Sampling method for sign changing contrast

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Abstract. We extend the applicability of the Generalized Linear Sampling Method (GLSM) [2] and the Factorization Method (FM)[14] to the case of inhomogeneities where the contrast change sign strictly inside the obstacle. Both methods give an exact characterization of the target shapes in term of the farfield operator (at a fixed frequency). One of the key ingredient to prove this exact characterization is based on a factorization of the farfield operator. This factorization involves three operators which should exhibit specific properties. This paper is concerned with the extension of the coercivity property required on one of them to the case of sign changing contrast both for isotropic and anisotropic scatters with possibly different supports for the isotropic and anisotropic parts. We fnally validate the method through some numerical tests in two dimensions.

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1. Introduction

We extend the applicability of the Generalized Linear Sampling Method (GLSM) [2] and the Factorization Method (FM)[14] to the case of inhomogeneities where the contrast change sign strictly inside the obstacle. Such an extension is quite natural because such properties of the contrast induce discreteness of the Interior Transmission Problem (ITP) and therefore range characterization [6] are available. Those range characterizations lead to Linear Sampling Method which has not a rigorous mathematical justification. In order to have a rigorous mathematical analysis we have to consider the FM and the GLSM that give an exact characterization of the target shapes in term of the farfield operator (at a fixed frequency). One of the key ingredient to prove this exact characterization is based on a factorization of the farfield operator. This factorization involves three operators which should exhibit specific properties. This paper is concerned with the extension of the coercivity property required on one of them to the case of sign changing contrast both for isotropic and anisotropic scatters with possibly different supports for the isotropic and anisotropic parts.

The main idea behind our method is to introduce an artificial contrast that will allow us to demonstrate that sign changing support induce a compact perturbation of the operator in the well study case of constant sign contrast. Therefore we will be able to demonstrate that the known assumptions on the contrasts usually stated on the whole domain are in fact only mandatory in a neighborhood of the boundary. A similar result for isotropic scatters have been obtain independently by [10].

This work clearly find its root in the study of the ITP for sign changing contrast [6, 15, 9, 4, 8, 13] however proving the coercivity needed involve also the properties of the field outside this obstacle. The main idea of our paper is to introduce an artificial contrast in order to isolate the contribution of the sign changing part of the contrast into a specific part of the field. Using regularity results of this part of the field we are able to demonstrate that the sign changing contrast only induce a compact perturbation of the operator from the well known case non sign changing part. As a direct consequence of our study of anisotropic scattering we also prove that using the GLSM framework extend the validity of sampling method for such medium with respect to the factorization method. We believe that our analysis could be applied straightforwardly to other type of perturbations, such as soundsoft or soundhard, of inhomogeneities to demonstrate coercivity in those cases [16].

The article is organized as follows. In Section 2 we introduce the scalar wave equation for orthotropic media and demonstrate that the farfield operator can be factorized in a similar way as for the isotropic case (although with some additional technicalities with respect to [5]). The obtained factorization does not require any correlation between the supports of the isotropic parameters and the anisotropic ones (which then may be different). In Section 3 we demonstrate the coercivity of the middle operator (denoted T) which is shown to hold true if the contrasts have fix (and compatible) sign in a neighborhood of the boundary of D. In Section 4 we conclude that the Generalized Linear Sampling Method or the Factorization Method could be apply in these cases. Finally in Section 5, we give some numerical illustration.

2. Model Problem

The model problem we are interested in is the scattering of scalar waves by an orthoptic medium. For a wave number $k > 0$, the total field solves the following scalar wave equation:

$$\mathrm{div}(A\nabla u) + k^2 n u = 0 \text{ in } \mathbb{R}^d$$

with $d = 2$ or 3 and with $n \in L^\infty(\mathbb{R}^d)$ denoting the refractive index such that the support of $n-1$ is included into $\overline{D}_n$ with D_n a bounded domain with Lipschitz boundary and connected complement and such that $\Im(n) \geq 0$. We assume that A is at least in $L^\infty(\mathbb{R}^d)^{d\times d}$ and that the support of $A - Id$ is included into $\overline{D}_A$ with D_A a bounded domain with Lipschitz boundary and connected complement and such that $\Im(A\zeta \cdot \bar{\zeta}) \leq 0$ and $\Re(A)\zeta \cdot \bar{\zeta} \geq c|\zeta|^2$ for $\zeta \in \mathbb{C}^d$ and for some positive constant c. We introduce a domain

D such that $D_n \cup D_A \subset D$ with D a bounded domain with Lipschitz boundary and connected complement. In the following we will assume that the simply connected components of D will either have boundary where A is not equal to Id or n is not equal to one if A is identically equal to Id. Therefore we will have $D_n \cup D_A = D$.

We are interested in the cases where the total field is generated by plane waves, $u^i(\theta, x) := e^{ikx\cdot\theta}$ with $x \in \mathbb{R}^d$ and $\theta \in \mathbb{S}^{d-1}$ and we denote by u^s the scattered field defined by

$$u^s(\theta, \cdot) = u(\theta, \cdot) - u^i(\theta, \cdot) \quad \text{in } \mathbb{R}^d,$$

which is assumed to be satisfying the Sommerfeld radiation condition,

$$\lim_{r\to\infty} \int_{|x|=r} \left| \frac{\partial u^s}{\partial r} - iku^s \right|^2 ds = 0.$$

Our data for the inverse problem will be formed by noisy measurements of so called farfield pattern $u^\infty(\theta, \hat{x})$ defined by

$$u^s(\theta, x) = \frac{e^{ik|x|}}{|x|^{(d-1)/2}} (u^\infty(\theta, \hat{x}) + O(1/|x|))$$

as $|x| \to \infty$ for all $(\theta, \hat{x}) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$. The goal is to be able to reconstruct D from these measurements (without knowing n and A). From those measurement we introduce the farfield operator $F : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$, defined by

$$Fg(\hat{x}) := \int_{\mathbb{S}^{d-1}} u^\infty(\theta, \hat{x}) g(\theta) ds(\theta), \ \hat{x} \in \mathbb{S}^{d-1}$$

Let us define for $\psi \in \{f \in L^2(D) \ s.t\ f|_{D_A} \in H^1(D_A)\}$, the unique function $w \in H^1_{\text{loc}}(\mathbb{R}^d)$ satisfying

$$\begin{cases} \operatorname{div}(A\nabla w) + nk^2 w = -k^2(n-1)\psi - \operatorname{div}((A - Id)\nabla\psi) \text{ in } \mathbb{R}^d, \\ \lim\limits_{r\to\infty} \displaystyle\int_{|x|=r} \left| \frac{\partial w}{\partial r} - ikw \right|^2 ds = 0. \end{cases} \tag{1}$$

By linearity of the forward scattering problem, Fg is nothing but the farfield pattern of w solution of (1) with $\psi = v_g$ in D , where

$$v_g(x) := \int_{\mathbb{S}^{d-1}} e^{ikx\cdot\theta} g(\theta) ds(\theta), \ g \in L^2(\mathbb{S}^{d-1}), \ x \in \mathbb{R}^d.$$

We introduce $X(D) = \{(f, g) \in L^2(D) \times L^2(D_A) \ s.t. \ g = \nabla f \ in \ D_A\}$, we identify X and its adjoint. Finally we consider the norm on $X(D)$ defined by

$$\|(f, g)\|_X = \|f\|_{L^2(D)} + \|g\|_{L^2(D_A)} = \|f\|_{H^1(D_A)} + \|f\|_{L^2(D_n)}$$

Now consider the (compact) operator $H : L^2(\mathbb{S}^{d-1}) \to X$ defined by

$$Hg := \begin{bmatrix} v_g|_{D_n} \\ \nabla v_g|_{D_A} \end{bmatrix}, \tag{2}$$

and the (compact) operator $G : \overline{\mathcal{R}(H)} \subset X \to L^2(\mathbb{S}^{d-1})$ defined by

$$G \begin{bmatrix} \psi \\ \nabla\psi \end{bmatrix} := w^\infty|_{\mathbb{S}^{d-1}} \tag{3}$$

where w^∞ is the farfield of $w \in H^1_{\text{loc}}(\mathbb{R}^d)$ solution of (1) and where $\overline{\mathcal{R}(H)}$ denotes the closure of the range of H in X. Then clearly

$$F = GH \tag{4}$$

One can also decompose G to get the second factorization of the farfield operator. More precisely, for the case under consideration, since the farfield pattern of w has the following expression [5]:

$$w^\infty(\hat{x}) = -\int_{D_n} e^{-iky.\hat{x}}(1-n)k^2(\psi(y)+w(y))dy - \int_{D_A}(A-Id)\nabla_y e^{-iky.\hat{x}} \cdot \nabla(\psi(y)+w(y))dy$$

one simply has $G = H^*T\psi$, where $H^* : X \to L^2(\mathbb{S}^{d-1})$ is the adjoint of H given by

$$H^* \begin{bmatrix} \varphi \\ \nabla\varphi \end{bmatrix}(\hat{x}) := \int_{D_n} e^{-iky.\hat{x}}\varphi(y)dy + \int_{D_A}(A-Id)\nabla_y e^{-iky.\hat{x}} \cdot \nabla\varphi dy,\ \varphi \in X,\ \hat{x} \in \mathbb{S}^{d-1},$$

and T: $X \to X$ is defined by

$$T \begin{bmatrix} \psi \\ \nabla\psi \end{bmatrix} := \begin{bmatrix} -k^2(1-n)(\psi+w) \\ -(A-Id)\cdot(\nabla\psi(y)+\nabla w(y)) \end{bmatrix}, \tag{5}$$

with $w \in H^1_{\text{loc}}(\mathbb{R}^d)$ being the solution of (1)(with $\psi_1 = \psi$ and $\psi_2 = \nabla\psi$). Finally we get

$$F = H^*TH, \tag{6}$$

Remark 1. *We remark that T is independent of the type of incident waves (either plane waves, point sources ...). We presented our results for plane waves but the properties of T presented in this paper remain true for other type of incident waves and measurements such as the one consider in [3].*

3. Key properties of the Factorization Operators

In the following we will review important properties of the operators involved in the factorization (4) and (6). First we will state the classical properties of H and G, in particular a range characterization of the obstacle D which is at the heart of both the GLSM and the FM. Then we will study the coercivity of T for sign changing contrasts.

3.1. Range characterization

First we assume that our obstacle D will be composed of several disjoint simply connected components. Those components will either have $A = Id$ and $n \neq 1$ or $A \neq Id$ in a neighborhood of their boundaries. The characterization of the obstacle D

in term of the range of G is based on the solvability of the interior transmission problem (for given regular boundary values f and g):

$$\begin{cases} \mathrm{div} A\nabla u + k^2 n u = 0 \quad \text{in } D, \\ \Delta v + k^2 v = 0 \quad \text{in } D, \\ (u - v) = f \quad \text{on } \partial D, \\ \frac{\partial}{\partial \nu}(u - v) = g \quad \text{on } \partial D, \end{cases} \tag{7}$$

where $(u, v) \in \mathcal{Y}(D)$ and $\mathcal{Y}(D)$ is a space of solutions that will be specified later. We will assume that the following hypothesis holds true.

Hypothesis 1. *We assume that $k^2 \in \mathbb{R}_+$ is such that problem (7) has a unique solution for all regular (to be specified later) functions f and g.*

This hypothesis and the interior transmission problem stated above are incomplete in the sense that we did not specify $\mathcal{Y}(D)$. This space actually depends on the properties of A and n. For example if we assume that $D_n \subset D_A = D$, (7) can be studied for $(u, v) \in \mathcal{Y}(D) = H^1(D) \times H^1(D)$. In this case we know from [4] that hypothesis 1 is for instance true if $A - Id$ and $n - 1$ have the same sign and do not change sign in a neighborhood of ∂D. The case where $D_A = \emptyset$ should be consider for $(u, v) \in L^2(D) \times L^2(D)$ such that $u - v \in H^2(D)$. In this case Hypothesis 1 is true if $n - 1$ does not change sign in a neighborhood of ∂D. The case where $n = 1$ in a neighborhood of ∂D has been less studied in the literature and the only case where we know that hypothesis 1 is true is when $A - Id$ does not change sign in all D and $n = 1$ in D. Finally when $A = Id$ in a neighborhood of ∂D, but not in all D, and $n - 1$ does not change sign in a neighborhood of ∂D, there is no clearly stated result in the literature about this case. Let us mention however that surface integral method applied to (7) (as proposed in [9]) would be an appropriate tool to study this case.

Lemma 1. *If hypothesis 1 holds true, we have that z is inside D if and only if $\phi_z(\hat{x}) := e^{-ik\hat{x}\cdot z} \in \mathcal{R}(G)$.*

We also give the following lemmas from [5]:

Lemma 2. *G is compact and if hypothesis 1 holds true, its range is dense in $L^2(\mathbb{S}^{d-1})$.*

Lemma 3. *The compact operator H has a dense range in the space $\{(f, g) \in L^2(D) \times L^2(D_A)$ s.t. $g = \nabla f$ in D_A and $\Delta f + k^2 f = 0$ in $D\} \subset X(D)$.*

3.2. Coercivity of the middle operator T

Our hypothesis on D implies that we can split the simply connected component into two categories. The first one is such that $A - Id$ does not equal zero on a neighborhood of the boundary and the second one is such that $A - Id = 0$ and $n - 1$ does not change sign in a neighborhood of the boundary. We will give a coercivity result for each of those two configurations and then merge them into a combined condition on n and A under which we have the coercivity of T defined in (5).

For both cases we will need the following equality.

Lemma 4. *We have the following identity, for $\psi = (\psi_1, \psi_2) \in X(D)$ and T defined in (5) :*

$$\Im\langle T\psi,\, \psi\rangle = k\int_{\mathbb{S}^{d-1}} |w^\infty|^2 - \int_{D_A} \Im(A)(\nabla w + \nabla\psi)\cdot(\overline{\nabla(w) + \nabla\psi}) + k^2\int_{D_n} \Im(n)|w+\psi|^2.$$

Proof. We recall that for any $\psi \in X(D)$ there exists a unique $w \in H^1_{loc}(\mathbb{R}^d)$ that solves (1). The definition of T, (5) gives:

$$\begin{array}{rl}\langle T\psi,\, \psi\rangle = & -\int_{D_A}(A-Id)(\nabla\psi+\nabla w)\cdot(\overline{\nabla\psi+\nabla w}) - k^2\int_{D_n}(1-n)|\psi+w|^2 \\ & +\int_{D_A}(A-Id)(\nabla\psi+\nabla w)\cdot(\overline{\nabla w}) + k^2\int_{D_n}(1-n)(\psi+w)\bar{w}\end{array} \quad (8)$$

Using (1) and integrating by parts over a ball B_R such that $D \subset B_R$ we have:

$$-\int_{B_R}\nabla w\cdot\overline{\nabla w} - k^2 w\overline{w} + \int_{\partial B_R}\frac{\partial w}{\partial r}\overline{w} = \int_{D_A}(A-Id)(\nabla w+\nabla\psi)\cdot\overline{\nabla w} + k^2\int_{D_n}(1-n)(w+\psi)\overline{w}$$

Substituting in (8), taking the imaginary part and letting R to $+\infty$ prove the lemma. □

3.2.1. The case where $\overline{D_A} \subsetneq D_n$ We first consider the case where $\overline{D_A} \subsetneq D_n$ which can be seen as an extension of the case $D_A = \emptyset$. The Herglotz wave operator reduces to $Hg = [v_g|_D, \nabla v_g|_{D_A}]$.

Hypothesis 2. *There exist $\alpha \geq 0$,$c > 0$ such that either $\Re(n-1) + \alpha\Im(n) \geq c$ or $\Re(1-n) + \alpha\Im(n) \geq c$ in a neighborhood of ∂D.*

Theorem 1. *If $\bar{D}_A \subsetneq D_n = D$ and hypothesis 2 and 1 holds true , then there exists μ such that the operator T defined in (5) verifies*

$$|\langle T\psi,\, \psi\rangle| \geq \mu\, \|\psi\|^2_{X(D)},$$

for all $\psi \in \overline{\mathcal{R}(H)}$.

Proof. We will proceed by a contradiction argument, therefore we assume:

$$\|\psi_\ell\|_{X(D)} = 1 \quad \Delta\psi_l + k^2\psi_l = 0 \text{ in } D \quad \text{ and } |(T\psi_\ell, \psi_\ell)| \to 0 \text{ as } \ell \to \infty.$$

Up to changing the initial sequence, one can assume that ψ_ℓ weakly converges to ψ in $L^2(D)$. one easily see that ψ satisfies

$$\Delta\psi + k^2\psi = 0 \text{ in } D.$$

We denoted $w_\ell \in H^2_{\text{loc}}(\mathbb{R}^d)$ the solution of

$$\begin{cases} \operatorname{div}(A\nabla w_\ell) + nk^2 w_\ell = -k^2(n-1)\psi_\ell - \operatorname{div}((A-Id)\nabla\psi_\ell) \text{ in } \mathbb{R}^d, \\ \displaystyle\lim_{r\to\infty}\int_{|x|=r}\left|\frac{\partial w_\ell}{\partial r} - ikw_\ell\right|^2 ds = 0. \end{cases} \quad (9)$$

w_ℓ converges weakly in $H^2_{\mathrm{loc}}(\mathbb{R}^d)$ and strongly in $H^1(D)$ to some $w \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$ that satifies with ψ equation (9).

Lemma 4 implies that $w_\ell^\infty \to 0$ in $L^2(\mathbb{S}^{d-1})$ and therefore $w^\infty = 0$. The Rellich theorem and the unique continuation principle imply that $w = 0$ outside D. Thus we have that $u = \psi + w$ and $v = \psi$ solve the interior transmission eigenvalue problem (7) with $f = g = 0$. Hypothesis 1 implies that that $\psi = w = 0$.

Our hypothesis on n allow us to introduce n_0 such that $n_0 = n$ in some domain $V \subset D$ and there exist $\alpha \geq 0$ and $c > 0$ such that either $\Re(n_0 - 1) + \alpha\Im(n_0) \geq c$ or $\Re(1 - n_0) + \alpha\Im(n_0) \geq c$ in D. We introduce $\Omega = supp(n_0 - n) \cup D_A$. By assumption we have that $\Omega \subsetneq D$ and we can choose V such that $V \cap \Omega = \emptyset$. We introduce the intermediate scattered field $u^s_{0,\ell} \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$ that satisfies :

$$\begin{cases} \Delta u^s_{0,\ell} + k^2 n u^s_{0,\ell} = -k^2(n_0 - 1)\psi_\ell \text{ in } \mathbb{R}^d, \\ \lim\limits_{r\to\infty} \int\limits_{|x|=r} \left| \frac{\partial(u^s_{0,\ell})}{\partial r} - ik(u^s_{0,\ell}) \right|^2 ds = 0. \end{cases} \tag{10}$$

We denoted by $u_{0,\ell} = u^s_{0,\ell} + \psi_\ell$ the associated total field. We also introduced the scattered field u^s_ℓ that satisfies :

$$\begin{cases} \operatorname{div}(A\nabla u^s_\ell) + k^2 n u^s_\ell = -\operatorname{div}((A - Id)\nabla u_{0,\ell}) - k^2(n - n_0)u_{0,\ell} \text{ in } \mathbb{R}^d, \\ \lim\limits_{r\to\infty} \int\limits_{|x|=r} \left| \frac{\partial(u^s_{0,\ell})}{\partial r} - ik(u^s_\ell) \right|^2 ds = 0. \end{cases} \tag{11}$$

Using the same argument as for w_ℓ we get that $u^s_{0,\ell}$ converges strongly to zero in $H^1(D)$. Since Ω is strictly included inside D, we have that $u_{0,\ell}$ is bounded in $H^2(\Omega)$ (by interior elliptic regularity). Therefore $u_{0,\ell} \in H^2(\Omega)$ converges strongly to zero in $H^1(\Omega)$ together with the continuity of the forward scattering problem for u^s_ℓ, we deduce that u^s_ℓ converges strongly to zero in $H^1_{loc}(\mathbb{R}^d)$. Finally the interior elliptic regularity implies that ψ_ℓ strongly converges to zero in $H^1(\Omega)$. Applying those strong convergence result to

$$\begin{aligned} \langle T\psi_\ell, \psi_\ell \rangle = \; & -\int_D k^2(1 - n_0)u_{0,\ell}\overline{\psi}_\ell - sign(\Re(1 - n_0)) \int_\Omega |\nabla\psi_\ell|^2 - \int_\Omega (A - Id)\nabla(u_{0,\ell} + u^s_\ell)\nabla\overline{\psi}_\ell \\ & -\int_D k^2(1 - n)u^s_\ell\overline{\psi}_\ell - \int_\Omega k^2(n - n_0)u_{0,\ell}\overline{\psi}_\ell + sign(\Re(1 - n_0)) \int_\Omega |\nabla\psi_\ell|^2. \end{aligned}$$

we deduce that the last four terms go to zero. The first two terms on the right hand side can be bounded from bellow :

$$\geq \| - sign(\Re(1 - n_0)) \int_\Omega |\nabla\psi_\ell|^2 - \int_D (1 - n_0)\psi_\ell\overline{\psi}_\ell\| - \|k^2 \int_D (1 - n_0)u^s_{0,\ell}\overline{\psi}_\ell\|$$

where the last term on the right hand side goes to zero (because of the strong convergence results), and using the assumption on n_0 we conclude that $\lim\limits_{\ell\to 0} \|\langle T\psi_\ell, \psi_\ell\rangle\| \geq k^2 c/2 > 0$, which is a contradiction. □

3.2.2. The case $D_n \subset D_A$ Without loss of generality we will consider the case where $D_n = \emptyset$ and $D = D_A$ rather than the case $D_n \subset D_A = D$ in order to lighten the notation. This is possible because thanks to compact embedding from H^1 to L^2, terms that come from contrast in n will go to zero in the proof similarly to the previous section. Therefore in the following $D_A = D$ and $\|\|_{X(D)}$ will be equal to the $\|\|_{H^1(D)}$.

Hypothesis 3. *A is C^1 in a neighborhood V of ∂D and if either of both conditions apply:*

- *there exists $\alpha \geq 0$, $c > 0$ such that $\Re(A - Id) - \alpha\Im(A) \geq c$ in V*
- *$\Re(A)$ is positive definite and there exists $\alpha \geq 0$, $c > 0$ and $0 < \eta \leq 1$ such that*

$$\Re(Id - A)X \cdot \overline{X} + (1-\eta)\Re(A)Y \cdot \overline{Y} - \alpha\Im(A)(X+Y) \cdot (\overline{X} + \overline{Y}) \geq cX \cdot \overline{X}$$

in V for all $X, Y \in \mathbb{C}^d$.

Theorem 2. *If $D_n = \emptyset$, $D = D_A$ and hypothesis 3 holds true, there exists μ such that the operator T defined in (5) verifies*

$$|\langle T\psi,\, \psi\rangle| \geq \mu \, \|\psi\|^2_{X(D)}$$

for all $\psi \in \overline{\mathcal{R}(H)}$.

Proof. We introduce A_0 such that $A_0 = A$ inside V a neighborhood of ∂D and A_0 verifies hypothesis 3 in all D. Since we suppose that A is C^1 inside V we can choose A_0 to be C^1 inside all D. We also introduce $\Omega = supp(A - A_0)$, by construction $\Omega \subsetneq D$.

We will proceed by a contradiction argument, therefore we assume:

$$\|\psi_\ell\|_{X(D)} = 1 \quad \text{ and } |(T\psi_\ell, \psi_\ell)| \to 0 \text{ as } \ell \to \infty$$

and that ψ_ℓ weakly converges in $H^1(D)$ to ψ that satisfies

$$\Delta\psi + k^2\psi = 0 \text{ in } D.$$

The solution w_ℓ satisfying (1) with $v = \psi_\ell$ weakly converges in $H^1(D)$ to $w \in H^1(\mathbb{R}^d)$ satisfying (1) with $v = \psi$.

Lemma 4 implies that $w^\infty \to 0$ in $\text{Ł}^2(\mathbb{S}^{d-1})$ and therefore $w^\infty = 0$. The Rellich theorem and unique continuation theorem imply that $w = 0$ outside D. Thus we have that $u = \psi + w$ and $v = \psi$ solve the interior transmission eigenvalue problem (7). Hypothesis 1 implies that that $\psi = w = 0$. Let us introduce the intermediate (scattered) field $u^s_{0,\ell}$ that solves:

$$\begin{cases} \operatorname{div}(A_0 \nabla u^s_{0,\ell}) + k^2 u^s_{0,\ell} = -\operatorname{div}((A_0 - Id)\nabla\psi_\ell) \text{ in } \mathbb{R}^d, \\ \lim\limits_{r\to\infty} \int\limits_{|x|=r} \left| \frac{\partial(u^s_{0,\ell})}{\partial r} - ik(u^s_{0,\ell}) \right|^2 ds = 0. \end{cases} \tag{12}$$

We denote by $u_{0,\ell} = u^s_{0,\ell} + \psi_\ell$ the total field. We also introduce u^s_ℓ that solves:

$$\begin{cases} \operatorname{div}(A\nabla u^s_\ell) + k^2 u^s_\ell = -\operatorname{div}((A - A_0)\nabla u_{0,\ell}) \text{ in } \mathbb{R}^d, \\ \lim\limits_{r\to\infty} \int\limits_{|x|=r} \left| \frac{\partial u^s_\ell}{\partial r} - iku^s_\ell \right|^2 ds = 0. \end{cases} \tag{13}$$

We have

$$\begin{aligned} |(T\psi_\ell, \psi_\ell)| &= \left|\int_D (A - Id)\nabla(u^s_\ell + u_{0,\ell})\nabla\overline{\psi}_\ell dx\right| \\ &= \left|\int_D (A_0 - Id)\nabla u_{0,\ell}\nabla\overline{\psi}_\ell dx + \int_D((A - Id)\nabla u^s_\ell\nabla\overline{\psi}_\ell dx + \int_\Omega (A - A_0)\nabla u_{0,\ell}\nabla\overline{\psi}_\ell d \right. \\ &\geq \left|\int_D (A_0 - Id)\nabla u_{0,\ell}\nabla\overline{\psi}_\ell dx\right| - \left|\int_D((A - Id)\nabla u^s_\ell\nabla\overline{\psi}_\ell dx + \int_\Omega (A - A_0)\nabla u_{0,\ell}\nabla\overline{\psi}\right. \\ &\geq \left|\int_D (A_0 - Id)\nabla u_{0,\ell}\nabla\overline{\psi}_\ell dx\right| - \left|\int_D (Id - A)\nabla u^s_\ell\nabla\overline{\psi}_\ell dx\right| - \left|\int_\Omega (A - A_0)\nabla u_{0,\ell}\nabla\overline{\psi}\right. \end{aligned}$$

Since $u_{0,\ell} \in H^1(D)$ satisfies $\operatorname{div}(A_0\nabla u_{0,\ell}) + k^2 u_{0,\ell} = 0$ in D, we infer by interior elliptic regularity that $u_{0,\ell} \in H^2(\Omega)$ (from [11] and the fact that A_0 is C^1). Due to compact embeddings from H^2 to H^1, we deduce that $u_{0,\ell}$ strongly converges to zero in $H^1(\Omega)$. For the same reasons we deduce that ψ_ℓ strongly converges to zero in $H^1(\Omega)$. By continuity of the forward scattering problem verified by u^s_ℓ and the strong convergence of $u_{0,\ell}$ in $H^1(\Omega)$, we deduce that u^s_ℓ strongly converges to zero in $H^1_{loc}(\mathbb{R}^d)$. We therefore deduce that for ℓ large enough (14) becomes:

$$|(T\psi_\ell, \psi_\ell)| \geq \frac{1}{2}\left|\int_D (A_0 - Id)\nabla u_{0,\ell}\nabla\overline{\psi}_\ell dx\right| \tag{15}$$

To treat $|(T_0\psi_\ell, \psi_\ell)| = \left|\int_D (A_0 - Id)\nabla u_{0,\ell}\nabla\overline{\psi}_\ell dx\right|$ we need to consider two cases depending on the compatibility of the sign of $A_0 - Id$ and Id (as in [7]). First we consider the case when there exist $\alpha \geq 0$ and $c > 0$ such that $\Re(A_0 - Id) - \alpha\Im(A_0) \geq c > 0$. Since $u^s_{0,\ell}$ solves (12) we deduce that:

$$\begin{aligned} (T_0\psi_\ell, \psi_\ell) = -\int_D (A_0 - Id)\nabla u_{0,\ell}\nabla\overline{u}_{0,\ell} + |u_{0,\ell}|^2 - \int_{\mathbb{R}^d} |\nabla u^s_{0,\ell}|^2 + |u^s_{0,\ell}|^2 + \int_D |u_{0,\ell}|^2 \\ + \int_{\mathbb{R}^d} |u^s_{0,\ell}|^2 + ik\int_{\mathbb{S}^{d-1}} |u^s_{0,\ell}|^2 \end{aligned} \tag{16}$$

The weak convergence of $u^s_{0,\ell}$ in $H^1_{loc}(\mathbb{R}^d)$ and $u_{0,\ell}$ in $H^1(D)$ imply the strong convergence in $L^2_{loc}(\mathbb{R}^d)$ and $L^2(D)$ respectively. Therefore the last three terms in the equality above go to zero. Moreover (15) implies that $|(T_0\psi_\ell, \psi_\ell)|$ go to zero. Therefore the first term in (16) goes also to zero and the hypothesis on A_0 implies that

$$\| - \int_D (A_0 - Id)\nabla u_{0,\ell}\nabla\overline{u}_{0,\ell} + |u_{0,\ell}|^2\| \geq c/2\, \|u_{0,\ell}\|_{H^1(D)}$$

Therefore $\|u_{0,\ell}\|^2_{H^1(D)}$ goes to zero as well as $\left\|u^s_{0,\ell}\right\|^2_{H^1_{loc}(\mathbb{R}^d)}$. This implies that $\|\psi_\ell\|^2_{H^1(D)} \to 0$ which is a contradiction.

Then we consider the case when $\Re(A_0)$ is positive definite. We cannot use (16) since the term involving $u_{0,\ell}$ and $u^s_{0,\ell}$ do not have the same sign. From the definition of T_0 we have:

$$(T_0\psi_\ell, \psi_\ell) = -\int_D (A_0 - Id)\nabla\psi_\ell\nabla\overline{\psi}_\ell - \int_D (A_0 - Id)\nabla u^s_{0,\ell}\nabla\overline{\psi}_\ell.$$

Using equation (1) verified by $u^s_{0,\ell}$ we have:

$$\begin{array}{rl}(T_0\psi_\ell,\psi_\ell) = & -\int_D (A_0 - Id)\nabla\psi_\ell\nabla\overline{\psi}_\ell + \int_{\mathbb{R}^d}\bar{A}_0\nabla\bar{u}^s_{0,\ell}\nabla u^s_{0,\ell} \\ & -2i\int_D \Im(A_0)\nabla u^s_{0,\ell}\nabla\overline{\psi}_\ell - ik\int_{\mathbb{S}^{d-1}}|u^\infty_{0,\ell}|^2 - \int_{\mathbb{R}^d} k^2\bar{u}^s_{0,\ell}u^s_{0,\ell}\end{array}$$

The last two terms go to zero because of regularity and compact embedding. The real part of the remainings term is

$$\begin{array}{l}\int_D (Id - \Re(A_0))\nabla\psi_\ell\nabla\overline{\psi}_\ell + \int_{\mathbb{R}^d}\Re(A_0)\nabla u^s_{0,\ell}\nabla\bar{u}^s_{0,\ell} \\ -i\int_D \Im(A_0)\nabla u^s_{0,\ell}\nabla(\overline{\psi}_\ell + \bar{u}^s_{0,\ell}) - \Im(A_0)\nabla\bar{u}^s_{0,\ell}\nabla(\psi_\ell + u^s_{0,\ell})\end{array}$$

and the imaginary part is

$$-\int_D \Im(A_0)\nabla(\psi_\ell + u^s_{0,\ell})\nabla(\overline{\psi}_\ell + \bar{u}^s_{0,\ell})$$

, both will go to zero. Those two terms can be combined through a positive parameter λ in order to form the following quantity:

$$\begin{array}{l}\int_D (Id - \Re(A_0))\nabla\psi_\ell\nabla\overline{\psi}_\ell + \int_D \Re(A_0)\nabla u^s_{0,\ell}\nabla\bar{u}^s_{0,\ell} - \lambda\int_D \Im(A_0)\nabla(\psi_\ell + u^s_{0,\ell})\nabla(\overline{\psi}_\ell + \bar{u}^s_{0,\ell}) \\ -i\int_D \Im(A_0)\nabla u^s_{0,\ell}\nabla(\overline{\psi}_\ell + \bar{u}^s_{0,\ell}) - \Im(A_0)\nabla\bar{u}^s_{0,\ell}\nabla(\psi_\ell + u^s_{0,\ell}) + \int_{\mathbb{R}^d\setminus D}\Re(A_0)\nabla u^s_{0,\ell}\nabla\bar{u}^s_{0,\ell}\end{array}$$

If we denote the quantity under the integral over D in this identity by $M(\nabla\psi_\ell, \nabla u^s_{0,\ell})$. We observe that :

$$\begin{array}{rl}M(X,Y) = & (I - \Re(A_0))X\cdot\bar{X} + (1-\eta)\Re(A_0)Y\cdot\bar{Y} - \lambda\Im(A_0)(X+Y)\cdot(\bar{X}+\bar{Y}) \\ & +|(\eta\Re(A_0))^{1/2}Y + i(\eta\Re(A_0))^{-1/2}\Im(A_0)(X+Y)|^2 - |(\eta\Re(A_0))^{-1/2}\Im(A_0)(X+Y)|^2\end{array}$$

Our assumption on A_0 implies that for

$$\lambda > \alpha + \sup_{x\in D}\|\Im(A_0(x))\| / (\eta\,\|\Re(A_0(x))\|)$$

we have

$$(T_0\psi_\ell,\psi_\ell) \geq c/2\,\|\psi_\ell\|^2_{H^1(D)}$$

This implies that $\|\psi_\ell\|^2$ goes to zero which is a contradiction. □

Remark 2. *One can weaken the regularity assumption on A in V (e.g. example piecewise C^1) as long as one obtain an interior regularity property (e.g. $u_{0,\ell}\in H^s(\Omega)$ where s is strictly larger than one) which implies strong convergence through compact embeddings [12].*

3.2.3. A final coercivity result We introduce $D = \bigcup_i D_n^i \cup \bigcup_i D_A^i$ where the D_i are simply connected disjoint component. We assume that $A - Id$ is not zero in the neighborhood of the boundary D_A^i and $A - Id$ equals zero in the neighborhood of the boundary D_n^i. With those notation and the result of Theorems 2 and 1 we can give the final result under Hypothesis 1 in the case of many disjoint scatter.

Theorem 3. *Assume A has C^1 regularity in $D_A^i \cap V$ and that either conditions apply :*

- *there exist $c > 0$ and $\alpha > 0$ such that either $\Re(A - Id) - \alpha\Im(A) \geq c > 0$ in $\bigcup_i D_A^i \cap V$ and $\Re(1 - n) + \alpha\Im(n) \geq c > 0$ in $\bigcup_i D_n^i \cap V$*
- *$\Re(A)$ is positive definite in $\bigcup_i D_A^i \cap V$ and there exists $\alpha \geq 0$, $c > 0$ and $0 < \eta \leq 1$ such that $\Re(n - 1) + \alpha Im(n) \geq c > 0$ in $\bigcup_i D_n^i \cap V$ and*

$$\Re(Id - A)X \cdot \overline{X} + (1 - \eta)\Re(A)Y \cdot \overline{Y} - \alpha\Im(A)(X + Y) \cdot (\overline{X} + \overline{Y}) \geq cX \cdot \overline{X}$$

in $\bigcup_i D_A^i \cap V$ for all $X, Y \in \mathbb{C}^d$.

We have that T defined by (5) verifies:

$$|\langle T\psi,\, \psi\rangle| \geq \mu \, \|\psi\|_X^2$$

where $\psi \in \overline{\mathcal{R}(H)}$.

Proof. We set $D_1 = \bigcup_i D_A^i$ and $D_2 = \bigcup_i D_n^i$. In this case we have that

$$\langle T\psi,\, \psi\rangle = \langle T\psi|_{D_1},\, \psi|_{D_1}\rangle + \langle T\psi|_{D_2},\, \psi|_{D_2}\rangle$$

By the linearity of the forward scattering problem, if we introduce the two total fields associated to the two incidents waves $\psi_1 = \psi|_{D_1}$ in D_1 and 0 in D_2 and $\psi_2 = \psi|_{D_2}$ in D_2 and 0 in D_1, denoted $u_1 = u_1^s + \psi_1$ and $u_2 = u_2^s + \psi_2$. Then we have:

$$\begin{aligned}\langle T\psi,\, \psi\rangle = \;& \langle T_1\psi_1,\, \psi_1\rangle_{D_1} + \langle T_2\psi_2,\, \psi_2\rangle_{D_2} - \int_{D_1} (A - Id)\nabla u_2^s \cdot \nabla\bar{\psi}_1 + k^2(1 - n)u_2^s\bar{\psi}_1 \\ & - \int_{D_2} k^2(1 - n)u_1^s\bar{\psi}_2 + (A - Id)\nabla u_1^s \cdot \nabla\bar{\psi}_2\end{aligned}$$

where T_1 and T_2 are the operators corresponding to D_1 and D_2 respectively. We clearly see that the last two terms go to zero (using a compactness argument). Therefore if T_1 and T_2 have the same sign, we obtain that T is coercive. The sign of T_1 and T_2 are given in the proofs of Theorems 2 and 1 respectively, which allows us to conclude. □

Remark 3. *In this article we concentrate on sign changing contrast but we believe that both the results and the methods of the proofs could be straightforwardly extend to inclusion of any kind (sound soft, sound hard, robin condition,...) strictly included inside the penetrable obstacle.*

4. Application to the GLSM and Factorization methods

4.1. Application to the GLSM

We recall that the farfield pattern of the green function Φ_z is given by,

$$\phi_z(\hat{x}) := e^{-ik\hat{x}\cdot z}$$

and that that lemma 1 give a range characterization of D. In order to use this range characterization the GLSM framework introduce the cost functional J_α, defined for $g \in L^2(\mathbb{S}^{d-1})$ by

$$J_\alpha(\phi_z, g) = \alpha|\langle Fg,\, g\rangle| + \alpha^{1-\gamma}|\langle Fg - \phi_z,\, g\rangle| + \|Fg - \phi_z\|^2 \tag{17}$$

where $\gamma \in]0, 1[$. We also introduce

$$j_\alpha(\phi_z) = \inf_{g\in L^2(\mathbb{S}^{d-1})} J_\alpha(\phi_z, g).$$

From the results of [2], [1] and [3] (partly based on lemmas 3, 2 and 1), we obtain the following characterization of D.

Theorem 4. *Assume that Hypothesis 1 and the hypothesis of theorem 3 hold true. For $z \in \mathbb{R}^d$ let us introduce $g^{z,\alpha}$ such that $J_\alpha(\phi_z, g^{z,\alpha}) \le j_\alpha(\phi_z) + p(\alpha)$ with $p(\alpha) = O(\alpha)$.*

Then $z \in D$ if and only if $\limsup_{\alpha\to 0} |\langle Fg^{z,\alpha},\, g^{z,\alpha}\rangle| < \infty$. Moreover, we have that the sequence of Herglotz wave functions associated with $g^{z,\alpha}$ converges strongly to the solution v of (7) with $(f, g) = (\Phi_z, \frac{\partial\Phi_z}{\partial\nu})$ as α goes to zero.

For the noisy case, consider $F^\delta : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ a compact operator such that:

$$\left\|F^\delta - F\right\| \le \delta.$$

Then consider for $\alpha > 0$ and $\phi \in L^2(\mathbb{S}^{d-1})$ the functional $J^\delta_\alpha(\phi, \cdot) : L^2(\mathbb{S}^{d-1}) \to \mathbb{R}$,

$$J^\delta_\alpha(\phi_z, g) = \alpha|\left\langle F^\delta g,\, g\right\rangle| + \alpha^{1-\gamma}\delta\,\|g\|^2 + \alpha^{1-\gamma}|\left\langle F^\delta g - \phi_z,\, g\right\rangle| + \left\|F^\delta g - \phi_z\right\|^2$$

where $\gamma \in]0, 1[$. We obtain the following asymptotic characterization of D.

Theorem 5. *Assume that the hypothesis of the previous theorem hold true. For $z \in \mathbb{R}^d$ let us denote by $g^{z,\alpha,\delta}$ the minimizer of $J^\delta_\alpha(\phi_z, \cdot)$ over $L^2(\mathbb{S}^{d-1})$.*
Then $z \in D$ if and only if $\limsup_{\alpha\to 0}\limsup_{\delta\to 0} |\left\langle F^\delta g^{z,\alpha,\delta},\, g^{z,\alpha,\delta}\right\rangle| + \alpha^{-\gamma}\delta\left\|g^{z,\alpha,\delta}\right\|^2 < \infty$.
Moreover, there exists $\delta_0(\alpha)$ such that for all $\delta(\alpha) \le \delta_0(\alpha)$, $Hg^{z,\alpha,\delta(\alpha)}$ converges strongly to the solution v of (7) with $(f, g) = (\Phi_z, \frac{\partial\Phi_z}{\partial\nu})$ as α goes to zero.

4.2. Application to the Factorization method

From [14] we have the following theorem for the factorization method:

Theorem 6. *For $F = H^*TH$, assume that :*

- *H^* is compact with dense range.*
- *$\Im(T)$ is compact and non negative on the range of H.*
- *$|\Re(T)|$ is one to one or $\Im(T)$ is strictly positive on $\overline{\mathcal{R}(H)}$*
- *$|\Re(T)| = C + K$, where K is compact and C is a self adjoint coercive operator.*

Then the range of the operators $F_{\#}^{1/2}$ and H^* coincide, where $F_{\#} = |\Re(F)| + \Im(F)$.

The first three assumptions are direct consequences of lemmas 3, 2 and 1. For the last assumption the application of the Factorization method is more restrictive than the GLSM as it relies on the fact that the real part of T have to be of the form "coercive + compact". In section 3 we have proven that T is actually of the form $T_0 + K$, where K is compact and T_0 extend assumptions on the contrast in a neighborhood of the boundary ∂D to all D. Therefore for the factorization method to work we need to find a set of assumptions on the contrasts inside all D that ensure that $|\Re(T_0)|$ is of the form "coercive+compact". Such hypothesis for D_A can be found in [7] (Theorem 4.8) and in [14] for D_n. Those results allow us to state the following theorem,

Theorem 7. *Assume that Hypothesis 1 holds true and A has C^1 regularity in $D_A^i \cap V$ and that either conditions apply :*

- *there exist $c > 0$ and $\alpha > 0$ such that either $\Re(A - Id) \geq c > 0$ in $\bigcup_i D_A^i \cap V$ and $\Re(1-n) \geq c > 0$ in $\bigcup_i D_n^i \cap V$*
- *there exists $\alpha \geq 0$, $c > 0$ and $0 < \eta \leq 1$ such that $\Re(n-1) + \alpha Im(n) \geq c > 0$ in $\bigcup_i D_n^i \cap V$ and $(Id - \Re(A) - \alpha|\Im(A)|$ is positive definite and $(\Re(A) - \frac{1}{\alpha}|\Im(A)| \geq 0$ for some $\alpha > 0$ in $\bigcup_i D_A^i \cap V$.*

Then z is inside D if and only if $\phi_z \in \mathcal{R}(F_{\#}^{1/2})$.

5. Numerical experiments

We restrict ourselves to the two dimensional isotropic case (A is the identity matrix) and will introduce the algorithms for the discrete version of the GLSM and FM. We indentify $\mathbb{S}^1$ with the interval $[0, 2\pi[$. In order to collect the data of the inverse problem we solve numerically (1) for N incident fields $u^i(\frac{2\pi j}{N}, \cdot)$, $j \in 0, ..., N-1$ using a finite element solver. The discrete version of F is the matrix $F := (u^\infty(\frac{2\pi j}{N}, \frac{2\pi k}{N})_{0 \leq j,k \leq N}$. We add some noise to the data to build a noisy farfield matrix F^δ where $(F^\delta)_{j,k} = F_{j,k}(1 + \sigma N_{j,k})$ for $\sigma > 0$ and $N_{j,k}$ an uniform complex random variable in $[-1, 1)^2$. Similarly we consider the discrete version of the green function $\Phi_z(j) = \phi_z(\frac{2\pi j}{N})$ for $j \in 0, ..., N$.

We apply both the factorization method and the GLSM to kite shape obstacle where with $n = 0.2$ except within a disk stricly inside the kite wehe $n = 2$. We choose $N = 100$ and a wavelength $\lambda = \frac{2\pi}{k} = 0.5$. We fix the regularization parameter α as explained in [2] for the GLSM and using the Morozov discrepancy principle for the factorization method. Figure 1 shows that there is no significant change in the ability of the methods to reconstruct the inclusion when the contrast changes sign or not. The axes of the figure are measured in λ.

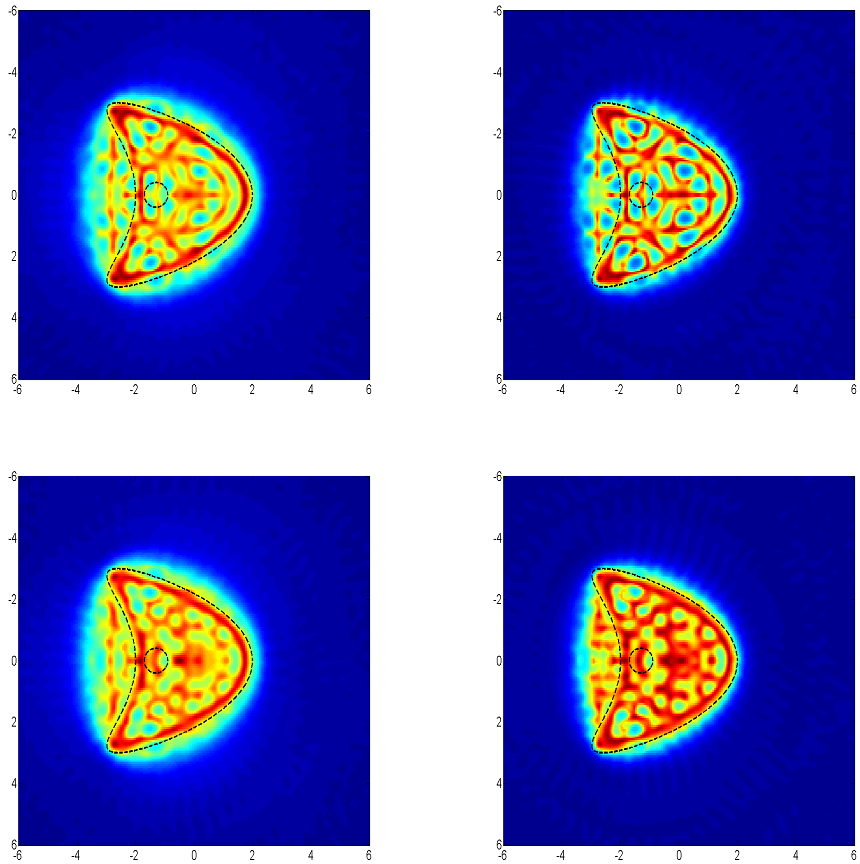

Figure 1. First line : Factorization method (left) and GLSM (right) without sign changing contrast. Second line : Factorization method (left) and GLSM (right) with sign changing contrast.

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