Z_0$ are not comparable. Following Proposition 2.4, the set $Z_0$ can be extended to a set $Z$ such that, following Proposition 2.5, the element $b = \sup(Z)$ is broadly strictly positive in $\mathcal{A}$. We conclude by setting $b_1 = \sup(Z \backslash Z_0)$. □

**Proposition 2.7** *Under the hypothesis and notations of Proposition 2.4, if* $x$ *is tightly comparable to* $0_{\mathcal{A}}$*, then for* $z \in \mathcal{A}$ *the element* $x.z$ *is tightly comparable to both* $0_{\mathcal{A}}$ *and* $x$*.*

*Proof.* Let $n \in \mathbb{N}$ be such that $z \preceq_{\mathcal{A}} n.\lambda$. Then we have $x.z \preceq_{\mathcal{A}} n.x$, but $n.x$ is tightly comparable to $0_{\mathcal{A}}$ and is tightly comparable to $x$. □

**Lemma 2.2** *For* $z$ *and* $z'$ *tightly comparable to* $0_{\mathcal{A}}$ *such that* $z$ *and* $z'$ *are no comparable, we have:*

1. *If* $0_{\mathcal{A}} \preceq_{\mathcal{A}} z$ *and* $0_{\mathcal{A}} \preceq_{\mathcal{A}} z'$*, then* $z + z' = \sup(\{z, z'\})$*.*
2. *If* $0_{\mathcal{A}} \preceq_{\mathcal{A}} z$ *and* $z' \preceq_{\mathcal{A}} 0_{\mathcal{A}}$*, then* $z + z' = \sup(\{z, z'\}) + \inf(\{z, z'\})$*.*

*Proof.* If $z$ and $z'$ are both strictly positive, clearly, $\sup(\{z, z'\}) \preceq_{\mathcal{A}} z + z'$. Now, from Proposition 2.2 and Lemma 2.3, we have $z + z' = \sup(\{z, z'\})$.

If $0_{\mathcal{A}} \preceq_{\mathcal{A}} z$ and $z' \preceq_{\mathcal{A}} 0_{\mathcal{A}}$, then $z = \sup(\{z, 0_{\mathcal{A}}\}) = \sup(\{z, z', 0_{\mathcal{A}}\}) = \sup(\{z, z'\})$. Similarly, $z' = \inf(\{z, z'\})$. □

**Proposition 2.8** *Under the hypothesis and notations of Proposition 2.4, let* $Y_1 \subset Z$ *and* $Y_2 \subset Z$*, with* $Y_1 \cap Y_2 = \emptyset$ *be any disjoint sets of elements tightly greater than* $0_{\mathcal{A}}$*. We denote* $y_1 = \sup(Y_1)$ *and* $y_2 = \sup(Y_2)$*. Then, we have:*

$$y_1 + y_2 = \sup(\{y_1, y_2\}) = \sup(Y_1 \cup Y_2) \text{ and } y_1.y_2 = 0_{\mathcal{A}}$$

*Proof.* The first relationship is due to Proposition 2.2 and from the fact that an element of $\mathcal{A}$ which is strictly tightly greater than $0_{\mathcal{A}}$ cannot be both less than $y_1$ and less than $y_2$.

The second relationship follows from the fact that, for $y_1.y_2$ to be non zero, due to Proposition 2.2, $y_1$ and $y_2$ would have to be both comparable to a non zero element, which is impossible from the definition of $Z$. □

**Lemma 2.3** *Let* $z, z'$ *and* $s$*, such that* $z, z'$ *and* $s$ *are all tightly strictly greater than* $0_{\mathcal{A}}$*, and are pairwise not comparable. Then,* $s$ *is not comparable to* $z + z'$*.*

*Proof.* If $z+z' \preceq_{\mathcal{A}} s$, then $s$ is comparable to both $z$ and $z'$, which contradicts our hypothesis. Now, suppose by *reductio ad absurdum* that $s \prec_{\mathcal{A}} z+z'$. let

$$z_0 = \sup\left(\left\{u \in \mathcal{A} \;\middle/\; \begin{array}{c} s \preceq_{\mathcal{A}} u \preceq_{\mathcal{A}} z+z' \\ u \text{ is tightly comparable to } 0_{\mathcal{A}} \end{array}\right\}\right)$$

By Remark 2.1, the interval $[s, z_0]$ is totally ordered.

By applying Lemma 2.1, there is a maximal totally ordered set $M \subset [x, z_0]$ such that $\inf(M) = 0_{\mathcal{A}}$ and $\sup(M) = z_0$, and any $u, u' \in M$ are tightly comparable. For any $u' \in M$ with $u' \prec_{\mathcal{A}} z_0$, we have, by considering (Remark 2.1) the infemum on $z$, that $[0_{\mathcal{A}}, u']$ is totally ordered. We conclude, by considering $z_0$ as the supremum on $u'$, that $z_0$ is tightly greater than $0_{\mathcal{A}}$.

At last, $z_0$ cannot be equal to $z+z'$, as in such a case $z$ and $z'$ would be comparable. Now, Lemma 2.1 contradicts the definition of $z_0$ as a supremum. □

**Proposition 2.9** *Let $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$ be a $\lambda-$multi-Archimedean partially ordered algebra, and let $x \in \mathcal{A}$. We the following two sets:*

$$Z = \left\{ z \in \mathcal{A} \;\middle/\; \begin{array}{c} 0_{\mathcal{A}} \preceq_{\mathcal{A}} z \preceq_{\mathcal{A}} x \textit{ and} \\ z \textit{ is tightly greater than } 0_{\mathcal{A}} \end{array} \right\}$$

$$Z' = \left\{ z \in \mathcal{A} \;\middle/\; \begin{array}{c} x \preceq_{\mathcal{A}} z \preceq_{\mathcal{A}} 0_{\mathcal{A}} \textit{ and} \\ z \textit{ is tightly less than } 0_{\mathcal{A}} \end{array} \right\}$$

*We set $s = \sup(Z)$ and $i = \inf(Z')$. Then, we have $s = \sup(\{x, 0_{\mathcal{A}}\})$ and $i = \inf(\{x, 0_{\mathcal{A}}\})$. At last, we have: $x = s + i$.*

*Proof.* Due to Proposition 2.2, applied between $i$ and $s$, we get

$$s - i = \sup\left(\left\{z \in \mathcal{A} \;\middle/\; \begin{array}{c} 0_{\mathcal{A}} \preceq_{\mathcal{A}} z \preceq_{\mathcal{A}} s-i \text{ and} \\ z \text{ is tightly greater than } 0_{\mathcal{A}} \end{array}\right\}\right) = (x - 2.i)$$

□

**Definition 2.2 (Finitely Genrated Multi-Archimedean Algebra)** *A general multi-Archimedean algebra is called finitely generated if and only if any set of pairwise not comparable elements is finite.*

**Definition 2.3 (Separable Multi-Archimedean Algebra)** *A general multi-Archimedean algebra is called separable if and only if any set of pairwise not comparable elements is countable.*

In the sequel, we assume that all the considered multi-Archimedean algebra are separable.

## 2.2 Ideals in a Multi-Archimedean Algebra

Till the end of this section, $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$ denotes a general multi-Archimedean partially ordered **unitary** algebra.

**Definition 2.4** *A sub-algebra or an ideal $\mathcal{B}$ of $\mathcal{A}$ is said to be closed if the order induced by $\prec_{\mathcal{A}}$ on $\mathcal{B}$ is Dedekind-complete.*

We may notice that the proof of Proposition 2.4 and Proposition 2.5 also work to prove the following:

**Remark 2.2** *Let $\mathcal{C}$ be a closed ideal of $\mathcal{A}$. Then, there exists a maximal set $Z \subset \mathcal{C}$ such that for any two elements $z, z' \in Z$ the elements $z$ and $z'$ are not comparable and all elements in $Z$ are tightly strictly positive. Moreover, the element $c = \sup(Z)$ is broadly strictly positive in the ordered algebra $(\mathcal{C}, \mathcal{R}, \preceq_{\mathcal{A}})$.*

**Proposition 2.10** *Let $\mathcal{C}$ be a complete ideal of a multi-Archimedean algebra $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$. Assume that for every $x \in \mathcal{C}$, all elements of $\mathcal{A}$ which are tightly between $0_{\mathcal{A}}$ and $x$ also belong to $\mathcal{C}$. Then we have:*

1. *The quotient $\mathcal{A}/\mathcal{C}$ is naturally provided with an order $\preceq_{\mathcal{A}/\mathcal{C}}$ such that the projection $\pi : \mathcal{A} : \mathcal{A}/\mathcal{C}$ from $\mathcal{A}$ to the quotient algebra $\mathcal{A}/\mathcal{C}$ is increasing.*

2. *This order provides $\mathcal{A}/\mathcal{C}$ with a $\pi(\lambda)-$multi-Archimedean algebra structure over $\mathcal{R}$.*

3. *We have an ordered algebra isomorphism:*

$$\Phi : \mathcal{A} \longrightarrow \mathcal{C} \times (\mathcal{A}/\mathcal{C})$$

   *through which we can construct a supplementary closed sub-algebra $\mathcal{S}$ of $\mathcal{C}$ in $\mathcal{A}$ (which we denote by $\mathcal{A} = \mathcal{C} \oplus \mathcal{S}$) such that $\mathcal{A} = \mathcal{C} \oplus \mathcal{S}$, and for every $x \in \mathcal{S}$, all elements of $\mathcal{A}$ which are tightly between $0_{\mathcal{A}}$ and $x$ also belong to $\mathcal{S}$.*

4. *Consequently, $\mathcal{C}$ is proved in turn to be multi-Archimedean.*

*Proof.* We consider a set $Z$ of generators which are tightly comparable to $0_{\mathcal{A}}$ and, as in Proposition 2.5 such that $\sup(Z) = \lambda$. We split the set $Z$ into $Z = Z_{\mathcal{C}} \cup \overline{Z_{\mathcal{C}}}$, with $Z_{\mathcal{C}} = Z \cap \mathcal{C}$ and $\overline{Z_{\mathcal{C}}} = Z \backslash Z_{\mathcal{C}}$. Let $c = \sup(Z_{\mathcal{C}}) \in \mathcal{C}$ and $d = \sup(\overline{Z_{\mathcal{C}}})$. Due to Proposition 2.8, we have $c + d = \lambda$.

For any element $u \in \mathcal{C}$ such that $0_{\mathcal{A}} \preceq_{\mathcal{A}} u$, due to Proposition 2.2, we have $u.d = 0_{\mathcal{A}}$. This is easily extended to any $u \in \mathcal{C}$ using Proposition 2.9.

We define in $\mathcal{Q} = \mathcal{A}/\mathcal{C}$ the order $\preceq_{\mathcal{Q}}$ defined by: $[x] \preceq_{\mathcal{Q}} [y]$ if and only $d.x \preceq_{\mathcal{A}} d.y$. If $x' = x + c_1$ and $y' = x + c_2$, with $c_1, c_2 \in \mathcal{C}$ are other representants, then we have $c_1.d = c_2.d = 0_{\mathcal{A}}$, so that the order on $\mathcal{Q}$ is well defined.

Let us show that the ordered algebra $(\mathcal{Q}, \mathcal{R}, \preceq_{\mathcal{Q}})$ is a $\lambda_Q-$multi-Archimedean algebra, with $\lambda_Q = [d]$. Let $[l] \in Q$, with $l \in \mathcal{A}$, and let $n \in \mathbb{N}$ be such that $l \preceq_{\mathcal{A}} n.\lambda = n(c + d)$. We have $[l] \preceq_{\mathcal{Q}} n.[d] = n.\lambda_Q$. □

**Corollary 2.1** *Under the hypothesis and notations of Proposition 2.10, if $0_{\mathcal{A}} \preceq_{\mathcal{A}} x \preceq_{\mathcal{A}} c$ and $c \in \mathcal{C}$, then $x \in \mathcal{C}$.*

Indeed, the projection onto $\mathcal{A}/\mathcal{C}$ being increasing, the projetction of the element $x$ is squeezed between $0_{\mathcal{A}}$ and the projection of $c$, also equal to $0_{\mathcal{A}}$.

**Proposition 2.11** *Assume that $\mathcal{A}$ is unitary. Let $\mathcal{B}$ be the ideal of $\mathcal{A}$ generated by $x_0$. Then, $x_0$ is a broadly strictly positive element in the multi-Archimedean algebra $\mathcal{B}$.*

*Proof.* Due to Proposition 2.8 and Proposition 2.6, there exists $x_1 \in \mathcal{A}$ such that $x_0 + x_1$ is broadly strictly positive in $\mathcal{A}$ and $x_0.x_1 = 0_{\mathcal{A}}$. Let $x \in \mathcal{B}$. The algebra $\mathcal{A}$ being multi-Archimedean, there exists $n \in \mathbb{N}$ such that $x \preceq_{\mathcal{A}} n.(x_0+x_1)$. We have $x \preceq_{\mathcal{B}} n.x_0$ since $[x_1] = 0$ in $\mathcal{B}$. We conclude (for exeample by considering $x = 1_{\mathcal{B}}$, that $x_0$ is broadly strictly positive in $\mathcal{B}$. □

**Proposition 2.12** *Assume that $\mathcal{A}$ is unitary. Let $\mathcal{B}$ be the ideal of $\mathcal{A}$ generated by $x_0$, with $x_0$ non-zero and tightly comparable to $0_{\mathcal{A}}$. Then, $\mathcal{B}$ is Archimedean, and is consequently (Proposition 1.1) isomorphic as an ordered algebra either to the usual structure on $\mathbb{Z}$ or to the usual structure on $\mathbb{R}$.*

*Proof.* Let $0_{\mathcal{B}} \prec_{\mathcal{B}} l$ and let $x \in \mathcal{B}$. Due to Proposition 2.8 and Proposition 2.6, there exists $l_1 \in \mathcal{A}$ such that $l + l_1$ is broadly strictly positive in $\mathcal{A}$ and $l.l_1 = 0_{\mathcal{A}}$. Since $x_0$ is tightly comparable to $0_{\mathcal{A}}$ and $l$ is of the form $z.x_0$, then $l$ is also tightly comparable to $0_{\mathcal{A}}$ and we have $[l_1] = 0_{\mathcal{B}}$. Let $n \in \mathbb{N}$ be such that $x \preceq_{\mathcal{A}} n.(l + l_1)$. We have $x \preceq_{\mathcal{B}} n.l$. □

## 2.3 Discrete Characteristics and Morphisms

**Definition 2.5** *Let $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$ be a multi-Archimedean unitary algebra over a ring $\mathcal{R}$. Note that the existence of a broadly strictly positive element is established in Proposition 2.5. We define the discrete characteristics of $\mathcal{A}$ the element of $\mathcal{A}$ defined by*

$$\chi(\mathcal{A}) = \inf\left(\{b \in \mathcal{A} \ / \ b \textit{ is broadly strictly positive}\}\right)$$

**Example 2.1** *We consider the following product multi-Archimedean algebras.*

- *For $\mathcal{A}_1 = \mathbb{Z} \times \mathbb{Z}$, we have $\chi(\mathcal{A}_1) = (1, 1)$.*
- *For $\mathcal{A}_2 = \mathbb{R} \times \mathbb{R}$, we have $\chi(\mathcal{A}_2) = (0, 0)$.*
- *For $\mathcal{A}_3 = \mathbb{R} \times \mathbb{Z}$, we have $\chi(\mathcal{A}_3) = (0, 1)$.*

**Remark 2.3** *Let $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$ be a unitary multi-Archimedean algebra and $\mathcal{B} \subset \mathcal{A}$ be a closed sub-algebra which contains $1_{\mathcal{A}}$ (note that we do not assume that $\mathcal{B}$ is an ideal) which contains at least one broadly strictly positive element of $\mathcal{A}$. Then the characteristics $\chi(\mathcal{B}) \in \mathcal{B}$ can naturally be defined as an element of $\mathcal{A}$ by*

$$\chi_{\mathcal{A}}(\mathcal{B}) = \inf\left(\{b \in \mathcal{B} \ / \ b \textit{ is broadly strictly positive in } \mathcal{A}\}\right)$$

*Furthermore, if $\mathcal{B}' \subset \mathcal{A}$ is another closed sub-algebra of $\mathcal{A}$ which contains at least one broadly strictly positive element of $\mathcal{A}$, then we have $\mathcal{B}' \subset \mathcal{B}$, if and only if $\chi(\mathcal{B}) \preceq_{\mathcal{A}} \chi(\mathcal{B}')$.*

*Proof.* If $\mathcal{B}' \subset \mathcal{B}$, then we have $\chi(\mathcal{B}) \preceq_{\mathcal{A}} \chi(\mathcal{B}')$ from the very definition. Conversely, if there exists $x \in \mathcal{B}' \backslash \mathcal{B}$ let us consider the ideal $\mathcal{C}$ generated by $x$. We may consider without loss of generality that $x$ is tightly comparable to $0_{\mathcal{A}}$. From Proposition 2.12 and Proposition 1.1, if $x \notin \mathcal{B}$, then $\mathcal{C} \bigcap \mathcal{B}$ is isomorphic to $\mathbb{Z}$ and $\mathcal{C} \bigcap \mathcal{B}'$ is isomorphic to $\mathbb{R}$, and we cannot have $\chi(\mathcal{B}) \preceq_{\mathcal{A}} \chi(\mathcal{B}')$. □

**Proposition 2.13** *Let* $(\mathcal{A}, \mathcal{R}, \preceq_\mathcal{A})$ *and* $(\mathcal{B}, \mathcal{R}, \preceq_\mathcal{A})$ *be multi-Archimedean algebras. Let* $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ *be an isomorphism of ordered algebras. Then we must have* $\varphi(\chi(\mathcal{A})) = \chi(\mathcal{B})$.

**Proposition 2.14** *Let* $\mathcal{A}_1$ *[respectively* $\mathcal{A}_2$*] be two product multi-Archimedean algebras of* $\lambda_1-$*bounded [respectively* $\lambda_2-$*bounded] elements in a Cartesian product of the for the form:*

$$\mathcal{A}_1 \subset \mathbb{Z}^{U_1} \times \mathbb{R}^{V_1} \textit{ and } \mathcal{A}_2 \subset \mathbb{Z}^{U_2} \times \mathbb{R}^{V_2}$$

*where* $U_1$*,* $V_1$*,* $U_2$ *and* $U_3$ *are sets, and* $\lambda_1$ *and* $\lambda_2$ *are broadly strictly positive elements in* $\mathbb{Z}^{U_1} \times \mathbb{R}^{V_1}$ *[respectively* $\mathbb{Z}^{U_2} \times \mathbb{R}^{V_2}$*]. Then, the ordered algebras* $(\mathcal{A}, \mathcal{R}, \preceq_\mathcal{A})$ *are isomorphic if and only if there is both a one to one correspondence between* $U_1$ *and* $U_2$ *and a one to one correspondence between* $V_1$ *and* $V_2$*.*

*Proof.* We see that $\chi(\mathcal{A}_1) = (1_{U_1}, 0_{V_1})$ and $\chi(\mathcal{A}_2) = (1_{U_2}, 0_{V_2})$, where $1_U$ denotes the function identically equal to 1 on $U$ and $0_V$ denotes the function identically equal to 0 on $V$. The result then follows from Proposition 2.13. □

## 2.4 Classification of Multi-Archimedean Algebras

Throughout this section, that multi-Archimedean algebra $\mathcal{A}$ is assumed to be unitary.

**Definition 2.6** *Let us consider* $G$ *the set of all elements of* $\mathcal{A}$ *which are either tightly strictly positive or tightly strictly negative. We consider the relation* $\rho$ *on* $G$ *such that* $\rho(x, y)$ *if and only if* $|x|$ *and* $|y|$ *are tightly comparable.*

*Then* $\rho$ *is easily seen to be an equivalence relation and the equivalence classes of* $\rho$ *are called the essential generators of* $\mathcal{A}$*. The class under* $\rho$ *of an element* $x$ *is denoted by* $[x]$*.*

**Proposition 2.15** *Let* $[x]$ *be an essential generator of* $\mathcal{A}$ *and let* $\mathcal{B} = [x] \bigcup \{0_\mathcal{A}\}$*. Then* $\mathcal{B}$ *is the ideal in* $\mathcal{A}$ *generated by* $x$*.*

*Proof.* We may assume w.l.o.g. that $0_\mathcal{A} \preceq_\mathcal{A} x$. Let $\mathcal{C}$ be the ideal in $\mathcal{A}$ generated by $x$. Let $y \in \mathcal{B}$. If $y \preceq_\mathcal{A} x$, Then, from Corollary 2.1, we have $y \in \mathcal{C}$. if $x$ is tightly between $0_\mathcal{A}$ and $y$, then by Proposition 2.11, there exists $n \in \mathbb{N}$ such that $y \preceq_\mathcal{A} n.x$, hence $y \in \mathcal{C}$.

Conversely, if $y \in \mathcal{C}$ is non zero, then there exists $z \in \mathcal{A}$ such that $y = x.z$. For some $n \in \mathbb{N}$, we have $z \preceq_\mathcal{A} n.1_\mathcal{A}$ so that $y = x.z \preceq_\mathcal{A} n.x \in \mathcal{B}$ and $y \in \mathcal{B}$. □

**Remark 2.4** *We have a natural identification of* $\mathbb{Z}^U \times \mathbb{R}^V$ *with the subset of* $\mathbb{R}^{U \cup V}$ *of maps which send elements of* $U$ *into* $\mathbb{Z}$*.*

**Theorem 2.1** *Any general multi-Archimedean algebra* $(\mathcal{A}, \mathcal{R}, \preceq_\mathcal{A})$ *is isomorphic, as an ordered ring, to the set of all* $\lambda-$*bounded elements a Cartesian product of the form* $\mathbb{Z}^U \times \mathbb{R}^V$*, where* $U$ *and* $V$ *are two sets,* $\lambda$ *is a broadly strictly positive element of* $\mathbb{Z}^U \times \mathbb{R}^V$*, and the ordered ring structure is defined as in Section 1.4.*

*Proof.* Let $U$ be the set of all essential generators $\beta$ of $\mathcal{A}$ such that $\{\beta\} \bigcup \{0_\mathcal{A}\}$ is isomorphic to $\mathbb{Z}$. For $\beta \in U$, let $\varphi_\beta$ be a choice of an isomorphism from $\beta \bigcup \{0_\mathcal{A}\}$ to $\mathbb{Z}$. Similarly, let $V$ be the set of all essential generators of $\mathcal{A}$ such that $\eta \bigcup \{0_\mathcal{A}\}$ is isomorphic to $\mathbb{R}$, and for $\eta \in V$, let $\varphi_\eta$ be a choice of an isomorphism from $\eta \bigcup \{0_{\mathcal{A}}\}$ to $\mathbb{R}$. We consider the map (defined using the identification set out in Remark 2.4):

$$\Phi : \begin{cases} G & \longrightarrow & \mathbb{Z}^U \times \mathbb{R}^V \\ x & \longmapsto & \Phi(x) : \beta \longmapsto \begin{cases} & (\varphi_\beta(x), 0_{\mathbb{R}}) \text{ if } \beta = [x] \in U, \\ & (0_{\mathbb{Z}}, \varphi_\beta(x)) \text{ if } \beta = [x] \in V, \\ \text{and} & (0_{\mathbb{Z}}, 0_{\mathbb{R}}) \text{ otherwise} \end{cases} \end{cases}$$

We also define $\Phi(0_{\mathcal{A}}) = (0_{\mathbb{Z}}, 0_{\mathbb{R}})$. Then $\Phi$ can be prolonged to a unique one to one morphism of ordered algebra

$$\Phi : \begin{cases} \mathcal{A} & \longrightarrow & \mathbb{Z}^U \times \mathbb{R}^V \\ x = \sup(\{z \in G \bigcup \{0\} \ / \ z \preceq_{\mathcal{A}} x\}) & \longmapsto & \sup(\{\Phi(z) \ / \ z \in G \text{ and } z \preceq_{\mathcal{A}} x\}) \end{cases}$$

If $\mathcal{A}$ id a $\lambda_{\mathcal{A}}-$multi-Archimedean algebra, then, by denoting $\lambda = \Phi(\lambda_{\mathcal{A}})$, then the image $\Phi(\mathcal{A})$ is the closed sub-algebra of $\lambda-$bounded elements in $\mathbb{Z}^U \times \mathbb{R}^V$. Since $\Phi$ is one to one, it defines an isomorphism onto its image. $\square$

**Theorem 2.2** *Any general multi-Archimedean algebra $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$ is isomorphic, as an ordered algebra, to the set of all $\lambda-$bounded elements a Cartesian product of the form:*

$$\left(\mathbb{Z}^U \times \mathbb{R}^V, \mathbb{Z}^{U_0} \times \mathbb{R}^{V_0}, \preceq\right)$$

*where $U, V, U_0, V_0$ are sets, where $\lambda$ is a broadly strictly positive element of $\mathbb{Z}^U \times \mathbb{R}^V$, and the multi-Archimedean structure is defined as in Section 1.4.*

*Proof.* We first define the isomorphism of an ordered ring on both $\mathcal{A}$ to $\mathbb{Z}^U \times \mathbb{R}^V$ and $\mathcal{R}$ to $\mathbb{Z}^{U_0} \times \mathbb{R}^{V_0}$ using Theorem 2.1. Then we define the action of $\mathbb{Z}^{U_0} \times \mathbb{R}^{V_0}$ by external product on $\mathbb{Z}^U \times \mathbb{R}^V$ and $\mathcal{R}$, by using the external product of elements of $\mathcal{R}$ by elements of $\mathcal{A}$ through the relevant isomorphisms, so as to create an obvious commutative diagram. $\square$

## 2.5 Fixed Denominator Rational Multi-Archimedean Algebra

In this section, we consider a **unitary** general multi-Archimedean algebra $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$.

Let $l \in \mathcal{A}$ (or possibly $l \in \mathcal{R}$, in which case we identify $l$ with the element $l.1_{\mathcal{A}}$ of $\mathcal{A}$) be a broadly strictly positive element. For $X \subset \mathcal{A}$, we consider the set

$$X/l = \left\{\frac{x}{l} \ / \ x \in X\right\}$$

Conversely, for $Y \subset \mathcal{A}/l$, we define $l.Y = \left\{x \in \mathcal{A} \ / \ \frac{x}{l} \in Y\right\}$. The set $\mathcal{A}/l$ is naturally in one to one correspondance with $\mathcal{A}$ through the map $x \longmapsto \frac{x}{l}$. The inverse map is the map which to some $y = \frac{x}{l} \in \mathcal{A}/l$ associates $l.y \stackrel{\text{def}}{=} x$.

We can be naturally provide $\mathcal{A}/l$ with operations: $(\mathcal{A}/l, \mathcal{R}, \preceq_{\mathcal{A}/l})$, by setting:

- $\frac{x}{l} + \frac{y}{l} = \frac{x+y}{l}$ for $x, y \in \mathcal{A}$.
- $\frac{x}{l}.y = \frac{x.y}{l}$ for $x, y \in \mathcal{A}$.

- $\frac{x}{l} \preceq_{\mathcal{A}/l} \frac{y}{l}$ if and only if $x \preceq_{\mathcal{A}} y$.

There is a natural one to one inclusion map:

$$i_* : \begin{cases} \mathcal{A} & \longrightarrow & \mathcal{A}/l \\ x & \longmapsto & \frac{l.x}{l} \end{cases}$$

which allows to define an inductive limit, which is provided with and ordered algebra structure, either based on all the sets $\mathcal{A}/l^n$ for $n \in \mathbb{N}$ for a fixed broadly strictly positive element $l$.

We can also define an inductive limit, which is provided with and ordered algebra structure, either based on all the sets $\mathcal{A}/l^n$ for all broadly strictly positive elements $l \in \mathcal{A}$.

**Proposition 2.16** *If $l$ is invertible in $\mathcal{A}$, then we have a commutative diagram*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\textit{Division by } l"] \arrow[d, "\textit{Multiplication by } l^{-1}"'] & \mathcal{A}/l \arrow[d, "\textit{Identity}"] \\ \mathcal{A} \arrow[r, "\textit{Inclusion } i_*"'] & \mathcal{A}/l \end{tikzcd}$$

Note that in the case when $l \in \mathcal{R}$, where $\mathcal{R}$ is a field and $\mathcal{A} = \mathcal{R}$ (for example $\mathcal{R} = \mathcal{A} = \mathbb{R}$), then $\mathcal{A}/l$ can be seen as $\mathcal{A}$ itself, and the natural isomorphism from $\mathcal{A}$ to $\mathcal{A}/l$ can be seen as an automorphism.

## 2.6 Integrals of Functions to Multi-Archimedean Algebras

In this section, we purpose to define the integral of a function from a measured space to a multi-Archimedean partially ordered algebra. We will proceed along the lines of classical definitions for an integral, first defining integrals for measurable functions with positive values, and generalizing to arbitrary measurable functions by decomposing them as the sum of functions with positive and negative values.

**Definition 2.7 (Borel $\sigma-$algebra)** *Given a general multi-Archimedean partially ordered unitary algebra $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$ over a Dedekind-complete multi-Archimedean abelian ring $\mathcal{R}$, we shall systematically provide the set $\mathcal{A}$ with the $\sigma-$algebra, which we call the Borel $\sigma-$algebra generated by open intervals, that is, generated by sets of the form*

$$I = \{x \in \mathcal{A} \ / \ x_1 \preceq_{\mathcal{A}} x \preceq_{\mathcal{A}} x_2\}$$

Due to our separability assumption and Proposition 2.4, easily see that interval with strict bounds defined by regular, tight or broad inequalities (Definition 1.9 and Definition 1.10) also belong to the Borel $\sigma-$algebra.

Given $l \in \mathcal{R}$, with $0_{\mathcal{R}} \prec_{\mathcal{R}} l$, the (see Section 2.5), the inclusion $i : \mathcal{A} \longrightarrow \mathcal{A}/l$ which to $x$ associates $l.\frac{x}{l}$ is measurable, as well as the natural isomorphism of division by $l$.

**Definition 2.8 (Measurable Function)** *Given a mseasurable $\sigma-$algebra $\Omega$ on a set $X$, we say that a function $f : X \longrightarrow \mathcal{A}$ is measurable if and only if the pre-image of any element in the Borel $\sigma-$algebra is an element of $\Omega$.*

**Definition 2.9 (Measure)** *A positive measurable function $\mu : X \longrightarrow \mathcal{R}_+/l$, where $l \in \mathcal{R}$, with $0_\mathcal{R} \prec_\mathcal{R} l$, is called a measure if and only if:*

- $\mu(\emptyset) = 0_{\mathcal{R}/l}$;
- $\mu(X)$ *is greater than or equal to* $0_{\mathcal{R}/l}$ *for all* $X \in \Omega/l$
- $\mu$ *is* $\sigma-$*additive, that is, for any countable family* $(A_n)_{n\in\mathbb{N}}$ *of pairwise disjoints sets we have*
$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mu(A_i).$$

The tricky point to define the intagral is that, since the range is only partially ordered, the notion of a “positive” or “negative” valued function is not immediately clear, and we shall use a couple of lemmas.

In the following lemma, we propose an alternative characterization of the notion of the absolute value of an element in a General Multi-Archimedean Algebra, introduced in Definition 2.1.

**Lemma 2.4** *Let $(\mathcal{A}, \mathcal{R}, \preceq_\mathcal{A})$ be a General multi-Archimedean partially ordered unitary algebra. Let $x \in \mathcal{A}$. Then there exists $u \in \mathcal{A}$, with $O_\mathcal{A} \preceq_\mathcal{A} u$ such that $x + |x| = 2u$.*

The proof follows directly from Proposition 2.9, where we have $|x| = s - i$ and $x = s + i$.

**Notation 2.1** *Under the hypothesis and notations of Lemma 2.4, we denote* $\frac{x+|x|}{2} \stackrel{\text{def}}{=} u$

**Remark 2.5** *Under the hypothesis and notations of Lemma 2.4, we see that any element in $\mathcal{A}$ is the difference between two positive elements.*

Indeed, using Notation 2.1, we have $x = \frac{x+|x|}{2} - \frac{(-x)+|-x|}{2}$.

**Definition 2.10 (Integral of a Positive Function)** *Let $(X, \Omega_X, \mu)$ be a measured space and $(\mathcal{A}, \mathcal{R}, \preceq_\mathcal{A})$ be a general multi-Archimedean partially ordered unitary algebra on a Dedekind-complete multi-Archimedean abelian ring $\mathcal{R}$. Let $f$ be a measurable function with domain $X$ and range*
$$\mathcal{A}_+ = \{y \in \mathcal{A} \ / \ 0_\mathcal{A} \preceq_\mathcal{A} y\}$$
*When we say that the function $f$ is integrable when the following set $S$ is bounded:*
$$S = \left\{ \sum_{i\in I} y_i\mu(A_i) \ \middle/ \ \begin{array}{c} I \text{ is finite, for all } i \in I, \text{ we have } A_i \in \Omega_\mathcal{A} \text{ and } y_i \in \mathcal{A} \\ (A_i)_{i\in I} \text{ is a partition of } X, \text{ and } \forall x \in A_i, y_i \preceq_\mathcal{A} f(x) \end{array} \right\}$$
*If $f$ is integrable, we set*
$$\int_X f(x)d\mu(x) \stackrel{\text{def}}{=} \sup(S)$$

**Definition 2.11 (Integrability and Integral)** *Let $f$ be a function with domain $\mathcal{A}$ and codomain $\mathcal{B}$. Using Notation 2.1, we define measurable functions $f_+$ and $f_-$ by:*
$$f_+(x) = \frac{f(x) + |f(x)|}{2} \ \text{and} \ f_-(x) = \frac{|f(x)| - f(x)}{2} \stackrel{\text{def}}{=} \frac{(-f(x)) + |(-f(x))|}{2}$$

*Note that we have a decomposition $f = (f_+ - f_-)$, and that $f_+$ and $f_-$ both have positive values, so that Definition 2.10 applies to them. We say that $f$ is integrable if and only if both $f_+$ and $f_-$ are integrable, and in that case we set*

$$\int_{\mathcal{A}} f(x)d\mu(x) \stackrel{\text{def}}{=} \int_{\mathcal{A}} f_+(x)d\mu(x) - \int_{\mathcal{A}} f_-(x)d\mu(x)$$

**Proposition 2.17** *If $f : X \longrightarrow \mathcal{A}$ is integrable and $a \in \mathcal{A}$, then $a.f$ is integrable and*

$$\int_{\mathcal{A}} a.f(x)d\mu(x) = a\int_{\mathcal{A}} f(x)d\mu(x).$$

*Proof.* We may assume without loss of generality that $f(x)$ is positive for all $x \in \mathcal{A}$ and $0 \preceq_{\mathcal{A}} a$. Due to Definition 2.10, the integral $\int_{\mathcal{A}} a.f(x)d\mu(x)$ is equal to $\sup(S_a)$ with

$$S_a = \left\{ \sum_{i\in I} y_i\mu(A_i) \middle/ \begin{array}{c} I \text{ is finite, for all } i \in I, \text{ we have } A_i \in \Omega_{\mathcal{A}} \text{ and } y_i \in \mathcal{A} \\ (A_i)_{i\in I} \text{ is a partition of } X, \text{ and } \forall x \in A_i, y_i \preceq_{\mathcal{A}} a.f(x) \end{array} \right\}$$

On the other hand, $a\int_{\mathcal{A}} f(x)d\mu(x)$ is equal to $a.\sup(S)$ with

$$S = \left\{ \sum_{i\in I} y_i\mu(A_i) \middle/ \begin{array}{c} I \text{ is finite, for all } i \in I, \text{ we have } A_i \in \Omega_{\mathcal{A}} \text{ and } y_i \in \mathcal{A} \\ (A_i)_{i\in I} \text{ is a partition of } X, \text{ and } \forall x \in A_i, y_i \preceq_{\mathcal{A}} f(x) \end{array} \right\}$$

Now, due to Corollary 2.1, if $y_i \preceq_{\mathcal{A}} a.f(x)$, then $y_i$ belongs to the ideal generated by $a$, and there exists $z_i \in \mathcal{A}$ such that $y_i = a.z_i$. Hence we have $\sup(S_a) = a\sup(S)$. □

**Proposition 2.18** *If $\mu(\{x \in \mathcal{A} \;/\; f_1(x) \neq f_2(x)\}) = 0_{\mathcal{R}}$, and $f_1$ and $f_2$ are integrable, then*

$$\int_{\mathcal{A}} f_1(x)d\mu(x) = \int_{\mathcal{A}} f_2(x)d\mu(x).$$

*Proof.* By difference, it is sufficient to prove the result if $f_1 = 0_{\mathcal{A}}$ and $0 \preceq_{\mathcal{A}} f_2(x)$ for all $x \in \mathcal{A}$. Let $Z = \{x \in \mathcal{A} \;/\; 0 \prec_{\mathcal{A}} f_2(x)\}$; we have $\mu(Z) = 0_{\mathcal{R}}$. For any $y$ such that $0_{\mathcal{A}} \prec_{\mathcal{A}} y$ and $A \in \Omega$ with $y \preceq_{\mathcal{A}} f_2(x)$ on $A$ we have either $y = 0_{\mathcal{A}}$ or $A \subset Z$ so that $\mu(A) = 0_{\mathcal{R}}$. We conclude from Definition 2.10 that the integral of $f_2$ is zero. □

**Proposition 2.19** *Let $f : X \longrightarrow \mathcal{A}$ be an integrable and positive function, that is, for all $x$ in $\mathcal{A}$, we have $0_{\mathcal{A}} \preceq f(x)$. If $\int_{\mathcal{A}} f(x)d\mu(x) = 0_{\mathcal{A}}$, then $\mu(\{x \in \mathcal{A}; f(x) \neq 0_{\mathcal{A}}\}) = 0_{\mathcal{A}}$.*

*Proof.* Let $(z_i)_{i\in\mathbb{N}}$ be a maximal family of pairwise non comparable elements of $\mathcal{A}$ (which is countable due to our separability assumption) as set out in Proposition 2.4. Following Proposition 2.5, we also assume that $\lambda = \sup(\{z_i \;/\; i \in \mathbb{N}\})$, where $\mathcal{A}$ is a $\lambda-$multi-Archimedean algebra.

Let $X = \{x \in \mathcal{A} \,/\, 0_{\mathcal{A}} \prec_{\mathcal{A}} f(x)\}$. For each $x \in X$, the ideal generated by $f(x)$ is unbounded, which means that there exists $n \in \mathbb{N}$ such that $n.f(x)$ is not smaller than $\lambda$. From the definition of the $z_i$'s, this means in turn that there exists $i \in \mathbb{N}$ such that $z_i \preceq_{\mathcal{A}} n.f(x)$ In other words, we have:

$$X \subset \bigcup_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} \{x \in \mathcal{A} \;/\; z_i \preceq_{\mathcal{A}} n.f(x)\}$$

If $0_{\mathcal{R}} \neq \mu(X)$, this implies by $\sigma-$additivity that there exist $n \in \mathbb{N}$ and $i \in \mathcal{N}$ such that $\mu(\{x \in \mathcal{A} \;/\; z_i \preceq_{\mathcal{A}} nf(x)\})$ is non zero. We conclude that $0_{\mathcal{A}} \prec_{\mathcal{A}} \mu(\{x \in \mathcal{A} \;/\; z_i \preceq_{\mathcal{A}} nf(x)\})$ and at last $0 \prec_{\mathcal{A}} z_i.\mu(\{x \in \mathcal{A} \;/\; z_i \preceq_{\mathcal{A}} nf(x)\}) \preceq_{\mathcal{A}} n.\int_{\mathcal{A}} f(x)d\mu(x)$. □

**Proposition 2.20** *Let us consider a measured space $(X, \Omega_X, \mu_X)$, where $\mu_X$ is a measure with values in $\mathcal{R}/l$, for a multi-Archimedean ring $\mathcal{R}$ and a broadly strictly positive element $l \in \mathcal{R}$. Let $(f_1, ..., f_d)$ be a function with $X$ as domain and with values in a product multi-Archimedean algebra $\prod_{i=1}^{i=d}(\mathcal{B}_i, \mathcal{R}, \preceq_{\mathcal{B}_i})$, provided with the Borel $\sigma-$algebra. Then $(f_1, ..., f_d)$ is integrable if and only if every $f_i$ are integrable.*

*Proof.* The proof is straightforward by noticing that intervals in the product multi-Archimedean algebra are exactly the cartesian products of intervals in the ordered algebras $\mathcal{B}_i$. We then follow the steps for the definition of the integral, through positive functions, and we see that everything works coordinate by coordinate. □

**Definition 2.12 (Product $\sigma-$algebra and Measure)** *Let $(X_1, \Omega_{X_1}, \mu_{X_1})$ and $(X_2, A, \Omega_{X_2}, \mu_{X_2})$ be measured spaces, with measures taking value in $\mathcal{R}/l_1$ and $\mathcal{R}/l_2$ respectively, for the same multi-Archimedean ring $\mathcal{R}$ and broadly strictly positive element $l_1, l_2 \in \mathcal{R}$. Then,*

- *The $\sigma-$algebra $\Omega_{X_1} \otimes \Omega_{X_2}$ generated by the family $\{\omega_1 \times \omega_2 \ / \ \omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_{X_2}\}$ is, classically, called the product $\sigma-$algebra on $X_1 \times X_2$.*
- *The map which to each set $\omega_1 \times \omega_2$, with $\omega_1 \in \Omega_{X_1}$ and $\omega_2 \in \Omega_{X_2}$ associates the element $\mu_{X_1}(\omega_1).\mu_{X_2}(\omega_2)$ of $\mathcal{R}/(l_1.l_2)$ can be extended in a unique manner to a $\sigma-$additive map on $\Omega_{X_1} \otimes \Omega_{X_2}$, and this $\sigma-$additive map, which is a measure with values in $\mathcal{R}/(l_1.l_2)$. This measure, denoted by $\mu_{X_1} \otimes \mu_{X_2}$, is called the product measure of $\mu_{X_1}$ and $\mu_{X_2}$.*

*These definition can clearly be extended to any cartesian product of a finite family $(X_1, \ldots, X_d)$ of measured spaces.*

**Remark 2.6** *Under the hypothesis and notations of Definition 2.12 above, if $X_1$ and $X_2$ are multi-Archimedean algebras provided with the Borel $\sigma-$algebra, then, since cartesian products of intervals in $X_1$ and $X_2$ are precisely intervals for the product multi-Archimedean algebra (i.e. provided with the coordinate by coordinate order, see Definition 1.20 and Theorem 1.4), the Borel $\sigma-$algebra on the product multi-Archimedean algebra is the same as the product $\sigma-$algebra $\sigma-$algebra $\Omega_{X_1} \otimes \Omega_{X_2}$.*

**Proposition 2.21** *(Fubini-Tonelli theorem) Let $(X_1, \Omega_{X_1}, \mu_{X_1})$ and $(X_2, \Omega_{X_2}, \mu_{X_2})$ be measured spaces, with $\mu_{X_1}$ and $\mu_{X_2}$ taking values in rigs of the form $\mathcal{R}l_1$ and $\mathcal{R}l_2$. Let $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$ be a multi-Archimedean unitary algebra over the same ring $\mathcal{R}$.*
*Let $f : X_1 \times X_2 \longrightarrow \mathcal{A}$ be an integrable function over the cartesian product $X_1 \times X_2$.*
*Then, the map $x_1 \longmapsto \int_{X_2} f(x_1, x_2) d\mu_{X_2}(y)$ is measurable over $X_1$, and we have:*

$$\int_{X_1 \times X_2} f(x_1, x_2) \left(d\mu_{X_1} \otimes \mu_{X_2}\right)(x_1, x_2) = \int_{X_1} \left( \int_{X_2} f(x_1, x_2) d\mu_{X_2}(x_2) \right) d\mu_{X_1}(x_1)$$

*Proof.* We first consider the case when $f$, is a constant function equal to $f_0$ on a cartesian product $\omega_2 \times \omega_2$ of two measurable sets with $\omega_1 \in \Omega_{X_1}$ and $\omega_2 \in \Omega_{X_2}$, and zero elsewhere. We may assume without loss of generality that $f_0 \neq 0_{\dashv}$.

Then for a fixed $x_1 \in X_1$, we have $f(x_1, x_2) = f_0$ if and only if $x_2 \in \{x_2 \in X_2 \ / \ (x_1, x_2) \in \omega_1 \times \omega_2\}$, a set which in our case is equal to $\omega_2$.. Then, $\int_{X_2} f(x_1, x_2) d\mu_{X_2}(x_2) = f_0 \int_{\omega_2} d\mu_{X_2}(x_2) = f_0 \mu_{X_2}(\omega_2)$.

On the other hand, $\int_{X_1\times X_2} f(x_1,x_2)\,(d\mu_{X_1}\otimes\mu_{X_2})\,(x_1,x_2) = f_0.\mu_{X_1}\otimes\mu_{X_2}(\omega_1\times\omega_2) = f_0\int_{\omega_2}\mu_{X_1}(\omega_1)d\mu_{X_2}(x_2)$ from the definitions.

To generalize the result for a general measurable function, by linearity, this remains true for positive fonctions $\sum_{i\in I} f_{\omega_i}$, where $I$ is finite, $\{\omega_i\ /\ i\in I\}$ is a partition of $X_1\times X_2$ and $\mathcal{A}_i\in\Omega_{X_1}\otimes\Omega_{X_2}$. From Definition 2.10, the result remains true for mesurable positive functions on $(\mathcal{A}_1\times\mathcal{A}_2,\Omega_{\mathcal{A}_1}\otimes\Omega_{\mathcal{A}_2})$. At last, from Definition 2.11, the result remains true for an arbitrary measurable function $f$. □

## 2.7 Normed Multi-Archimedean Algebras, Functional Norms

**Definition 2.13** *We call a norm over a multi-Archimedean algebra $(\mathcal{A},\mathcal{R},\preceq_{\mathcal{A}})$ on a multi-Archimedean ring $\mathcal{R}$ a function $N:\mathcal{A}\longrightarrow\mathcal{R}/l$ for some broadly strictly positive $l\prec_{\mathcal{R}} 0_{\mathcal{R}}$ with the following properties:*

1. $0_{\mathcal{R}}\preceq_{\mathcal{R}/l} N(x)$ *for any* $x\in\mathcal{A}$*;*
2. $N(x+y)\preceq_{\mathcal{R}/l} N(x)+N(y)$ *for all* $x,y\in\mathcal{A}$*;*
3. $N(r.x)=|r|.N(x)$ *for all* $r\in\mathcal{R}$ *and* $x\in\mathcal{A}$*;*
4. *If* $N(x)=0_{\mathcal{R}}$*, then* $x=0_{\mathcal{A}}$*;*

*The norm is said to be compatible with the order, in addition to the conditions 1. to 4. above, we have:*

5. *if* $0_{\mathcal{A}}\preceq_{\mathcal{A}} x\preceq_{\mathcal{A}} y$ *in* $\mathcal{A}$ *then* $0_{\mathcal{R}}\preceq_{\mathcal{R}/l} N(x)\preceq_{\mathcal{R}/l} N(y)$ *in* $\mathcal{R}$

*The norm is called an algebra norm if, in addition to the conditions 1. to 4. above, we have:*

6. $N(x.y)\preceq_{\mathcal{R}/l} N(x).N(y)$ *for all* $x,y\in\mathcal{A}$*;*

*The norm is called a multi-Archimedean norm if it is both compatible with the order and an algebra norm, that is, if it satisfies all conditions 1. to 6.*

We often denote by $||x||$ the norm of an element $x\in\mathcal{A}$, instead of a notation of the form $N(x)$. In that case the norm itself is denoted by $||.||$.

**Remark 2.7** *If a subset $X$ of $\mathcal{A}$ is bounded for the order, then by definition it is bounded for any norm which is compatible with the order. The converse is also true.*

*Proof.* Let us suppose that $r$ in $\mathcal{R}$ is such that for all $x$ in $X$, we have $||x||\preceq_{\mathcal{R}} r$. Since $\mathcal{R}$ is multi-Archimedean property, there exists $n\in\mathbb{N}$ such that $r\preceq_{\mathcal{R}} n.1_{\mathcal{R}}$. For $x\in X$, we have $x\preceq_{\mathcal{R}} n.1_{\mathcal{A}}$, which shows the $X$ is bounded for the order. □

**Definition 2.14** *Let $(\mathcal{X},\Omega_{\mathcal{X}},\mu_X)$ be a measurable space, where $\mu_l$ has values in a ring of the form $\mathcal{R}/l$, and let $(\mathcal{A},\mathcal{R},\preceq_{\mathcal{A}})$ be a multi-Archimedean algebra over the ring $\mathcal{R}$. Let $||.||$ be a norm on $\mathcal{A}$, and $\alpha\in\mathbb{N}^*$. Let $f:\mathcal{X}\longrightarrow\mathcal{A}$ be a measurable function. We say that $f$ has finite $\alpha-$norm on a measurable subset $\omega\subset\mathcal{X}$ if the following integral exists and is finite:*

$$||f||_\alpha=\int_X ||(f(x))^\alpha||d\mu_l(x)$$

*This integral is then called the $1-$norm of $f$ on $X$.*

**Remark 2.8** *From Proposition 2.19, if* $||f||_\alpha = 0_{\mathcal{R}}/l$*, then* $\mu_l(\{x \in \mathcal{A} \ / \ (f(x))^\alpha \neq 0_{\mathcal{R}/l}\}) = 0_{\mathcal{R}}/l$*. Using Proposition 2.2 we can see that this implies that* $\mu_l(\{x \in \mathcal{A} \ / \ f(x) \neq 0_{\mathcal{R}/l}\}) = 0_{\mathcal{R}}/l$ *This allows us to consider* $||f||_\alpha$ *as a norm according to Definition 2.13, Point 4, if we consider the space of equivalence classes of functions which are equal except possibly on a zero measure subset of* $X$*.*

**Definition 2.15** *Let* $(\mathcal{X}, \Omega_{\mathcal{X}}, \mu_X)$ *be a measurable space, where* $\mu_l$ *has values in a ring of the form* $\mathcal{R}/l$*, and let* $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$ *be a multi-Archimedean algebra over the ring* $\mathcal{R}$*. Let* $||.||$ *be a norm on* $\mathcal{A}$*. Let* $f : \mathcal{X} \longrightarrow \mathcal{A}$ *be a measurable function. We say that* $f$ *has finite* $\infty-$*norm on a measurable subset* $X \subset \mathcal{A}$ *if there exists a subset* $\omega$ *of* $X$ *with zero measure such that if* $x \longmapsto ||f(x)||$ *has an upper bound on* $X\backslash\omega$*. We then denote*

$$||f||_\infty = \inf_{\omega \subset X, \mu_l(\omega) = 0_{\mathcal{R}}} \left( \sup_{X\backslash\omega} ||f(x)||. \right)$$

*This defines a norm* $||.||_\infty$ *according to Definition 2.13.*

**Notation 2.2** *Let* $(\mathcal{X}, \Omega_{\mathcal{X}}, \mu_X)$ *be a measurable space, where* $\mu_l$ *has values in a ring of the form* $\mathcal{R}/l$*, and let* $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$ *be a multi-Archimedean algebra over the ring* $\mathcal{R}$*. Let* $||.||$ *be a norm on* $\mathcal{A}$*. Let* $\alpha \in \mathbb{N}^* \cup \{\infty\}$*. We denote by* $\mathfrak{L}_\alpha(X, \mathcal{A}, \mu_l)$ *the space of all equivalence classes of measurable functions up to difference on a zero measure set from* $X$ *to* $\mathcal{A}$ *with finite* $\alpha-$*norm. This space is naturally provided with the norm* $||.||_\alpha$*.*

# 3 Analyzable Spaces

## 3.1 Definition of an Analyzable Space

**Definition 3.1** *An analyzable space over a multi-Archimedean ring $\mathcal{R}$ is a tuple, $(\mathcal{A}, \mathcal{R}, \Omega_{\mathcal{A}}, \mu_l, \preceq_{\mathcal{A}}$ ), where $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$ is a unitary complete multi-Archimedean algebra, $\Omega_{\mathcal{A}}$ is the Borel $\sigma-$algebra on $\mathcal{A}$, and $\mu_l$ is either a translation-invariant $(\mathcal{R}/l)-$valued, or a translation-invariant $\mathbb{R}/l-$valued measure on $\Omega_{\mathcal{A}}$, that is, for any $\omega \in \Omega_{\mathcal{A}}$ and $x \in \mathcal{A}$, the measure $\mu_{\mathcal{A}}(\omega+x) = \mu_{\mathcal{A}}(\{y+x \ / \ y \in \omega\})$ is equal to $\mu_{\mathcal{A}}(\omega)$.*

*The analyzable space $(\mathcal{A}, \mathcal{R}, \Omega_{\mathcal{A}}, \mu_l, \preceq_{\mathcal{A}})$ is called standard if the measure $\mu_l$ is $\mathbb{R}/l-$valued (or equivalently $\mathbb{R}-$valued), in which case it it a measure in the most usual sense of the word. The analyzable space $(\mathcal{A}, \mathcal{R}, \Omega_{\mathcal{A}}, \mu_l, \preceq_{\mathcal{A}})$ is called pure if the measure $\mu_l$ is $\mathcal{R}/l-$valued, in which case it is not a measure in the most usual sense of the word, except of course when $\mathcal{R} = \mathbb{R}$.*

*The analyzable space $(\mathcal{A}, \mathcal{R}, \Omega_{\mathcal{A}}, \mu_l, \preceq_{\mathcal{A}})$ is called essential if any nontrivial ideal of $\mathcal{A}$ has non-zero measure. The analyzable space is called descent if any non-trivial ideal of $\mathcal{A}$ has a subset with non-zero measure.*

In the sequel, all analyzable spaces are assumed to be both essential and descent.

**Remark 3.1** *We remind the reader (Proposition 2.12) that the ideal generated by a nonzero element which is tightly comparable to $0_{\mathcal{A}}$ is either isomorphic to $\mathbb{Z}$ or to $\mathbb{R}$ as an ordered algebra. In an essential and descent analyzable space, the measure on the ideal generated by a nonzero element which is tightly comparable to $0_{\mathcal{A}}$, as a translation-invariant measure, is proportional (through an isomorphism) either to the usual uniform discrete measure (in the case of $\mathbb{Z}$), or to the usual Lebesgue measure (in the case of $\mathbb{R}$).*

**Example 3.1 (Standard Analyzable Space)** *Let us consider the ring $\mathcal{R} = \mathbb{Z}^d$, with $d \in \mathbb{N}^*$, provided with the multi-Archimedean product structure constructed from the usual $(\mathbb{Z}, \mathbb{Z}, \leq)$, that is: coordinate by coordinate addition, multiplication, and partial order.*

*Let $\mathcal{A} = \left(\mathbb{R}^{\mathcal{R}}\right)_\lambda$ be the space of $\lambda-$bounded maps from the discrete grid $\mathcal{R}$ with range $\mathbb{R}$, which is also provided with the product multi-Archimedian algebra structure, where $\lambda : \mathcal{R} \longrightarrow \mathbb{R}$ is a map. For $r \in \mathcal{R}$, we consider the element $\alpha(r)$ of $\mathcal{A}$ defined by*

$$(\alpha(r))\,(x) = r.\left\lfloor \frac{x}{l} \right\rfloor,$$

*where $l \in \mathbb{Z}^d$ is broadly strictly positive (i.e. all of its coordinates are positive). The map $\alpha(r)$ is piecewise constant: it is constant on some rectangle parallelepipedic polytopes with edge length equal to the $l_a$ for $a = 1, \ldots, d$. We denote $v = \Pi_{a=1}^d l_a$ the volume of that polytope.*

*For $r \in \mathcal{R}$ and $f \in \mathcal{A}$, we consider the external product $r.f \in \mathcal{A}$ which is equal to the value by value product $\alpha(r).f$ in $\mathcal{A}$. This defines a multi-Archimedean algebra structure $(\mathcal{A}, \mathcal{R}, \preceq_{\mathcal{A}})$.*

*In order to define a measure on the Borel $\sigma-$algebra $\Omega_{\mathcal{A}}$ on $\mathcal{A}$, we first define its value on intervals for the partial order:*

$$\mu_l([f_1, f_2[) = \frac{1}{v} \int_{\mathbb{Z}^d} |f_2(x) - f_1(x)|\, dp(x)$$

*where $p(x)$ is a bounded $\mathbb{R}-$valued measure on $\mathbb{Z}^d$ (for example a probability with an integrable density such as a normalized Gaussian sampling), and the absolute value in $\mathcal{A}$ is defined* *coordinate by coordinate, which coincides with the notion set out in Definition 2.1. This notion can be shown to extend to a $\sigma-$additive function on finite unions of intervals, and at last to extend on $\Omega_{\mathcal{A}}$ to a measure with values in $\mathcal{R}/v$. This defines an analyzable space $(\mathcal{A}, \mathcal{R}, \Omega_{\mathcal{A}}, \mu_v, \preceq_{\mathcal{A}})$.*

**Example 3.2 (Pure Analyzable Space)** *Let us consider the ring $\mathcal{R} = \mathbb{Z} \times \mathbb{R}$, provided with the multi-Archimedean product structure constructed from the usual $(\mathbb{Z}, \mathbb{Z}, \leq)$ and $(\mathbb{R}, \mathbb{R}, \leq)$, that is: coordinate by coordinate addition, multiplication, and partial order.*

*Let $\mathcal{A} = U^d \times \mathbb{R}^V$, where $U$ and $V$ are countable sets, provided with the product multi-Archimedian ring structure. For $r = (r_D, R_C) \in \mathcal{R}$ and $x = (x_D, x_C) \in \mathcal{A}$, with $x_D \in \mathbb{Z}^U$ and $x_C \in \mathbb{R}^V$, we consider the external product $r.x = (r_D.x_D, r_C.x_C)$.*

*In order to define a measure on the Borel $\sigma-$algebra $\Omega_{\mathcal{A}}$ on $\mathcal{A}$, we first define its value on intervals for the partial order (using the identification set out in Remark 2.4):*

$$\mu_l([x, y[) = \left( \sum_{\beta \in U} (y(\beta) - x(\beta)) p(\{\beta\}), \sum_{\beta' \in V} (y(\beta') - x(\beta')) p(\{\beta'\}) \right)$$

*where $p$ is a positive measure with non-zero density on $U \cup V$, with values in $\mathbb{R}$, taking only values in $\mathbb{Z}/l$ on subsets of $U$. The measure on the Borel $\sigma-$algebra is defined by $\sigma-$additivity.*

*This defines a pure analyzable space $(\mathcal{A}, \mathcal{R}, \Omega_{\mathcal{A}}, \mu_v, \preceq_{\mathcal{A}})$. In general, the first coordinate of the measure taking values in a discrete ring, the measure of an interval $[x, y[$, with $x \preceq y$, can be finite only when $x$ and $y$ differ by a finite number of values on $U$.*

## 3.2 Convolutions In Analyzable Spaces

**Definition 3.2** *Let $(\mathcal{A}, \mathcal{R}, \Omega_{\mathcal{A}}, \mu_l, \preceq_{\mathcal{A}})$ be an analyzable space over an ordered abelian ring $\mathcal{R}$ and $(\mathcal{B}, \mathcal{R}, \preceq_{\mathcal{B}})$ multi-Archimedean partially ordered unitary algebra on the same ring. Let $K : \mathcal{A} \longrightarrow \mathcal{B}$ (or $K : \mathcal{A} \longrightarrow \mathcal{R}$, which can be identified to the $\mathcal{B}-$valued function $a \longmapsto K(a).1_{\mathcal{B}}$) be an integrable function, and $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a measurable function. We define the convolution product $K \star f : \mathcal{A} \longmapsto \mathcal{B}$ of $f$ by $K$ by setting for $x \in \mathcal{A}$:*

$$K \star f(x) = \int_{\mathcal{A}} f(t) K(x - t) d\mu_l(t)$$

## 3.3 Integration on Intervals

**Definition 3.3** *Let $(\mathcal{A}, \mathcal{R}, \Omega_{\mathcal{A}}, \mu_l, \preceq_{\mathcal{A}})$ be an analyzable space over a ring $\mathcal{R}$. Let $l_1, l_2 \in \mathcal{A} \cup \{-\infty, +\infty\}$. The Interval of $\mathcal{A}$ between $l_1$ and $l_2$, denoted by $[l_1, l_2]_{\mathcal{A}}$ (or simply $[l_1, l_2]$ for short if no confusion can arise), is defined by*

$$[l_1, l_2]_{\mathcal{A}} = \{x \in \mathcal{A} \ / \ l_1 \preceq_{\mathcal{A}} x \preceq_{\mathcal{A}} l_2\}$$

*We define similarly broadly semi-open bounded or unbounded intervals, using the broad strict order $<_{\mathcal{A}}$ on $\mathcal{A}$:*

$$[l_1, l_2[_{\mathcal{A}} = \{x \in \mathcal{A} \ / \ l_1 \preceq_{\mathcal{A}} x <_{\mathcal{A}} l_2\}$$

*At last, we define partially open intervals using the regular strict:*

$$[l_1, l_2(_{\mathcal{A}} = \{x \in \mathcal{A} \ / \ l_1 \preceq_{\mathcal{A}} x \prec_{\mathcal{A}} l_2\}$$

*We shall often omit the subscript $\mathcal{A}$ to denote $[l_1, l_2[$ when no ambiguity can arise.*

**Remark 3.2** *In* $\mathbb{Z}^2$ *we have* $](0,0),(1,1)] = \{(1,0),(0,1),(1,1)\}$*, however* $(1,0)$ *and* $(0,1)$ *are not broadly strictly greater than* $(0,0)$*.*

**Lemma 3.1** *Let* $(\mathcal{A}, \mathcal{R}, \Omega_{\mathcal{A}}, \mu_l, \preceq_{\mathcal{A}})$ *be an analyzable space.*
*Let* $l \in \mathcal{A}$ *be broadly strictly greater than* $0_{\mathcal{A}}$*. Then we have the following partition of* $\mathcal{A}$*:*

$$\mathcal{A} = \bigcup_{s \in \mathbb{Z}} [s.l, (s+1).l[_{\mathcal{A}}$$

**Notation 3.1** *Let* $(\mathcal{A}, \mathcal{R}, \Omega_{\mathcal{A}}, \mu_l, \preceq_{\mathcal{A}})$ *be an analyzable space and* $(\mathcal{B}, \mathcal{R}, \preceq_{\mathcal{B}})$ *be a multi-Archimedean partially ordered unitary algebra on the same ring* $\mathcal{R}$*. Let* $f$ *be a function with domain* $\mathcal{A}$ *and codomain* $\mathcal{B}$*.*
*Let* $I = [l_1, l_2[$ *be a (possibly unbounded) interval in* $\mathcal{A}$*. We denote*

$$\int_{l_1}^{l_2} f(x) d\mu_l(x) = \int_{[l_1, l_2[} f(x) d\mu_l(x)$$

## 3.4 Ordinary Differentiation

In this section, we consider $(\mathcal{A}, \mathcal{R}, \Omega_{\mathcal{A}}, \mu_l, \preceq_{\mathcal{A}})$ and $(\mathcal{B}, \mathcal{R}, \Omega_{\mathcal{B}}, \mu'_{l'}, \preceq_{\mathcal{B}})$ two analyzable spaces over a ring $\mathcal{R}$.

**Definition 3.4** *(Integral Based Primitive Operator) Let* $f \in \mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_l)$*. We define the integral based primitive of* $f$ $\mathcal{I}_{\mu_l}(f) : \mathcal{A} \mapsto \mathcal{B}$*, by*

$$\left(\mathcal{I}_{\mu_l}(f)\right)(x) = \int_{\{t <_{\mathcal{A}} x\}} f(\beta) d\mu_l(\beta)$$

**Lemma 3.2** *Let* $f \in \mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_l)$*. Then,* $\mathcal{I}_{\mu_l}(f)$ *is integrable over* $\mathcal{A}$*, so that the operator* $\mathcal{I}_{\mu_l}$ *sends* $\mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_l)$ *into itself.*

*Proof.* First we assume that $f$ has positive values, that is, for all $x \in \mathcal{A}$ we have $0_{\mathcal{B}} \preceq_{\mathcal{B}} f(x)$. Let us denote $F = \mathcal{I}_{\mu_l}(f)$. Since the measure $\mu_l$ is translation-invariant, the function $F$ is increasing and defined on all $\mathcal{A}$, then $F$ is bounded on any bounded subset of $\mathcal{A}$ From the definition of integrable positive functions the function $F$ is integrable on any bounded subset of $\mathcal{A}$.

Now, let us consider the following map, which is clearly integrable for the product measure $\mu_l \otimes \mu_l$:

$$G : \left\{ \begin{array}{ccc} \mathcal{A} \times \mathcal{A} & \longrightarrow & \mathcal{B}/2 \\ (x, y) & \longmapsto & \frac{f(x) + f(y)}{2} \end{array} \right.$$

First suppose that $f = f_0.\mathbb{1}_\omega$ is some constant $f_0$ multiplied by the characteristic function of a measurable set $\omega \in \Omega_{\mathcal{A}}$. Then, since $F(x) \preceq_{\mathcal{B}} \mu_l(\omega)$, we have:

$$\int_{\mathcal{A}} F(x) d\mu_l(x) \preceq_{\mathcal{B}} f_0 \left(\mu_l(\omega)\right)^2 \preceq_{\mathcal{B}} \int_{\mathcal{A} \times \mathcal{A}} G(x) d(\mu_l \otimes \mu_l)(x)$$

This shows that $F$ is integrable. We can easily generalize this by linearity to measurable linear combinations of characteristics functions, which proves our statement for a positive function $f$. The general case follows immediately from the definition of the integral through integrals for positive functions. $\square$

**Definition 3.5** *Let $\Phi : \mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_l) \longrightarrow \mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_{\mathcal{A}})$ be a linear operator. We say that $\Phi$ commutes with the integral based primitive operator if for any $f \in \mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_l)$, we have*

$$\mathcal{I}_{\mu_l}(\Phi(f)) = \Phi(\mathcal{I}_{\mu_l}(f))$$

.

**Remark 3.3** *Let $f \in \mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_l)$. If $\mathcal{I}_{\mu_l}(f)$ is constant, then $f$ is null almost everywhere.*

**Lemma 3.3** *Let $f \in \mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_l)$ and $H \in \mathfrak{L}_\infty(\mathcal{A}, \mathcal{B}, \mu_l)$. Then $\mathcal{I}_{\mu_l}(H \star f) = (H \star (\mathcal{I}_{\mu_l}(f)))$*

*Proof.*

$$\begin{aligned}
(\mathcal{I}_{\mu_l}(H \star f))(x) &= \int_{\{t <_{\mathcal{A}} x\}} H \star f(t) d\mu_l(t) = \int_{\{t <_{\mathcal{A}} x\}} \left( \int_{\mathcal{A}} H(s) f(t-s) d\mu_l(s) \right) d\mu_l(t) \\
&= \int_{\mathcal{A}} H(s) \left( \int_{\{t <_{\mathcal{A}} x\}} f(t-s) d\mu_l(t) \right) d\mu_l(s) \\
&= \int_{\mathcal{A}} H(s) \left( \int_{\{t <_{\mathcal{A}} x-s\}} f(t) d\mu_l(t) \right) d\mu_l(s) = (H \star (\mathcal{I}_{\mu_l}(f)))(x)
\end{aligned}$$

□

**Definition 3.6 (Ordinary Differentiation Operator)** *Let $\delta : \mathcal{I}_{\mu_l}(\mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_l)) \longmapsto \mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_l)$ be a linear operator. We say that $\delta$ is an ordinary differentation operator if*

$$\delta \circ \mathcal{I}_{\mu_l} = Id_{\mathcal{I}_{\mu_l}(\mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_l))} \text{ and } \delta \circ \mathcal{I}_{\mu_l} = Id_{\mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_l)}$$

**Proposition 3.1** *Let $\delta : \mathcal{I}_{\mu_l}(\mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_l)) \longmapsto \mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_l)$ be a differentiation operator. Then $\delta(f)$ is the zero function if and only if $f$ is constant on $\mathcal{A}$ almost anywhere.*

*Proof.* Let $f$ be a constant function. Then we have $\mathcal{I}_{\mu_l}(\delta(f)) = \delta(\mathcal{I}_{\mu_l}(f)) = f$. So $\mathcal{I}_{\mu_l}(\delta(f))$ is constant, which implies (Proposition 2.19) that $\delta(f)$ is zero almost anywhere. Conversely, if $\delta(f)$ is zero almost anywhere, obviously, $\mathcal{I}_{\mu_l}(\delta(f))$ is equal to $0_{\mathcal{B}}$. □

**Lemma 3.4** *Let $\Phi : \mathcal{I}_{\mu_l}(\mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_l)) \longmapsto \mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_l)$ be a linear operator which commutes with the integral based primitive operator and is zero for functions for the form $\mathcal{I}_{\mu_l}(\varphi)$. Then $\Phi$ is zero on $\mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_l)$. In other words, for all $f \in \mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_l)$, we have $\mu_l(\{x \in \mathcal{A}; \Phi(f)(x) \neq 0_{\mathcal{B}}\}) = 0_{\mathcal{R}}$.*

*Proof.* Let us suppose by *reductio ad absurdum* that for some $f \in \mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_l)$, we have $\mu_l(\{x \in \mathcal{A} \ / \ \Phi(f)(x) \neq 0_{\mathcal{B}}\}) \neq 0_{\mathcal{R}}$. Then $\Phi(\mathcal{I}_{\mu_l}(f)) = \mathcal{I}_{\mu_l}(\Phi(f))$ is not a constant function. This shows $\Phi(\mathcal{I}_{\mu_l}(f))$ not almost anywhere zero, which contradicts our hypothesis on $\Phi$. □

As a direct application of Lemma 3.4, we obtain:

**Proposition 3.2 (Uniqueness of The Ordinary Differentiation Operator)** *If $\delta_1$ and $\delta_2$ are two differentiation operators on $\mathcal{I}_{\mu_l}(\mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_l))$, then they are almost equal in the following sense: for all $f \in \mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_l)$, we have $\mu_l(\{x \in \mathcal{A} \ / \ \Phi_1(f)(x) \neq \Phi_2(f)(x)\}) = 0_{\mathcal{R}}$.*

**Definition 3.7 (Canonical System of Generators)** *Assume that $\mathcal{A} \subset \mathbb{Z}^U \times \mathbb{R}^V$ and $\mathcal{B} \subset \mathbb{Z}^{U'} \times \mathbb{R}^{V'}$ are products multi-Archimedean algebra in the sense of Definition 1.20. For $\beta \in U \cup V$, we consider $e_\beta$ the element of $\mathcal{A}$ which is defined (using the identification set out in Remark 2.4) depending on two cases:*

1. *If $\beta \in U$, then the map $e_\beta$ sends sends all element of $U$ onto $0_\mathbb{Z}$ except $\beta$ which is sent onto $1_\mathbb{Z}$, and sends all elements of $V$ onto $0_\mathbb{R}$.*
2. *If $\beta \in V$, then the map $e_\beta$ sends sends all element of $U$ onto $0_\mathbb{Z}$, and sends all elements of $V$ onto $0_\mathbb{R}$, except $u$ which is sent onto $1_\mathbb{R}$.*

*The family $(e_\beta)_{\beta \in U \cup V}$ of elements of $\mathcal{A}$ is called the canonical family of generators of $\mathcal{A}$. We can of course define similarly the canonical family $(e'_{\beta'})_{\beta' \in U' \cup V'}$ of generators of $\mathcal{B}$.*

We use the notations $e_\beta$ and $e'_{\beta'}$ from Definition 3.7 throughout the remainder of this section.

**Definition 3.8 (Canonical Projections)** *Assume that $\mathcal{A} \subset \mathbb{Z}^U \times \mathbb{R}^V$ is a product multi-Archimedean algebra in the sense of Definition 1.20. Let us consider $\beta \in U \cup V$. Then there exists a unique algebra morphism $p_\beta : \mathcal{A} \longrightarrow \mathcal{A}$, which sends $1_\mathcal{A}$ on $e_\beta$, and such that for $r \in \mathcal{R}$ we have $p_\beta(r.\beta) = r.\beta$ (i.e. which lets $e_\beta$ invariant). Note that this morphism can be expressed as the multiplication by $e_\beta$.*

**Proposition 3.3** *Assume that $\mathcal{A} = \mathbb{R}^V$ and $\mathcal{B} = \mathbb{R}^{V'}$ is a product multi-Archimedean algebra in the sense of Definition 1.20. We remind the reader that we assume $\mathcal{A}$ separable, which implies that $V$ is countable.*

*Consider a function $F : \mathcal{A} \longrightarrow \mathcal{B}$, and, for $\beta \in V$ and $\beta' \in V'$, define*

$$f_{\beta,\beta'} : \left\{ \begin{array}{ccc} \mathbb{R}^{\{\beta\}} \subset \mathcal{A} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & p_{\beta'}(f(x)) \end{array} \right.$$

*Then $F$ belongs to $\mathcal{I}_{\mu_l}(\mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_l))$ if and only if each of the functions $f_{\beta,\beta'}$ belong to $\mathcal{I}_\mu(\mathfrak{L}_1((\mathbb{R}, \mathbb{R}, \mu))$, where $\mu$ is the usual Lebesgue measure on $\mathbb{R}$. In other words, the functions $f_{\beta,\beta'}$ have bounded variations.*

*As a corollary, there exists a ordinary differentiation operator on $\mathcal{I}_{\mu_l}(\mathfrak{L}_1(\mathcal{A}, \mathcal{B}, \mu_l))$, which if given by the gradient operator in the usual sense in $\mathbb{R}$, for each coordinate $\beta' \in V'$ in the range.*

*Proof.* Follows directly from the definitions. □

**Proposition 3.4** *Assume that $\mathcal{A} = \mathbb{Z}^U$ and $\mathcal{B} = \mathbb{Z}^{U'} \times \mathbb{R}^{V'}$ are products multi-Archimedean algebra in the sense of Definition 1.20. We remind the reader that we assume $\mathcal{A}$ separable, which implies that $U$ is countable. By removing a zero-measure ideal if necessary, we may assume that for each $i \in U$, the measure of $\mu_f\left(\mathbb{Z}^{\{i\}}\right)$ on each copy of $\mathbb{Z}$ is non-zero, which implies by translation invariance that it is proportional to the usual discrete measure on $\mathbb{Z}$.*

*Consider a function $F : \mathcal{A} \longrightarrow \mathcal{B}$, and, for $\beta \in U$ and $\beta' \in U' \cup V'$, define*

$$f_{\beta,\beta'} : \left\{ \begin{array}{ccc} \mathbb{Z}^{\{\beta\}} \subset \mathcal{A} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & p_{\beta'}(f(x)) \end{array} \right.$$

*Then $F$ belongs to $\mathcal{I}_{\mu_l}(\mathfrak{L}_1(\mathcal{A},\mathcal{B},\mu_l))$ if and only if each of the functions $f_{\beta,\beta'}$ belong to $\mathcal{I}_{\mu}(\mathfrak{L}_1((\mathbb{Z},\mathbb{R},\mu))$, where $\mu$ is the usual discrete measure on $\mathbb{Z}$. In other words, the functions $f_{\beta,\beta'}$ can be any sequence, either from $\mathbb{Z}$ to $\mathbb{R}$, or from $\mathbb{Z}$ to $\mathbb{Z}$, depending on whether $\beta' \in U'$ or $\beta' \in V'$.*

*As a corollary, there exists a ordinary differentiation operator on $\mathcal{I}_{\mu_l}(\mathfrak{L}_1(\mathcal{A},\mathcal{B},\mu_l))$, which if given by the gradient operator in the sense of finite differences in $\mathbb{Z}$. In other words, $(\delta(F))(x) = F(x) - F(x-1_{\mathcal{A}})$.*

*Proof.* Follows directly from the definitions. □

We remind the reader that, from Theorem 2.2, any separable multi-Archimedean algebra is isomorphic to the set of $\lambda-$bounded a product of the form $\mathbb{Z}^U \times \mathbb{R}^V$, where $U$ and $V$ are countable.

**Proposition 3.5 (Existence of a Ordinary Differentiation Operator)** *There exists an ordinary differentiation operator on $\mathcal{I}_{\mu_l}(\mathfrak{L}_1(\mathcal{A},\mathcal{B},\mu_l))$.*

*Proof.* This result follows directly from Proposition 2.19, which shows that $\mathcal{I}_{\mu_l}$ is one to one on $\mathfrak{L}_1(\mathcal{A},\mathcal{B},\mu_l)$, so that, for any $F \in \mathcal{I}_{\mu_l}(\mathfrak{L}_1(\mathcal{A},\mathcal{B},\mu_l))$ we can define a unique $\delta(F) \in \mathfrak{L}_1(\mathcal{A},\mathcal{B},\mu_l)$ such that $\mathcal{I}_{\mu_l}(\delta(F)) = F$. □

**Definition 3.9** *A map $f : \mathcal{A} \longrightarrow \mathcal{B}$ is said to be derivable if it belongs to $\mathcal{I}_{\mu_l}(\mathfrak{L}_1(\mathcal{A},\mathcal{B},\mu_l))$.*

## 3.5 Symmetric Derivative Operator

We assume in this section that there exists an ordinary differentiation operator for $\mathcal{A}$ to $\mathcal{B}$.

**Notation 3.2** *Given a map $f \in \mathfrak{L}_1(\mathcal{A},\mathcal{B},\mu_l)$ and $x, x_0 \in \mathcal{A}$, we call mirror of $f$ at $x_0$, and we denote by $\widehat{f}_{x_0}$ the function defined by $\widehat{f}_{x_0}(x) = f(2.x_0 - x)$.*

**Definition 3.10 (Derivative Operator)** *We call the symmetric derivative operator, or simply derivative operator for short, the linear operator define by:*

$$\delta_S : \left\{ \begin{array}{ccc} \mathcal{I}_{\mu_l}(\mathfrak{L}_1(\mathcal{A},\mathcal{B},\mu_l)) & \longrightarrow & \mathfrak{L}_1(\mathcal{A},\mathcal{B}/2,\mu_l) \\ f & \longrightarrow & \delta_S(f) : \left\{ \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{B}/2 \\ x_0 & \longmapsto & (\delta_S(f))(x_0) = \frac{(\delta(f))(x_0) - \delta\left(\widehat{f}_{x_0}\right)(x_0)}{2} \end{array} \right. \end{array} \right.$$

**Proposition 3.6** *Under the hypothesis and notation of Proposition 3.3, namely: $\mathcal{A} = \mathbb{R}^U$ and $\mathcal{B} = \mathbb{R}^{V'}$ is a product multi-Archimedean algebra in the sense of Definition 1.20, then, we have $\delta_S = \delta$. In other words, in purely continuous multi-Archimedean algebras, the derivative operator coincides with the ordinary differentiation operator, which coincides with the usual gradient of functions with bounded variations.*

*Proof.* Indeed, we know that the ordinary differentiation operator $\delta$ is the usual derivative on each coordinate. So, we have $\left(\delta \widehat{f}_{x_0}\right)(x_0) = -\delta_f(x_0)$, so that $(\delta_S(f))(x_0) = \frac{(\delta f)(x_0)+(\delta f)(x_0)}{2} = (\delta f)(x_0)$. □

**Proposition 3.7** *Under the hypothesis and notation of Proposition 3.4, namely:* $\mathcal{A} = \mathbb{Z}^U$ *and* $\mathcal{B} = \mathbb{Z}^{U'} \cup \mathbb{Z}^{V'}$ *is a product multi-Archimedean algebra in the sense of Definition 1.20. Then we have:*

$$(\delta_S(f))(x_0) = \frac{f(x_0 + 1_\mathcal{A}) - f(x_0 - 1_\mathcal{A})}{2}$$

*Proof.* Indeed, we know that the ordinary differentiation operator $\delta$ is the usual finite difference on each coordinate. So, we have $\left(\delta \widehat{f}_{x_0}\right)(x_0) = f(x_0 + 1_\mathcal{A}) - f(x_0)$, so that $(\delta_S(f))(x_0) = \frac{f(x_0+1_\mathcal{A}) - f(x_0 - 1_\mathcal{A})}{2}$. □

## 3.6 Polynomials and Power Functions in Analyzable Spaces

In the remainder of this section, we consider $(\mathcal{A}, \mathcal{R}, \Omega_\mathcal{A}, \mu_l, \preceq_\mathcal{A})$ and $(\mathcal{B}, \mathcal{R}, \Omega_\mathcal{B}, \mu'_{l'}, \preceq_\mathcal{B})$ two analyzable spaces over a ring $\mathcal{R}$. We assume (up to an isomorphism) that $\mathcal{A} = \mathbb{Z}^U \times \mathbb{R}^V$ and $\mathcal{B} = \mathbb{Z}^{U'} \times \mathbb{R}^{V'}$.
We use the notations $e_\beta$ and $e'_{\beta'}$ from Definition 3.7 throughout the remainder of this section.

**Definition 3.11 (Canonical Morphisms from $\mathcal{A}$ to $\mathcal{B}$)** *Under the hypothesis and notations of Definition 3.7, let us consider* $\beta \in U \cup V$ *and* $\beta' \in U' \cup V'$*, such that either* $\beta \in U$ *or* $\beta' \in V'$*. Then, there exists a unique algebra morphism which sends* $1_\mathcal{A}$ *on* $\beta'$*, and such that for* $r \in \mathcal{R}$ *we have* $X_{\beta,\beta'}(r.\beta) = r.\beta'$ *(i.e. which sends* $e_\beta$ *onto* $e'_{\beta'}$*).*
*This morphism can be expressed using the projections introduced in Definition 3.8:*

$$X_{\beta,\beta'} : \left\{ \begin{array}{ccc} \mathcal{A} \subset \mathbb{R}^{U \cup V} & \longrightarrow & \mathcal{B} \\ x & \longmapsto & \left((p_\beta(x))(\beta)\right).\beta' \end{array} \right.$$

*Note that for* $x \in \mathcal{A}$*, we have* $(p_\beta(x))(\beta) \in \mathbb{Z}$ *or* $(p_\beta(x))(\beta) \in \mathbb{R}$*, depending on whether* $\beta \in U$ *or* $\beta \in V$*.*

**Definition 3.12 (Formal Monomial from $\mathcal{A}$ to $\mathcal{B}$)** *Let* $\mathbb{N}_c^{U \cup V}$ *be the set of maps* $\sigma : U \cup V \longrightarrow \mathbb{N}$ *with finite support, that is, such that the set*

$$\mathbf{supp}(\sigma) = \{\beta \in U \cup V \ / \ \sigma(\beta) \neq 0_\mathbb{N}\}$$

*A formal monomial* $m$ *from* $\mathcal{A}$ *to* $\mathcal{B}$ *is a choice, for each range axis* $\beta' \in U' \cup V'$*, of an element of* $m(\beta') \in \mathbb{N}_c^{U \cup V}$*, the support of which is contained in* $U$ *when* $\beta' \in U'$*. In other words, a map:*

$$m : \left\{ \begin{array}{ccl} U' \cup V' & \longrightarrow & \mathbb{N}_c^{U \cup V} \\ \beta' & \longmapsto & m(\beta') : U \cup V \longrightarrow \mathbb{N} \text{ such that} \\ & & \text{if } \beta' \in U', \text{ then } \mathbf{supp}(m(\beta')) \subset U \end{array} \right.$$

*The degree of the monomial* $m$ *on a coordinate* $\beta' \in U' \cup V'$ *is the maximum of the map* $m(\beta')$ *over its finite support.*

**Definition 3.13 (Monomial Map Associated to a Formal Monomial)** *Let* $m$ *be a formal monomial from* $\mathcal{A}$ *to* $\mathcal{B}$*. We define (using the identification set out in Remark 2.4) the monomial map* $\pi_m$ *associated to* $m$ *by:*

$$\pi_m : \left\{ \begin{array}{ccl} \mathcal{A} = \mathbb{Z}^U \times \mathbb{R}^V & \longrightarrow & \mathcal{B} = \mathbb{Z}^{U'} \times \mathbb{R}^{V'} \\ x & \longmapsto & \pi_m(x) : \left\{ \begin{array}{ccc} U' \cup V' & \longrightarrow & \mathbb{R} \\ \beta' & \longmapsto & \displaystyle\prod_{\beta \in \mathbf{supp}(m(\beta'))} (X_{\beta,\beta'}(x))^{m(\beta')} \end{array} \right. \end{array} \right.$$

**Notation 3.3** *Let $m$ be a formal monomial from $\mathcal{A}$ to $\mathcal{B}$. The monomial map can be denoted by:*

$$\pi_m = \sum_{\beta' \in U' \cup V'} \prod_{\beta \in U \cup V} X_{\beta,\beta'}^{m(\beta')} \tag{1}$$

*using the convention that the product by elements of the form $X_{\beta,\beta'}^{m(\beta')}$ when $m(\beta') = 0_{\mathbb{N}}$ leave the result unchanged, so that each product in Equation (1) has in fact a finite number of factors. Moreover, the sum in Equation (1), which has potentially an infinite number of terms, takes place in disjoint ideals (generated by each of the $e'_{\beta'}$), so that it is weel defined without any problem of convergence.*

**Definition 3.14 (Polynomial maps from $\mathcal{A}$ to $\mathcal{B}$)** *The $\mathcal{B}-$valued polynomial functions with degree $K \in \mathbb{N}^{U' \cup V'}$ over $\mathcal{A}$ are the linear combinations of the monomial maps with degrees less than or equal to $K$, as introduced in Definition 3.13.*

**Definition 3.15 (Power Functions on Positive Elements)** *Let $\alpha \in \mathcal{A}$ be a broadly strictly positive element. Let $0_{\mathcal{A}} \preceq_{\mathcal{A}} x$. We identify, through the classification theorem, the elements $\alpha$ and $x$ with elements of $\mathbb{Z}^U \times \mathbb{R}^V$. we denote by $x^{[\alpha]}$ the element of $\mathcal{A} = \mathbb{R}^{U \cup V}$ obtained by composing the map $x$ by the map "$\alpha$'s power" on each image of canonical generator in $\mathcal{A}$.*

# 4 Digital Differentiation

In the sequel of this section, we consider the following structures. Let $d \in \mathbb{N}$ and, for $a = 1, \ldots, d$, let $\mathcal{A}_a$ be an analyzable space over a ring $\mathcal{R}$. We denote

$$\mathcal{M} = \prod_{a=1}^{d} \mathcal{A}_a \text{ and } \mathcal{M}' \subset \mathbb{Z}^{U'} \times \mathbb{R}^{V'}$$

For $a \in \{1, \ldots, d\}$, we consider $e_a$ the element of $\mathcal{M}$, the $i^{th}$ coordinate of which is $1_{\mathcal{A}_i}$ if $i = a$, and $0_{\mathcal{A}_i}$ otherwise. We denote by $\mathcal{Z}_d$ the sub-algebra of $\mathcal{M}$ generated by the $e_a$'s, for $a = 1, \ldots, d$.

We assume that $\mathcal{M}'$ is the set of all $\lambda'-$bounded elements of $\mathbb{Z}^{U'} \times \mathbb{R}^{V'}$ for some broadly strictly positive element $\lambda' \in \mathbb{Z}^{U'} \times \mathbb{R}^{V'}$. For $\beta' \in U' \cup V'$, we consider $e'_{\beta'}$ the canonical element of $\mathcal{M}'$, as defined in Definition 3.7.

For the sake of simplicity, we assume that $\mathcal{R}$ is either the ring $\mathbb{Z}$ or the ring $\mathbb{R}$. Moreover, we assume that the analyzable space $\mathcal{M}$ is finitely generated, so that we assume without loss of generality that either $\mathcal{A}_a = \mathbb{Z}$ or $\mathcal{A}_a = \mathbb{R}$ for $a = 1, \ldots, d$. Generalized statements in the case of general separable analizable spaces are given in Section 8, and the corresponding proofs are left to the reader.

## 4.1 Rapidely Decreasing and Moderately Increasing Multi-sequences

**Definition 4.1** *Let $\mathbf{u}$ be a multi-sequence in ${\mathcal{M}'}^{\mathcal{Z}_d}$. We say that $\mathbf{u}$ is rapidly decreasing if and only if for any polynomial function $\pi\mathcal{Z}_d \longrightarrow \mathcal{M}'$, the function $I \longmapsto \pi(I)u(I)$ is bounded. We denote by $\mathcal{D}[\mathcal{Z}_d, \mathcal{M}']$ the set of rapidly decreasing multi-sequences in ${\mathcal{M}'}^{\mathcal{Z}_d}$.*

**Remark 4.1** *The space $\mathcal{D}[\mathcal{Z}_d, \mathcal{M}']$ of rapidly decreasing multi-sequences is stable under inner addition, inner multiplication, and stable under multiplication by a polynomial function.*

**Lemma 4.1** *Let $\mathbf{u}$ a rapidly decreasing multi-sequence and let $\pi$ be an $\mathcal{M}'-$valued polynomial function on $\mathcal{M}$. For $I \in \mathcal{Z}_{d-1}$ and $i \in \mathcal{A}_d$, let us denote by $u(I,i)$ [resp. $\pi(I,i)$] the image under $\mathbf{u}$ [resp. under $\pi$] of the concatenation of $I$ and $(i)$. Then the multi-sequence defined on $\mathcal{Z}_{d-1}$ by*

$$s_d(I) = \sum_{i \in \mathcal{A}_d} |u(I,i)||\pi(I,i)|$$

*is well defined and bounded on $\mathcal{Z}_{d-1}$.*

*Proof.* It is sufficient to prove this property when $\pi$ is a monomial and, due to Remark 4.1, it is sufficient to prove it for polynomials of degree 0. In other words, we just need to show that the sum of the values of the multi-sequence $\mathbf{u}$ itself is absolutely convergent.

First we prove that for $d \geq 1$, the sum:

$$s_d(I) = \sum_{i \in \mathcal{A}_d} u(I,i)$$

is well defined for $I \in \mathcal{Z}_{d-1}$, and that the multi-sequence $\mathbf{s_d}$ itself is rapidly decreasing on $\mathcal{Z}_{d-1}$. Since $\mathbf{u}$ is rapidly decreasing, we can find $K > 0_{\mathcal{M}'}$ such that for $I \in \mathcal{Z}_{d-1}$ and $i \in \mathcal{A}_d$, we have $u(I,i) \preceq K$ and $i^2 u(I,i) \preceq K$. For $N \in \mathbb{N}^*$, we have

$$\begin{array}{rcl} (N!)^2 \sum_{i=1}^{N} |u(I,i)| & \preceq & (N!)^2 |u(I,1)| + \sum_{i=2}^{N} i^2 |u(I,i)|.\frac{(N!)^2 1_{\mathcal{M}'}}{i^2} \\ & \preceq & (N!)\,(K + K.2) \end{array}$$

Note that the expression $\frac{N!}{i^2}1_{\mathcal{M}'}$ denotes a well defined element of the algebra $\mathcal{M}'$ over the ring $\mathcal{R}$. Indeed, by expanding the expression of $(N!)^2$ and simplifying by $i^2$ to get an integer value, which is then multiplied by $1_{\mathcal{M}'}$ in the algebra $\mathcal{M}'$.

Hence, we get $\sum_{i\in\mathbb{N}^*}|u(I,i)|$ is well defined and bounded by $3K$. By a similar argument for $i<0$, we get that $\sum_{i\in\mathcal{A}_d}|u(I,i)|$ is well defined and bounded on $\mathcal{Z}_{d-1}$. □

**Lemma 4.2** *Let us consider the multi-sequence* $\mathbf{v}$ *defined on* $\mathcal{Z}_{d-1}$ *by* $v(I)=\sum_{i\in\mathcal{A}_d}u(I,i)$, *which is well-defined due to from Lemma 4.1, Then,* $\mathbf{v}$ *is a rapidly decreasing multi-sequence.*

*Proof.* Let $\pi$ be a polynomial function on $\mathcal{Z}_{d-1}$. Then, by considering $\pi$ as a function on $\mathcal{Z}_d$ (which does not depend on the $d^{th}$ coordinate), we get by Remark 4.1 that the multi-sequence $I\longmapsto\pi(I)u(I)$ is rapidly decreasing. From Lemma 4.1, we get that the multi-sequence $I\longmapsto\pi(I)v(I)$ on $\mathcal{Z}_{d-1}$ is bounded, which proves that $\mathbf{v}$ is rapidly decreasing. □

**Lemma 4.3** *For any rapidly decreasing multi-sequence* $\mathbf{u}$ *and any polynomial* $\pi$ *on* $\mathcal{Z}_d$, *the following series is absolutely convergent:*

$$\sum_{I\in\mathcal{Z}_d}\pi(I)u(I)$$

*In particular, the multi-sequence* $\mathbf{u}\pi$ *is bounded.*

The proof follows immediately by induction using Lemma 4.1 and Lemma 4.2.

**Proposition 4.1** *For any rapidly decreasing multi-sequence* $\mathbf{u}$ *and any polynomial* $\pi$ *on* $\mathcal{Z}_d$, *the multi-sequence* $\mathbf{u}\pi$ *is rapidly decreasing.*

**Definition 4.2** *Let* $\mathcal{I}$ *be a sub-algebra of* $\mathcal{M}$ *containing* $\mathcal{Z}_d$ *(typically,* $\mathcal{I}=\mathcal{Z}_d$ *or* $\mathcal{I}=\mathcal{M}$*). Let* $\mathbf{u}$ *be a function in* $\mathcal{M}'^{\mathcal{I}}$. *We say that* $\mathbf{u}$ *is moderately increasing if and only if there exist a bounded subset* $B$ *of* $\mathcal{I}$ *and a polynomial* $\pi$ *on* $\mathcal{I}$ *such that for any* $I\in\mathcal{I}\backslash B$ *we have* $|u(I)|\preceq|\pi(I)|$. *We denote by* $\mathcal{P}[\mathcal{I},\mathcal{M}']$ *the set of moderately increasing multi-sequences in* $\mathcal{M}'^{\mathcal{I}}$.

**Remark 4.2** *The product of a rapidly decreasing multi-sequence by a moderately increasing multi-sequence is rapidly decreasing.*

**Remark 4.3** *The space* $\mathcal{P}[\mathcal{I},\mathcal{M}']$ *of* $\mathcal{M}'-$*valued moderately increasing multi-sequences over a sub-algebra* $\mathcal{I}$ *of* $\mathcal{M}$ *is stable under inner addition, inner multiplication, and multiplication by a polynomial.*

## 4.2 Digital Differentiation, Tensor Products

First, we introduce a few notations about multi-indices.

**Notation 4.1** *Let* $\mathcal{P}=\prod_{a=1}^{d}\mathcal{B}_a$ *be a Cartesian product of* $d$ *analyzable spaces (e.g. the Cartesian product* $\mathcal{P}$ *can be* $\mathbb{Z}^d$ *or* $\mathcal{Z}_d$ *over the ring* $\mathbb{Z}$, *or possibly* $\mathcal{M}$ *or an ideal* $\mathcal{I}$ *over the ring* $\mathcal{R}$*). Let* $(I(a))_{a=1,\dots,d}\in\mathcal{P}$ *be a multi-index. We shall use the following notations:*

1. *For* $a\in\{1,\dots,d\}$ *and for* $j\in\mathbb{Z}$ *or* $j\in\mathcal{R}$ *or* $j\in\mathcal{B}_a$, *we denote by* $L(a,j)$ *the element in* $\mathcal{P}$, *all coordinates of which are zero, except the* $a$*'s coordinate which is equal to* $j.1_{\mathcal{B}_a}$.

2. *For $v \in \mathcal{P}$, for $a \in \{1,\dots,d\}$ and $j \in \mathbb{Z}$ or $j \in \mathcal{R}$ or $j \in \mathcal{B}_a$, we denote $v^{(a,j)} = v + L(a,j)$ the element obtained from $v$ by adding $j.1_{\mathcal{B}_a}$ to the $a$'s coordinate.*

3. *For $u_a \in \mathcal{B}_a$, we denote by $u_a 1_{\mathcal{P}}$ the product of the unit element $1_{\mathcal{P}}$ of $\mathcal{P}$ with the element of $\mathcal{P}$, identified with $u_a$, all coordinates of which are the unit element, except for the $a$'s coordinate which is equal to $u_a$. If no ambiguity can occur, we shall omit the unit $1_{\mathcal{P}}$ and simply denote by $u_a$ this element of $\mathcal{P}$.*

4. *We denote $|I| = \sum_{i=1,\dots,i} |I(a)|$ (with $|I(a)| = I(a)$ if $I(a) \geq 0$ and $|I(a)| = -I(a)$ if $I(a) < 0$), which is called the order of $I$.*

5. *Given $\alpha \in \mathbb{Z}^d$, following Definition 3.15, we denote $I^\alpha = \prod_{a=1}^d ((I(a))^{\alpha_a} 1_{\mathcal{P}})$, which is called the $\alpha$'s power of $I$ (possibly in a sub-ring of the product of the fields of fractions over the ring $\mathcal{B}_a$).*

6. *Given $\alpha \in \mathbb{R}^d$, following Definition 3.15, we denote by $I^{[\alpha]}$, which is called the coordinate by coordinate $\alpha$'s power of $I$ the vector, the coordinates of which (possibly in a sub-ring of the product of the fields of fractions over the ring $\mathcal{B}_a$) are given by $I^{[\alpha]} = \prod_{a=1}^d ((I(a))^{\alpha_a})$.*

7. *we denote $I! = \prod_{a=1}^d (I(a)!)$, where $I(a)! = \prod_{i \in \mathbb{N}, i.1_{\mathcal{A}_a} \preceq I(a)} (i.1_{\mathcal{P}})$ The element $I! \in \mathcal{P}$ is called the factorial of $I$.*

8. *If $(J(a))_{a=1,\dots,d}$ is another multi-index, we denote by $((IJ)(a))_{a=1,\dots,d}$ the multi-sequence with $(IJ)(a) = I(a)J(a)$, which is called the product of $I$ and $J$.*

9. *If, for $a = 1,\dots,d$, the algebra $\mathcal{B}_a$ is provided with an analyzable space structure and $\preceq_a$ is the order underlying this analyzable space structure, and if $(J(a))_{a=1,\dots,d}$ is another multi-index, we denote by $\preceq$ the binary relation, which is a partial order, such that $I \preceq J$ if and only if for $a = 1\dots,d$ we have $I(a) \preceq_a J(a)$. In Section 1.3, we called this partial order the coordinate by coordinate order.*

10. *If $(J(a))_{a=1,\dots,d}$ is another multi-index, we denote by $\prec$ the binary relation such that $I \prec J$ if and only if for $I \preceq J$ and $I \neq J$. In Section 1.3, we said in that case that $I$ is a strict lower bound of $J$.*

11. *If $(J(a))_{a=1,\dots,d}$ is another multi-index, we denote by $<$ the binary relation such that $I < J$ if and only if for all $a = 1\dots,d$ we have $I(a) < J(a)$ (i.e. the element $I(a)$ is broadly strictly less than $J(a)$ in $\mathcal{A}_a$, Definition 1.10). In Section 1.3, we said in that case that $I$ is a broad strict lower bound of $J$.*

12. *We denote by $0$ the multi-sequence with $d$ coordinates equal to $0$, and by $1$ the multi-sequence with $d$ coordinates equal to $d$. Note that the dimension $d$ of these vectors can be omitted as, due to the context, no ambiguity will arise in practice.*

13. *If $(J(a))_{a=1,\dots,d}$ is another multi-index with $J \preceq I$, we denote by $\binom{I}{J}$ the element of the ring $\mathcal{R}$ defined by:*

$$\binom{I}{J} = \prod_{a=1}^{d} \binom{I(a)}{J(a)}$$

*where* $\binom{I(a)}{J(a)}$ *is the binomial coefficient defined as usual using the Pascal induction formula:*

$$\binom{I(a)}{J(a)} = 1 \text{ if } (I(a) = J(a) \text{ or } J(a) = 0_{\mathcal{B}_a}),$$

$$\binom{I(a)}{J(a)} = 0 \text{ if } (I(a) < J(a) \text{ or } J(a) < 0_{\mathcal{B}_a}),$$

$$\text{and } \binom{I(a)}{J(a)} = \binom{I(a) - 1\mathcal{B}_a}{J(a) - 1\mathcal{B}_a} + \binom{I(a) - 1\mathcal{B}_a}{J(a)} \text{ otherwise}$$

*The element* $\binom{I}{J}$ *is called the multi-dimensional binomial coefficient of* $J$ *from* $I$*.*

**Remark 4.4 (Multidimensional Pascal Formula)** *Using Notation 4.1, we get for* $a = 1, \ldots, d$ *the following multi-dimensional version of the Pascal Formula:*

$$\binom{I}{J} = \binom{I^{(a,-1)}}{J^{(a,-1)}} + \binom{I^{(a,-1)}}{J}$$

### 4.2.1 Digital Differentiation Masks and their Tensor Products

We now introduce a notion of digital differentiations.

**Definition 4.3 (Digital differentiation Mask)** *Let* $\omega \in \mathbb{N}^d$*. A (*$d-$*dimensional) digital* $\omega-$*differentiation mask is a multi-sequence* $\mathbf{u} = (u(I))_{I \in \mathcal{Z}_d} \in \mathcal{M}'^{\mathcal{Z}_d}$ *with finite support, satisfying the following properties:*

1. *For all* $k \in \mathbb{N}^d$ *with* $0 \preceq k_a \preceq \omega_a$ *and* $k \neq \omega$*, we have:*

$$\sum_{I \in \mathcal{Z}_d} \left( \prod_{a=1}^{d} (I(a))^{k_a} \right) u(I) = 0_{\mathcal{M}'} \tag{2}$$

2. 

$$\sum_{I \in \mathcal{Z}_d} \left( \prod_{a=1}^{d} (I(a))^{\omega_a} \right) u(I) = \prod_{a=1}^{d} \left( (-1_{\mathcal{M}'})^{\omega_a} \omega_a! \right) \tag{3}$$

**Remark 4.5** *Using Notation 4.1, we can rewrite Definition 4.3 above saying that* $\mathbf{u}$ *is a (*$d-$*dimensional) digital* $\omega-$*differentiation mask if and only if we have:*

$$\sum_{I \in \mathcal{Z}_d} I^k u(I) = 0_{\mathcal{M}'} \text{ for } 0 \preceq k \prec \omega \tag{4}$$

*and*

$$\sum_{I \in \mathcal{Z}_d} I^\omega u(I) = (-1_{\mathcal{M}'})^\omega \omega! \tag{5}$$

**Definition 4.4 (Extended Digital differentiation Mask)** *Let* $\omega \in \mathbb{N}^d$*. An extended (*$d-$*dimensional) digital* $\omega-$*differentiation mask is a rapidly decreasing multi-sequence* $\mathbf{u} = (u(I))_{I \in \mathcal{Z}_d} \in \mathcal{M}'^{\mathcal{Z}_d}$ *with finite support, satisfying Formula (4) and Formula (5).*

In the sequel, unless otherwise specified, we shall say an $\omega-$differentiation mask as a shorthand for an extended ($d-$dimensional) digital $\omega-$differentiation mask.

**Definition 4.5 (Tensor Product of Masks or Functions)** *For $a \in \{1, \ldots, d\}$, let $\mathcal{I}_a$ be a sub-algebra of $\mathcal{M}_a$ which contains $1_{\mathcal{A}_a}\mathcal{A}_a$ (typically $\mathcal{I}_a = 1_{\mathcal{A}_a}\mathcal{A}_a$ or $\mathcal{I}_a = \mathcal{A}_a$), and let $\mathbf{u}_a = (u_a(I))_{I \in \mathcal{I}_a} \in \mathcal{M}'^{\mathcal{I}_a}$ be a sequence with a one-dimensional domain $\mathcal{I}_a$. We denote $\mathcal{I} = \prod_{a=1}^{d} \mathcal{I}_a$ and $\mathbf{u} = (u(I))_{I \in \mathcal{I}} \in \mathcal{M}'^{\mathcal{I}}$ the multi-sequence defined by*

$$u(I) = \prod_{a=1}^{d} u_a(I(a))$$

*The function $\mathbf{u}$ is called the tensor product of the function $\mathbf{u}_a$ for $a = 1, \ldots, d$. We denote by $\bigotimes_{a=1}^{d} \mathbf{u}_a \in \mathcal{M}'^{\mathcal{I}}$ the tensor product $\mathbf{u}$.*

**Definition 4.6 (Isotropic Multi-Sequence)** *Let us consider a multi-sequence $\mathbf{u} = (u(I))_{I \in \mathcal{Z}_d} \in \mathcal{M}'^{\mathcal{Z}_d}$. Let $a_1 \in \{1 \ldots, d\}$. For $I \in \mathcal{Z}_d$ and $i_1 \in \mathcal{A}_{a_1}$, we consider*

$$(\tau_{a_1}(I, i_1))(a) = \begin{cases} I(a) \text{ if } a \neq a_1 \\ i_1 \text{ if } a = a_1 \end{cases}$$

*thus defining an element $\tau_{a_1}(I, i_1)$ in $\mathcal{M}'^{\mathcal{Z}_d}$. The multi-sequence $\mathbf{u}$ is called isotropic if and only if for any $I \in \mathcal{Z}_d$, any $a_1 \in \{1 \ldots, d\}$, any $i_1 \in \mathcal{A}_{a_1}$, we have:*

$$u(\tau_{a_1}(I, J(a_1)))\, u(\tau_{a_1}(J, I(a_1))) = u(I)\, u(J)$$

**Proposition 4.2** *Let $\mathbf{u} = (u(I))_{I \in \mathcal{Z}_d} \in \mathcal{M}'^{\mathcal{Z}_d}$ be a multi-sequence. If $\mathbf{u}$ is a tensor product of one-dimensional sequences, that is, there exist $\mathbf{u}_a = (u_a(I))_{I \in \mathcal{A}_a} \in \mathcal{M}'^{\mathcal{A}_a}$ such that $\mathbf{u} = \bigotimes_{a=1}^{d} \mathbf{u}_a$. Then, it is isotropic.*

*Proof.* Assume that $\mathbf{u} = \bigotimes_{a=1}^{d} \mathbf{u}_a$ and consider $I \in \mathcal{Z}_d$ and $a_1 \in \{1 \ldots, d\}$.

$$\begin{aligned}
&u(\tau_{a_1}(I, J(a_1)))\, u(\tau_{a_1}(J, I(a_1))) \\
&= \left(\prod_{a=1}^{d} u_a(\tau_{a_1}(I, J(a_1))(a))\right)\left(\prod_{a=1}^{d} u_a(\tau_{a_1}(J, I(a_1))(a))\right) \\
&= u_{a_1}(J(a_1))\left(\prod_{a \neq a_1} u_a(I(a))\right) u_{a_1}(I(a_1))\left(\prod_{a \neq a_1} u_a(J(a)))\right) \\
&= \left(\prod_{a=1}^{d} u_a(I(a))\right)\left(\prod_{a=1}^{d} u_a(J(a))\right) \\
&= u(I)u(J)
\end{aligned}$$

□

**Theorem 4.1** *A multi-sequence $\mathbf{u} = (u(I))_{I \in \mathcal{Z}_d} \in \mathcal{M}'^{\mathcal{Z}_d}$ is an isotropic digital $\omega-$differentiation mask if and only if, for $a = 1, \ldots, d$, there exist one-dimensional $\omega_a-$differentiation masks $\mathbf{u}_a = (u_a(I))_{I \in \mathcal{Z}_1} \in \mathcal{M}'^{\mathcal{Z}_1}$ such that $\mathbf{u} = \bigotimes_{a=1}^{d} \mathbf{u}_a$.*

*Proof.* Let us first prove by induction that a tensor product of $d$ one-dimensional $\omega_a-$differentiation masks is an isotropic differentiation mask. We already know from Proposition 4.2 that a tensor product of $d$ one-dimensional masks is isotropic. We prove the result by induction on $d$. For $d = 1$, there is nothing to prove. Let us assume the result true for $d-1$ one-dimensional $\omega_a-$differentiation masks, and consider, for $a = 1, \dots, d$, a one-dimensional $\omega_a-$differentiation mask $\mathbf{u}_a = (u_a(I))_{I\in\mathcal{Z}_1} \in \mathcal{M}'^{\mathcal{Z}_1}$ . Let $k \in \mathbb{N}^d$ with $0 \preceq k_a \preceq \omega_a$ and $k \neq \omega$. We have:

$$\begin{aligned}
\textstyle\sum_{I\in\mathcal{Z}_d}\left(\prod_{a=1}^d (I(a))^{k_a}\right)u(I) &= \textstyle\sum_{I\in\mathcal{Z}_d}\left(\prod_{a=1}^d (I(a))^{k_a}\right)\left(\prod_{a=1}^d u_a(I(a))\right)\\
&= \textstyle\sum_{i\in\mathcal{A}_d}\sum_{I\in\mathcal{Z}_{d-1}}\left(i^{k_d}\prod_{a=1}^{d-1}(I(a))^{k_a}\right)\left(u_d(i)\prod_{a=1}^{d-1}u_a(I(a))\right)\\
&= \left(\textstyle\sum_{i\in\mathcal{A}_d}(i^{k_d}u_d(i))\right)\left(\textstyle\sum_{I\in\mathcal{Z}_{d-1}}\left(\prod_{a=1}^{d-1}(I(a))^{k_a}\left(\bigotimes_{a=1}^{d-1}\mathbf{u}_a\right)(I)\right)\right)\\
&= 0_{\mathcal{M}'}
\end{aligned}$$

The last equality follows from our induction hypothesis, either applied to the one-dimensional $\omega_d-$differentiation mask $\mathbf{u}_d$, either to the $(d-1)-$dimensional differentiation mask $\bigotimes_{a=1}^{d-1}\mathbf{u}_a$, depending on which of the coordinates of $k$ differs form the corresponding coordinate of $\omega$. Similarly,

$$\begin{aligned}
\textstyle\sum_{I\in\mathcal{Z}_d}\left(\prod_{a=1}^d (I(a))^{\omega_a}\right)u(I) &= \textstyle\sum_{I\in\mathcal{Z}_d}\left(\prod_{a=1}^d (I(a))^{\omega_a}\right)\left(\prod_{a=1}^d u_a(I(a))\right)\\
&= \left(\textstyle\sum_{i\in\mathcal{A}_d}(i^{\omega_d}u_d(i))\right)\left(\textstyle\sum_{I\in\mathcal{Z}_{d-1}}\left(\prod_{a=1}^{d-1}(I(a))^{\omega_a}\left(\bigotimes_{a=1}^{d-1}\mathbf{u}_a\right)(I)\right)\right)\\
&= \textstyle\prod_{a=1}^{d-1}\left((-1_{\mathcal{M}'})^{\omega_a}\omega_a!\right)\left((-1_{\mathcal{M}'})^{\omega_d}\omega_d!\right) = \prod_{a=1}^{d-1}\left((-1_{\mathcal{M}'})^{\omega_a}\omega_a!\right)
\end{aligned}$$

Conversely, let us consider an isotropic digital $\omega-$differentiation mask $\mathbf{u} = (u(I))_{I\in\mathcal{Z}_d}$. Again, we prove the result by induction. If $d = 1$ there is nothing to prove. Assume the result true for a $(d-1)-$dimensional isotropic mask. For $I \in \mathcal{Z}_{d-1}$ and $i \in \mathcal{A}_d$, we set $u(I, i)$ the value of the $d-$dimensional multi-sequence $\mathbf{u}$ evaluated on $(I(1), \dots, I(d-1), i)$. For $i \in \mathcal{A}_d$, we set:

$$u_d(i) = \sum_{I\in\mathcal{Z}_{d-1}}\left(\frac{\prod_{a=1}^{d-1}(I(a))^{\omega_a}}{\prod_{a=1}^{d-1}(-1)^{\omega_a}\omega_a!}u(I,i)\right)$$

and, for $I \in \mathcal{Z}_{d-1}$,

$$u^{(d-1)}(I) = \sum_{i\in\mathcal{A}_d}\left(\frac{i^{\omega_d}u(I,i)}{(-1)^{\omega_d}\omega_d!}\right)$$

Both multi-sequences $\mathbf{u}_d$ and $\mathbf{u}^{(d-1)}$ are clearly isotropic. We show that $\mathbf{u} = \mathbf{u}^{(d-1)} \otimes \mathbf{u}_d$, that is: $u^{(d-1)}(J)u_d(j) = u(J, j)$. Indeed,

$$\begin{aligned}
u^{(d-1)}(J)u_d(j) &= \left(\textstyle\sum_{i\in\mathcal{A}_d}\left(\frac{i^{\omega_d}u(J,i)}{(-1)^{\omega_d}\omega_d!}\right)\right)\left(\textstyle\sum_{I\in\mathcal{Z}_{d-1}}\left(\frac{\prod_{a=1}^{d-1}(I(a))^{\omega_a}}{\prod_{a=1}^{d-1}(-1)^{\omega_a}\omega_a!}u(I,j)\right)\right)\\
&= \textstyle\sum_{i\in\mathcal{A}_d}\sum_{I\in\mathcal{Z}_{d-1}}\left(\frac{\prod_{a=1}^{d}(I(a))^{\omega_a}}{\prod_{a=1}^{d}(-1)^{\omega_a}\omega_a!}u(J,i)u(I,j)\right)\\
&= \textstyle\sum_{i\in\mathcal{A}_d}\sum_{I\in\mathcal{Z}_{d-1}}\left(\frac{\prod_{a=1}^{d}(I(a))^{\omega_a}}{\prod_{a=1}^{d}(-1)^{\omega_a}\omega_a!}\left(u(J,j)u(I,i)\right)\right)\\
&= u(J,j)\textstyle\sum_{i\in\mathcal{A}_d}\sum_{I\in\mathcal{Z}_{d-1}}\left(\frac{\prod_{a=1}^{d}(I(a))^{\omega_a}}{\prod_{a=1}^{d}(-1)^{\omega_a}\omega_a!}u(I,i)\right)\\
&= u(J,j)
\end{aligned}$$

Let us now prove that, $\mathbf{u}_d$ is a one-dimensional differentiation mask. Let us set $k_a = \omega_a$ for

$a = 1, \dots, d-1$. Let $0 \preceq k_d < \omega_d$, we have:

$$\begin{array}{rcl}\sum_{i\in\mathcal{A}_d} i^{k_d} u_d(i) & = & \sum_{i\in\mathcal{A}_d} i^{k_d} \left(\sum_{I\in\mathcal{Z}_{d-1}} \left(\frac{\prod_{a=1}^{d-1}(I(a))^{\omega_a}}{\prod_{a=1}^{d-1}(-1)^{\omega_a}\omega_a!}\right) u(I,i)\right) \\ & = & \frac{1}{\prod_{a=1}^{d-1}(-1)^{\omega_a}\omega_a!} \sum_{I\in\mathcal{Z}_d} \left(\prod_{a=1}^{d}(I(a))^{k_a}\right) u(I) \\ & = & 0_{M'}\end{array}$$

Similarly,

$$\begin{array}{rcl}\sum_{i\in\mathcal{A}_d} i^{\omega_d} u_d(i) & = & \sum_{i\in\mathcal{A}_d} i^{\omega_d} \left(\sum_{I\in\mathcal{Z}_{d-1}} \left(\frac{\prod_{a=1}^{d-1}(I(a))^{\omega_a}}{\prod_{a=1}^{d-1}(-1)^{\omega_a}\omega_a!}\right) u(I,i)\right) \\ & = & \frac{1}{\prod_{a=1}^{d-1}(-1)^{\omega_a}\omega_a!} \sum_{I\in\mathcal{Z}_d} \left(\prod_{a=1}^{d}(I(a))^{\omega_a}\right) u(I) \\ & = & (-1)^{\omega_d}\omega_d!\end{array}$$

Now we prove that that, if we set $\omega^{(d-1)} = (\omega_1, \dots, \omega_{d-1})$, $\mathbf{u}^{(d-1)}$ is a $(d-1)-$dimensionnal $\omega^{(d-1)}-$differentiation mask. The result then follows from our induction hypothesis. Let $k \in \mathbb{N}^{d-1}$ with $0 \preceq k_a \preceq \omega_a^{(d-1)}$ for $a = 1, \dots, d-1$ and $k \neq \omega^{(d-1)}$. We have

$$\begin{array}{rcl}\sum_{I\in\mathcal{Z}_{d-1}} \left(\prod_{a=1}^{d-1}(I(a))^{k_a}\right) u^{(d-1)}(I) & = & \sum_{I\in\mathcal{Z}_{d-1}} \left(\prod_{a=1}^{d-1}(I(a))^{k_a}\right) \sum_{i\in\mathcal{A}_d} \left(\frac{i^{\omega_d}u(I,i)}{(-1)^{\omega_d}\omega_d!}\right) \\ & = & \sum_{I\in\mathcal{Z}_d} \left(\prod_{a=1}^{d}(I(a))^{k_a}\right) u(I) \\ & = & 0_{\mathcal{M}'}\end{array}$$

Similarly,

$$\begin{array}{rcl}\sum_{I\in\mathcal{Z}_{d-1}} \left(\prod_{a=1}^{d-1}(I(a))^{\omega_a}\right) u^{(d-1)}(I) & = & \sum_{I\in\mathcal{Z}_{d-1}} \left(\prod_{a=1}^{d-1}(I(a))^{\omega_a}\right) \sum_{i\in\mathcal{A}_d} \left(\frac{i^{\omega_d}u(I,i)}{(-1)^{\omega_d}\omega_d!}\right) \\ & = & \frac{1}{(-1)^{\omega_d}\omega_d!} \sum_{I\in\mathcal{Z}_d} \left(\prod_{a=1}^{d}(I(a))^{\omega_a}\right) \\ & = & \prod_{a=1}^{d-1} \left((-1_{\mathcal{M}'})^{\omega_a}\omega_a!\right)\end{array}$$

□

In the sequel of this paper, all the considered differentiation masks are assumed to be isotropic.

### 4.2.2 Convolution and Differentiation Operators

**Definition 4.7 (Convolution Product)** *Let $\mathbf{u}$ be a multi-sequence in ${\mathcal{M}'}^{\mathcal{Z}_d}$. Let $\mathcal{I}$ be a sub-algebra of $\mathcal{M}$ which contains $\mathcal{Z}_d$ (typically, $\mathcal{I} = \mathcal{Z}_d$ or $\mathcal{I} = \mathcal{M}$) and $\mathbf{v} : \mathcal{I} \longrightarrow \mathcal{M}'$ be a function. We say that $\mathbf{u}$ and $\mathbf{v}$ are convolvable if the following sum is absolutely convergent for any $N \in \mathcal{I}$:*

$$(\mathbf{u} \star \mathbf{v})(N) = \sum_{I\in\mathcal{Z}_d} u(I)v(N-I)$$

*The multi-sequence $\mathbf{u} \star \mathbf{v}$ thus defined is then called the convolution product of $\mathbf{u}$ and $\mathbf{v}$.*

**Proposition 4.3** *For $i = 1 \dots, m$, let $\mathbf{u}_i$ and $\mathbf{v}_i$ be two multi-sequences on a network $\mathcal{Z}^{(m)}$ in a Cartesian product of analyzable spaces $\mathcal{M}^{(m)}$, with values in $\mathcal{M}'$. Then, we have*

$$\left(\bigotimes_{i=1}^{m} \mathbf{u}_i\right) \star \left(\bigotimes_{i=1}^{m} \mathbf{v}_i\right) = \bigotimes_{i=1}^{m} (\mathbf{u}_i \star \mathbf{v}_i)$$

*Proof.* We prove the result for $m = 2$ and, by associativity of the tensor product and of the convolution product, the result follows by an immediate induction.

$$\begin{array}{rcl}((u_1 \otimes u_2) \star (v_1 \otimes v_2))(n_1, n_2) & = & \sum_{(i_1,i_2)\in\mathbb{Z}^{(m_1)}\times\mathbb{Z}^{(m_2)}} (u_1 \otimes u_2)(i_1, i_2)(v_1 \otimes v_2)((n_1, n_2) - (i_1, i_2)) \\ & = & \sum_{(i_1,i_2)\in\mathbb{Z}^{(m_1)}\times\mathbb{Z}^{(m_2)}} (u_1(i_1)u_2(i_2)v_1(n_1 - i_1)v_2(n_2 - i_2)) \\ & = & \sum_{(i_1,i_2)\in\mathbb{Z}^{(m_1)}\times\mathbb{Z}^{(m_2)}} (u_1(i_1)v_1(n_1 - i_1))(u_2(i_2)v_2(n_2 - i_2)) \\ & = & \sum_{i_1\in\mathbb{Z}^{(m_1)}} (u_1(i_1)v_1(n_1 - i_1)) \sum_{i_2\in\mathbb{Z}^{(m_2)}} (u_2(i_2)v_2(n_2 - i_2)) \\ & = & ((u_1 \star v_1) \otimes (u_2 \star v_2))(n_1, n_2)\end{array}$$

□

**Proposition 4.4** *For $i = 1\dots, m$, let $\mathbf{u}_i$ be a multi-sequence on a network $\mathcal{Z}^{(i)}$ in a Cartesian product of analyzable spaces $\mathcal{M}^{(i)}$, with values in $\mathcal{M}'$. Then,*

1. *Suppose that for $i = 1, \dots, m$, the multi-sequence is an $\omega^{(i)}-$differentiation mask. Then, the tensor product $\bigotimes_{i=1}^m \mathbf{u}_i$ is an $\omega-$differentiation mask, where $\omega$ is the concatenation of the vectors $\omega_i$ for $i = 1\dots, m$.*
2. *Conversely, if we assume that $\mathbf{u} = \bigotimes_{i=1}^m \mathbf{u}_i$ is an $\omega-$differentiation mask on $\mathcal{Z} = \prod_{i=1}^m \mathcal{Z}^{(i)}$, where $\omega$ is the concatenation of the vectors $\omega_i$ for $i = 1\dots, m$. then $\mathbf{u}_i$ is an $\omega^{(i)}-$differentiation mask for each $i \in \{1, \dots, m\}$.*

*Proof.* **1)** We prove the first part of the result for $m = 2$ and, by associativity of the tensor product and vector concatenation, the result follows by an immediate induction. Let $\omega^{(1)} = (\omega_1^{(1)}, \dots, \omega_{d_1}^{(1)})$ and $\omega^{(2)} = (\omega_1^{(2)}, \dots, \omega_{d_2}^{(2)})$. Let $k_1 \in \mathbb{N}^{d_1}$ and $k_2 \in \mathbb{N}^{d_2}$ Let $k$ be the concatenation of $k_1$ and $k_2$.

$$\begin{array}{l}\sum_{I\in\mathcal{Z}_d} \left(\prod_{a=1}^{d_1+d_2} (I(a))^{k_a}\right) (u_1 \otimes u_2)(I) \\ = \sum_{I\in\mathcal{Z}_{d_1}\times\mathcal{Z}_{d_2}} \left(\prod_{a=1}^{d_1+d_2} (I(a))^{k_a}\right) u_1(I_1)u_2(I_2)(I) \\ = \left(\sum_{I_1\in\mathcal{Z}_{d_1}} \left(\prod_{a=1}^{d_1} (I_1(a))^{k_a}\right) u_1(I_1)\right) \left(\sum_{I_2\in\mathcal{Z}_{d_2}} \left(\prod_{a=1}^{d_2} (I_2(a))^{k_a}\right) u_2(I_2)(I)\right)\end{array}$$

Then, depending on whether $k = \omega$ or not, we get Equation (2) or Equation (3).

**2)** To prove the converse, observe that $\mathbf{u}$ is not identically zero. Let $I \in \mathcal{Z}$ be such that $u(I) \neq 0_{\mathcal{M}'}$ and let $i \in \{1, \dots, m\}$. By restricting $\mathbf{u}(I)$ to the elements $I \in \mathcal{Z}$ of the product $\mathcal{Z}$ in which only the $i^{th}$ coordinate varies, we obtain a multi-sequence on $\mathcal{Z}^{(i)}$ which is proportional to $\mathbf{u}_i$. Then, Definition 4.4 applied to this restriction of $\mathbf{u}$ immediately yields Equation 2 and Equation 3 for $\mathbf{u}_i$. □

**Definition 4.8 (Differentiation Operator)** *Let $\mathbf{u}$ be a differentiation mask with finite support. Let $\mathcal{I}$ be a sub-algebra of $\mathcal{M}$ with contains $\mathcal{Z}_d$. The $\omega$-differentiation operator associated to $\mathbf{u}$ over $\mathcal{M}^{\mathcal{I}}$ is the function $\Delta_{\mathbf{u}}$ with domain $\mathcal{M}'^{\mathcal{I}}$ and co-domain $\mathcal{R}^{\mathcal{I}}$ defined by*

$$\Delta_{\mathbf{u}} : \left\{ \begin{array}{ccc} \mathcal{M}'^{\mathcal{I}} & \longrightarrow & \mathcal{M}'^{\mathcal{I}} \\ \mathbf{v} & \longmapsto & \Delta_{\mathbf{u}}(\mathbf{v}) = \mathbf{u} \star \mathbf{v} \end{array} \right.$$

**Definition 4.9 (Extended Differentiation Operator)** *Let $\mathbf{u}$ be a rapidly decreasing differentiation mask. Let $\mathcal{I}$ be a sub-algebra of $\mathcal{M}$ with contains $\mathcal{Z}_d$. The (extended) $\omega$-differentiation operator associated to $\mathbf{u}$ over the space of moderately increasing functions $\mathcal{P}[\mathcal{I}, \mathcal{M}']$, with co-domain $\mathcal{P}[\mathcal{I}, \mathcal{M}']$, is defined by*

$$\Delta_{\mathbf{u}} : \left\{ \begin{array}{ccc} \mathcal{P}[\mathcal{I}, \mathcal{M}'] & \longrightarrow & \mathcal{P}[\mathcal{I}, \mathcal{M}'] \\ \mathbf{v} & \longmapsto & \Delta_{\mathbf{u}}(\mathbf{v}) = \mathbf{u} \star \mathbf{v} \end{array} \right.$$

**Remark 4.6** *Note that the fact that the image $\Delta_{\mathbf{u}}(\mathbf{v})$ with a rapidly decreasing sequence $\mathbf{u}$ and a moderately increasing function $\mathbf{v}$ lies in $\mathcal{P}[\mathcal{I},\mathcal{M}']$ requires a justification, which is given in Proposition 4.7 shown below.*

In the sequel, if no ambiguity can arise, we shall assume without mentioning this hypothesis, either that differentiation masks have finite support, or the differentiation masks are rapidly decreasing and the corresponding differentiation operators are applied only to moderately increasing functions.

**Proposition 4.5** *Let $\mathbf{u} = (u(I))_{i\in\mathcal{Z}_d}$ be an $\omega$-derivative mask and $\mathbf{v} = (v(I))_{i\in\mathcal{Z}_d}$ be an $\omega'$-derivative mask. Then $\mathbf{u}\star\mathbf{v}$ is an $\omega+\omega'$-derivative mask.*

*Proof.* We prove the one-dimensional case. The general case follows from Theorem 4.1 and Proposition 4.4.

Let $0 \preceq k \preceq \omega$. Then

$$\begin{array}{rcl}
\sum\limits_{n\in\mathbb{Z}} n^k(\mathbf{u}\star\mathbf{v})(n) & = & \sum_{n\in\mathbb{Z}}(i+(n-i))^k\sum_{i\in\mathbb{Z}}u(i)v(n-i) \\
& = & \sum_{n\in\mathbb{Z}}\sum_{i\in\mathbb{Z}}\sum_{p=0}^{k}\binom{k}{p}i^p(n-i)^{k-p}u(i)v(n-i) \\
& = & \sum_{p=0}^{k}\binom{k}{p}\left(\sum_{n\in\mathbb{Z}}\sum_{i\in\mathbb{Z}}i^pu(i)(n-i)^{k-p}v(n-i)\right) \\
& = & \sum_{p=0}^{k}\binom{k}{p}\sum_{j\in\mathbb{Z}}\sum_{i\in\mathbb{Z}}i^pu(i)j^{k-p}v(j) \\
& = & \sum_{p=0}^{k}\binom{k}{p}\left(\sum_{j\in\mathbb{Z}}j^{k-p}v(j)\right)\left(\sum_{i\in\mathbb{Z}}i^pu(i)\right)
\end{array}$$

This is zero except if $k=\omega+\omega'$ and in this case all the terms are zero except if $p=\omega$ and in this case the sum is $\binom{\omega+\omega'}{\omega'}(-1)^\omega\omega!(-1)^{\omega'}\omega'! = (-1)^{\omega+\omega'}(\omega+\omega')!$ □

## 4.3 Differential Operators and Polynomials

We consider the monomial and polynomial maps as defined in Definition 3.13 and Definition 3.14, and we introduce here some simplified notations in our case when $\mathcal{M}'$ is finitely generated.

**Definition 4.10 (Canonical Monomials $X_{a,\beta'}$ from $\mathcal{A}_a$ to $\mathcal{M}'$)** *For $a \in \{1,\dots,d\}$, and for each canonical generator $e'_{\beta'}$ of $\mathcal{M}'$, we consider $X_{a,\beta'}$ the canonical morphism (Definition 3.11) from $\mathcal{A}_a$ to $\mathcal{M}'$ such that the image of the unit element $1_{\mathcal{A}_a}$ in $\mathcal{A}_a$ is the unit element $e'_{\beta'}$ in $\mathcal{M}'$.*
*The maps $\mathbf{p}_{k,\beta'}$, for $k\in\mathbb{N}^d$ with $\sum_{a=1}^d k_a \preceq \delta$, defined by:*

$$\mathbf{p}_{k,\beta'} = \prod_{a=1}^{d} X_{a,\beta'}^{k_a} \tag{6}$$

*The arbitrary linear combinations of such monomials $\mathbf{p}_{k,\beta'}$, for a potentially infinite number different coordinates $\beta'\in U'\cup V'$ of the range $\mathcal{M}'$, are called monomial maps from $\mathcal{A}_a$ to $\mathcal{M}'$.*

**Definition 4.11** *The $\mathcal{M}'-$valued polynomial functions with degree $\delta\in\mathbb{N}$ over $\mathcal{M}$ are the linear combinations of the monomials $\mathbf{p}_k$ introduced in Definition 4.10.*

In other words, using Notation 4.1, any $\mathcal{M}'-$valued polynomial $\mathbf{p}$ with degree $\delta$ function over $\mathcal{M}$ is of the form

$$P(x) = \sum_{\beta' \in U' \cup V'} \sum_{k \in \mathbb{N}^d, |k| \preceq \delta} \lambda_k X^k_{\beta'}(x) \tag{7}$$

where $\lambda_k \in \mathcal{M}'$, and $X_{\beta'} = \bigotimes_{a=1}^{d} X_{a,\beta'}$. The $\lambda_k$'s, for $|k| \preceq \delta$ are called the *coefficients of the polynomial* $\mathbf{p}$ *for the basis of the* $\mathbf{p}_k$.

**Proposition 4.6** *Let* $\mathbf{p}$ $\mathcal{M}'-$*valued polynomial function over* $\mathcal{M}$ *as defined in Equation (6). Let* $\mathbf{u} = \bigotimes_{a=1}^{m} \mathbf{u}_a$ *be an* $\omega-$*differentiation operator, with* $\omega \in \mathbb{N}^d$*. Then, we have*

$$(\Delta_{\mathbf{u}}(\mathbf{p})) = \sum_{i=0}^{\delta} \sum_{\substack{k \in \mathbb{N}^d \\ k_1 + \ldots + k_d = i}} \left( \lambda_k \prod_{a=1}^{d} \binom{k_a}{k_a - \omega_a} \right) \prod_{a=1}^{d} X^{k_a - \omega_a}_{a,\beta'} \tag{8}$$

In other words, differentiation operators act on polynomial functions like usual partial derivative operators on (say) usual polynomials over $\mathbb{R}^d$.

**Remark 4.7** *Equation (8) can be rewritten using Notation 4.1 to obtain:*

$$(\Delta_{\mathbf{u}}(\mathbf{p}))\,(n) = \sum_{k \in \mathbb{N}^d, |k| \preceq \delta} \lambda_k \binom{k}{k - \omega} X^{k-\omega}_{\beta} \tag{9}$$

*Moreover, in the latter sum, only the multi-indices* $k$ *such that* $\omega \preceq k$ *contribute with a non-zero term.*

*Proof.* Due to Proposition 4.3, the definition of monomials as tensor products of $1D$ monomials, and Proposition 4.4, it is sufficient to prove the result for $d = 1$. By linearity, it is also sufficient to prove it for a monomial $\mathbf{p} = n^k$ Then,

$$\begin{aligned} (\Delta_{\mathbf{u}}(\mathbf{p}))\,(n) &= \textstyle\sum_{i \in \mathcal{A}_a} u(i)(n-i)^k \\ &= \textstyle\sum_{i \in \mathcal{A}_a} u(i) \sum_{l=0}^{k} \binom{k}{l} n^l (-i)^{k-l} \\ &= \textstyle\sum_{l=0}^{k} \binom{k}{l} \left( \sum_{i \in \mathcal{A}_a} u(i)(-i)^{k-l} \right) n^l \end{aligned}$$

Now, from Definition 4.4, the sum $\sum_{i \in \mathcal{A}_a} u(i)(-i)^{k-l}$ is equal to $0_{\mathcal{M}'}$ if $k - l < \omega_1$, and equal to $((-1)^{\omega_1}\omega_1!)$ if $k - l = \omega_1$. Hence, for $k \preceq \omega_1$, $(\Delta_{\mathbf{u}}(\mathbf{p}))\,(n) = \binom{k}{k-\omega_1}((-1)^{\omega_1}\omega_1!)n^{k-\omega_1}$.

At last, we prove the result for any $k > \omega_1$ by induction. Suppose it is true for $k - 1$, and set $\mathbf{v} = (v(n))_{n \in \mathcal{Z}_1}$, with $v(n) = \sum_{s \preceq n} u(s)$. Then we have $\mathbf{u} = \Delta_- \star \mathbf{v}$, where $\Delta_-$ is a finite difference $(1)-$differentiation mask (specifically: $\Delta_- \star \mathbf{v}(n) = v(n) - v(n-1) = u(n)$). It can be seen that the mask $\mathbf{v}$ is an $\omega_1 - 1$ differentiation mask. Furthermore, the differential $\Delta_{\mathbf{u}}(\mathbf{p})$, which is a $(1)-$differential differentiation mask applied to the $(\omega - 1)-$differential $\Delta_{\mathbf{v}}(\mathbf{p})$ which is constant (equal either to $((-1)^{\omega_1}\omega_1!)n^0$ if $k - 1 = \omega_1$ or, by induction hypothesis, identically zero otherwise), is also zero. □

**Lemma 4.4** *Let* $\mathbf{u}$ *be a rapidly decreasing function in* $\mathcal{M}'^{\mathcal{A}_a}$*, with* $a \in \{1, \dots, d\}$*. Let* $\mathbf{p}$ *be a polynomial with degree* $k$ *on a sub-algebra* $\mathcal{I}_a$ *of* $\mathcal{A}_a$ *containing* $1_{\mathcal{A}_a}\mathcal{A}_a$*. Then, there exists a polynomial function* $\pi$ *over* $\mathcal{I}_a$ *with degree* $k$ *such that:*

$$\mathbf{u} \star \mathbf{p} = \pi$$

*Proof.* It is sufficient to prove the result for the monomial with degree $k$ in $\mathbf{p}$. Hence we may assume w.l.o.g. that $p(i) = i^k$. We have: $\mathbf{u} \star \mathbf{p}(n) = \sum_{i \in \mathcal{A}_a} u(i)(n-i)^k$. Now,

$$\begin{array}{rcl} \sum_{i \in \mathcal{A}_a} u(i)(n-i)^k & = & \sum_{i \in \mathcal{A}_a} u(i) \sum_{l=0}^{k} \binom{k}{l} n^l (-i)^{k-l} \\ & = & \sum_{l=0}^{k} \binom{k}{l} \left( \sum_{i \in \mathcal{A}_a} (-i)^{k-l} u(i) \right) n^l \end{array}$$

Due to Lemma 4.3, if we set $\pi(n) = \sum_{l=0}^{k} \binom{k}{l} \left( \sum_{i \in \mathcal{A}_a} (-i)^{k-l} u(i) \right) n^l$, the value $\pi(n)$ is well defined. The function $\pi$ thus defined is a polynomial function of $n$, and we have $\mathbf{u} \star \mathbf{p} \preceq \pi$. $\square$

**Lemma 4.5** *Let* $\mathbf{u} = \bigotimes_{a=1}^{d} \mathbf{u}_i$ *be a rapidly decreasing function in* $\mathcal{M}'^{\mathcal{Z}_d}$*, with* $a \in \{1, \dots, d\}$*. Let* $\mathbf{p}$ *be a polynomial with degree* $\delta \in \mathbb{N}$ *over a sub-algebra* $\mathcal{I}$ *of* $\mathcal{M}$ *containing* $\mathcal{Z}_d$*. Then, there exists a polynomial function* $\pi$ *over* $\mathcal{I}$ *with degree* $\delta$ *over* $\mathcal{Z}_d$ *such that:*

$$\mathbf{u} \star \mathbf{p} = \pi$$

*Proof.* Follows directly from Proposition 4.3, Remark 4.8, and Lemma 4.4. $\square$

**Remark 4.8** *Let* $\mathbf{u} = \bigotimes_{a=1}^{d} \mathbf{u}_a \in \mathcal{M}'^{\mathcal{Z}_d}$ *be a tensor product of non identically zero sequences. Then,* $\mathbf{u}$ *is rapidly decreasing if and only if* $\mathbf{u}_a$*'s is rapidly decreasing for each* $a \in \{1, \dots, d\}$*.*

*Proof.* The "if part" is an immediate consequence of Lemma 4.5. The "only if" part is easily proved by distinguishing between the case when $\mathbf{u}$ is identically zero, in which case the result is obvious, and the case when $\mathbf{u}$ is not identically zero, in which case a restriction of $\mathbf{u}$ to a subset of $\mathcal{Z}_d$ where only one coordinate varies, which is proportional to $\mathbf{u}_a$, is seen to be rapidly decreasing. $\square$

Similarly, we see:

**Remark 4.9** *Let* $\mathbf{u} = \bigotimes_{a=1}^{d} \mathbf{u}_a \in \mathcal{M}'^{\mathcal{Z}_d}$ *be a tensor product of non identically zero sequences. Then,* $\mathbf{u}$ *is moderately increasing if and only if* $\mathbf{u}_a$*'s is moderately increasing for each* $a \in \{1, \dots, d\}$*.*

Hence we have the following:

**Proposition 4.7** *Let* $\mathbf{u} = \bigotimes_{a=1}^{d} \mathbf{u}_a$ *be a rapidly decreasing multi-sequence in* $\mathcal{M}'^{\mathcal{Z}_d}$*, with* $a \in \{1, \dots, d\}$*. The convolution product* $\mathbf{u}$ *with a moderately increasing function over a sub-algebra* $\mathcal{I}$ *of* $\mathcal{M}$ *containing* $\mathcal{Z}_d$ *is always defined and is moderately increasing on* $\mathcal{I}$*.*

# 5 Multigrid Convergence for Differentials

## 5.1 Taylor Formula With Multiple Integral Remainder

The purpose of this section is to provide upper bounds for the difference between a digital derivative of a sampled (and quantized) signal, with possible errors on the values. We shall need a specific form of the Taylor Formula, in which we have an explicit form for the remainder, as in the integral form for the remainder. However, the formula we prove and use does not require that all partial derivatives of a given order be available or bounded. Instead, we assume that partial derivatives exist at different orders on the different variables, as, for example, in the tensor product of a $C^2$ function by a $C^1$ function, for which the differential of order $(2,1)$ exists and is continuous, but neither the differential of order $(1,2)$, nor the differential of order $(2,2)$ exist in general.

## 5.2 Notations

**Definition 5.1** *Let $x^{(1)} \preceq x^{(2)}$ be two element of $\mathcal{M}$. We denote by $[x^{(1)}, x^{(2)}[$ the interval for the broad strict order, which is defined as set of all $T \in \mathcal{M}$ such that $x^{(1)} \preceq_{\mathcal{M}} T <_{\mathcal{M}} x^{(2)}$ (i.e. each cordinate $T_a$ of $T$ is greater than or equal to $x^{(1)}_a$ and (broadly strictly) less than $x^{(2)}_a$).*

*Let $X \in \{1, \dots, d\}$ be a set of indices. We denote $\overline{X} = \{1, \dots, d\} \backslash X$ the complement of $X$. We consider the following subsets of $\mathcal{M}$:*

$$\mathcal{M}_X = \prod_{a \in X} \mathcal{A}_a \text{ and } \mathcal{M}_{\overline{X}} = \prod_{a \in \overline{X}} \mathcal{A}_a$$

$$\mathcal{C}_X(x^{(1)}, x^{(2)}) = \prod_{a \in X} [x^{(1)}_a, x^{(2)}_a[ \text{ and } \mathcal{C}_{\overline{X}}(x^{(1)}, x^{(2)}) = \prod_{a \in \overline{X}} [x^{(1)}_a, x^{(2)}_a[$$

*We have a clear identification through a natural isomorphism: $Id_X : \mathcal{M}_X \times \mathcal{M}_{\overline{X}} \longrightarrow \mathcal{M}$. We denote by $T_X$ and [respectively $T_{\overline{X}}$] the projections of an element $T \in \mathcal{M}$ onto $\mathcal{M}_X$ [respectively $\mathcal{M}_{\overline{X}}$]. In that way, a function $f : [x^{(1)}, x^{(2)}[ \longmapsto \mathcal{M}'$ can also be identified to a function*

$$f_X : \begin{cases} \mathcal{C}_X(x^{(1)}, x^{(2)}) \times \mathcal{C}_{\overline{X}}(x^{(1)}, x^{(2)}) & \longmapsto \quad \mathcal{M}' \\ (T, U) & \longrightarrow \quad f(Id_X(T, U)) \quad = f(T + U) \end{cases}$$

*The sets $\mathcal{C}_X(x^{(1)}, x^{(2)})$ [respectively $\mathcal{C}_{\overline{X}}(x^{(1)}, x^{(2)})$] is called the $X-$face of the cube $[x^{(1)}, x^{(2)}[$ [respectively the $\overline{X}-$face of the cube $[x^{(1)}, x^{(2)}[$ ].*

*For each $a \in \{1, \dots, d\}$, we consider $dt_a$ the measure on $\mathcal{A}_a$ underlying the analyzable space structure. We consider $dT_X = \bigotimes_{a=1}^{d} dt_a$ the product measure on $\mathcal{M}_X$. At last, we consider the operator*

$$Int_X : \begin{cases} \mathfrak{L}_1([x^{(1)}_a, x^{(2)}_a[, \mathcal{M}') & \longmapsto \quad \mathfrak{L}_1(C_{\overline{X}}, \mathcal{M}') \\ f & \longmapsto \quad Int_X(f) \end{cases}$$

*with, for $f \in \mathfrak{L}_1([x^{(1)}, x^{(2)}[, \mathcal{M}')$ and $U \in C_{\overline{X}}$,*

$$(Int_X(f))(U) = \int_{\mathcal{C}_X(x^{(1)}, x^{(2)})} f(T_X + U) dT_X$$

*The function $Int_X(f)$ is called the partial integral of $f$ over the $X-$slices of the cube $[x^{(1)}, x^{(2)}[$. By convention, if $X = \emptyset$, the integral $\int_{\mathcal{C}_X(x^{(1)}, x^{(2)})} f_X(T_X, U) dT_X$ is defined equal to $f(U)$, so that $Int_\emptyset(f) = f$.*

**Notation 5.1** *Let $X \subset \{1,\dots,d\}$. We denote by $\mathbb{1}_X \in \mathbb{N}^d$ the vector such that for $a = 1,\dots,d$, the coordinate $(\mathbb{1}_X)_a$ is equal to $1_\mathbb{N}$ if $a \in X$, and is equal to $0_\mathbb{N}$ otherwise.*

## 5.3 Taylor Formula with Multiple Integral Remainder in $\mathbb{R}^d$

We assume in this section that $\mathcal{A}_a = \mathbb{R}$ for $a = 1,\dots,d$, so that $\mathcal{M} = \mathbb{R}^d$.

**Theorem 5.1 (Purely Continuous Taylor Formula with Multiple Integral Remainder)** *Let $f : \mathcal{M} \longrightarrow \mathcal{M}'$ be a map and let $\omega \in \mathbb{N}^d$. We assume that the partial differentials $f^{(J)}$ of the map $f$ exist and are continuous for all $J \in \mathbb{N}^d$ with $0 \preceq J \preceq \omega + 1_{\mathbb{N}^d}$. Then, using Notation 4.1, we have the following identity, for $x$ and $x^{(0)}$ in $\mathcal{M}$:*

$$f(x) = \sum_{X\subset\{1,\dots,d\}} \sum_{J\in\{\omega_X\}\times[0,\omega_{\overline{X}}]} \int_{\mathcal{C}_X(x^{(0)},x)} f^{(J+\mathbb{1}_X)}\left(T_X + x^{(0)}_{\overline{X}}\right) \frac{\left(x - x^{(0)}_{\overline{X}} - T_X\right)^J}{J!} dT_X \qquad (10)$$

*where, in accordance with Definition 5.1, the set $\{\omega_X\}\times[0,\omega_{\overline{X}}]$ denotes the set of all $J \in \mathbb{N}^d$ such that $0 \preceq J \preceq \omega$ and such that $J_a = \omega_a$ for all $a \in X$. This identity is called the Purely Continuous Taylor formula with multiple integral form for the remainder.*

*Proof.* We prove the result by induction on $d$. For $d = 1$, there are two possible subsets $X \subset \{1,\dots,d\}$: $X = \emptyset$ and $X = \{1\}$.
*The term for $X = \emptyset$* yields

$$\sum_{J\in[0,(\omega_1)]} f^{(J+\mathbb{1}_\emptyset)}\left(x^{(0)}_{\{1\}}\right) \frac{\left(x - x^{(0)}_{\{1\}}\right)^J}{J!} = \sum_{j=0}^{\omega_1} f^{(j)}(x^{(0)})\frac{(x-x^{(0)})^j}{j!}$$

*The term for $X = \{1\}$* yields

$$\sum_{J\in\{\omega_1\}} \int_{\mathcal{C}_{\{1\}}(x^{(0)},x)} f^{(J+\mathbb{1}_{\{1\}})}\left(T_{\{1\}}\right) \frac{\left(x - T_{\{1\}})\right)^J}{J!} dT_{\{1\}} = \int_{x^{(0)}}^{x} f^{(\omega_1+1)}(T)\frac{(x-T)^{\omega_1}}{\omega_1!}dT$$

Hence Equation (10) corresponds for $d = 1$ to the usual Taylor Theorem with Integral Remainder in $1D$

$$f(x) = \sum_{j=0}^{\omega_1} f^{(j)}(x^{(0)})\frac{(x-x^{(0)})^j}{j!} + \int_{x^{(0)}}^{x} f^{(\omega_1+1)}(T)\frac{(x-T)^{\omega_1}}{\omega_1!}dT$$

which is proved as usual.

Now, assume that the result is true in dimension $d-1$, with $d \geq 2$. We consider the element $x_{\{1,\dots,d-1\}} + x^{(0)}_{\{d\}}$ of $\mathcal{M}$, all coordinates of which are equal to those of $x$, except the $d^{th}$ coordinate which is equal to $x^{(0)}_d$

From the $1D$ case, dealt with above, applied to the value of $f(x) = f\left(x_{\{1,\dots,d-1\}} + x_{\{d\}}\right)$ expressed through the Taylor development of $f$ at the point $x_{\{1,\dots,d-1\}} + x^{(0)}_{\{d\}}$, we get:

$$\begin{aligned} f\left(x_{\{1,\dots,d-1\}} + x_{\{d\}}\right) &= \sum_{j=0}^{\omega_d} f^{(j\mathbb{1}_{\{d\}})}\left(x_{\{1,\dots,d-1\}} + x^{(0)}_{\{d\}}\right) \frac{\left(x_{\{1,\dots,d-1\}} + x_{\{d\}} - x^{(0)}_{\{d\}}\right)^{j\mathbb{1}_{\{d\}}}}{j!} \\ &\quad + \int_{x^{(0)}_{\{d\}}}^{x_{\{d\}}} f^{((\omega_d+1)\mathbb{1}_{\{d\}})}(x_{\{1,\dots,d-1\}} + T_{\{d\}}) \frac{\left(x_{\{1,\dots,d-1\}} + x_{\{d\}} - T_{\{d\}}\right)^{\omega_d\mathbb{1}_{\{d\}}}}{\omega_d!} dT_{\{d\}} \end{aligned}$$

From our induction hypothesis applied, for $j = 1, \dots, \omega_d + 1$, to the function

$$g_j : \left\{ \begin{array}{ccc} \mathcal{M}_{\{1,\dots,d-1\}} & \longrightarrow & \mathcal{M}' \\ y & \longmapsto & f^{(j\mathbb{1}_{\{d\}})}\left(y + x^{(0)}_{\{d\}}\right) \end{array} \right.$$

we have:

$$\begin{array}{rcl} f^{(j\mathbb{1}_{\{d\}})}\left(x_{\{1,\dots,d-1\}} + x^{(0)}_{\{d\}}\right) & = & \displaystyle\sum_{X\subset\{1,\dots,d-1\}} \sum_{J\in\{\omega_X\}\times[0,\omega_{\overline{X}}]} \int_{\mathcal{C}_X\left(x^{(0)}_{\{1,\dots,d-1\}}, x_{\{1,\dots,d-1\}}\right)} \\ & & \displaystyle f^{(J+\mathbb{1}_X + j\mathbb{1}_{\{d\}})}\left(T_X + x^{(0)}_{\overline{X}} + x^{(0)}_{\{d\}}\right) \frac{\left(x_{\{1,\dots,d-1\}} - x^{(0)}_{\overline{X}} - T_X)\right)^J}{J!} dT_X \end{array}$$

Note that, as opposed to our statement in Equation (10), in the latest formula, $\overline{X}$ denotes the complement of $X$ in $\{1, \dots, d-1\}$, as it is an application of our induction hypothesis in dimension $d-1$. By substituting the latest expression for $f^{(j\mathbb{1}_{\{d\}})}\left(x_{\{1,\dots,d-1\}} + x^{(0)}_{\{d\}}\right)$ (substitution which is also valid, by changing $x^{(0)}_{\{d\}}$ for $T_{\{d\}}$, to evaluate $f^{((\omega_d+1)\mathbb{1}_{\{d\}})}\left(x_{\{1,\dots,d-1\}} + T_{\{d\}}\right)$), into the expression of $f(x) = f\left(x_{\{1,\dots,d-1\}} + x_{\{d\}}\right)$ above, we obtain:

$$\begin{array}{rcl} f(x) & = & \left[ \displaystyle\sum_{j=0}^{\omega_d} \sum_{X\subset\{1,\dots,d-1\}} \sum_{J\in\{\omega_X\}\times[0,\omega_{\overline{X}}]} \int_{\mathcal{C}_X\left(x^{(0)}_{\{1,\dots,d-1\}}, x_{\{1,\dots,d-1\}}\right)} f^{(J+\mathbb{1}_X + j\mathbb{1}_{\{d\}})}\left(T_X + x^{(0)}_{\overline{X}} + x^{(0)}_{\{d\}}\right) \right. \\ & & \left. \displaystyle\frac{\left(x_{\{1,\dots,d-1\}} - x^{(0)}_{\overline{X}} - T_X\right)^J}{J!} \frac{\left(x_{\{1,\dots,d-1\}} + x_{\{d\}} - x^{(0)}_{\{d\}}\right)^{j\mathbb{1}_{\{d\}}}}{j!} dT_X \right] \\ & & + \left[ \displaystyle\sum_{X\subset\{1,\dots,d\}, d\in X} \sum_{J\in\{\omega_X\}\times[0,\omega_{\overline{X}}]} \int_{\mathcal{C}_X\left(x^{(0)}, x_{\{1,\dots,d-1\}} + x^{(0)}_{\{d\}}\right)} \int_{x^{(0)}_d}^{x_d} \right. \\ & & \displaystyle f^{(J+\mathbb{1}_X + (\omega_d+1)\mathbb{1}_{\{d\}})}\left(T_{\{d\}} + T_X + x^{(0)}_{\overline{X}} + x^{(0)}_{\{d\}}\right) \frac{\left(x_{\{1,\dots,d-1\}} - x^{(0)}_{\overline{X}} - T_X)\right)^J}{J!} \\ & & \left. \displaystyle\frac{\left(x_{\{1,\dots,d-1\}} + x_{\{d\}} - T_{\{d\}}\right)^{\omega_d\mathbb{1}_{\{d\}}}}{\omega_d!} dT_{\{d\}} dT_X \right] \\ & = & \displaystyle\sum_{X\subset\{1,\dots,d\}} \sum_{J\in\{\omega_X\}\times[0,\omega_{\overline{X}}]} \int_{\mathcal{C}_X(x^{(0)},x)} f^{(J+\mathbb{1}_X)}\left(T_X + x^{(0)}_{\overline{X}}\right) \frac{\left(x - x^{(0)}_{\overline{X}} - T_X)\right)^J}{J!} dT_X \end{array}$$

In the latest expression, $\overline{X}$ now denotes the complement of $X$ in $\{1, \dots, d\}$, and not in $\{1, \dots, d-1\}$ as previously. □

## 5.4 Digitization, Quantization, Noise Models

In the sequel of this section, we consider a map $f : \mathcal{M} \longmapsto \mathcal{M}'$, and a map $\Gamma : \mathcal{Z}_d \longmapsto \mathcal{M}'$ on the discrete network $\mathcal{Z}_d$. We shall make extensive use of Notation 4.1 for exponentiation notations, as well as orders in $\mathcal{M}$ and $\mathcal{M}'$.

Let $h = (h_a)_{a\in\{1,\dots,d\}} \in \mathcal{M}$, with $0_{\mathcal{A}_a} < h_a$, be a strictly positive vector representing some digitization step in the domain of $f$. For $K = (k_a)_{a=1,\dots,d} \in \mathbb{R}^d$, we consider the element $h^{[K]} \in \mathcal{M}$. As in the definition of monomials (Definition 4.10) let $(h')^{[K]}_a \in \mathcal{M}'$ the image of

$h_a^{[K]}$ by the unique morphism of algebra sending $1_{\mathcal{A}_a}$ to $1_{\mathcal{M}'}$. Since $h_a > 0_{\mathcal{A}_a}$, we also have $(h')_a^{[K]} > 0_{\mathcal{M}'}$. At last, we denote $h' = (h')^{[1]}$, corresponding to the case when $k_a = 1$ for all $a \in \{1, \dots, d\}$.

By abuse, we shall write $h$ instead of $h'$ in some formulas, having in mind that, when considered as an element of $\mathcal{M}'$, a monomial function has been applied to the element $h \in \mathcal{M}$.

**Definition 5.2** *We say that the map $\Gamma$ is a digitization of $f$ with error $\varepsilon_{h,h'} : \mathcal{Z}_d \longrightarrow \mathcal{M}'$ if for any $N \in \mathcal{Z}_d$, setting as usual $(N.h)(a) = N(a)h_a$, and considering the element $(h'\Gamma(N)) = \left(\prod_{a=1}^d h'_a\right)(\Gamma(N))$ of $\mathcal{M}'$, we have:*

$$h'\Gamma(N) = f(N.h) + \varepsilon_{h,h'}(N) \tag{11}$$

**Definition 5.3 (Vector Valued Infinite Norm for Functions)** *Let $X \subset \mathcal{M}$ and let $g : X \longrightarrow \mathcal{M}'$ be a bounded function. The infinte norm of $g$, denoted by $\|g\|_\infty$ the vector in $\mathcal{M}'$ is defined as follows. For $a' \in \{1, \dots, d'\}$, we denote $N_{a'} = \sup_{x \in X} (|(g(x))_{a'}|)$ the upper bound of the $(a')^{th}$ coordinate of $g(x)$ in $\mathcal{M}'$. Now, we set*

$$\|g\|_\infty = (N_{a'})_{a'=1,\dots,d'}$$

We consider the following particular models for the errors $\varepsilon_{h,h'}$ on the values:

- *Exact Values:* In this model, the values are known exactly:

  $$\varepsilon_{h,h'} \equiv 0_{M'}$$

  Note that, although this model has been the most widely used in approximation theory, this value error model is not very realistic from an Information Sciences point of view.

- *Uniform Noise (or Uniform Bias) on Values:* In this model, the error $\varepsilon_{h,h'}$ on the values is uniformly bounded by some constant which depends on the quantization step $h'$. In our model, however, this bound can be asymptotically greater that $h'$. Namely we assume here that (see Notation 4.1 for the coordinates by coordinates exponentiation, denoted with brackets notation)

  $$0 \preceq |\varepsilon_{h,h'}(I)| \preceq K(h')^{[\alpha]}$$

  where $\alpha \in \mathbb{R}^d$ with $0 < \alpha_a \preceq 1$ for all $a \in \{1, \dots, d\}$, and $K$ is a positive constant. Note that this error can also have some bias, in the sense that the average noise value (or expected value) could be non-zero.

- *Quantization of Values:* In this model, the errors $\varepsilon_{h,h'}$ on the values is uniformly bounded by $\frac{1}{2}h'$. This is a particular case of uniform noise with $\alpha = 1$, and corresponds to the case when some basic quantization has been obtained by rounding-off the exactly known values of the function, for example for digital storage. This case is equivalent to $\Gamma(I) = \left[\frac{f(Ih)}{h'}\right]$. A variant is when quantization has been obtained by an integer part (floor case): $0 \preceq \varepsilon_{h,h'}(I) < h'$ , which is equivalent to $\Gamma(I) = \lfloor \frac{f(Ih)}{h'} \rfloor$.

- *Stochastic Noise on Values*: In this model, the errors $\varepsilon_{h,h'}(I)$ on the different values for $I \in \mathcal{Z}_d$ are independent random variables with expected value 0 and standard deviation $\sigma(h')$, converging to 0 along with $h'$. In that case, Equation 11 implies that the values $\Gamma(I)$, for $I \in \mathcal{Z}_d$ also are defined as independent random variables.

## 5.5 Basic Error Decomposition and Upper Bounds

### 5.5.1 Errors Related to Sampling and to Input Values

In order to show that the digital $\omega-$differentiation of a digitization $\Gamma$ of a real function $f$ provides an estimate for the continuous derivative $f^{(\omega)}$ of $f$, we would like to evaluate, at each sample point $N \in \mathcal{Z}_d$, the difference between the digital differentiation $\frac{1}{(h')^{[\omega-1]}}(\Delta_{\mathbf{u}}^{\omega} \star \Gamma)(N)$ (where, as usual in this context, the product [resp. exponentiation] between two $d-$dimensional vectors is a coordinate by coordinate product [resp. exponentiation]) of the digitized signal and the value of the usual $\omega^{th}$ partial derivative $f^{(\omega)}(Nh)$ of $f$. This difference may easily be decomposed from Equation (11) and Definition 4.4 into the sum

$$\frac{1}{(h')^{[\omega-1]}}(\Delta_{\mathbf{u}}^{\omega} \star \Gamma)(N) - f^{(\omega)}(Nh) = ES_{\omega}(f, h, h', \Gamma, \mathbf{u}, N) + EV_{\omega}(f, h, h', \Gamma, \mathbf{u}, N) \tag{12}$$

where

$$ES_{\omega}(f, h, h', \Gamma, \mathbf{u}, N) = \left( \frac{1}{(h')^{[\omega]}} \sum_{I \in \mathcal{Z}_d} u(I) f\left((N-I)h\right) \right) - f^{(\omega)}(Nh) \tag{13}$$

is called the *sampling error*, and

$$EV_{\omega}(f, h, h', \Gamma, \mathbf{u}, N) = \frac{1}{(h')^{[\omega]}} \sum_{I \in \mathcal{Z}_d} u(I) \varepsilon_{h,h'}(N-I) \tag{14}$$

is called the *(input) values error*. As their names imply, the sampling error is due to the fact that we only know about the values of $f$ at some grid points, and the values error is due to the fact that we do not know the exact values of $f$ at sample points.

The sampling error is a real values sequence. Under the uniform bias hypothesis, the values error is also a real valued sequence, but under the stochastic hypothesis, the values error is a sequence of random variable.

### 5.5.2 Upper Bound for the Sampling Error

In the following lemma, we show that the sampling error can be bounded independently from the error on input values, using the mask values, the norm of the partial derivatives of $f$ with order higher than $\omega^{th}$, and a the digitization step. The immediate consequences are some convergence results in the case when exact values of the function at sample points are known.

**Lemma 5.1** *Let us assume that the partial derivative $f^{(K)}$ exists and is continuous on $\mathcal{M}$, for every $K = (k_a)_{a=1,\dots,d} \in \mathbb{N}^d$ with $k_a \geq 1 + \omega_a$ for $a = 1, \dots, d$. Let $\mathbf{u}$ be a digital $\omega-$differentiation mask with convergence order $\rho$. Let $S = (s_1, \dots, s_d)$ with $s_a = max\{\omega_a, 1 + \rho_a\}$ for $a = 1, \dots, d$. Let $\Gamma$ be a digitization of $f$ with error $\varepsilon_{h,h'} : \mathcal{Z}_d \longrightarrow \mathcal{M}'$. Suppose that $f^{(s)}$ is bounded on $\mathbb{R}$. Then for all $N \in \mathcal{Z}_d$,*

$$ES_{\omega}(f, h, h', \Gamma, \mathbf{u}, N) \preceq \sum_{X \subset \{1,\dots,d\}, X \neq \emptyset} ||f^{(\omega + \mathbb{1}_X)}||_{\infty} \sum_{I \in (\mathcal{Z}_d)_X} |I^{\omega_X + \mathbb{1}_X} u(I_X)| \frac{h^{\mathbb{1}_X}}{\omega_X!} \tag{15}$$

*Moreover, if we consider a lowest order approximation when all coordinates of $h$ tend to zero at the same speed (e.g. constant ratio), the error can be approximated by the sum for $X$ with cardinality 1, which yields:*

$$ES_{\omega}(f, h, h', \Gamma, \mathbf{u}, N) = O\left( \sum_{a=1}^{d} h_a \, ||f^{(\omega + \mathbb{1}_{\{a\}})}||_{\infty} \sum_{I \in \mathcal{A}_a} |I^{\omega_a + 1} u(I_X)| \right) \tag{16}$$

*Proof.* From the Taylor formula with Integral Remainder (see Theorem 5.1, the sum involved in Equation (13) can be written

$$\begin{array}{rcl}\sum_{I\in\mathcal{Z}_d} u(I) f\left((N-I)h\right) & = & \sum_{I\in\mathcal{Z}_d} u(I) \Big[ \sum_{X\subset\{1,\dots,d\}} \sum_{J\in\{\omega_X\}\times[0,\omega_{\overline{X}}]} \int_{\mathcal{C}_X(Nh,(N-I)h)} \\ & & f^{(J+\mathbb{1}_X)}\left(T_X+(Nh)_{\overline{X}}\right) \frac{\left((N-I)h-(Nh)_{\overline{X}}-T_X)\right)^J}{J!} dT_X \Big]\end{array}$$

Now, for $X\subset\{1,\dots,d\}$ and $J\in\{\omega_X\}\times[0,\omega_{\overline{X}}$, we have

$$\begin{array}{rcl}\frac{\left((N-I)h-(Nh)_{\overline{X}}-T_X)\right)^J}{J!} & = & \frac{(((N-I)h)_X-T_X))^{J_X}}{J_X!}\frac{\left(((N-I)h)_{\overline{X}}-(Nh)_{\overline{X}})\right)^{J_{\overline{X}}}}{J_{\overline{X}}!} \\ & = & \frac{(((N-I)h)_X-T_X))^{J_X}}{J_X!}\frac{\left(-I_{\overline{X}}\right)^{J_{\overline{X}}} h_{\overline{X}}^{J_{\overline{X}}}}{J_{\overline{X}}!}\end{array}$$

Hence

$$\begin{array}{rcl}\sum_{I\in\mathcal{Z}_d} u(I) f\left((N-I)h\right) & = & \sum_{X\subset\{1,\dots,d\}} \sum_{J\in\{\omega_X\}\times[0,\omega_{\overline{X}}]} h_{\overline{X}}^{J_{\overline{X}}} \Big[\sum_{I\in\mathcal{Z}_d} u(I)\frac{\left(-I_{\overline{X}}\right)^{J_{\overline{X}}}}{J_{\overline{X}}!} \\ & & \int_{\mathcal{C}_X(Nh,(N-I)h)} f^{(J+\mathbb{1}_X)}\left(T_X+(Nh)_{\overline{X}}\right)\frac{(((N-I)h)_X-T_X)^{J_X}}{J_X!}dT_X\Big]\end{array}$$

Yet, since $u$ is a tensor product due to Theorem 4.1, for $X\subset\{1,\dots,d\}$, we have $u(I)=u(I_X)u(I_{\overline{X}})$, where $I_X(a)=I(a)$ if $a\in X$ and $I_X(a)=1_{\mathcal{A}_a}$ otherwise (and similarly for $I_{\overline{X}}$). Furthermore, due to $I_X\mapsto u(I_X)$ is an $\omega_X-$differentiation mask, and $I_{\overline{X}}\mapsto u(I_{\overline{X}})$ is an $\omega_{\overline{X}}-$differentiation mask. Therefore,

$$\begin{array}{rcl}\sum_{I\in\mathcal{Z}_d} u(I) f\left((N-I)h\right) & = & \sum_{X\subset\{1,\dots,d\}} \sum_{J\in\{\omega_X\}\times[0,\omega_{\overline{X}}]} h_{\overline{X}}^{J_{\overline{X}}} \Big[\sum_{I\in\mathcal{Z}_d} u(I_X)u(I_{\overline{X}})\frac{\left(-I_{\overline{X}}\right)^{J_{\overline{X}}}}{J_{\overline{X}}!} \\ & & \int_{\mathcal{C}_X(Nh,(N-I)h)} f^{(J+\mathbb{1}_X)}\left(T_X+(Nh)_{\overline{X}}\right)\frac{(((N-I)h)_X-T_X)^{J_X}}{J_X!}dT_X\Big] \\ & = & \sum_{X\subset\{1,\dots,d\}} \sum_{J\in\{\omega_X\}\times[0,\omega_{\overline{X}}]} h_{\overline{X}}^{J_{\overline{X}}} \Big[\sum_{I\in(\mathcal{Z}_d)_{\overline{X}}} u(I_{\overline{X}})\frac{\left(-I_{\overline{X}}\right)^{J_{\overline{X}}}}{J_{\overline{X}}!}\Big] \\ & & \Big[\sum_{I\in(\mathcal{Z}_d)_X} u(I_X)\int_{\mathcal{C}_X(Nh,(N-I)h)} f^{(J+\mathbb{1}_X)}\left(T_X+(Nh)_{\overline{X}}\right) \\ & & \frac{(((N-I)h)_X-T_X)^{J_X}}{J_X!}dT_X\Big] \\ & = & \sum_{X\subset\{1,\dots,d\}} h\overline{X}^{\omega_{\overline{X}}} \sum_{I\in(\mathcal{Z}_d)_X} u(I_X) \\ & & \int_{\mathcal{C}_X(Nh,(N-I)h)} f^{(\omega+\mathbb{1}_X)}\left(T_X+(Nh)_{\overline{X}}\right)\frac{(((N-I)h)_X-T_X)^{\omega_X}}{\omega_X!}dT_X\end{array}$$

The last equality comes from the fact that, due to the fact that $I_{\overline{X}}\mapsto u(I_{\overline{X}})$ is an $\omega_{\overline{X}}-$differentiation mask (hence satisfies Equation (4) and Equation (5), all terms of the sums over $I\in(\mathcal{Z}_d)_{\overline{X}}$ between brackets are zero except for the term with $J_{\overline{X}}=\omega_{\overline{X}}$, form which Equation (5) holds. Finally,

Now, using the expression for the sampling error (Equation 13), the term of the latest sum corresponding to $X=\emptyset$ cancels out with $-f^{(\omega)}(Nh)$, and we provide an upper bound for the remaining sum for $X\neq\emptyset$:

$$ES_\omega(f,h,h',\Gamma,\mathbf{u},N)\preceq \sum_{X\subset\{1,\dots,d\},X\neq\emptyset} ||f^{(\omega+\mathbb{1}_X)}||_\infty \sum_{I\in(\mathcal{Z}_d)_X} |I^{\omega_X+\mathbb{1}_X}u(I_X)|\,\frac{h_X^{\omega_X+\mathbb{1}_X}h_{\overline{X}}^{(h')^\omega\omega_{\overline{X}}}}{\omega_X!}$$

from which the result follows by simplification by $(h')^\omega$. □

**Remark 5.1** *Lemma 5.1 shows that the sampling error tends to zero along with h for a fixed function and a fixed differentiation mask.*

### 5.5.3 Upper Bound for the Input Values Error

The following lemma gives an upper bound for the error related to uniform noise or uniform bias on the values at sample points (see Section 5.4).

**Lemma 5.2** *Let* $\mathbf{u}$ *be a digital* $\omega-$*differentiation mask with convergence order* $\rho$*. Let us assume that* $\Gamma$ *is a digitization of* $f$ *with errors on input values* $\varepsilon_{h,h'}$ *such that* $\|\varepsilon_{h,h'}\|_\infty \preceq K(h')^{[\alpha]}$ *with* $0 < \alpha_a \preceq 1$ *for* $a = 1, \ldots, d$*, which satisfies the uniform noise/bias error model. Then, for all* $N \in \mathcal{Z}_d$*,*

$$|EV_\omega(f, h, h', \Gamma, \mathbf{u}, N)| \preceq \frac{K}{(h')^{[\omega-\alpha]}} \left( \sum_{I \in \mathcal{Z}_d} |u(I)| \right)$$

*Proof.* We derive an upper bound for the values error from its expression in Equation (14):

$$|EV_\omega(f, h, h', \Gamma, \mathbf{u}, N)| \preceq \frac{\|\varepsilon_{h,h'}\|_\infty}{(h')^{[\omega]}} \sum_{I \in \mathcal{Z}_d} |u(I)| \preceq \frac{K}{(h')^{[\omega-\alpha]}} \left( \sum_{I \in \mathcal{Z}_d} |u(I)| \right)$$

□

The following lemma gives an upper bound for the error related to statistic noise with expected values 0 on the values at sample points.

**Lemma 5.3** *Let* $\mathbf{u}$ *be a digital* $\omega-$*differentiation mask. Assume that* $\Gamma$ *is a digitization of* $f$ *with error on input values* $\varepsilon_{h,h'}$ *following the stochastic noise model. In other words, the* $\varepsilon_{h,h'}(N)$*'s for all* $N \in \mathcal{Z}_d$ *are independent random variable with expected value* 0 *and standard deviation* $\sigma(h, h')$*.*

*Then for all* $N \in \mathcal{Z}_d$*, the random variable* $\frac{1}{(h')^{[\omega-1]}}(\Delta_{\mathbf{u}}^\omega \star \Gamma)(N) - f^{(\omega)}(Nh)$*, defined after the independent random variables* $\Gamma(N)$*, has expected value* $ES_\omega(f, h, h', \Gamma, \mathbf{u}, n)$ *and standard deviation* $\frac{\sigma(h,h')}{(h')^{[\omega]}} \left( \sum_{I \in \mathcal{Z}_d} (u(I))^2 \right)^{\frac{1}{2}}$*.*

In other words, and roughly speaking, the global error is in this case statistically close to the sampling error.

*Proof.* From Equation (12) and Equation (14), for a fixed $N \in \mathcal{Z}_d$, the random variable $\frac{1}{(h')^{[\omega-1]}}(\Delta_{\mathbf{u}}^\omega \star \Gamma)(N) - f^{(\omega)}(Nh)$ is equal to the sum of the constant random variable $ES_\omega(f, h, h', \Gamma, \mathbf{u}, N)$ and the random variable defined by

$$EV_\omega(f, h, h', \Gamma, \mathbf{u}, N) = \frac{1}{(h')^{[\omega]}} \sum_{I \in \mathcal{Z}_d} u_I \varepsilon_{h,h'}(N - I).$$

By linearity of expected values, its expected value is $ES_\omega(f, h, h', \Gamma, \mathbf{u}, N)$, which shows the first part of the statement.

Since the random variables $\varepsilon_{h,h'}(N - I)$ are assumed to be independent, and the series $\sum_{I \in \mathcal{Z}_d} u_i$ is assumed to be absolutely convergent, the variance of $EV_\omega(f, h, h', \Gamma, \mathbf{u}, N)$ is equal to the sum for $I \in \mathcal{Z}_d$ of the variances of $\frac{u(I)}{(h')^{[\omega]}} \varepsilon_{h,h'}(N - I)$ which, for standard the deviation, yields $\left( \frac{|u(I)|}{(h')^{[\omega]}} \sigma(h, h') \right)^2$. □

**Remark 5.2** *Note that for a fixed mask, the values error (or its standard deviation) generally does not converge to zero when $h'$ converges to 0. We shall propose below a way to make it tend to zero by adapting the mask to the digitization step (see Theorem 5.2 and Theorem 5.3 below).*

## 5.6 Skipping Masks: Cheap Multigrid Convergence

The idea is to adapt the mask to the step of digitization, in order to get $\frac{1}{(h')^{[\omega-\alpha]}}\left(\sum_{I\in\mathcal{Z}_d}|u(I)|\right)$ converging to zero along with $h$. For limiting the complexity of computation, we set the number of non zero coefficients of the mask fixed.

**Definition 5.4** *Let $L=(l_a)_{a=1,\dots,d}\in\left(\mathcal{R}_+^*\right)^d$ be a vector with $d$ coordinates which are strictly positive elements of the base ring. We consider the following map:*

$$\begin{cases} \mathcal{M} & \longrightarrow & \mathcal{M}/L=\prod_{a=1}^d(\mathcal{A}_a/l_a) \\ (t_a)_{a=1,\dots,d} & \longmapsto & (t_a/l_a)_{a=1,\dots,d} \end{cases}$$

*Then, this map is an of analyzable spaces isomorphism, and is called called the division by $L$ operation.*

In the sequel of this section, $L=(l_a)_{a=1,\dots,d}\in(\mathcal{Z}_d)^d$ with each coordinate $l_a>0_{\mathcal{M}}$ and $l_a$ multiple of $1_{\mathcal{M}}$. We call the vector $L$ the *skipping step* for our masks.

**Definition 5.5 (Skipping Masks)** *Let $\mathbf{u}$ be an $\omega-$differentiation mask. The corresponding $\omega-$differentiation $L-$skipping mask $\mathbf{u}_L$ is defined by $\mathbf{u}_l(I)=\frac{1}{L^{[\omega]}}u(\frac{I}{L})$ if for all $a\in\{1,\dots,d\}$ the coordinate $l_a$ divides $I(a)$, and equal to 0 in all other cases.*

**Remark 5.3** *For $K\in\mathbb{N}^d$, we have*

$$\sum_{I\in\mathcal{Z}_d} I^K\mathbf{u}_L(I)=L^{[K-\omega]}\sum_{I\in\mathcal{Z}_d} I^K\mathbf{u}(I).$$

*Therefore, the mask $\mathbf{u}_L$ is an $\omega-$differentiation mask as well as $\mathbf{u}$.*

*We also have*

$$\sum_{I\in\mathcal{Z}_d}|\mathbf{u}_L(I)|=\frac{1}{L^{[\omega]}}\sum_{I\in\mathcal{Z}_d}|\mathbf{u}(I)|.$$

This allows a convenient choice of $L$, depending on $h$, which yields a values error which converges to zero, either using Lemma 5.2 or Lemma 5.3. This is formalized in the following theorems, which specify the skipping step $L(h)$ to use as a function of the sampling step.

### 5.6.1 Uniform Multigrid Convergence with Uniform Noise or Bias

**Theorem 5.2** *Let $\mathbf{u}$ be an $\omega-$differentiation mask with and $\mathbf{u}_L$ the corresponding $\omega-$differentiation $L-$skipping mask with skips of length L. Suppose that $f:\mathcal{M}\longrightarrow\mathcal{M}'$ is a $C^{\omega+1}$ (we remind the reader that $(\omega+1)_a=\omega_a+1$ for all $a$) function. This means that the partial derivatives $f^{(J)}$ exist and are continuous for all $0\preceq J\preceq\omega+1$, and $f^{(\omega+\mathbb{1}_X)}$ is bounded for any $X\in\{1,\dots,d\}$.*

*Let $\alpha \in ]0,1]^d$, $K \in \mathcal{R}_+^*$ and let $h$ and $h'$ be defined as at the beginning of Section 5.4. Suppose $\Gamma : \mathcal{Z}_d \longrightarrow \mathcal{Z}_d$ is such that $|h'\Gamma(I) - f(hI)| \preceq Kh^{[\alpha]}$ for all $I \in \mathcal{Z}_d$ (which corresponds to our uniform noise/bias input values errors model).*
*Then, using the skipping steps $L(h) = \left\lfloor h^{[-1+\frac{\omega\alpha}{\omega+1}]} \right\rfloor$, we have:*

$$\left| \left( \frac{1}{(h')^{[\omega-1]}} \Delta_{\mathbf{u}_{L(h)}} \star \Gamma \right)(N) - f^{(\omega)}(Nh) \right| \in O(h^{[\frac{\alpha}{\omega+1}]})$$

*Proof.* First, we give an upper bound for the values error. From Lemma 5.2 and definitions, we have $\left|EV(f,h,h',\Gamma,\mathbf{u}_{L(h)},n)\right| \preceq \frac{K}{(L(h))^{[\omega]}(h')^{[\omega-\alpha]}} \sum_{I\in\mathcal{Z}_d} |u(I)|$. If $L(h) = \left\lfloor h^{[-1+\frac{\alpha}{\omega+1}]} \right\rfloor$, it is easy to check that $\dfrac{1}{(L(h))^{[\omega]}(h')^{[\omega-\alpha]}} \preceq \dfrac{h^{[\alpha-\frac{\omega\alpha}{\omega+1}]}}{1-h^{[\omega-\frac{\omega\alpha}{\omega+1}]}}$, which is $O(h^{[\frac{\alpha}{\omega+1}]})$.

We now turn to the sampling error. Let us consider the upper bounds provided by Lemma 5.1. We could use Equation (15) for a more explicit bound for the error, but we chose for the sake of simplicity to use Equation (16) instead. Also using Remark 5.3 we get:

$$ES(f,h,h',\Gamma,\mathbf{u}_{l(h)},n) = O\left( \sum_{a=1}^{d} h_a \left(L(h)\right)_a^{(\omega_a+1)-\omega} \right)$$

Now, with $L(h) = \left\lfloor h^{[-1+\frac{\omega\alpha}{\omega+1}]} \right\rfloor$, we obtain $ES(f,h,h',\Gamma,\mathbf{u}_{l(h)},n) = O\left(h^{[\frac{\alpha}{\omega+1}]}\right)$ □

### 5.6.2 Stochastic Multigrid Convergence with Stochastic Noise

**Theorem 5.3** *Let $\mathbf{u}$ be a $\omega$-differentiation mask and let $\mathbf{u}_L$ be the corresponding $\omega-$differentiation $L-$skipping mask. Suppose that $f : \mathcal{M} \longrightarrow \mathcal{M}'$ is a $C^{\omega+1}$ function, and $f^{(\omega+\mathbb{1}_X)}$ is bounded for all $X \subset \{1,\ldots,d\}$ Let $\alpha \in ]0,1]^d$, let $K \in \mathcal{M}'$, with $K > 0_{\mathcal{M}'}$, and let $h$ and $h'$ be defined as at the beginning of Section 5.4. Let $\Gamma$ be a digitization of $f$ with step $h$ and a stochastic noise $\varepsilon_{h,h'}$ with expected value $0_{\mathcal{M}'}$, and standard deviation $\sigma(h,h') \preceq Kh^{[\alpha]}$.*

*Then for skipping steps $L(h) = \left\lfloor h^{[1-\frac{\alpha}{\omega+1}} \right\rfloor$, and for $N \in \mathcal{Z}_d$, the random variable*

$$\left( \frac{1}{(h')^{[\omega-1]}} \Delta_{\mathbf{u}_{L(h)}} \star \Gamma \right)(N) - f^{(\omega)}(Nh)$$

*has an expected value and a standard deviation which are $O(h^{[\frac{\alpha}{\omega+1}]})$.*

The proof is similar to that of Theorem 5.2, but using Lemma 5.3 instead of Lemma 5.2.

# 6 Locally Analytical Functions

All along this section, we consider again the notations $\mathcal{R}$, $\mathcal{A}_a$, $\mathcal{M}$ and $\mathcal{M}'$ defined in Section 4. Moreover, $\mathcal{I}$ denotes a sub-algebra of $\mathcal{M}$ containing $\mathcal{Z}_d$ (typically $\mathcal{I} = \mathcal{Z}_d$ or $\mathcal{I} = \mathcal{M}$). At last $\Delta$ is a differentiation operator on functions from $\mathcal{I}$ to $\mathcal{M}'$.

We shall also use the following notions and notations concerning shift in vectors and functions, as well as division by positive vectors:

**Definition 6.1** *Let $\Phi : \mathcal{I} \longrightarrow \mathcal{M}'$ be a function. Given $L = (l_a)_{a=1,\dots,d} \in \mathcal{R}^d$ a vector with $d$ coordinates which are elements of the base ring. We identify the vector $L$ with the element $(l_a.1_{\mathcal{A}_a})_{a=1,\dots,d}$ of $\mathcal{M}$. We thus define $\tau^L(\Phi)$ the $L-$shift of $\Phi$ which to $T \in \mathcal{I}$ associates*

$$\left(\tau^L(\Phi)\right)(T) = \Phi(T + L)$$

We remind the reader of Definition 5.4, in which the definition of the (coordinate by coordinate) division by a vector $L = (l_a)_{a=1,\dots,d} \in \left(\mathcal{R}_+^*\right)^d$ is presented. In the sequel of this section, $L = (l_a)_{a=1,\dots,d} \in \left(\mathcal{R}_+^*\right)^d$ denotes a vector with $d$ coordinates which are strictly positive elements of the base ring.

**Definition 6.2** *For $a \in 1, \dots, d$, let $\Delta_a$ be a differentiation operator over the analyzable space $\mathcal{A}_a$, with values in $\mathcal{M}'$. For any function $f : \mathcal{M} \longrightarrow \mathcal{M}'$, if for $T = (t_a)_{a=1,\dots,d}$ the function $f_{a,T} : \mathcal{A}_a \longrightarrow \mathcal{M}'$ which to $t \in \mathcal{A}_a$ associates $f(T^{(a,t-t_a)}$ is differentiable relatively to $\Delta_a$, we denote*

$$\frac{\partial}{\partial t_a}(T) = \left(\Delta_a\left(f_{a,T}\right)\right)(T)$$

*Moreover, this value is called the partial derivative of $f$ with respect to (the $a^{th}$ coordinate) $t_a$ at the point $T$.*

## 6.1 Definition of Differential $B-$Splines Families

**Definition 6.3** *Let us consider a family $\mathcal{D} = (D_{I,S,P,R,L})$ of functions from $\mathcal{I}/L$ to $\mathcal{M}'/L^R$, where, roughly speaking,*

- *$S = (s_a)_{a=1,\dots,d} \in \mathcal{Z}_d$ is a shift factor, through which the parameter $T = (t_a)_{a=1,\dots,d}$ of functions is translated.*
- *$L = (l_a)_{a=1,\dots,d} \in \left(\mathcal{R}_+^*\right)^d$ denotes a vector with $d$ coordinates which are strictly positive elements of the base ring, and determines a partition of $\mathcal{I}$ into intervals $[S.L, (S+1)L[$.*
- *$R = (r_a)_{a=1,\dots,d} \in \mathbb{N}^d$ is a blunder order, or smoothing order, which determines the regularity of elements of $\mathcal{D}$, as functions on $\mathcal{I}$.*
- *$P = (p_a)_{a=1,\dots,d} \in \mathbb{N}^d$ denotes the primitive order, which represents the number of times the primitive operator was applied, in the respective dimensions, relative to the differentiation operator $\Delta$, on the corresponding function with $P = 0$.*
- *$\delta_R = (\delta_{R,a})_{a=1,\dots,d}$ denotes the dimension of the space $\mathcal{D} = (D_{I,0,0,R,L})_{0 \preceq I \preceq \delta_R}$, of the functions in the family $\mathcal{D}$ for a fixed $L$. The index $L$ is omitted in the notation $\delta_R$ because, in the families we present in this paper, the dimension $\delta_R$ does not depend on $L$.*

*Now defining precisely, using Notation 4.1, we say that the family $\mathcal{D}$ is a Differential B−spline Family of functions with respect to $\Delta$ if and only if it satisfies the four following properties, valid for all $S \in \mathbb{Z}_d$, $P \in \mathbb{N}^d$, $R \in \mathbb{N}^d$ and any vector $L \in \left(\mathcal{R}_+^*\right)^d$ with $d$ coordinates which are strictly positive:*

1. *Differential Property: for $T = (t_a)_{a=1,\dots,d}$, we have*

$$\frac{\partial}{\partial t_a}(D_{I,S,P^{(a,1)},R,L})(T) = (p_a + 1)D_{I,S,P,R,L}(T)$$

2. *Commutation with Finite Differences Property:*

$$D_{I,S,P,R^{(a,1)},L} = \frac{1}{l_a(p_a+1)(r_a+1)}(D_{I,S,P^{(a,1)},R,L} - D_{I,S^{(a,1)},P^{(a,1)},R,L})$$

3. *Shift Property:*

$$D_{I,S^{(a,-1)},P,R,L} = \tau^{L(a,l_a)}\left(D_{I,S,P,R,L}\right)$$

4. *Partition of Unity Property: For all $T \in \mathcal{I}$ and for $P = 0$,*

$$\sum_{S\in\mathcal{Z}_d}\sum_{0\preceq I\preceq\delta_R} D_{I,S,0,R,L}(T) = \frac{1_{\mathcal{M}'}}{L^{[R]}}$$

## 6.2 Generic Construction from Partitions of Unity

**Definition 6.4** *A function $F : \mathcal{I} \longrightarrow \mathcal{M}'$ is said to be eventually zero when the coordinates tend to $-\infty$ if there exists $U \in \mathcal{I}$ such that $F(T) = 0_{\mathcal{M}'}$ for $T \preceq U$.*

Let us consider a family $\mathcal{D} = (D_{I,0,0,0,1})$ of functions from $\mathcal{I}$ to $\mathcal{M}'$ such that: For all $T \in \mathcal{I}$, we have the *partition of unity property*:

$$\sum_{S\in\mathcal{Z}_d}\sum_{0\preceq I\preceq\delta_O} D_{I,0,0,0,1}(T) = 1_{\mathcal{M}'}$$

We extend the family $\mathcal{D}$ to a complete family (also denoted by $\mathcal{D} = (D_{I,S,P,R,L})$) as follows.

1. For $L = (l_a)_{a=1,\dots,d} \in \left(\mathcal{R}_+^*\right)^d$ a vector with $d$ coordinates which are strictly positive elements of the base ring, we set:

$$D_{I,S,0,0,L}(T) = \frac{1}{L^{[R]}} D_{I,0,0,0,1}(\frac{T}{L} - S)$$

2. We define by induction on $P = (p_a)_{a=1,\dots,d} \in \mathbb{N}^d$ the function $D_{I,S,P,0,L}$, by setting for $T = (t_a)_{a=1,\dots,d}$:

$$D_{I,S,P^{(a,1)},0,L}(T) = \int_{-\infty}^{t_a} D_{I,S,P,0,L}\left(T^{(a,u-t_a)}\right)du$$

Note that the integral is well defined for a function which is eventually zero when the coordinates tend to $-\infty$. Furthermore, if $D_{I,S,0,0,L}$ is eventually zero when the coordinates tend to $-\infty$, then so is $D_{I,S,P,0,L}$ for any $P \in \mathbb{N}^d$.

3. At last we define by induction on $R = (r_a)_{a=1,\dots,d} \in \mathbb{N}^d$ the function $D_{I,S,P,R,L}$, by setting:

$$D_{I,S,P,R^{(a,1)},L} = \frac{1}{l_a(p_a+1)(r_a+1)}(D_{I,S,P^{(a,1)},R,L} - D_{I,S^{(a,1)},P^{(a,1)},R,L})$$

Then, we have the following result, which follows from the definition by a straightforward induction:

**Proposition 6.1** *The family $\mathcal{D} = (D_{I,S,P,R,L})$ is a Differential $B-$spline family.*

## 6.3 Generalized Cox-de-Boor Formula

**Theorem 6.1 (Generalized Cox-de-Boor Relation)** *Let $\mathcal{D} = (D_{I,S,P,R,L}))$ is be a differential $B-$spline family. For all $S \in \mathbb{Z}_d$, $R \in \mathbb{N}^d$, for any vector $L \in \left(\mathcal{R}_+^*\right)^d$ with $d$ coordinates which are strictly positive, for any $a \in \{1,\dots,d\}$ and for any $T \in \mathcal{I}$, we have:*

$$D_{I,S,0,R^{(a,1)},L}(T) = \frac{t_a - s_a}{l_a(r_a+1)} D_{I,S,0,R,L}(T) + \frac{s_a + r_a - t_a}{l_a(r_a+1)} D_{I,S^{(a,1)},0,R,L}(T)$$

*Proof.* We prove the result by induction on $R$. For $R = 0$ and any $P\mathbb{N}^d$, we have

$$\begin{array}{rcl}
D_{I,S,P,R^{(a,1)},L}(T) & = & \frac{1}{l_a(p_a+1)(r_a+1)}\left(D_{I,S,P^{(a,1)},R,L}(T) - D_{I,S^{(a,1)},P^{(a,1)},R,L}(T)\right) \\
& = & \frac{1}{l_a(r_a+1)}\left(\int_{-\infty}^{t_a} D_{I,S,P,R,L}\left(T^{(a,u_a-t_a)}\right)du_a - \int_{-\infty}^{t_a} D_{I,S^{(a,1)},P,R,L}\left(T^{(a,u_a-t_a)}\right)du_a\right) \\
& = & \frac{1}{l_a(r_a+1)}\left(\left[(u_a - s_a)D_{I,S,P,R,L}(T^{(a,u_a-t_a)})\right]_{u_a=-\infty}^{u_a=t_a}\right. \\
& & \left. \qquad - \left[(u_a - r_a - s_a)D_{I,S^{(a,1)},P,R,L}(T^{(a,u_a-t_a)})\right]_{u_a=-\infty}^{u_a=t_a}\right) \\
& & -\frac{1}{l_a(r_a+1)}\left(\int_{-\infty}^{t_a}(u_a - s_a)\frac{\partial}{\partial u_a}(D_{I,S,P,R,L})\left(T^{(a,u_a-t_a)}\right)du_a\right. \\
& & \left. \qquad - \int_{-\infty}^{t_a}(u_a - r_a - s_a)\frac{\partial}{\partial u_a}\left(D_{I,S^{(a,1)},P,R,L}\left(T^{(a,u_a-t_a)}\right)\right)du_a\right) \\
& = & \frac{t_a-s_a}{l_a(r_a+1)}D_{I,S,P,R,L}(T) + \frac{s_a-r_a-t_a}{l_a(r_a+1)}D_{I,S^{(a,1)},P,R,L}(T) \\
& & +\frac{1}{l_a(r_a+1)}\int_{-\infty}^{t_a}\left((u_a - s_a)D_{I,S,P^{(a,-1)},R,L}(T^{(a,u_a-t_a)})\right. \\
& & \left. \qquad + (s_a + r_a - u_a)D_{I,S^{(a,1)},P^{(a,-1)},R,L}(T^{(a,u_a-t_a)})\right)du_a
\end{array}$$

Now, for $P = 0$ as in our statement, we have $D_{I,S,P^{(a,-1)},R,L} \equiv 0_{\mathcal{M}'}$ and $D_{I,S^{(a,1)},P^{(a,-1)},R,L} \equiv 0_{\mathcal{M}'}$ due to the differential property in Definition 6.3, which completes the proof. □

## 6.4 Generalized Power Series and Analytical Functions

**Note on the draft version.** The remainder of this section is somewhat sketch for lack of time. The final version of this draft ought to contain more about generalized power series, especially as solutions to linear partial differential equations.

Let $\mathbf{D} = (D_{I,S,P,R,L})$ be a differential $B-$spline family with respect to a differentiation operator $\Delta$ over $\mathcal{M}$, which is obtained by tensor product of differentiation operators $\Delta_a$ for $a = 1,\dots,d$. For $\omega = (\omega_a)_{a=1,\dots,d} \in \mathbb{N}^d$, we denote by $\Delta^{(\omega)}$ the partial derivative of order $\omega_a$ using $\Delta_a$ on $\mathcal{A}_a$.

For the sake of simplicity, we assume that the functions $D_{I,0,0,R,L}$ have bounded support, namely that $\mathrm{supp}(D_{I,0,0,R,L}) \subset [-m(R), m(R)]$ for some positive element $m(R) \in \mathcal{I}$. We also assume that $D_{I,0,0,R,L}(T) > 0_{\mathcal{M}'}$ for all $T \in \mathcal{I}$, which implies, from the partition of unity property, that $||D_{I,0,0,R,L}||_\infty \preceq \frac{1}{L^{[R]}}$.

Note, however, that the content of this paper regarding analytical functions and their applications might work also for such functions families of functions with rapidly decreasing derivatives of all orders, such as constructed as in Section 6.2 using partitions of unity, as well as for some families of non positive functions.

**Lemma 6.1** *For $P \in \mathbb{N}^d$ and $b > 0_{\mathcal{I}}$, the suppremum $M$ of $||D_{I,S,P,R,L}(T)||_\infty$ for $x$ element of an interval $[-b,b] \subset \mathcal{I}$ is less that or equal to*

$$(|S| + 2m(R))^{[P]}$$

*Proof.* Let $T$ be an element of an interval $[A,B] \subset \mathcal{I}$. Since $||D_{I,0,0,R,L}||_\infty \preceq 1$, the result is true for $P = 0$. We then show the result by induction on $P$. Assuming it is true for $P$, we see that $||D_{I,S,P^{(a,+1)},R,L}(T)||_\infty = ||\int_S^T D_{I,S,P,R,L}(U)||_\infty dU \preceq \int_0^{|S|+2m(R)} (|S| + 2m(R))^{[P]} dU = (|s| + 2m(R))^{[P^{(a,+1)}]}$. $\square$

**Lemma 6.2** *Let $\mathbf{c} = (c_{I,S,P})$, for $0 \preceq I \preceq \delta_R$, $S \in \mathcal{Z}_d$, and $P \in \mathbb{N}^d$ be a family of elements of $\mathcal{R}$ such that*

$$\sum_{S\in\mathcal{Z}_d} \sum_{0\preceq P\preceq N} \sum_{0\preceq I\preceq\delta_R} ||c_{I,S,P}||_A (|S| + 2m(R))^P \tag{17}$$

*is absolutely convergent when all coordinates of $N \in \mathbb{N}^d$ tend $+\infty$. Then for any $R \in \mathbb{N}$ and $S \in \mathcal{Z}_d$, the sum*

$$\mathcal{S}_{\mathbf{c},\mathbf{D},N}(T) = \sum_{S\in\mathcal{Z}_d} \sum_{0ă\preceq P\preceq N} \sum_{0\preceq I\preceq\delta_R} c_{I,S,P} D_{I,S,P,R,L}(T) \tag{18}$$

*also converges when all coordinated of $N \to +\infty$.*

**Definition 6.5** *Under the assumptions of Lemma 6.2, the coefficients $\mathbf{c} = (c_{I,S,P})$ are said to define a convergent generalized power series relative to $\Delta$ and $\mathbf{D}$ Moreover, the limit for $n \to +\infty$ for the sums considered in Equation (18) is called the sum of the generalized power series relative to $\Delta$ and $\mathbf{D}$ with coefficients $(c_{I,S,P})$, with scaling factor $L$, with blending order $R$*

$$\mathcal{S}_{\mathbf{c},\mathbf{D}}(T) = \sum_{S\in\mathcal{Z}_d} \sum_{0\preceq P} \sum_{0\preceq I\preceq\delta_R} c_{I,S,P} D_{I,S,P,R,L}(T) \tag{19}$$

**Definition 6.6** *A function $f$ from $\mathcal{I}$ to $\mathcal{M}'$ is called a generalized analytical function relative to $\Delta$ and $\mathbf{D}$ if is can be expressed as a generalized power series relative to $\Delta$ and $\mathbf{D}$ for some coefficients $(c_{I,S,P})$, with scaling factor $L$, with blending order $R$*

**Proposition 6.2 (Differentiation of Generalized Analytical Functions)** *Let $R \geq 1_{\mathbb{N}}^d$, let $\mathbf{c} = (c_{I,S,P})$, for $0 \preceq I \preceq \delta_R$, $S \in \mathcal{Z}_d$, and $P \in \mathbb{N}^d$ be a family of elements of $\mathcal{R}$ which define a convergent generalized power series relative to $\Delta$ and $\mathbf{D}$. Then the function $\mathcal{S}_{\mathbf{c},\mathbf{D}}$ is differentiable for $\Delta$ (i.e. $\Delta(\mathcal{S}_{\mathbf{c},\mathbf{D}})$ exists) and we have:*

$$\Delta_a (\mathcal{S}_{\mathbf{c},\mathbf{D}})(T) = \sum_{S\in\mathcal{Z}_d} \sum_{0\preceq P=0} \sum_{0\preceq I\preceq\delta_R} c_{I,S,P^{(a,+1)}} D_{I,S,P,R,L}(T) \tag{20}$$

*Proof.* From the differential property of the differential $B-$spline family $\mathbf{D}$, we get for $N \in \mathbb{N}^d$ that the sum $\mathcal{S}_{\mathbf{c},\mathbf{D},N}(T)$ defined in Equation (18), as a function of $T \in \mathcal{I}$, is derivable for $\Delta$ and its derivative is

$$\Delta_a\left(\mathcal{S}_{\mathbf{c},\mathbf{D},N}\right)(T) = \sum_{S\in\mathcal{Z}_d}\sum_{0\preceq P\preceq N}\sum_{0\preceq I\preceq\delta_R} c_{i,s,p+1}D_{I,S,P,R,L}(T)$$

Furthermore, we see that the coefficients $\mathbf{a}' = (c_{i,s,p+1})$ for $0 \preceq I \preceq \delta_R$, $S \in \mathcal{Z}_d$, and $P \in \mathbb{N}^d$ defines a convergent generalized power series relative to $\Delta$ and $\mathbf{D}$, that is, the series $\sum_{D\in\mathcal{Z}_d}\sum_{0\preceq P\preceq N}\sum_{0\preceq I\preceq\delta_R} ||c_{I,S,P^{(a,1)}}||_\infty(|s|+2m(R))^{[P]}$ converges. Hence the series of Equation (20) above converges when all coordinates of $N$ tend to $\to +\infty$, and, using the continuity of the operator $\Delta$, by taking the limit when $N \to +\infty$ we get our result. □

By an immediate induction on Proposition 6.2, we get the following

**Theorem 6.2** *Let $R \geq 1^d_{\mathbb{N}}$, Let $\mathbf{c} = (c_{I,S,P})$, for $0 \preceq I \preceq \delta_R$, $S \in \mathcal{Z}_d$, and $P \in \mathbb{N}^d$ be a family of elements of $\mathcal{R}$ which define a convergent generalized power series relative to $\Delta$ and $\mathbf{D}$. Then, for any $\omega \in \mathcal{N}^d$, the function $\mathcal{S}_{\mathbf{c},\mathbf{D}}$ is $\omega-$differentiable for $\Delta$ (i.e. $\Delta^\omega(\mathcal{S}_{\mathbf{c},\mathbf{D}})$ exists) and we have:*

$$\Delta^\omega(\mathcal{S}_{\mathbf{c},\mathbf{D}})(T) = \sum_{s\in\mathbb{Z}}\sum_{p=0}^{+\infty}\sum_{0\preceq I\preceq\delta_R} c_{I,S,P+\omega}D_{I,S,P,R,L}(T) \tag{21}$$

## 6.5 Solutions of Linear Differential Equations

Le us consider a linear partial differential equation of the form:

$$\sum_{0\preceq J\preceq K}\alpha_J(T)\left(\Delta^{(J)}(f)\right)(T) = 0 \tag{22}$$

where $K \in \mathbb{N}^d$. Let us look for generalized analytical functions which are solutions.

So, as in Section 6.4, let $\mathbf{D} = (D_{I,S,P,R,L})$ be a differential $B-$spline family with respect to a differentiation operator $\Delta$ over $\mathcal{M}$. We assume, as has been proven for some differential $B-$spline families in section 6.4, that Definition 6.5 holds, as well as Theorem 6.2.

Let $\mathbf{c} = (c_{I,S,P})$, for some $R \geq 1_{\mathbb{N}^d}$, let $0 \preceq I \preceq \delta_R$, $S \in \mathcal{Z}_d$, and $P \in \mathbb{N}^d$ be a family of elements of $\mathcal{R}$ which define a convergent generalized power series relative to $\Delta$ and $\mathbf{D}$. From Theorem 6.2, it is sufficient that the coefficients $(c_{I,S,P})$ satisfy the following linear equations, for every $T = (t_a)_{a=1,\ldots,d}$:

$$\sum_{0\preceq J\preceq K}\alpha_j(T)c_{I,S,P+J}D_{I,S,P,R,L}(T) = 0 \tag{23}$$

**Example 6.1** *In the one dimensional case ($d = 1$) real case $\mathcal{R} = \mathcal{M} = \mathcal{M}' = \mathbb{R}$. Let us consider the equation $\Delta(f) = f$ (which is classically solved to get the exponential function $T \to e^T$). Equation (23) yields:*

$$c_{I,S,P+1} = c_{I,S,P}$$

*We therefore get the following family of solutions, for any given $P \in \mathbb{N}$ and $L > 0$:*

$$\exp_{\mathbf{N}}(T) \stackrel{\text{def}}{=} \sum_{S\in\mathbb{Z}}\sum_{P=0}^{+\infty} c_{0,S,0}N_{0,S,P,R,L}(T)$$

*Where $(c_{0,S,0})_{S\in\mathbb{Z}}$ is an arbitrary sequence. This example was implemented, to get the results presented on Figure 1.*

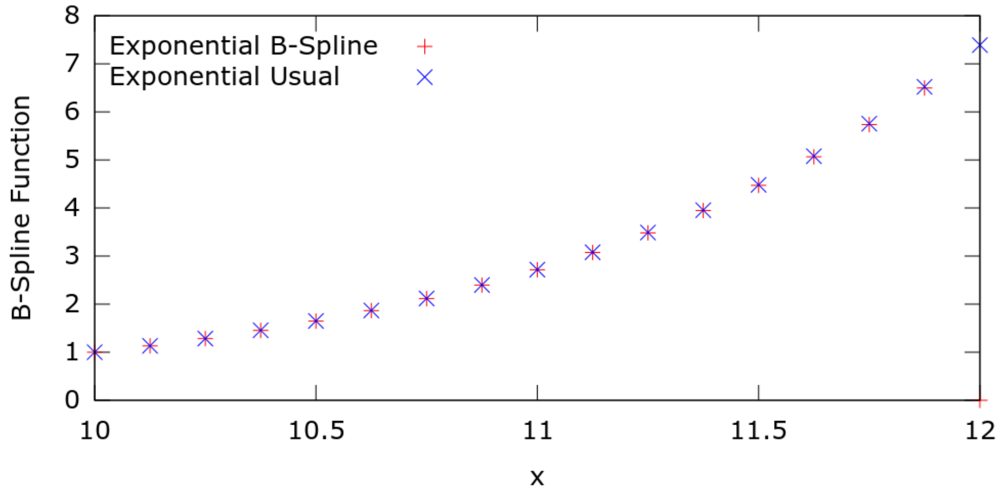


Figure 1: The results of a partial sum (for a finite $N$) of $B-$splines obtained as in Example 6.1. The graph superimposes perfectly with the usual exponential $e^x$.

The final version of the paper ought to provide more about linear partial differential equations, as how to choose solutions of the linear equations in Equation (23) to obtain an integer only and drift-free solution.

# 7 Bernstein-Based Differential $B-$Splines

## 7.1 Bézier Functions and Bernstein polynomials Basics

Note that the following definition uses multidimensional binomial coefficients and exponents following Notation 4.1, as well as polynomial functions introduced in Section 4.3.

**Definition 7.1** *Let $R \in \mathbb{N}^d$. For $I \in \mathbb{N}^d$ with $0 \preceq I \preceq R$, we consider the $\mathcal{M}'-$valued polynomial with degree $R$ which defines for $T \in \mathcal{I}$ the element*

$$B_{I,R}(T) = \binom{R}{I} (T^I 1_{\mathcal{M}'}) \left(1_{\mathcal{M}'} - T 1_{\mathcal{M}'}\right)^{R-I}$$

*These (in the framework of $\mathbb{R}-$vector spaces well-known) polynomials are called the Bernstein polynomials with degree $R$ from $\mathcal{I}$ to $\mathcal{M}'$.*

In the sequel, unless otherwise specified, we shall say Bernstein polynomials or Bernstein functions as a shorthand for Bernstein polynomials from $\mathcal{I}$ to $\mathcal{M}'$. The Bernstein polynomials with degree $R$ constitute, as formal polynomials, a basis of the vector space of polynomials with degree less than or equal to $R$. We shall often omit the $1_{\mathcal{M}'}$ factors if no ambiguity can arise, thus writing:

$$B_{I,R}(T) = \binom{R}{I} T^I \left(1_{\mathcal{M}'} - T\right)^{R-I}$$

**Remark 7.1 (Partition of Unity Property)** *By developing $(T+(1-T))^R$ we obtain $\sum_{0 \preceq I \preceq R} B_{I,R}(T) = 1$ for all $T \in\in \mathcal{I}$*

From the Pascal formula (remark 4.4) for binomial coefficients, we derive a similar formula about Bernstein polynomials:

**Proposition 7.1**

$$B_{I,R}(T) = (1_{\mathcal{M}'} - t_a 1_{\mathcal{M}'}) B_{I,R^{(a,-1)}}(T) + (t_a 1_{\mathcal{M}'}) B_{I^{(a,-1)},R^{(a,-1)}}(T) \tag{24}$$

By omitting the unit $1_{\mathcal{M}'}$, we can equivalently write:

$$B_{I,R}(T) = (1_{\mathcal{A}_a} - t_a) B_{I,R^{(a,-1)}}(T) + t_a B_{I^{(a,-1)},R^{(a,-1)}}(T) \tag{25}$$

*Proof.*

$$\begin{array}{rcl} B_{I,R}(T) & = & \binom{R}{I} T^I \left(1_{\mathcal{M}'} - T\right)^{R-I} \\ & = & \left[\binom{R^{(a,-1)}}{I^{(a,-1)}} + \binom{R^{(a,-1)}}{I}\right] T^I \left(1_{\mathcal{M}'} - T\right)^{R-I} \\ & = & t_a \binom{R^{(a,-1)}}{I^{(a,-1)}} T^{I^{(a,-1)}} \left(1_{\mathcal{M}'} - T\right)^{R^{(a,-1)}-I^{(a,-1)}} + (1_{\mathcal{M}'} - t_a) \binom{R^{(a,-1)}}{I} T^I \left(1_{\mathcal{M}'} - T\right)^{R^{(a,-1)}-I} \\ & = & (1_{\mathcal{M}'} - t_a 1_{\mathcal{M}'}) B_{I,R^{(a,-1)}}(T) + (t_a 1_{\mathcal{M}'}) B_{I^{(a,-1)},R^{(a,-1)}}(T) \end{array}$$

□

**Definition 7.2** *Let $R \in \mathbb{N}^d$. Let $\mathbf{P} = (P_I)_{0 \preceq I \preceq R}$ be a multi-sequence of points in $\mathcal{M}'$. We define, for $T = (t_a)_{a=1,\dots,d} \in \mathcal{I}$, the image of $T$ under the Bézier function $B_{\mathbf{P}} : \mathcal{I} \longrightarrow \mathcal{M}'$ with control points $\mathbf{P}$ by*

$$B_{\mathbf{P}}(T) = \sum_{0 \preceq I \preceq R} P_I B_{I,R}(T)$$

Now, if we wan to compute partial differentials for Bernstein polynomials, we consider the (in $\mathbb{R}^d$ classical) formula for Bernstein polynomials are concerned. For $T = (t_a)_{a=1,\dots,d}$, we have:

$$\frac{\partial}{\partial t_a} B_{I,R}(T) = r_a (B_{I^{(a,-1)},R^{(a,-1)}}(T) - B_{I,R^{(a,-1)}}(T)) \tag{26}$$

**Proposition 7.2** *As in the usual case of Bézier functions over $\mathbb{R}$, the partial derivative of a Bézier function from $\mathcal{I}$ to $\mathcal{M}'$ $B_{\mathbf{P}} : \mathcal{I} \longrightarrow \mathcal{M}'$ with control points $(P_I)_{0 \preceq I \preceq R}$ can be expressed as follows, denoting $T = (t_a)_{a=1,\dots,d}$:*

$$\frac{\partial}{\partial t_a} B_{\mathbf{P}}(T) = r_a \sum_{0 \preceq I \preceq R^{(a,-1)}} (P_{I^{(a,1)}} - P_I) B_{I,R^{(a,-1)}}(T)$$

*which is the Bézier function with control points $(P'_I)_{0 \preceq I \preceq R^{(a,-1)}}$, where $P'_I = r_a(P_{I^{(a,1)}} - P_I)$.*

*Proof.*

$$\begin{aligned}
\tfrac{\partial}{\partial t_a} B_{\mathbf{P}}(T) &= r_a \textstyle\sum_{0 \preceq I \preceq R} P_I \left( B_{I^{(a,-1)},R^{(a,-1)}}(T) - B_{I,R^{(a,-1)}}(T) \right) \\
&= r_a \left( \textstyle\sum_{0 \preceq I \preceq R} P_I \, B_{I^{(a,-1)},R^{(a,-1)}}(T) - \textstyle\sum_{0 \preceq I \preceq R} P_I \, B_{I,R^{(a,-1)}}(T) \right) \\
&= r_a \textstyle\sum_{0 \preceq I \preceq R^{(a,-1)}} (P_{I^{(a,1)}} - P_I) B_{I,R^{(a,-1)}}(T)
\end{aligned}$$

□

## 7.2 Scaled Bézier Function Associated to a Sequence

**Notation 7.1** *Let $u$ be an element of $\mathcal{M}/L$ (see Definition 5.4). We consider the floor of $u$, denoted by $\lfloor u \rfloor$, which is the greatest element (considering the coordinate by coordinate partial order on $\mathcal{M}$) of $\mathcal{Z}_d$ such that $L\left(\frac{\lfloor u \rfloor}{L}\right)$ is less than or equal to $L\left(\frac{u}{L}\right)$ in $\mathcal{M}/L$ (considering the coordinate by coordinate partial order on $\mathcal{M}/L$).*

**Definition 7.3** *Let $L \in \left(\mathcal{R}_+^*\right)^d$ be a vector with $d$ coordinates which are strictly positive elements of the base ring. Let $S \in \mathcal{Z}_d$. For $R \in \mathbb{N}^d$ and $I \in \mathbb{N}^d$ with $0 \preceq I \preceq R$, we introduce the $S-$shifted $L-$scaled Bernstein polynomials with degree $R$, with values in $\mathcal{M}'/L^R$, by:*

$$B_{I,S,R,L}(T) = \begin{cases} B_{I,R}\left(\frac{T}{L} - S\right) & \text{if } \frac{T}{L} \in [S, S + 1_{\mathcal{M}}[ \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the value $B_{I,S,R,L}(T)$ can be non-zero only for $S = \left\lfloor \frac{T}{L} \right\rfloor$. Using the caracteristics function of an interval, we can also write $B_{I,S,R,L}(T) = B_{I,R}\left(\frac{T}{L} - S\right) \mathbb{1}_{[s,\, s+1_{\breve{a}M}[}$.

**Remark 7.2 (Shift Property)**

$$\left(\tau^{L(a,l_a)}\left(B_{I,S,R,L}\right)\right)(T) = B_{I,S,R,L}(T^{(a,l_a)}) = B_{I,S^{(a,-1)},R,L}(T)$$

In the remainder of this section $(\Gamma(S))_{S\in\mathcal{Z}_d}$ is a multi-sequence with values in $\mathcal{M}'$, and $L=(l_a)_{a=1,\dots,d}$ is an element of $\left(\mathcal{R}^*_+\right)^d$.

**Definition 7.4** *For $R\in\mathbb{N}^d$, the $L-$scaled (piecewise) Bézier function with degree $R$ associated to $\Gamma$ is defined for $T\in\mathcal{M}$ by:*

$$\left(\mathcal{B}^{(0)}_{L,R}(\Gamma)\right)(T)=\sum_{S\in\mathcal{Z}_d}\sum_{0\preceq I\preceq R}\Gamma\left(L(I-S)\right)B_{I,S,R,L}(T)$$

Note that in the previous definition, due to the definition of $B_{I,S,R,L}$, for a given value of $I$, only one value of $S$ (namely $S=\left\lfloor\frac{T}{L}\right\rfloor$) contributes to the double sum $\left(\mathcal{B}^{(0)}_{L,R}(\Gamma)\right)(T)$, so that, in fact, at most $(|R|+d)$ terms are non-zero for a given $T$.

**Proposition 7.3 (Commutation with the Shift)** *For $R\in\mathbb{N}^d$ and $I\in\mathbb{N}^d$ with $0\preceq I\preceq R$, for $T=(t_a)_{a=1,\dots,d}\in\mathcal{M}$, we have:*

$$\left(\mathcal{B}^{(0)}_{L,R}(\Gamma)\right)(\tau^{L(a,l_a)}(T))=\left(\mathcal{B}^{(0)}_{L,R}(\tau^{-L(a,l_a)}(\Gamma))\right)(T)$$

*Proof.*

$$\begin{array}{rcl}\left(\mathcal{B}^{(0)}_{L,R}(\Gamma)\right)(\tau^{L(a,l)}(T)) & = & \sum_{S\in\mathcal{Z}_d}\sum_{0\preceq I\preceq R}\Gamma\left(L(I-S)\right)B_{I,S,R,L}(T^{(a,l_a)})\\ & = & \sum_{S\in\mathcal{Z}_d}\sum_{0\preceq I\preceq R}\Gamma\left(L(I-S)\right)B_{I,S^{(a,-1)},R,L}(T)\\ & = & \sum_{S\in\mathcal{Z}_d}\sum_{0\preceq I\preceq R}\Gamma\left(L(I-S^{(a,1)})\right)B_{I,S,R,L}(T)\\ & = & \sum_{S\in\mathcal{Z}_d}\sum_{0\preceq I\preceq R}\Gamma\left(L(S-I)-0^{(a,l_a)})\right)B_{I,S,R,L}(T)\\ & = & \left(\mathcal{B}^{(0)}_{L,R}(\tau^{-L(a,l_a)}(\Gamma))\right)(T)\end{array}$$

□

**Proposition 7.4 (De Casteljau Property on Sequences)** *Using the elements $L(a,j)$ for $a=1,\dots,d$ and $j\in\mathcal{A}_a$, as well as $R^{(a,-1)}\in\mathbb{N}^d$, defined in Notation 4.1, we have for $T=(t_a)_{a=1,\dots,d}$:*

$$\left(\mathcal{B}^{(0)}_{L,R}(\Gamma)\right)(T)=\left(1-\left(\frac{t_a}{l_a}-\left\lfloor\frac{t_a}{l_a}\right\rfloor\right)\right)\left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\Gamma)\right)(T)+\left(\frac{t_a}{l_a}-\left\lfloor\frac{t_a}{l_a}\right\rfloor\right)\left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\tau^{L(a,l_a)}(\Gamma))\right)(T)$$

*In the equation, we omitted $1_{\mathcal{M}'}$ when multiplying $\mathcal{M}'-$valued polynomials by $\left(\frac{t_a}{l_a}-\left\lfloor\frac{t_a}{l_a}\right\rfloor\right)1_{\mathcal{M}}$ or $\left(1_{\mathcal{A}_a}-\left(\frac{t_a}{l_a}-\left\lfloor\frac{t_a}{l_a}\right\rfloor\right)\right)1_{\mathcal{M}}$, seen as degree zero monomials (Definition 4.10). This notation is also valid for $R=0$ if we use the convention that $\mathcal{B}^{(0)}_{L,R^{(a,j)}}=0$ if $r_a+j<0$.*

*Proof.*

$$\begin{array}{rcl}\left(\mathcal{B}^{(0)}_{L,R}(\Gamma)\right)(T) & = & \sum_{S\in\mathcal{Z}_d}\sum_{0\preceq I\preceq R}\Gamma\left(L(I-S)\right)B_{I,R}(\frac{T}{L}-S)\mathbb{1}_{[S,S+1_{\mathcal{M}}[}\left(\frac{T}{L}\right)\\ & = & \sum_{S\in\mathcal{Z}_d}\sum_{0\preceq I\preceq R}\ \Gamma\left(L(I-S)\right)\\ & & .\left[\left(1_{\mathcal{A}_a}-\left(\frac{t_a}{l_a}-s_a\right)\right)B_{I,R^{(a,-1)}}(\frac{T}{L}-S)\right.\\ & & \left.+\left(\frac{t_a}{l_a}-s_a\right)B_{I^{(a,-1)},R^{(a,-1)}}(\frac{T}{L}-S)\right].\mathbb{1}_{[S,S+1[}\left(\frac{T}{L}\right)\end{array}$$

The last equality follows from Equation 25. Now, taking into account that the only value of $S$ for which $\frac{T}{L} \in [S, S+1[$, which implies that $\frac{t_a}{l_a} - s_a = \frac{t_a}{l_a} - \left\lfloor \frac{t_a}{l_a} \right\rfloor$, and then by changing the index $I$ to $I^{(a,-1)}$ in the sum, we get:

$$\begin{array}{rcl} \left(\mathcal{B}^{(0)}_{L,R}(\Gamma)\right)(T) & = & \left(1_{\mathcal{A}_a} - \frac{t_a}{l_a} + \left\lfloor \frac{t_a}{l_a} \right\rfloor)\right) \sum_{S\in\mathcal{Z}_d} \sum_{0\preceq I\preceq R} \Gamma\left(L(I-S)\right) B_{I,S,R^{(a,-1)},L}(T) \\ & & + \left(\frac{t_a}{l_a} - \left\lfloor \frac{t_a}{l_a} \right\rfloor\right) \sum_{S\in\mathcal{Z}_d} \sum_{0^{(a,-1)}\preceq I\preceq R^{(a,-1)}} \Gamma\left(L(I^{(a,1)}-S)\right) B_{I,S,R^{(a,-1)},L}(T) \\ & = & \left(1_{\mathcal{A}_a} - \frac{t_a}{l_a} + \left\lfloor \frac{t_a}{l_a} \right\rfloor)\right) \left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\Gamma)\right)(T) + \left(\frac{t_a}{l_a} - \left\lfloor \frac{t_a}{l_a} \right\rfloor\right) \left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\tau^{(a,l_a)}(\Gamma))\right)(T) \end{array}$$

The indices $I = 0^{(a,-1)}$ and $I = R$ yielding a zero term because out of range for the Bernstein polynomials. □

Now, we derive the following from Proposition 7.3 and Proposition 7.4:

**Proposition 7.5 (De Casteljau Property on Functions)** *We have for* $T = (t_a)_{a=1,\dots,d}$*:*

$$\left(\mathcal{B}^{(0)}_{L,R}(\Gamma)\right)(T) = \left(1 - \left(\frac{t_a}{l_a} - \left\lfloor \frac{t_a}{l_a} \right\rfloor\right)\right)\right) \left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\Gamma)\right)(T) + \left(\frac{t_a}{l_a} - \left\lfloor \frac{t_a}{l_a} \right\rfloor\right) \left(\tau^{-L(a,l_a)}(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\Gamma))\right)(T)$$

### 7.2.1 Derivative of the Scaled Bézier Function

As far as Bernstein polynomials are concerned, we get the partial differential form Euqation 26. We derive from this that, for $s = \lfloor t \rfloor$, we have

$$\frac{\partial}{\partial t_a} B_{I,S,R,L}(T) = \frac{1}{l_a} \frac{\partial}{\partial t_a} B_{I,R}(\frac{T}{L} - S) = \frac{r_a}{l_a} \left(B_{I^{(a,-1)},S,R^{(a,-1)},L}(T) - B_{I,S,R^{(a,-1)},L}(T)\right) \qquad (27)$$

so that

**Proposition 7.6 (Differentiation and Finite Differences of Sequences)** *For* $0 \preceq R = (r_a)_{a=1,\dots,d}$*, the function* $\mathcal{B}^{(0)}_{L,R}(\Gamma)$ *is* $C^{R-1}$ *on* $\mathcal{M}$ *and we have:*

$$\frac{\partial}{\partial t_a} \left(\mathcal{B}^{(0)}_{l,r}(\Gamma)\right)(T) = \frac{r_a}{l_a} \left[\left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\tau^{(a,-l_a)}(\Gamma))\right)(T) - \left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\Gamma)\right)(T)\right]$$

*Proof.* First we prove the result for all $t \in \mathcal{M} \backslash \mathcal{Z}_d$, on which the curve $\mathcal{B}^{(0)}_{l,r}(\Gamma)$ is easily seen to be polynomial, hence infinitely differentiable.

$$\begin{array}{rcl} \frac{\partial}{\partial t_a}\left(\mathcal{B}^{(0)}_{L,R}(\Gamma)\right)(T) & = & \sum_{S\in\mathcal{Z}_d} \sum_{0\preceq I\preceq R} \Gamma\left(L(S-I)\right) \frac{\partial}{\partial t_a} B_{I,S,L,R}(T) \\ & = & \sum_{S\in\mathcal{Z}_d} \sum_{0\preceq I\preceq R} \Gamma\left(L(S-I)\right) \frac{r_a}{l_a} \left(B_{I^{(a,-1)},S,R^{(a,-1)},L}(T) - B_{I,S,R^{(a,-1)},L}(T)\right) \\ & = & \sum_{S\in\mathcal{Z}_d} \sum_{0^{(a,-1)}\preceq I\preceq R^{(a,-1)}} \Gamma\left(L(S - I^{(a,1)})\right) \frac{r_a}{l_a} \left(B_{I,S,R^{(a,-1)},L}(T)\right) \\ & & - \sum_{S\in\mathcal{Z}_d} \sum_{0\preceq I\preceq R} \Gamma\left(L(S-I)\right) \frac{r_a}{l_a} \left(B_{I,S,R^{(a,-1)},L}(T)\right) \\ & = & \sum_{S\in\mathcal{Z}_d} \sum_{0\preceq I\preceq R^{(a,-1)}} \Gamma\left(L(S-I) - 0^{(a,l_a)}\right) \frac{r_a}{l_a} \left(B_{I,S,R^{(a,-1)},L}(T)\right) \\ & & - \sum_{S\in\mathcal{Z}_d} \sum_{0\preceq I\preceq R} \Gamma\left(L(S-I)\right) \frac{r_a}{l_a} \left(B_{I,S,R^{(a,-1)},L}(T)\right) \\ & = & \frac{r_a}{l_a} \left[\left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\tau^{(a,-l_a)}(\Gamma))\right)(T) - \left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\Gamma)\right)(T)\right] \end{array}$$

Now, for $R = 0$, we have $B^{(0)}_{L,R}(T) = \sum_{s\in\mathcal{Z}_d} \Gamma(-LS) \mathbb{1}_{[SL,(SL+L)[}(T)$. Consequently, $B^{(0)}_{L,1}(\Gamma)$ is $C^0$ (we remind the reader that the vector 1 is here considered as having *all* its coordinates equal to 1. The result follows by induction on $1 \preceq R$. □

**Proposition 7.7 (Differentiation and Finite Differences of Functions)** *For $0 \preceq R = (r_a)_{a=1,\ldots,d}$, the curve $\mathcal{B}^{(0)}_{L,R}(\Gamma)$ is $C^R$ on $\mathcal{M}$ and we have:*

$$\frac{\partial}{\partial t_a}\left(\mathcal{B}^{(0)}_{L,R}(\Gamma)\right)(T) = \frac{r_a}{l_a}\left[\left(\tau^{(a,l_a)}\left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\Gamma)\right)\right)(T) - \left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\Gamma)\right)(T)\right]$$

**Definition 7.5** *Let $\Phi : \mathbb{Z} \longrightarrow E$ be a sequence, or $\Phi : \mathbb{R} \longrightarrow E$ be a function. We define the finite difference masks:*

- $\left(\Delta^{(a,l)}_{-}(\Phi)\right)(S) = \frac{1}{l}(\Phi(S) - \Phi(S^{(a,-l)}))$;
- $\left(\Delta^{(a,l)}_{+}(\Phi)\right)(S) = \frac{1}{l}(\Phi(S^{(a,l)}) - \Phi(S))$.

**Notation 7.2** *For $\omega \in \mathbb{N}$ with $0 \preceq \omega \preceq R$, we denote by $\mathcal{B}^{(\omega)}_{L,R}(\Gamma)$ the function on $\mathcal{M}$ defined as the differential of order $\omega$ of $\mathcal{B}^{(0)}_{L,R}(\Gamma)$:*

$$\mathcal{B}^{(\omega)}_{L,R}(\Gamma) = \left(\mathcal{B}^{(0)}_{l,r-\omega}(\Gamma)\right)^{(\omega)}$$

Therefore, Proposition 7.6 and Proposition 7.7 can be restated as:

**Proposition 7.8** *The first order partial derivatives of $\mathcal{B}_{l,r}(\Gamma)$ can be computed in two ways through finite differences:*

- *On the sequence by* $\frac{\partial}{\partial t_a}\left(\mathcal{B}_{L,R}(\Gamma)\right)(T) = -r_a\left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\Delta^{(a,l_a)}_{-}(\Gamma)\right)(T)$;
- *On the function by* $\frac{\partial}{\partial t_a}\left(\mathcal{B}_{L,R}(\Gamma)\right)(T) = r_a\Delta^{(a,l_a)}_{+}\left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\Gamma)\right)(T)$

The following immediately follows by induction:

**Proposition 7.9** *for $R \geq 0$ and $\omega \in \mathbb{N}^d$ with $\omega \preceq R$, we can compute the differential with order $\omega$ of $\mathcal{B}_{L,R}(\Gamma)$, by applying an $\omega-$differentiation mask either to the sequence $\Gamma$ by*

$$\left(\mathcal{B}^{(\omega)}_{L,R}(\Gamma)\right) = \frac{R!}{(R-\omega)!}\mathcal{B}^{(0)}_{L,R\omega}\left(\left(-1_{\mathcal{M}'}\right)^{|\omega|}\Delta^{\omega}_{-}(\Gamma)\right)$$

*or to the function $\mathcal{B}_{L,R}(\Gamma)$ itself by*

$$\left(\mathcal{B}^{(\omega)}_{L,R}(\Gamma)\right) = \frac{R!}{(R-\omega)!}\left(\Delta^{\omega}_{+}\right)\left(\mathcal{B}^{(0)}_{L,R-\omega}(\Gamma)\right)$$

## 7.3 Bernstein Based Differential $B-$Splines Family

**Definition 7.6** *We consider, for $P \in \mathbb{N}^d$, for $I \in \mathcal{Z}_d$, for $S \in \mathcal{Z}_d$, for $R \in \mathbb{N}^d$, a function $B_{I,S,P,R,L} \in \mathcal{M}'^{\mathcal{M}}$, based on the function $B_{I,S,R,L}$ defined in Definition 7.3, by the following inductive definition:*

- $B_{I,S,0,R,L} = \frac{1}{L^{[R]}}B_{I,S,R,L}$
- *For $P \geq 0$ and $T = (t_a)_{a=1,\ldots,d}$, we set:*

$$D_{I,S,P^{(a,1)},R,L}(T) = (p_a+1)\int_{\infty}^{t_a} D_{I,S,P,R,L}(T^{(a,u-t_a)})du$$

*The family of piecewise polynomial functions thus defined is called the Bernstein-based differential $B-$spline family.*

**Theorem 7.1** *Bernstein-based differential $B-$spline family is a differential $B-$spline family as defined through Definition 6.3.*

The proof follows directly

- from Definition 7.6 which yields the differential property;
- from Equation (27), which can be integrated, and generalized for all $P \in \mathbb{N}^d$ gives us the commutation with the finite differences property;
- from Remark 7.2 which gives us the shift property;
- and from Remark 7.1 which gives us the partition of unity property.

# 8 Generalized Statements

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