Optimal monopoly pricing with congestion and random utility via partial mass transport

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Abstract

We consider a bilevel optimization framework corresponding to a monopoly spatial pricing problem: the price for a set of given facilities maximizes the profit (upper level problem) taking into account that the demand is determined by consumers' cost minimization (lower level problem). In our model, both transportation costs and congestion costs are considered, and the lower level problem is solved via partial transport mass theory. The partial transport aspect of the problem comes from the fact that each consumer has the possibility to remain out of the market. We also generalize the model and our variational analysis to the stochastic case where utility involves a random term.

Keywords: monopoly spatial pricing, partial optimal mass transport, congestion, random utility.

1. Introduction

Since the classical work of Hotelling [1], spatial pricing issues have received a lot of attention. Many generalizations and variants of Hotelling's competitive model where firms compete both in locations and prices have been studied in literature (see e.g. [2] and the references therein). In the present paper, we consider a monopoly situation but allow for general transport costs, congestion effects and possible randomness in the consumers' utility.

In our model, there is a fixed finite set of locations at which the monopoly can sell an homogeneous good to a continuum of consumers, distributed according to a given spatial distribution μ. Our aim is to analyze profit maximizing spatial pricing. The profit maximization can naturally (as in Mallozzi and Passarelli di Napoli [3]) be viewed as a special instance of bilevel optimization. Indeed, consumer's demands at each facility location is determined by their cost minimizing behavior, based not only on price but also on travelling cost and congestion or queing (as in Crippa, Jimenez and Pratelli [4]) effects. We call the consumer's demand stage, given the price system, the lower level and the profit maximization with respect to the price the upper level. The lower level problem can be seen as an equilibrium partition problem, in the

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same spirit as the generalized market area problem ([5], [6]) where the production levels and the distribution patterns at n plants are determined simultaneously to satisfy the demand distributed over a given region.

Our analysis of the lower level problem with congestion is very similar to the variational/mass transport approach of [4], with one important difference in the fact that, in our model, we do not impose that the market is fully covered, i.e. that the total demand is the mass of μ. Indeed, in our model, consumers have a reservation cost, corresponding to the option of not purchasing the good anywhere and then paying zero cost. It may then well be the case that some consumers remain out of the market and this effect is actually even strenghtened by congestion effects. It is also important to allow the market not to be fully covered since it might be too costly for the monopoly hence non-optimal for the upper level problem. We show nevertheless that the analysis of [4] easily extends to the not covered case provided one allows partial optimal transport (see for instance Figalli [7] for a detailed analysis of partial optimal transport, in particular for a quadratic cost). The importance of partial optimal transport in optimal/equilibrium partition problems was clearly emphasized in the recent work of Wolansky [8] who introduced a new cooperative approach to partition games (but did not consider congestion effects). This quite general framework enables us to go one step further and prove an existence result for the upper level. Note that, in our upper level problem, the demands for some facilities can vanish, so if we imagine that the finite set of feasible facilities for the monopoly is a very fine discretization of the whole urban region, the upper level problem also determines the effective optimal operating locations for the monopoly. Deeper theoretical or numerical investigations of optimal prices are left for future research.

Most realistic economic situations involve some stochastic effects (see e.g. [9], for a random utility scheme in a competitive facility problem). Another contribution of our paper is to allow for some randomness (or heterogeneity) in consumers' utilities and to show how the variational approach to the lower level problem can be extended to this noisy setting.

The organization of the paper is the following: the model is described in section 2 and some tractable examples are presented in section 3. The lower level problem is shown to be equivalent to a convex variational problem in section 4, we deduce an existence result for the monopolit's upper level problem in section 5. Our analysis is extended to the random utility case in section 6. Some technical results from optimal partial transport and convex duality are gathered in the appendix.

2. The model

We consider an urban area given by $\Omega \subset \mathbb{R}^d$, a bounded domain (i.e. open connected) of $\mathbb{R}^d$, the density of population/customers in this region is given by a probability measure $\mu \in \mathcal{P}(\overline{\Omega})$ which captures the potential spatial distribution of demand. We are interested in the profit-maximizing pricing policies of a monopoly operating at N given distinct locations $y_1, \cdots, y_N \in \overline{\Omega}^N$. Each customer is assumed to purchase either 1 or 0 quantity of the good sold by the monopoly at one of the locations $y_1, \cdots, y_N$. The demand for the good at each location $y_1, \cdots, y_N$ results from the cost-minimizing behavior of customers which we now describe. First (and this is in contrast with the model of [4] for instance), we assume that customers also have the option of not purchasing the good then getting a reservation cost of 0. If, on the contrary, a customer from x decides to purchase the good from the monopoly at y_j, her cost will be the sum of a transport cost $c(x, y_j)$, a congestion (or queing cost) cost $h_j(\omega_j)$ where ω_j is the demand at location j net of a utility u_j for purchasing the good at a price p_j. Prices p_j and demands ω_j's are the main unknowns to be determined from the monopoly and customers' rational behaviors.

In addition to the city Ω and the locations $y_1, \cdots, y_N$, the data of the model are the transport cost c, the customers distributions μ, the congestion functions h_j, the vector of utilities $\boldsymbol{u} := (u_1, \cdots, u_N)$ (in the sequel, we shall always use bold letters to denote vectors) and the monopoly's production cost function C. We shall always assume the following:

- $c \in C^0(\overline{\Omega} \times \overline{\Omega}, \mathbb{R}_+)$,
- each congestion function $h_j : [0,1] \to \mathbb{R}_+$ is continuous and increasing,
- c and μ satisfy the nondegeneracy conditions: for every $\alpha \in \mathbb{R}$ and for every $i, j \in \{1, \cdots, N\}^2$ with $i \neq j$ one has

$$\mu(\{x \in \overline{\Omega} \ : \ c(x, y_j) = \alpha\}) = \mu(\{x \in \overline{\Omega} \ : \ c(x, y_i) - c(x, y_j) = \alpha\}) = 0, \tag{1}$$

- denoting

$$\Delta_N := \{\boldsymbol{\omega} = (\omega_1, \cdots, \omega_N) \in \mathbb{R}_+^N \ : \ \sum_{i=1}^N \omega_i \leq 1\}$$

  and by C: $\Delta_N \to \mathbb{R}_+$ the monopoly's cost function, C is lsc on Δ_N (a reasonable form for C is $C(\boldsymbol{\omega}) := \Phi(\sum_{i=1}^N \omega_i) + \sum_{i=1}^N \Phi_i(\omega_i)$ where Φ represents the production cost and Φ_i represents a location-dependent operating cost which may naturally involve a fixed cost and thus be lsc but not necessarily continuous at 0).

Given a price system $\boldsymbol{p} := (p_1, \cdots, p_N)$ and demands $\boldsymbol{\omega} := (\omega_1, \cdots, \omega_N)$, agents located at x, can either stay where they are and pay 0 reservation cost or purchase the good at location j for the total cost $c(x, y_j) + h_j(\omega_j) + p_j - u_j$. If $\min_{j=1,\cdots,N}\{c(x, y_j) + h_j(\omega_j) + p_j - u_j\} > 0$, agents located at x just stay out of the market, if one the contrary $\min_{j=1,\cdots,N}\{c(x, y_j) + h_j(\omega_j) + p_j - u_j\} < 0$, agents located at x choose to purchase the good at a location y_i for which $\min_{j=1,\cdots,N}\{c(x, y_j) + h_j(\omega_j) + p_j - u_j\} = c(x, y_i) + h_i(\omega_i) + p_i - u_i$. Thanks to the nondegeneracy condition (1), the sets of customers which are indifferent between either purchasing the good or not or between purchasing it optimally at two distinct locations are negligible. This implies that given the prices $\boldsymbol{p}$, the demand vector $\boldsymbol{\omega} = (\omega_1, \cdots, \omega_N)$ has to fulfill for every i the following consistency relations:

$$\omega_i = \mu(D_i(\boldsymbol{p}, \boldsymbol{\omega}) \cap D(\boldsymbol{p}, \boldsymbol{\omega})) \tag{2}$$

where

$$D(\boldsymbol{p},\boldsymbol{\omega}) := \{x \in \overline{\Omega} \ : \ \min_{j=1,\cdots,N}\{c(x,y_j) + h_j(\omega_j) + p_j - u_j\} < 0\}, \tag{3}$$

and

$$D_i(\boldsymbol{p},\boldsymbol{\omega}) := \cap_{j\neq i}\{x \in \overline{\Omega} \ : \ c(x,y_i) + h_i(\omega_i) + p_i - u_i < c(x,y_j) + h_j(\omega_j) + p_j - u_j\}. \tag{4}$$

The monopoly's problem then consists in maximizing its profit:

$$\Pi(\boldsymbol{p},\boldsymbol{\omega}) := \boldsymbol{p}\cdot\boldsymbol{\omega} - C(\boldsymbol{\omega})$$

subect to the condition that $\boldsymbol{\omega}$ and $\boldsymbol{p}$ are linked by the consistency conditions (2)-(3)-(4). This is a typical instance of bi-level program, we shall refer to the conditions (2)-(3)-(4) as the lower level (or equilibrium constraint) and shall call the upper level the profit maximization problem:

$$\sup_{\boldsymbol{p}\in\mathbb{R}_+^N} \sup\{\Pi(\boldsymbol{p},\boldsymbol{\omega}) \ : \ \omega \in \mathrm{LL}(\boldsymbol{p})\} \tag{5}$$

where $\mathrm{LL}(\boldsymbol{p})$ denotes the set of $\boldsymbol{\omega} \in \Delta_N$ for which (2)-(3)-(4) hold. The fact that the lower-level problem has a unique solution $\boldsymbol{\omega}(\boldsymbol{p})$ which can be found very conveniently by variational mass transport arguments was first emphasized by Crippa, Jimenez and Pratelli in [4] in the case where customers do not have the option of not purchasing the good at all. We shall see in section 4 that these arguments can be adapted to our framework by using partial optimal transport, let us also remark that our assumptions on the transport cost are more general than in [4].

3. Motivating examples

Before going further, we start with some simple examples with one or two facility locations on an interval and with a uniform distribution of consumers. Our primary aim is to emphasize that the market is in general not fully covered when prices optimize profit, we illustrate this in a simple linear city model. We then consider a toy model with a random demand, the general form of such models will be addressed in section 6.

- *A single facility case*

  We consider here the linear city case where $\Omega = (0,1)$, a uniform distribution of customers, the Euclidean distance as transport cost and a single facility located at the center of the city $y_1 = \frac{1}{2}$. Here, in the single facility case, we omit the index 1 in the notation. The congestion cost is taken proportional to the demand, i.e. $h(\omega) = \alpha\omega$ so that the strength of congestion is captured by the parameter $\alpha \geq 0$ and the utility derived from purchasing the good is given by $u > 0$. A direct computation shows that the demand $\omega(p)$ is given for $p < u$ (otherwise it is zero) by

$$\omega(p) = \begin{cases} \frac{2(u-p)}{1+2\alpha} \text{ if } u - p - \alpha \leq \frac{1}{2} \\ 1 \text{ otherwise} \end{cases}$$

Assuming that the production cost is quadratic: $C(\omega) = \frac{\lambda}{2}\omega^2$, the profit maximization can be solved explicitly and leads to two cases:

- if $u < 1 + 2\alpha + \lambda$ then the optimal prices, demand and profits are given by

$$p^* = \frac{1+2\alpha+2\lambda}{2+4\alpha+2\lambda}u,\ \omega^* = \frac{u}{1+2\alpha+\lambda},\ \Pi^* = \frac{u^2}{2+4\alpha+2\lambda},$$

- if $u \geq 1 + 2\alpha + \lambda$, the maximizing profit situation corresponds to

$$p^* = u - \alpha - \frac{1}{2},\ \omega^* = 1,\ \Pi^* = u - \alpha - \frac{1+\lambda}{2}.$$

In other words, the market is not fully covered if u is small compared to production and congestion costs i.e. whenever $u < 1 + 2\alpha + \lambda$.

- *Uncovered market with two facilities*

Consider now the case where the two facilities are located at the extreme points of the city $y_1 = 0$ and $y_2 = 1$. The transportation cost function is again the Euclidean distance and the congestion cost for each facility $h_i(\omega_i) = \alpha\omega_i$ proportional to the demand for this facility ($\alpha \geq 0$ as before). For simplicity we assume the same utility $u_1 = u_2 = u$ for both facilities. For a price vector $\boldsymbol{p} := (p_1, p_2)$ the demand $\boldsymbol{\omega} = (\omega_1, \omega_2)$ is given by

$$\omega_1 = \mu(D_1(\boldsymbol{p},\boldsymbol{\omega}) \cap D(\boldsymbol{p},\boldsymbol{\omega})), \omega_2 = \mu(D_2(\boldsymbol{p},\boldsymbol{\omega}) \cap D(\boldsymbol{p},\boldsymbol{\omega})) \tag{6}$$

where

$$D_1(\boldsymbol{p},\boldsymbol{\omega}) = \{x \in (0,1)\ :\ x < \frac{1}{2} + \frac{p_2 - p_1}{2(1+\alpha)}\}. \tag{7}$$

The demands are given by

$$\begin{aligned} &\omega_1 = \mu(D_1(\boldsymbol{p},\boldsymbol{\omega}) \cap D(\boldsymbol{p},\boldsymbol{\omega})) \\ = \ &\mu(\{x \in (0,1)\ :\ x < \frac{1}{2} + \frac{p_2 - p_1}{2(1+\alpha)}\} \cap \{x \in (0,1)\ :\ x < \frac{u - p_1}{1+\alpha}\}) \end{aligned} \tag{8}$$

$$\begin{aligned} \omega_2 &= \mu(D_2(\boldsymbol{p},\boldsymbol{\omega}) \cap D(\boldsymbol{p},\boldsymbol{\omega})) \\ = \quad & \mu(\{x \in (0,1) \;:\; x > \frac{1}{2} + \frac{p_2 - p_1}{2(1+\alpha)}\} \cap \{x \in (0,1) \;:\; x > 1 + \frac{p_2 - u}{1+\alpha}\}). \end{aligned} \tag{9}$$

Denote $L = 2u - (1+\alpha)$. For a vector price such that $p_1 + p_2 > L$ the market is not totally served since $\frac{u-p_1}{1+\alpha} < \frac{1}{2} + \frac{p_2-p_1}{2(1+\alpha)}$ and also $\frac{1}{2} + \frac{p_2-p_1}{2(1+\alpha)} < 1 + \frac{p_2-u}{1+\alpha}$. The demands are thus given by

$$\omega_1 = \begin{cases} \frac{u-p_1}{1+\alpha} & \text{if } L \leq p_1 + p_2 \\ \frac{1}{2} + \frac{p_2-p_1}{2(1+\alpha)} & \text{if } p_1 + p_2 < L \end{cases}$$

$$\omega_2 = \begin{cases} \frac{u-p_2}{1+\alpha} & \text{if } L \leq p_1 + p_2 \\ \frac{1}{2} + \frac{p_1-p_2}{2(1+\alpha)} & \text{if } p_1 + p_2 < L \end{cases}$$

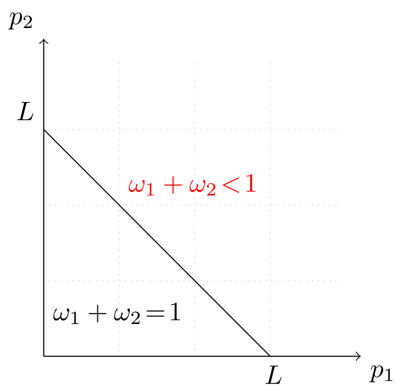

Figure 1: Uncovered market with two facilities.

In case of zero cost for the monopoly $C(\boldsymbol{\omega}) = 0$, the upper level problem reads

$$\sup_{\boldsymbol{p} \in \mathbb{R}^N_+} \boldsymbol{p} \cdot \boldsymbol{\omega} = \sup_{\boldsymbol{p} \in \mathbb{R}^N_+} \begin{cases} p_1\big(\frac{u-p_1}{1+\alpha}\big) + p_2\big(\frac{u-p_2}{1+\alpha}\big) & \text{if } L \leq p_1 + p_2 \\ p_1\big(\frac{1}{2} + \frac{p_2-p_1}{2(1+\alpha)}\big) + p_2\big(\frac{1}{2} + \frac{p_1-p_2}{2(1+\alpha)}\big) & \text{if } p_1 + p_2 < L. \end{cases}$$

If the congestion effect is large i.e. $u \leq 1+\alpha$ the optimal prices are $(\frac{u}{2}, \frac{u}{2})$ and the optimal profit obtained is $\frac{u^2}{2(1+\alpha)}$. Note that for $u = 1+\alpha$ the optimal price system is $(\frac{u}{2}, \frac{u}{2}) = (\frac{L}{2}, \frac{L}{2})$ and the market is totally served since the utility is sufficiently high to compensate the congestion effect.

- *A case with random utility*

Now, let us consider the case where the utility is no longer deterministic but contains a random component. For the sake of simplicity in our example, we just take zero congestion and production costs, assume again $\Omega = (0,1)$ with uniformly distributed customers and a single facility located at

$y = 0$. If we take a deterministic utility equal to $\frac{1}{2}$, the optimal price is unique equal to $\frac{1}{4}$ and the optimal profit is $\frac{1}{16}$. Now consider that the utility is random of the form $1 - \varepsilon$ with ε uniformly distributed on $[0, 1]$, the average utility is $\frac{1}{2}$ as in the previous deterministic case. Consumers located at x purchase the good whenever $x+p-1+\varepsilon < 0$, which has probability $F(1-p-x)$ where F denotes the cdf of the uniform distribution on $[0, 1]$, the demand thus takes the form $p \mapsto \omega(p) := \int_0^1 F(1-p-x)\mathrm{d}x$. If we take $p = \frac{1}{3}$, a direct computation gives that $\omega(\frac{1}{3}) = \frac{2}{9}$ yielding a profit $\frac{2}{27}$ larger than the optimal profit $\frac{1}{16}$ in the deterministic case. This example shows that the presence of randomness may actually increase the monopoly profits and should therefore be taken into account both at the lower and upper levels, this is what we shall do in section 6.

4. Solving the lower level by partial optimal transport

The goal of this section is to show that given $\boldsymbol{p} \in \mathbb{R}^N$, there exists a unique $\boldsymbol{\omega} \in \Delta_N$ which satisfies the lower-level problem $\mathrm{LL}(\boldsymbol{p})$. The main ingredient in this result is the use of a variational problem which involves a partial optimal transport problem (we refer the reader to the Appendix for details). First, let us set some notations, given $\boldsymbol{\omega} \in \Delta_N$ let us define $\boldsymbol{h}(\boldsymbol{\omega}) := (h_1(\omega_1), \cdots, h_N(\omega_N))$ and for $\boldsymbol{b} = (b_1, \cdots, b_N) \in \mathbb{R}^N$ let us introduce the (open in $\overline{\Omega}$) cells:

$$\boldsymbol{A}(\boldsymbol{b}) := \{x \in \overline{\Omega} \ : \ \min_{i=1,\cdots,N} \{c(x, y_i) - b_i\} < 0\} \tag{10}$$

and for every $i = 1, \cdots, N$:

$$\boldsymbol{A}_i(\boldsymbol{b}) := \cap_{j \neq i} \{x \in \overline{\Omega} \ : c(x, y_i) - b_i < c(x, y_j) - b_j\}. \tag{11}$$

So that one can rewrite the lower level requirement $\boldsymbol{\omega} \in \mathrm{LL}(\boldsymbol{p})$ as

$$\omega_i = \mu(\boldsymbol{A}_i(\boldsymbol{u} - \boldsymbol{p} - \boldsymbol{h}(\boldsymbol{\omega})) \cap \boldsymbol{A}(\boldsymbol{u} - \boldsymbol{p} - \boldsymbol{h}(\boldsymbol{\omega})), \ i = 1, \cdots, N. \tag{12}$$

For each i, let us introduce the strictly convex function H_i by

$$H_i(t) := \int_0^t h_i(s)\mathrm{d}s, \ \forall t \geq 0.$$

Let us define for every $\boldsymbol{\omega} \in \Delta_N$, the optimal partial transport cost:

$$\mathrm{M}_-(\boldsymbol{\omega}) := \begin{cases} \mathrm{MK}_-(c, \mu, \sum_{i=1}^N \omega_i \delta_{y_i}) \text{ if } \boldsymbol{\omega} \in \Delta_N \\ +\infty \text{ otherwise} \end{cases}$$

where $\mathrm{MK}_-(c, \mu, \nu)$ is, as defined in (22), the value of the partial optimal transport problem for the cost c between μ and ν (a positive measure with total mass less than 1). We refer to the Appendix for a detailed study of this function but it is easy to see that it is convex with respect to ν so that M_{-} is convex. Let us then fix $\boldsymbol{p} \in \mathbb{R}^N$ and define for every $\boldsymbol{\omega} \in \Delta_N$

$$J_{\boldsymbol{p}}(\boldsymbol{\omega}) := \mathrm{M}_{-}(\boldsymbol{\omega}) + (\boldsymbol{p} - \boldsymbol{u}) \cdot \boldsymbol{\omega} + \sum_{i=1}^{N} H_i(\omega_i) \tag{13}$$

Since $J_{\boldsymbol{p}}$ is a strictly convex (as the sum of the transport cost, which is convex and a strictly convex congestion cost) and lsc function, it admits a unique minimizer $\boldsymbol{\omega}(\boldsymbol{p})$ on the convex compact set Δ_N, moreover it is easy to check that $\boldsymbol{p} \mapsto \omega(\boldsymbol{p})$ is continuous by compactness and strict convexity.

Theorem 1. *Let $\boldsymbol{p} \in \mathbb{R}^N$ then $\boldsymbol{\omega}$ satisfies (12) if and only if $\boldsymbol{\omega}$ minimizes $J_{\boldsymbol{p}}$ over Δ_N i.e. $\boldsymbol{\omega} = \boldsymbol{\omega}(\boldsymbol{p})$. In particular* $\mathrm{LL}(p)$ *has one and only one solution $\boldsymbol{\omega}(\boldsymbol{p})$ and it depends continuously on $\boldsymbol{p}$.*

Proof. Note that $J_{\boldsymbol{p}}$ is the sum of the convex lsc function M_{-} and a convex and differentiable function whose gradient is $\boldsymbol{h} + \boldsymbol{p} - \boldsymbol{u}$, hence $\boldsymbol{\omega}$ minimizes $J_{\boldsymbol{p}}$ if and only if $0 \in \partial J_{\boldsymbol{p}}(\omega)$ i.e. $\boldsymbol{u} - \boldsymbol{p} - \boldsymbol{h}(\boldsymbol{\omega}) \in \partial \mathrm{M}_{-}(\boldsymbol{\omega})$ which is equivalent to the fact that $\boldsymbol{b} := \boldsymbol{u} - \boldsymbol{p} - \boldsymbol{h}(\boldsymbol{\omega})$ solves the dual of $\mathrm{MK}_{-}(c, \mu, \sum_{i=1}^{N} \omega_i \delta_{y_i})$ (see (28) in the Appendix) which is equivalent (see Lemma 5 in the Appendix) to the requirement that

$$\omega_i = \mu(\boldsymbol{A}_i(\boldsymbol{u} - \boldsymbol{p} - \boldsymbol{h}(\boldsymbol{\omega})) \cap \boldsymbol{A}(\boldsymbol{u} - \boldsymbol{p} - \boldsymbol{h}(\boldsymbol{\omega}))),\ i = 1, \cdots, N,$$

which is exactly (12). □

5. The upper level problem

Thanks to Theorem 1, we may rewrite the upper level problem (5) as

$$\sup_{\boldsymbol{p} \in \mathbb{R}_+^N} \pi(\boldsymbol{p}) \text{ where } \pi(\boldsymbol{p}) := \boldsymbol{p} \cdot \boldsymbol{\omega}(\boldsymbol{p}) - C(\boldsymbol{\omega}(\boldsymbol{p})) \tag{14}$$

and $\boldsymbol{\omega}(\boldsymbol{p})$ denotes the minimizer of $J_{\boldsymbol{p}}$ given by (13). To prove existence, it is useful to observe that one can impose a bound on prices:

Lemma 1. *Define $M := 1 + \max_{j=1,\cdots,N} u_j$ then for every $\boldsymbol{p} \in \mathbb{R}_+^N$ if we define $\widetilde{\boldsymbol{p}} = (\widetilde{p}_1, \cdots, \widetilde{p}_N)$ by $\widetilde{p}_i := \min(p_i, M)$ then $\boldsymbol{\omega}(\widetilde{\boldsymbol{p}}) = \boldsymbol{\omega}(\boldsymbol{p})$.*

Proof. First observe that since transport costs and congestion costs are nonnegative if $p_i \geq M$ then $\omega_i(p) = 0$ and therefore $\widetilde{\boldsymbol{p}} \cdot \boldsymbol{\omega}(\boldsymbol{p}) = \boldsymbol{p} \cdot \boldsymbol{\omega}(\boldsymbol{p})$. Set $\boldsymbol{\omega} := \boldsymbol{\omega}(\boldsymbol{p})$, to show that $\boldsymbol{\omega} = \boldsymbol{\omega}(\widetilde{\boldsymbol{p}})$ it is enough to show that $\widetilde{\boldsymbol{b}} := \boldsymbol{u} - \widetilde{\boldsymbol{p}} - \boldsymbol{h}(\boldsymbol{\omega})$ solves the dual of $\mathrm{MK}_{-}(c, \mu, \sum_{i=1}^{N} \omega_i \delta_{y_i})$ (see (28) in the Appendix). Setting $\boldsymbol{b} := \boldsymbol{u} - \boldsymbol{p} - \boldsymbol{h}(\boldsymbol{\omega})$, we first have

$$\mathrm{M}_{-}(\boldsymbol{\omega}) = \int_{\Omega} \min(0, \min_{i=1,\cdots,N} (c(x, y_i) - b_i)) \mathrm{d}\mu(x) + \boldsymbol{b} \cdot \boldsymbol{\omega}$$

but, by construction, $\widetilde{\boldsymbol{b}} \cdot \boldsymbol{\omega} = \boldsymbol{b} \cdot \boldsymbol{\omega}$ and

$$\min(0, \min_{i=1,\cdots,N}(c(x,y_i) - b_i)) = \min(0, \min_{i=1,\cdots,N}(c(x,y_i) - \widetilde{b}_i))$$

so that $\boldsymbol{\omega} = \boldsymbol{\omega}(\widetilde{\boldsymbol{p}})$. $\square$

We then have

Proposition 1. *The upper level program (14) admits at least one solution.*

Proof. It follows from Lemma 1 that the supremum of π on $\mathbb{R}_+^N$ is the same as the supremum of π over the compact set $[0, M]^N$. But since π is usc thanks to the lsc of C and the continuity of $\boldsymbol{p} \mapsto \boldsymbol{\omega}(\boldsymbol{p})$, π achieves its maximum over $[0, M]^N$ hence on $\mathbb{R}_+^N$.

$\square$

Remark 1. *The upper level problem can also be formulated in terms of quantities rather than prices. Indeed, we have seen that*

$$\boldsymbol{\omega} = \boldsymbol{\omega}(\boldsymbol{p}) \iff \boldsymbol{u} - \boldsymbol{p} \in \partial \mathrm{M}_-(\boldsymbol{\omega}) + \boldsymbol{h}(\boldsymbol{\omega}) \iff \boldsymbol{p} \in \boldsymbol{u} - (\partial \mathrm{M}_- + \boldsymbol{h})(\boldsymbol{\omega})$$

and $\boldsymbol{u} - (\partial \mathrm{M}_- + \boldsymbol{h})(.)$ has a closed graph. So one can write the upper level as an optimization problem on $\boldsymbol{\omega}$ (which lives on a compact set) as

$$\sup_{\boldsymbol{\omega} \in \Delta_N} (\boldsymbol{u} - \boldsymbol{h}(\boldsymbol{\omega})) \cdot \boldsymbol{\omega} - C(\boldsymbol{\omega}) - \inf_{\boldsymbol{q} \in \partial \mathrm{M}_-(\boldsymbol{\omega})} \boldsymbol{q} \cdot \boldsymbol{\omega}.$$

6. Extension to the random utility case

We now consider the case where the utility may be random, i.e. in addition to the deterministic utility level u_i there is some noisy component ε_i, where for $i = 1, \cdots, N$, ε_i is a random variable defined on some probability space $(\mathcal{A}, \mathcal{F}, \mathbb{P})$. We then set $\boldsymbol{\varepsilon} := (\varepsilon_1, \cdots, \varepsilon_N)$ and assume that the noise is integrable and satisfies a nonatomicity assumption:

$$\boldsymbol{\varepsilon} \in L^1((\mathcal{A}, \mathcal{F}, \mathbb{P}), \mathbb{R}^N), \ \mathbb{P}(\varepsilon_i - \varepsilon_j = t) = 0 = \mathbb{P}(\varepsilon_i = t), \ \forall t \in \mathbb{R}, \forall i \neq j. \tag{15}$$

Given a vector of net utilities $\boldsymbol{u} \in \mathbb{R}^N$ to which we add the noise $\boldsymbol{\varepsilon}$ we define, similarly to what we did in section 4, the following sets (which are now random):

$$\boldsymbol{A}(\boldsymbol{b} + \boldsymbol{\varepsilon}) := \{x \in \overline{\Omega} \ : \ \min_{i=1,\cdots,N}\{c(x,y_i) - b_i - \varepsilon_i\} < 0\} \tag{16}$$

and for every $i = 1, \cdots, N$:

$$\boldsymbol{A}_i(\boldsymbol{b}+\boldsymbol{\varepsilon}) := \cap_{j\neq i}\{x \in \overline{\Omega}\ : c(x, y_i) - b_i - \varepsilon_i < c(x, y_j) - b_j - \varepsilon_j\}. \tag{17}$$

Given a price system $\boldsymbol{p}$, the total utility of an agent purchasing the good at location y_i is now random and given by $c(x, y_i) + h_i(\omega_i) + p_i - u_i - \varepsilon_i$ where ω_i is the total demand for location y_i and $h_i(\omega_i)$ represents the congestion cost. The lower-level problem, consists, given $\boldsymbol{p}$, in finding $\boldsymbol{\omega} = (\omega_1, \cdots, \omega_N) \in \Delta_N$ such that for every i, ω_i coincides with the average demand for location i, in the random setting, this reads as:

$$\omega_i = \mathbb{E}\Big(\mu(\boldsymbol{A}_i(\boldsymbol{u}+\boldsymbol{\varepsilon}) \cap \boldsymbol{A}(\boldsymbol{u}+\boldsymbol{\varepsilon}))\Big),\ i = 1, \cdots, N,\ \text{with } \boldsymbol{b} := \boldsymbol{u} - \boldsymbol{p} - \boldsymbol{h}(\boldsymbol{\omega}). \tag{18}$$

We can adapt the variational approach of section 4 to this random setting, and in particular define a noisy analogue of the partial transport cost M_- which we define for $\boldsymbol{\omega} \in \Delta_N$ by:

$$\mathrm{NM}_-(\boldsymbol{\omega}) := \sup_{\boldsymbol{b}\in\mathbb{R}^N}\{\boldsymbol{b}\cdot\boldsymbol{\omega} + \int_\Omega \mathbb{E}\Big(\min(0, \min_i(c(x, y_i) - b_i - \varepsilon_i))\Big)\mathrm{d}\mu(x)\} \tag{19}$$

we refer to the appendix for a detailed study of the convex and lsc functional NM_- and in particular a dual formula. For fixed $\boldsymbol{p} \in \mathbb{R}^N$, define for every $\boldsymbol{\omega} \in \Delta_N$

$$\mathrm{NJ}_{\boldsymbol{p}}(\boldsymbol{\omega}) := \mathrm{NM}_-(\boldsymbol{\omega}) + (\boldsymbol{p} - \boldsymbol{u})\cdot\boldsymbol{\omega} + \sum_{i=1}^N H_i(\omega_i) \tag{20}$$

Since $\mathrm{NJ}_{\boldsymbol{p}}$ is a strictly convex and lsc function, it admits a unique minimizer $\boldsymbol{\omega}(\boldsymbol{p})$ on the convex compact set Δ_N, again it is easy to see that it implies that $\boldsymbol{p} \mapsto \omega(\boldsymbol{p})$ is continuous.

Theorem 2. *Let $\boldsymbol{p} \in \mathbb{R}^N$ then $\boldsymbol{\omega}$ solves the noisy lower-level problem (18) if and only if $\boldsymbol{\omega}$ minimizes $\mathrm{NJ}_{\boldsymbol{p}}$ over Δ_N. In particular (18) has one and only one solution $\boldsymbol{\omega}(\boldsymbol{p})$ and it depends continuously on $\boldsymbol{p}$.*

Proof. As in the proof of theorem 1, $\boldsymbol{\omega}$ minimizes $\mathrm{NJ}_{\boldsymbol{p}}$ if and only if $0 \in \partial\mathrm{NJ}_{\boldsymbol{p}}(\omega)$ i.e. $\boldsymbol{u} - \boldsymbol{p} - \boldsymbol{h}(\boldsymbol{\omega}) \in \partial\mathrm{NM}_-(\boldsymbol{\omega})$ which, by duality (see the Appendix and in particular Lemma 6 and (38)-(39) for details) is equivalent to the fact that $\boldsymbol{b} := \boldsymbol{u} - \boldsymbol{p} - \boldsymbol{h}(\boldsymbol{\omega})$ solves the concave maximization problem

$$\sup_{\boldsymbol{b}\in\mathbb{R}^N}\{\boldsymbol{b}\cdot\boldsymbol{\omega} + V(\boldsymbol{b})\} \tag{21}$$

where

$$V(\boldsymbol{b}) := \int_\Omega \mathbb{E}\Big(\min(0, \min_i(c(x, y_i) - b_i - \varepsilon_i))\Big)\mathrm{d}\mu(x)$$

but, thanks to (15) and Lebesgue's dominated convergence theorem, it is easy to see that V is differentiable with

$$\frac{\partial V}{\partial b_i}(\boldsymbol{b}) = -\mathbb{E}\Big(\mu(\boldsymbol{A}_i(\boldsymbol{u}+\boldsymbol{\varepsilon}) \cap \boldsymbol{A}(\boldsymbol{u}+\boldsymbol{\varepsilon}))\Big).$$

The necessary and sufficient optimality condition for (21) therefore is

$$\omega_i = \mathbb{E}\Big(\mu(\boldsymbol{A}_i(\boldsymbol{u}+\boldsymbol{\varepsilon}) \cap \boldsymbol{A}(\boldsymbol{u}+\boldsymbol{\varepsilon}))\Big),\ i = 1, \cdots, N,$$

so that (18) is a necessary and sufficient condition for $\boldsymbol{\omega}$ to minimize $\mathrm{NJ}_{\boldsymbol{p}}$.

$\square$

Remark 2. *If we assume that the noises are upper bounded $\varepsilon_i \leq \lambda$, almost surely and for every i, for some nonnegative constant λ, then, arguing exactly as in lemma 1, and setting $M := 1 + \lambda + \max_{j=1,\cdots,N} u_j$ one can see that changing a price system $\boldsymbol{p} \in \mathbb{R}^N_+$ into $\widetilde{\boldsymbol{p}} = (\widetilde{p}_1, \cdots, \widetilde{p}_N)$ with $\widetilde{p}_i := \min(p_i, M)$ then one still has $\boldsymbol{\omega}(\widetilde{\boldsymbol{p}}) = \boldsymbol{\omega}(\boldsymbol{p})$ in the noisy setting and $\boldsymbol{p} \cdot \boldsymbol{\omega}(\boldsymbol{p}) = \widetilde{p} \cdot \boldsymbol{\omega}(\boldsymbol{p})$ so that both $\boldsymbol{p}$ and $\widetilde{\boldsymbol{p}}$ give the same profit. The upper level profit maximization problem can therefore again be brought down to a maximization problem on a compact set, existence of at least one optimal price directly follows.*

7. Appendix

7.1. On the partial optimal transport problem

Duality

In this appendix, we gather some useful results on the partial optimal mass transport problems which we have used in the paper. These results are all more or less folklore in the field, we recall them for the sake of completeness and for the reader's convenience, we refer to [7] for further results in particular on the regularity of optimal partial transport.

Let $\nu \in \mathcal{M}_+(\overline{\Omega})$ (ν a (positive) Borel measure on $\overline{\Omega}$) such that $\nu(\overline{\Omega}) \leq 1$, the optimal partial transport problem between μ (that is a probability) and ν (with mass less than 1) is then defined as

$$\mathrm{MK}_-(c, \mu, \nu) := \inf_{\gamma \in \Pi_-(\mu,\nu)} \int_{\overline{\Omega}\times\overline{\Omega}} c(x, y)\mathrm{d}\gamma(x, y) \tag{22}$$

where

$$\Pi_-(\mu, \nu) := \{\gamma \in \mathcal{P}(\overline{\Omega} \times \overline{\Omega})\ \ \pi_{1\#}\gamma \leq \mu,\ \pi_{2\#}\gamma = \nu\} \tag{23}$$

and $\pi_1(x, y) = x$, $\pi_2(x, y) = y$ so that $\Pi_-(\mu, \nu)$ is the set of Borel measures on $\overline{\Omega} \times \overline{\Omega}$ with second marginal ν and first marginal less than μ. Note that $\Pi_-(\mu, \nu)$ is a convex and weakly $*$-compact set of measures so that the continuity of c ensures that (22) admits a solution. As a linear programming problem, (22) also admits the following dual expression:

$$\mathrm{MK}_-(c, \mu, \nu) := \sup\Big\{ \int_{\Omega} a\mathrm{d}\mu + \int_{\Omega} b\mathrm{d}\nu\ :\ a, b \in C^0(\overline{\Omega}), a \leq 0,\ a \oplus b \leq c\Big\} \tag{24}$$

where $a \oplus b$ denotes $(x, y) \mapsto a(x) + b(y)$. We omit the proof of this duality formula which can easily be deduced from Fenchel-Rockafellar's theorem [10]. Note that for fixed b, the constraint on a can be rewritten as

$$a \leq \min(0, b^c) \text{ where } b^c(x) := \min_y \{c(x,y) - b(y)\}$$

so that the dual can be rewritten in terms of b only as

$$\sup_{b \in C^0(\overline{\Omega})} \Big\{ \int_{\overline{\Omega}} \min(0, b^c) \mathrm{d}\mu + \int_{\overline{\Omega}} b \mathrm{d}\nu \Big\} \tag{25}$$

or similarly in terms of a only as

$$\sup_{a \in C^0(\overline{\Omega}), a \leq 0} K(a) \text{ where } K(a) := \int_{\overline{\Omega}} a \mathrm{d}\mu + \int_{\overline{\Omega}} a_c \mathrm{d}\nu \tag{26}$$

where $a_c(y) := \min_x \{c(x,y) - a(x)\}$. Now let us explain why (26) (hence (24)) admits a solution:

Lemma 2. *The dual problem (26) admits solutions.*

Proof. Take $a \in C^0(\overline{\Omega})$ with $a \leq 0$ and set $\hat{a} = \min(0, (a_c)^c)$, it is well known (see for instance [11]) that $(a_c)^c \geq a$ and that $((a_c)^c)_c = a_c$ but since $a \mapsto a^c$ is order reversing and $\hat{a} \leq (a_c)^c$ we have $\hat{a}_c \geq ((a_c)^c)_c = a_c$ so that $K(\hat{a}) \geq K(a)$. Define then $\tilde{a} = \hat{a} - \max \hat{a}$ then $K(\tilde{a}) = K(\hat{a}) - \max \hat{a}(1 - \nu(\overline{\Omega})) \geq K(\hat{a}) \geq K(a)$. We then take a maximizing sequence (a_n) for (26), by replacing it as previously by $\tilde{a}_n$, we obtain another maximizing sequence such that $\max \tilde{a}_n = 0$ for every n and which is uniformly equicontinuous (because the set $\{\min(0, (a_c)^c), a \in C^0(\overline{\Omega})\}$ is uniformly equicontinuous). It thus follows from Ascoli-Arzela's theorem that $\tilde{a}_n$ has a subsequence which converges uniformly to some nonpositive continuous function a and it is then clear that a solves (26).

$\square$

Semi discrete case

We now consider the case where $\nu := \sum_{i=1}^N \omega_i \delta_{y_i}$ where $\boldsymbol{\omega} \in \Delta_N$ and assume that the cost c and the probability μ satisfy the nondegeneracy condition (1) and simply set

$$\mathrm{M}_-(\boldsymbol{\omega}) := \mathrm{MK}_-(c, \mu, \sum_{i=1}^N \omega_i \delta_{y_i}).$$

In this semi discrete setting, the duality formula (25) can be written in terms of a single vector of prices $\boldsymbol{b} = (b_1, \cdots, b_N)$ at the locations $(y_1, \cdots y_N)$ and takes the form:

$$\mathrm{M}_-(\boldsymbol{\omega}) = \sup_{\boldsymbol{b} \in \mathbb{R}^N} \Big\{ \int_{\overline{\Omega}} \min(0, \min_{i=1,\cdots,N} (c(x, y_i) - b_i)) \mathrm{d}\mu(x) + \sum_{i=1}^N b_i \omega_i \Big\} \tag{27}$$

It is now convenient to introduce the (convex and continuous) function

$$E(\boldsymbol{b}) := - \int_{\overline{\Omega}} \min(0, \min_{i=1,\cdots,N} (c(x, y_i) - b_i)) \mathrm{d}\mu(x), \ \forall \boldsymbol{b} \in \mathbb{R}^N$$

and to observe that

Lemma 3. *The Legendre transform of E, E^* has the form*

$$E^*(\boldsymbol{\omega}) = \begin{cases} \mathrm{M}_-(\boldsymbol{\omega}) \text{ if } \boldsymbol{\omega} \in \Delta_N \\ +\infty \text{ otherwise.} \end{cases}$$

Proof. If $\boldsymbol{\omega} \in \Delta_N$, this is just the duality formula (27), it remains to prove that $E^*(\boldsymbol{\omega}) = +\infty$ if one of the ω_i's is negative (but it is clear by taking $b_j = 0$ for $j \neq i$ and $b_i \to -\infty$) or if $\sum_{i=1}^N \omega_i > 1$ (but again this is clear by taking $\boldsymbol{b} = (\lambda, \cdots \lambda)$, letting $\lambda \to \infty$). ◻

From this lemma, we immediately deduce that for $\boldsymbol{b}$ and $\boldsymbol{\omega}$ in $\mathbb{R}^N$, we have the equivalence:

$$\boldsymbol{b} \in \partial E^*(\boldsymbol{\omega}) \iff \boldsymbol{\omega} \in \partial E(\boldsymbol{b}) \iff \boldsymbol{\omega} \in \Delta_N \text{ and } \boldsymbol{b} \text{ solves (27).} \tag{28}$$

It remains to characterize the solutions of $\mathrm{MK}_-(c, \mu, \sum_{i=1}^N \omega_i \delta_{y_i})$ and its dual (27):

Lemma 4. *Assume (1). Let $\boldsymbol{b}$ solve (27) then the partial optimal transport problem* $\mathrm{MK}_-(c, \mu, \sum_{i=1}^N \omega_i \delta_{y_i})$ *admits a unique solution which is given by*

$$\gamma := \Big(\mathrm{id}, \sum_{i=1}^N \mathbf{1}_{\boldsymbol{A}_i(\boldsymbol{b})} y_i\Big)_\# (\mathbf{1}_{\boldsymbol{A}(\boldsymbol{b})}\mu), \tag{29}$$

where $A_i(\boldsymbol{b})$ and $A(\boldsymbol{b})$ are given by (11) and (10).

Proof. Let $\gamma \in \Pi_-(\mu, \sum_{i=1}^N \omega_i \delta_{y_i})$ solve $\mathrm{MK}_-(c, \mu, \sum_{i=1}^N \omega_i \delta_{y_i})$ then defining $\theta := \pi_{1\#}\gamma$, one can write γ is the form

$$\gamma(\mathrm{d}x, \mathrm{d}y) = \theta(\mathrm{d}x) \otimes \Big(\sum_{i=1}^N \alpha_i(x)\delta_{y_i}(\mathrm{d}y)\Big), \text{ with } \alpha_i \geq 0, \ \sum_{i=1}^N \alpha_i = 1 \ \theta\text{-a.e.} \tag{30}$$

and since $\pi_{2\#}\gamma = \sum_{i=1}^N \omega_i \delta_{y_i}$, one has

$$\int_{\overline{\Omega}} \alpha_i \mathrm{d}\theta = \omega_i, \text{ for } i = 1, \cdots, N. \tag{31}$$

It now follows from (27) that

$$\int_{\overline{\Omega}\times\overline{\Omega}} c \, \mathrm{d}\gamma = \sum_{i=1}^N \int_{\overline{\Omega}} \alpha_i(x)c(x, y_i)\mathrm{d}\theta(x) = \int_{\overline{\Omega}} a \, \mathrm{d}\mu + \sum_{i=1}^N b_i\omega_i \tag{32}$$

where $a(x) := \min(0, \varphi)$ with $\varphi(x) := \min_i\{c(x, y_i) - b_i\}$. By construction we have $c(x, y_i) \geq \varphi(x) + b_i$ hence, thanks to (31)

$$\sum_{i=1}^N \int_{\overline{\Omega}} \alpha_i(x)c(x, y_i)\mathrm{d}\theta(x) \geq \int_{\overline{\Omega}} \varphi \mathrm{d}\theta + \sum_{i=1}^N b_i\omega_i \tag{33}$$

but since $\varphi \geq a$ and $\theta \leq \mu$ and $a \leq 0$, we have

$$\int_{\overline{\Omega}\times\overline{\Omega}} c \, \mathrm{d}\gamma \geq \int_{\overline{\Omega}} a \mathrm{d}\theta + \boldsymbol{b}\cdot\boldsymbol{\omega} \geq \int_{\overline{\Omega}} a \mathrm{d}\mu + \boldsymbol{b}\cdot\boldsymbol{\omega}. \tag{34}$$

With (32), we deduce that all inequalities in (33) and (34) should be equalities. Thanks to (1), equality in (33) firstly implies that $\alpha_i = \mathbf{1}_{\boldsymbol{A}_i(\boldsymbol{p})}$ μ-a.e. hence also θ-a.e.. Secondly, having two equalities in (34) means that $a = \varphi$, i.e. $\varphi \leq 0$ on $\mathrm{Supp}(\theta)$ and $a = 0$, i.e. $\varphi \geq 0$ on $\mathrm{Supp}(\mu - \theta)$, since $\mu(\{\varphi = 0\}) = 0$ thanks to (1). This implies that $\theta = \mathbf{1}_{\{\varphi<0\}}\mu = \mathbf{1}_{\boldsymbol{A}(\boldsymbol{p})}\mu$ which proves (29).

□

Lemma 5. *Assume (1). Let $\boldsymbol{b} \in \mathbb{R}^N$ then $\boldsymbol{b}$ solves (27) if and only if*

$$\omega_i = \mu(\boldsymbol{A}_i(\boldsymbol{b}) \cap \boldsymbol{A}(\boldsymbol{b})) \text{ for } i = 1, \cdots, N \tag{35}$$

where $\boldsymbol{A}_i(\boldsymbol{b})$ and $\boldsymbol{A}(\boldsymbol{b})$ are given by (11) and (10).

Proof. Necessity follows from Lemma 4, indeed if $\boldsymbol{b}$ solves (27), the second marginal of γ given by (29) being $\sum_{i=1}^N \omega_i \delta_{y_i}$ directly gives (35). Conversely, assume that $\boldsymbol{b}$ satisfies (35) and define γ by (29), then $\gamma \in \Pi_-(\mu, \sum_{i=1}^N \omega_i \delta_{y_i})$ and a direct computation gives

$$\begin{aligned}\int_{\overline{\Omega}\times\overline{\Omega}} c \, \mathrm{d}\gamma &= \sum_{i=1}^N \int_{\boldsymbol{A}_i(\boldsymbol{b})\cap\boldsymbol{A}(\boldsymbol{b})} c(x, y_i) \mathrm{d}\mu(x) \\ &= \int_{\overline{\Omega}} \min(0, \min_{j=1,\cdots,N}(c(x, y_j) - b_j)) \, \mathrm{d}\mu + \sum_{i=1}^N b_i \omega_i\end{aligned}$$

which, by duality, implies that γ solves $\mathrm{MK}_-(c, \mu, \sum_{i=1}^N \omega_i \delta_{y_i})$ and $\boldsymbol{b}$ solves (27).

□

7.2. Optimal partial transport with noise

In what follows the random vector $\boldsymbol{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_N) \in L^1((\mathcal{A}, \mathcal{F}, \mathbb{P}), \mathbb{R}^N)$ satisfies (15). Given $\boldsymbol{\omega} \in \Delta_N$, our starting point for studying the noisy lower-level problem (18) is the following maximization problem whose value defines the function NM_-:

$$\mathrm{NM}_-(\boldsymbol{\omega}) := \sup_{\boldsymbol{b}\in\mathbb{R}^N} \{\boldsymbol{b}\cdot\boldsymbol{\omega} + V(\boldsymbol{b})\} \tag{36}$$

where

$$V(\boldsymbol{b}) := \int_{\overline{\Omega}} \mathbb{E}\Big(\min(0, \min_i(c(x, y_i) - b_i - \varepsilon_i))\Big) \mathrm{d}\mu(x) \tag{37}$$

Obviously, NM_- is convex lsc on Δ_N. Moreover, since μ is a probability measure,

$$V(\boldsymbol{b}) \leq \max\{c(x, y_i)\ :\ x \in \overline{\Omega},\ i = 1, \cdots, N\} + \mathbb{E}|\boldsymbol{\varepsilon}| - \max_i b_i,$$

and then, for every $\boldsymbol{\omega} \in \Delta_N$

$$\mathrm{NM}_-(\boldsymbol{\omega}) \leq \max\{c(x, y_i)\ :\ x \in \overline{\Omega},\ i = 1, \cdots, N\} + \mathbb{E}|\boldsymbol{\varepsilon}|$$

which implies that NM_- is also bounded on Δ_N. Clearly V is concave and 1-Lipschitz, the Legendre transform of $\mathrm{NE} := -V$ is characterized by:

Lemma 6. *The Legendre transform of* NE, NE^* *is*

$$\mathrm{NE}^*(\boldsymbol{\omega}) = \begin{cases} \mathrm{NM}_-(\boldsymbol{\omega}) \ \textit{if } \boldsymbol{\omega} \in \Delta_N \\ +\infty \ \textit{otherwise.} \end{cases}$$

Proof. If $\boldsymbol{\omega} \in \Delta_N$, this is just the definition of $\mathrm{NM}_-(\boldsymbol{\omega})$ given in (36) and it can be proved that $\mathrm{NE}^*(\boldsymbol{\omega}) = +\infty$ if one of the ω_i's is negative similarly to Lemma 3. $\square$

From this lemma, we immediately deduce that for $\boldsymbol{b}$ and $\boldsymbol{\omega}$ in $\mathbb{R}^N$, we have the equivalence:

$$\boldsymbol{b} \in \partial \mathrm{NM}_-(\boldsymbol{\omega}) \iff \boldsymbol{\omega} \in \partial \mathrm{NE}(\boldsymbol{b}) \tag{38}$$

but assumption (15) and Lebesgue's dominated convergence theorem ensure that V is differentiable so that the previous conditions are also equivalent to:

$$\boldsymbol{\omega} + \nabla V(\boldsymbol{b}) = 0 \iff \boldsymbol{\omega} \in \Delta_N \text{ and } \boldsymbol{b} \text{ solves (36).} \tag{39}$$

Eventhough it is not essential for the proof of Theorem 2, we would like to emphasize here the fact that $\mathrm{NM}_-(\boldsymbol{\omega})$ can be expressed by a dual expression for (36) which is connected to a partial mass transport problem in the sense that it can be expressed as an infimum over the set of subplans $\Pi_-(\mu, \sum_{i=1}^N \omega_i \delta_{y_i})$. Let $Y := \{y_1, \cdots, y_N\}$ and endow $C^0(\overline{\Omega} \times Y) \simeq C^0(\overline{\Omega})^N$ with the uniform norm, for $\psi \in C^0(\overline{\Omega} \times Y)$, define then

$$G(\psi) := \int_{\overline{\Omega}} \mathbb{E}(\max_{i=1,\cdots,N} (\psi(x, y_i) + \varepsilon_i)_+) \mathrm{d}\mu(x)$$

and observe that G is a convex and 1-Lipschitz function on $C^0(\overline{\Omega} \times Y)$. For $\boldsymbol{b} \in \mathbb{R}^N$ define $\Lambda \boldsymbol{b} \in C^0(\overline{\Omega} \times Y)$ by $(\Lambda \boldsymbol{b})(x, y_i) := b_i$, we then have

$$\mathrm{NE}(\boldsymbol{b}) = -V(\boldsymbol{b}) = G(\Lambda \boldsymbol{b} - c),\ \forall \boldsymbol{b} \in \mathbb{R}^N$$

so that

$$\mathrm{NM}_-(\boldsymbol{\omega}) = \sup_{\boldsymbol{b}\in\mathbb{R}^N} \{\boldsymbol{b}\cdot\boldsymbol{\omega} - G(\Lambda b - c)\}$$

which by an application of Fenchel-Rockafellar duality theorem [10] can be rewritten as

$$\mathrm{NM}_-(\boldsymbol{\omega}) = \inf_{\gamma\in\mathcal{M}(\overline{\Omega}\times Y)\,:\,\Lambda^*\gamma=\sum_{i=1}^N \omega_i\delta_{y_i}} \Big\{ \int_{\Omega\times Y} c\mathrm{d}\gamma + G^*(\gamma)\Big\}$$

we then observe that $\Lambda^*\gamma = \pi_{2\#}\gamma$ and that if $G^*(\gamma) < +\infty$ then necessarily $\gamma \in \mathcal{M}_+(\overline{\Omega}\times Y)$ and $\pi_{1\#}\gamma \leq \mu$. In other words, the set of joint measures γ in the domain of G^* and such that $\pi_{2\#}\gamma = \sum_{i=1}^N \omega_i\delta_{y_i}$ is included in the set of subplans $\Pi_-(\mu, \sum_{i=1}^N \omega_i\delta_{y_i})$ hence

$$\mathrm{NM}_-(\boldsymbol{\omega}) = \inf_{\gamma\in\Pi_-(\mu,\sum_{i=1}^N \omega_i\delta_{y_i})} \Big\{ \int_{\Omega\times Y} c\mathrm{d}\gamma + G^*(\gamma)\Big\}.$$

Note that this is a strictly convex perturbation (G^* is strictly convex since G is differentiable) of the partial mass transport problem $\mathrm{MK}_-(c, \mu, \sum_{i=1}^N \omega_i\delta_{y_i})$ where the additional penalization term G^* comes from the noise on the utility.

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