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**Abstract**

This paper addresses the problem of estimating, in the presence of random censoring as well as competing risks, the extreme value index of the (sub)-distribution function associated to one particular cause, in the heavy-tail case. Asymptotic normality of the proposed estimator (which has the form of an Aalen-Johansen integral, and is the first estimator proposed in this context) is established. A small simulation study exhibits its performances for finite samples. Estimation of extreme quantiles of the cumulative incidence function is also addressed.

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## 1. Introduction

The study of duration data (lifetime, failure time, re-employment time...) subject to random censoring is a major topic of the domain of statistics, which finds applications in many areas (in the sequel we will, for convenience, talk about lifetimes to refer to these observed durations, but without restricting our scope to lifetime data analysis). In general, the interest lies in obtaining informations about the central characteristics of the underlying lifetime distribution (mean lifetime or survival probabilities for instance), often with the objective of comparing results between different conditions under which the lifetime data are acquired. In this work, we will address the problem of inferring about the (upper) tail of the lifetime distribution, for data subject both to random (right) censoring and competing risks.

Suppose indeed that we are interested in the lifetimes of $n$ individuals or items, which are subject to $K$ different causes of death or failure, and to random censorship (from the right) as well. We are particularly interested in one of these causes (this main cause will be considered as cause number $k$ thereafter, where $k \in \{1, \ldots, K\}$), and we suppose that all causes are exclusive and are likely to be dependent on the others. The censoring time is assumed to be independent of the different causes of death or failure and of the observed lifetime itself. However, since the other causes (different from the $k$-th cause of interest) generally cannot be considered as independent of the main cause, in no way they can be included in the censoring mechanism. This prevents us from relying on the basic independent censoring statistical framework, and we are thus in the presence of what is called a competing risks framework (see Moeschberger and Klein (1995)).

For instance, if a patient is suffering from a very serious disease and starts some treatment, then the final outcome of the treatment can be death due to the main disease, or death due to other causes (nosocomial infection for instance). And censoring can occur due to loss of follow up or end of the clinical study. Another example, in a reliability experiment, is that the failure of some mechanical system can be due to the failure of a particular subpart, or component, of the system : since separating the different components for studying the reliability of only one of them is generally not possible, accounting for these different competing causes of failure is necessary. Another field where competing risks often arise are labor economics, for instance in re-employment studies (see Fermanian (2003) for practical examples).

One way of formalising this is to say that we observe a sample of $n$ independent couples $(Z_i, \xi_i)_{1\leqslant i\leqslant n}$ where

$$Z_i = \min(X_i, C_i), \quad \delta_i = \mathbb{I}_{X_i \leqslant C_i}, \quad \xi_i = \begin{cases} 0 & \text{if } \delta_i = 0, \\ \mathscr{C}_i & \text{if } \delta_i = 1. \end{cases}$$

The i.i.d. samples $(X_i)_{i\leqslant n}$ and $(C_i)_{i\leqslant n}$, of respective continuous distribution functions $F$ and $G$, represent the lifetimes and censoring times of the individuals, and are supposed to be independent. For convenience, we will suppose in this work that they are non-negative. The variables $(\mathscr{C}_i)_{i\leqslant n}$ form a discrete sample with values in $\{1, \ldots, K\}$, and represent the causes of failure or death of the $n$ individuals or items. It is important to note that these causes are observed only when the data is uncensored (*i.e.* when $\delta_i = 1$), therefore we only observe the $\xi_i$'s, not the complete $\mathscr{C}_i$'s.

One way of considering the failure times $X_i$ is to write

$$X_i = \min(X_{i,1}, \ldots, X_{i,K}),$$

where the variable $X_{i,k}$ is a (rather artificial) variable representing the imaginary latent lifetime of the $i$-th individual when the latter is only affected by the $k$-th cause (the other causes being absent). This viewpoint may be interesting in its own right, but we will not keep on considering it in the sequel, one reason being that such variables $X_{i,1}, \ldots, X_{i,K}$ cannot be realistically considered as independent, and their respective distributions are of no practical use or interpretability (as explained and demonstrated in the competing risks literature, these distributions are in fact not statistically identifiable, see Tsiatis (1975) for example).

The object of interest is the probability that a subject dies or fails after some given time $t$, due to the $k$-th cause, for high values of $t$. This quantity, denoted by

$$\bar{F}^{(k)}(t) = \mathbb{P}[\, X > t \,, \mathscr{C} = k \,],$$

is related to the so-called *cumulative incidence function* $F^{(k)}$ defined by

$$F^{(k)}(t) = \mathbb{P}[\, X \leqslant t \,, \mathscr{C} = k \,].$$

Note that $\bar{F}^{(k)}(t)$ is not equal to $1 - F^{(k)}(t)$, but to $\mathbb{P}(\mathscr{C} = k) - F^{(k)}(t)$, because $F^{(k)}$ is only a sub-distribution function. However we have $\bar{F}^{(k)}(t) = \int_t^\infty dF^{(k)}(u)$. In the sequel, the notation $\bar{S}(.) = S(\infty) - S(.)$ will be used, for any non-decreasing function $S$.

In this paper, we are interested in investigating the behaviour of $\bar{F}^{(k)}(t)$ for large values of $t$. This amounts to statistically study extreme values in a context of censored data under competing risks, and will lead us to consider some extreme value index $\gamma_k$ related to $\bar{F}^{(k)}$, which will be defined in a few lines. Equivalently, the object of interest is the high quantile $x_p^{(k)} = (\bar{F}^{(k)})^{-}(p) = \inf\{ x \in \mathbb{R} \,;\, \bar{F}^{(k)}(x) \geqslant p \}$ when $p$ is close to 0, which can be interpreted as follows (in the context of lifetimes of individuals or failure times of systems) : in the presence of the other competing causes, a given individual (or item) will die (or fail), due to cause $k$ after such a time $x_p^{(k)}$, only with small probability $p$. A nonparametric inference for quantiles of fixed (and therefore not extreme) order, in the competing risk setting, has been already proposed in Peng and Fine (2007).

One way of addressing this problem could be through a parametric point of view (see Crowder (2001) for further methods in the competing risk setting), however, the non-parametric approach is the most common choice of people faced with data presenting censorship or competing risks. Of course, the standard Kaplan-Meier method for survival analysis does not yield valid results for a particular risk if failures from other causes are treated as censoring times, because the other causes cannot always be considered independent of the particular cause of interest.

The commonly used nonparametric estimator of the cumulative incidence function $F^{(k)}$ is the so-called Aalen-Johansen estimator (see Aalen and Johansen (1978), or Geffray (2009) equation (7)) defined by

$$F_n^{(k)}(t) = \sum_{Z_i \leqslant t} \frac{\delta_i \mathbb{I}_{\mathscr{C}_i = k}}{n\bar{G}_n(Z_i^-)},$$

where $\bar{G}_n$ denotes the standard Kaplan-Meier estimator of $G$ (and $\bar{G}_n(t^-)$ denotes $\lim_{s\uparrow t} \bar{G}_n(s)$), so that we can introduce the following estimator for $\bar{F}^{(k)}$ :

$$\bar{F}_n^{(k)}(t) = \sum_{Z_i > t} \frac{\delta_i \mathbb{I}_{\mathscr{C}_i = k}}{n\bar{G}_n(Z_i^-)}.$$

But if the value $t$ considered is so high that only very few (if any) observations $Z_i$ (such that $\mathscr{C}_i = k$) exceed $t$, then this purely nonparametric approach will lead to very unstable estimations $\bar{F}_n^{(k)}(t)$ of $\bar{F}^{(k)}(t)$. This is why a semiparametric approach is desirable, and the one we will consider here is the one inspired by classical extreme value theory.

First note that in this paper, we will only consider situations where the underlying distributions $F$ and $G$ of the variables $X$ and $C$ are supposed to present power-like tails (also commonly named heavy tails), and we will focus on the evaluation of the order of this tail. Our working hypothesis will be thus that the different functions $\bar{F}^{(k)}$ (for $k = 1, \ldots, K$) as well as $\bar{G} = 1 - G$ belong to the Fréchet maximum domain of attraction. In other words, we assume that they are (see Definition 1 in the Appendix) regularly varying at infinity, with respective negative indices $-1/\gamma_1, \ldots, -1/\gamma_K$ and $-1/\gamma_C$

$$\forall 1 \leqslant k \leqslant K, \quad \forall x > 0, \; \lim_{t\to+\infty} \bar{F}^{(k)}(tx)/\bar{F}^{(k)}(t) = x^{-1/\gamma_k} \quad \text{and} \quad \lim_{t\to+\infty} \bar{G}(tx)/\bar{G}(t) = x^{-1/\gamma_C}. \tag{1}$$

Consequently, $\bar{F} = 1-F = \sum_{k=1}^{K} \bar{F}^{(k)}$ and $\bar{H} = \bar{F}\bar{G}$ (the survival function of $Z$) are regularly varying (at $+\infty$) with respective indices $-1/\gamma_F$ and $-1/\gamma$, where $\gamma_F = \max(\gamma_1, \ldots, \gamma_K)$ and $\gamma$ satisfies $\gamma^{-1} = {\gamma_F}^{-1} + {\gamma_C}^{-1}$ (these relations are constantly used in this paper).

The estimation of $\gamma_F$ has been already studied in the literature, as it corresponds to the random (right) censoring framework, without competing risks. We can cite Beirlant et al. (2007) and Einmahl et al. (2008), where the authors propose to use consistent estimators of $\gamma$ divided by the proportion of non-censored observations in the tail, or Worms and Worms (2014), where two Hill-type estimators are proposed for $\gamma_F$, based on survival analysis techniques. However, our target here is $\gamma_k$ (for a fixed $k = 1, \ldots, K$) and the point is that there seems to be no way to deduce an estimator of $\gamma_k$ from an estimator of $\gamma_F$. Note that the useful trick used in Beirlant et al. (2007) and Einmahl et al. (2008) to construct an estimator of $\gamma_F$ does not seem to be extendable to this competing risks setting. To the best of our knowledge, our present paper is the first one addressing the problem of estimating the cause-specific extreme value index $\gamma_k$ .

Considering assumption (1), it is simple to check that, for a given $k$, we have

$$\lim_{t\to+\infty} \frac{1}{\bar{F}^{(k)}(t)} \int_t^{+\infty} \log(u/t)\, dF^{(k)}(t) \;=\; \gamma_k.$$

It is therefore most natural to propose the following (Hill-type) estimator of $\gamma_k$ , for some given threshold value $t_n$ (assumptions on this threshold are detailed in the next section) :

$$\widehat{\gamma}_{n,k} \;=\; \int \widehat{\phi}_n(u) dF_n^{(k)}(u) \quad \text{where} \quad \widehat{\phi}_n(u) \;=\; \frac{1}{\bar{F}_n^{(k)}(t_n)} \log\left(\frac{u}{t_n}\right) \mathbb{I}_{u>t_n},$$

which can be also written as

$$\widehat{\gamma}_{n,k} = \frac{1}{n\bar{F}_n^{(k)}(t_n)} \sum_{i=1}^n \frac{\log(Z_i/t_n)}{\bar{G}_n(Z_i^-)} \mathbb{I}_{\xi_i=k}\mathbb{I}_{Z_i>t_n} = \frac{1}{n\bar{F}_n^{(k)}(t_n)} \sum_{Z_{(i)}>t_n} \frac{\log(Z_{(i)}/t_n)}{\bar{G}_n(Z_{(i-1)})} \delta_{(i)} \mathbb{I}_{\mathscr{C}_{(i)}=k},$$

where $Z_{(1)} \leqslant \ldots \leqslant Z_{(n)}$ are the ordered random variables associated to $Z_1, \ldots, Z_n$, and $\delta_{(i)}$ and $\mathscr{C}_{(i)}$ are the censoring indicator and cause number which correspond to the order statistic $Z_{(i)}$. It is clear that this estimator is a generalisation of one of the estimators proposed in Worms and Worms (2014), in which the situation $K = 1$ (with only one cause of failure/death) was considered. The asymptotic result we prove in the present work is then valid in the situation studied in the latter, where only consistency was proved and a random threshold was used.

Our paper is organized as follows: in Section 2, we state the asymptotic normality result of the proposed estimator, and of a corresponding estimator of an extreme quantile of the cumulative incidence function. Section 5 is devoted to the proofs. In Section 3, we present some simulations in order to illustrate finite sample behaviour of our estimator. Some technical aspects of the proofs are postponed to the Appendix.

## 2. Assumptions and Statement of the results

The central limit theorem which is going to be proved has the rate $\sqrt{v_n}$ where $v_n = n\bar{F}_n^{(k)}(t_n)\bar{G}(t_n)$ and $t_n$ is a threshold tending to $\infty$ with the following constraint

$$v_n \overset{n\to\infty}{\longrightarrow} +\infty \quad \text{such that} \quad n^{-\eta_0} v_n \overset{n\to\infty}{\longrightarrow} +\infty \quad \text{for some } \eta_0 > 0. \tag{2}$$

If we note $l_k$ the slowly varying function associated to $\bar{F}^{(k)}$ ($i.e.$ such that $\bar{F}^{(k)}(x) = x^{-1/\gamma_k} l_k(x)$ in condition (1)), the second order condition we consider is the classical $SR2$ condition for $l_k$ (see Bingham, Goldie and Teugels (1987)),

$$\forall x > 0,\; \frac{l_k(tx)}{l_k(t)} - 1 \;\overset{t\to\infty}{\sim}\; h_{\rho_k}(x)\; g(t) \quad (\forall x > 1), \tag{3}$$

where $g$ is a positive measurable function, slowly varying with index $\rho_k \leqslant 0$, and $h_{\rho_k}(x) = \frac{x^{\rho_k}-1}{\rho_k}$ when $\rho_k < 0$, or $h_{\rho_k}(x) = \log x$ when $\rho_k = 0$.

**Theorem 1.** *Under assumptions* (1), (2) *and* (3), *if there exists* $\lambda \geqslant 0$ *such that* $\sqrt{v_n}g(t_n) \overset{n\to\infty}{\longrightarrow} \lambda$, *and if* $\gamma_k < \gamma_C$ *then we have*

$$\sqrt{v_n}(\widehat{\gamma}_{n,k} - \gamma_k) \;\overset{d}{\longrightarrow}\; \mathscr{N}(\lambda m, \sigma^2) \qquad \text{as } n \to \infty$$

*where*

$$m = \begin{cases} \frac{\gamma_k^2}{1-\gamma_k\rho_k} & \text{if } \rho_k < 0, \\ \gamma_k^2 & \text{if } \rho_k = 0, \end{cases} \quad \text{and} \quad \sigma^2 = \frac{\gamma_k^2}{(1-r)^3}\left((1+r^2) - 2cr\right),$$

*with* $c = \lim_{x\to\infty} \bar{F}^{(k)}(x)/\bar{F}(x) \in [0,1]$ *and* $r = \gamma_k/\gamma_C \in ]0,1[$.

**Remark 1.** *Note that when* $\gamma_k < \gamma_F$, *then* $c = 0$, *and, when* $\gamma_k = \gamma_F$ *and* $c = 1$ *(for instance when there is only one cause of failure/death), then* $\sigma^2$ *reduces to* $\gamma_F^2/(1-r)$.

**Proposition 1.** *Under assumptions* (1) *and* (2), *we have*

$$\widehat{\gamma}_{n,k} \;\overset{\mathbb{P}}{\longrightarrow}\; \gamma_k \qquad \text{as } n \to \infty.$$

**Remark 2.** *The condition* $\gamma_k < \gamma_C$ *(weak censoring) is not necessary for the consistency of* $\widehat{\gamma}_{n,k}$.

Now, concerning the estimation of an extreme quantile $x^{(k)}_{p_n}$ (of order $p_n$ tending to 0) associated to $\bar F^{(k)}$, we propose the usual Weissman-type estimator (in this heavy tailed context), associated to the threshold $t_n$ used in the estimation of $\gamma_k$,

$$\hat x^{(k)}_{p_n,t_n} = t_n \left( \frac{\bar F^{(k)}_n(t_n)}{p_n} \right)^{\hat\gamma_{n,k}},$$

where $p_n$ is assumed to satisfy the constraint $p_n = o\big(\bar F^{(k)}(t_n)\big)$. Remind that by definition $\bar F^{(k)}(x^{(k)}_{p_n}) = p_n$, and thus the definition of this estimator is based on the fact that, by the assumed regular variation of $\bar F^{(k)}$, the ratio $\bar F^{(k)}(x^{(k)}_{p_n})/\bar F^{(k)}(t_n)$ is close to $(x^{(k)}_{p_n}/t_n)^{-1/\gamma_k}$.

**Corollary 1.** *Under the assumptions of Theorem 1, if in addition $\rho_k < 0$ (in (3)) and $d_n = \bar F^{(k)}(t_n)/p_n \to \infty$ satisfies the condition*

$$\sqrt{v_n}\,/\log(d_n) \overset{n\to\infty}{\longrightarrow} \infty, \tag{4}$$

*then (with $\lambda$, $m$ and $\sigma^2$ being defined in the statement of Theorem 1)*

$$\frac{\sqrt{v_n}}{\log(d_n)} \left( \frac{\hat x^{(k)}_{p_n,t_n}}{x^{(k)}_{p_n}} - 1 \right) \overset{d}{\longrightarrow} \mathcal{N}(\lambda m, \sigma^2) \quad \text{as } n \to \infty.$$

## 3. Simulations

In this section, a small simulation study is conducted in order to illustrate the finite-sample behaviour of our new estimator in some simple cases, and discuss the main issues associated with the competing risks setting.

For simplicity, we focus on the situation with two competing risks ($K = 2$), also called causes below, and our aim is the extreme value index $\gamma_1$ associated to the first cause. Data are generated from one of the following two models : for $c_1$, $c_2$ non-negative constants satisfying $c_1 + c_2 = 1$, we consider the following (sub-)distribution for each cause-specific function $\bar F^{(k)}$ ($k \in \{1,2\}$) :

– Fréchet : $\bar F^{(k)}(t) = c_k \ \exp(-t^{-1/\gamma_k})$, for $t \geqslant 0$ ;

– Burr : $\bar F^{(k)}(t) = c_k \ (1 + t^{\tau_k}/\beta)^{-1/(\gamma_k\tau_k)}$, for $t \geqslant 1$, where $\tau_k > 0$, $\beta > 0$.

The lifetime $X$, of survival function $\bar F = \bar F^{(1)} + \bar F^{(2)}$, is generated by the inversion method (with numerical computation of $\bar F^{-1}$). Censoring times are then generated from a Fréchet or a Burr distribution :

$$\bar G(t) = \exp(-t^{-1/\gamma_C}) \ (t \geqslant 0) \qquad \text{or} \qquad \bar G(t) = (1 + t^{\tau_C}/\beta)^{-1/(\gamma_C\tau_C)} \ (t \geqslant 1).$$

In this section, we consider (as it is often done in simulation studies) that the threshold $t_n$ used in the definition of our new estimator $\hat\gamma_{n,1}$ is taken equal to $Z_{(n-k_n)}$ (i.e. we consider it as random). One aim of this section is to show how our estimator (with random threshold)

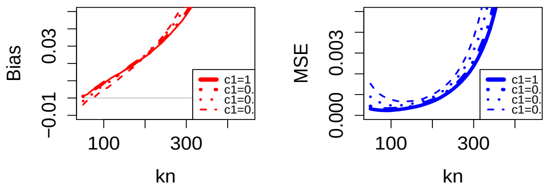
$$\hat\gamma_1 = \frac{1}{n\bar F^{(1)}_n(Z_{(n-k_n)})} \sum_{i=1}^{k_n} \frac{\log(Z_{(n-i+1)}/Z_{(n-k_n)})}{\bar G_n(Z_{(n-i,n)})} \delta_{(n-i+1)} \mathbb{I}_{\mathscr{C}_{(n-i+1)}=1}$$

of $\gamma_1$ behaves when the proportion $c_1$ of cause 1 events varies : we consider $c_1 \in \{1, 0.9, 0.7, 0.5\}$, the case $c_1 = 1$ corresponding to the simple censoring framework, without competing risk.

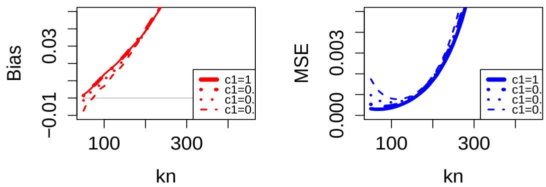
Another aim is to illustrate the impact of dependency between the causes, when estimating the tail. The starting point is that, if cause 2 could be considered independent of cause 1, then we could (and would) include it in the censoring mechanism and we would be in the simple random censoring setting, without competing risk. In this case, it would be possible to estimate $\gamma_1$ by one of the following two estimators, the first one being proposed in Beirlant et al. (2007) (a Hill estimator weighted with a constant weight), and the second one in Worms and Worms (2014) (a Hill estimator weighted with varying Kaplan-Meier weights):

$$\hat\gamma_1^{(BDFG)} = \frac{1}{k_n} \sum_{i=1}^{k_n} \frac{1}{\hat p_1} \log(Z_{(n-i+1)}/Z_{(n-k_n)}) \tag{5}$$

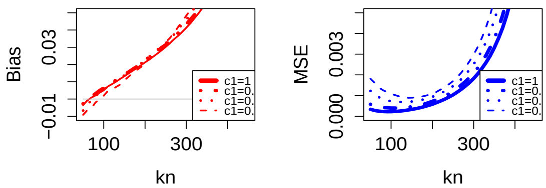
$$\hat\gamma_1^{(KM)} = \frac{1}{n\bar F_{n,b}(Z_{(n-k_n)})} \sum_{i=1}^{k_n} \frac{\delta_{(n-i+1)} \mathbb{I}_{\mathscr{C}_{(n-i+1)}=1}}{\bar G_{n,b}(Z_{(n-i,n)})} \ \log(Z_{(n-i+1)}/Z_{(n-k_n)}), \tag{6}$$



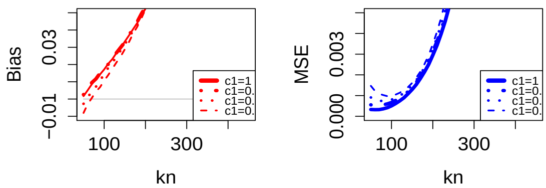

(a) Fréchet case, $\gamma_1 = 0.1$, $\gamma_2 = 0.25$, $\gamma_C = 0.3$



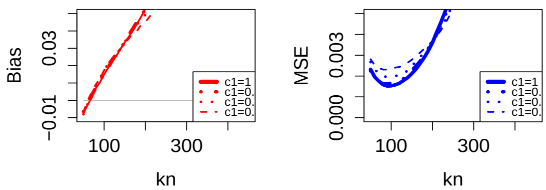

(b) Same case as (a) but for Burr distribution



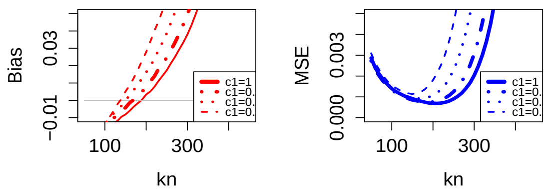

(c) Fréchet case, $\gamma_1 = 0.1$, $\gamma_2 = 0.25$, $\gamma_C = 0.2$




(d) Same case as (c) but for Burr distribution

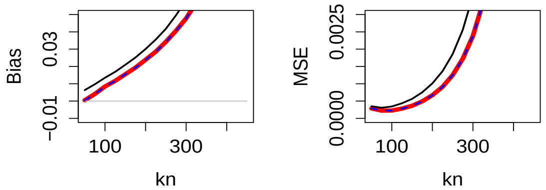



(e) Fréchet case, $\gamma_1 = 0.25$, $\gamma_2 = 0.1$, $\gamma_C = 0.45$

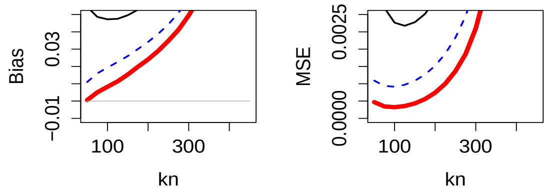



(f) Same case as (e) but for Burr distribution

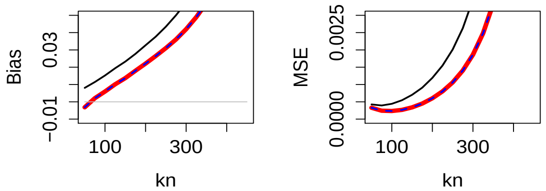
Figure 1: Comparison of bias and MSE (respectively left and right in each subfigure) of $\widehat{\gamma}_{n,k}$ for different values of $c_1$ ; in figures $(a)$, $(c)$ and $(e)$, $X$ and $C$ are Fréchet distributed, but in figures $(b)$, $(d)$ and $(f)$ they are Burr distributed.



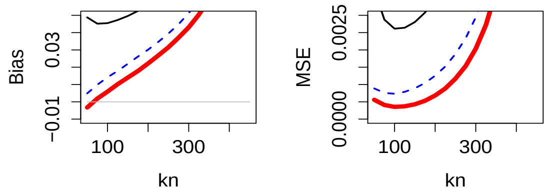

(a) Fréchet case, $\gamma_1 = 0.1$, $\gamma_2 = 0.25$, $\gamma_C = 0.3$, and $c_1 = 1$



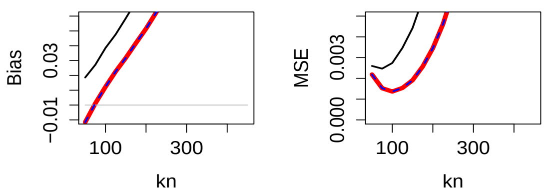

(b) Case (a) but with $c_1 = 0.9$



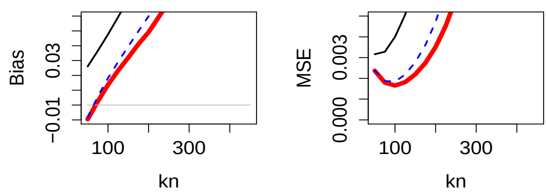

(c) Fréchet case, $\gamma_1 = 0.1$, $\gamma_2 = 0.25$, $\gamma_C = 0.2$, and $c_1 = 1$




(d) Case (c) but with $c_1 = 0.9$




(e) Fréchet case, $\gamma_1 = 0.1, \gamma_2 = 0.25, \gamma_C = 0.45$, and $c_1 = 1$




(f) Case (e) but with $c_1 = 0.9$

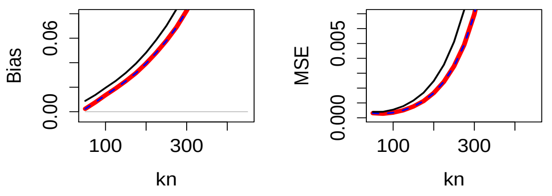
Figure 2: Comparison of bias and MSE (respectively left and right in each subfigure) for $\hat{\gamma}_{n,k}$ (plain thick), $\hat{\gamma}_1^{(BDFG)}$ (plain thin) and $\hat{\gamma}_1^{(KM)}$ (dashed), for Fréchet distributed $X$ and $C$.

where, in Equation (5), $\hat{p}_1 = \frac{1}{k_n}\sum_{i=1}^{k_n} \delta_{(n-i+1)}\mathbb{I}_{\mathscr{C}_{(n-i+1)}=1}$, and in Equation (6), the Kaplan Meier estimators $\bar{F}_{n,b}$ and $\bar{G}_{n,b}$ are based on the $\tilde{\delta}_i = \delta_i \mathbb{I}_{\mathscr{C}_i=1}$. These two estimators consider the uncensored lifetimes associated to cause 2 as independent censoring times. Comparing our new estimator with these latter two estimators, when $c_1 < 1$, will empirically prove that considering cause 2 as a competing risk independent of cause 1 has a great (negative) impact on the estimation of $\gamma_1$. Note that when $c_1 = 1$, the new estimator $\hat{\gamma}_1$ and $\hat{\gamma}_1^{(KM)}$ are exactly the same (therefore the thick and dashed lines in sub-figures (a), (c) and (e) of Figures 2 and 3 are overlapping, identical).

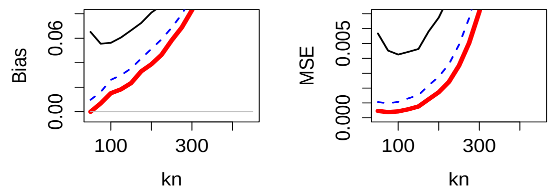
We address these two aims for each set-up (Fréchet, or Burr), by generating 2000 datasets of size 500, with three configurations of the triplet $(\gamma_1, \gamma_2, \gamma_C)$ : $(0.1, 0.25, 0.3)$ ($\gamma_1 < \gamma_2$, moderate censoring $\gamma_C > \gamma_F$), $(0.1, 0.25, 0.2)$ ($\gamma_1 < \gamma_2$, heavy censoring $\gamma_C < \gamma_F$), or $(0.25, 0.1, 0.45)$ ($\gamma_1 > \gamma_2$, moderate censoring $\gamma_C < \gamma_F$). Median bias and mean squared error (MSE) of the different estimators are plotted against different values of $k_n$, the number of excesses used. When Burr distributions are simulated, the parameter $\beta$ is taken equal to 1, and the parameters $(\tau_1, \tau_2, \tau_C)$ are taken equal to $(12, 6, 5)$ in configurations 1 and 2, and to $(6, 12, 5)$ in configuration 3.

Figure 1 illustrates the behaviour of our estimator when $c_1$ varies. In terms of bias and MSE, we can see that the first configuration is a little better than the second one, which is itself much better than the third one. We observed this phenomenon in many other cases, not reported here : our estimator behaves best when it is the smallest parameter $\gamma_k$ which is estimated, and when the censoring is not too strong. Our simulations also show that the quality of our estimator (especially in terms of the MSE) diminishes with $c_1$.

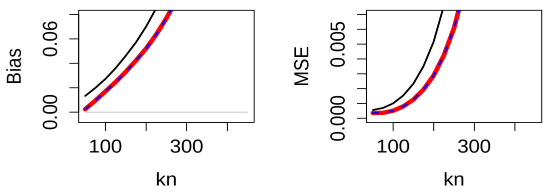
Figures 2 and 3 present the comparison between our new estimator and the ones described in (5) and (6). A general conclusion (confirmed by other simulations not reported here) is that $\hat{\gamma}_1^{(BDFG)}$ and $\hat{\gamma}_1^{(KM)}$



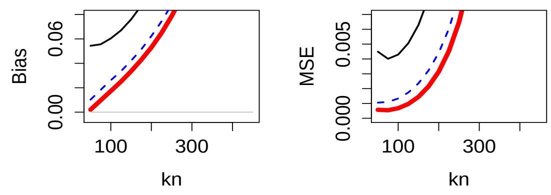

(a) Burr case, $\gamma_1 = 0.1$, $\gamma_2 = 0.25$, $\gamma_C = 0.3$, and $c_1 = 1$



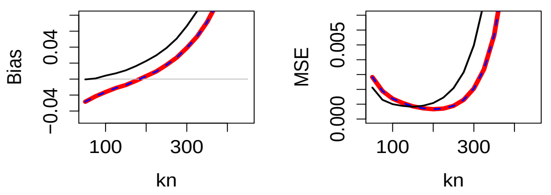

(b) Case (a) but with $c_1 = 0.9$



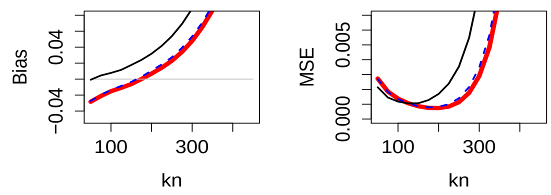

(c) Burr case, $\gamma_1 = 0.1$, $\gamma_2 = 0.25$, $\gamma_C = 0.2$, and $c_1 = 1$




(d) Case (c) but with $c_1 = 0.9$




(e) Burr case, $\gamma_1 = 0.1, \gamma_2 = 0.25, \gamma_C = 0.45$, and $c_1 = 1$




(f) Case (e) but with $c_1 = 0.9$

Figure 3: Comparison of bias and MSE (respectively left and right in each subfigure) for $\hat{\gamma}_{n,k}$ (plain thick), $\hat{\gamma}_1^{(BDFG)}$ (plain thin) and $\hat{\gamma}_1^{(KM)}$ (dashed), for Burr distributed $X$ and $C$.

behave worse in most cases, even for a value of $c_1$ of 0.9, which is only a slight modification of the situation without competing risk ($c_1 = 1$). Therefore, a contamination of the cause 1 distribution by another cause rapidly yield inadequate estimations of $\gamma_1$ if dependency between causes is ignored ; this conclusion is true for both $\hat{\gamma}_1^{(BDFG)}$ and $\hat{\gamma}_1^{(KM)}$, but to a greater extent for $\hat{\gamma}_1^{(BDFG)}$ . In the third configuration $(\gamma_1, \gamma_2, \gamma_C) = (0.25, 0.1, 0.45)$, the improvement provided by $\hat{\gamma}_1$ (with respect to $\hat{\gamma}_1^{(KM)}$) becomes notable when $c_1$ drops below 0.7.

## 4. Conclusion

In this paper, we consider heavy tailed lifetime data subject to random censoring and competing risks, and use the Aalen-Johansen estimator of the cumulative incidence function to construct an estimator for the extreme value index associated to the main cause of interest. To the best of our knowledge, this is the first estimator proposed in this context. Its asymptotic normality is proved and a small simulation study exhibiting its finite-sample performance shows that accounting for the dependency of the different causes is important, but that the bias can be particularly high. Estimating second order tail parameters would then be interesting in order to reduce this bias. A first step towards this aim could be to study the following moments

$$M_n^{(\alpha)} = \frac{1}{n\bar{F}_n^{(k)}(t_n)} \sum_{i=1}^{n} \frac{\log^{\alpha}(Z_i/t_n)}{\bar{G}_n(Z_i^-)} \mathbb{I}_{\xi_i=k} \mathbb{I}_{Z_i>t_n},$$

which asymptotic behaviour can be derived following the same lines as in the proof of Theorem 1.

## 5. Proofs

This section is essentially devoted to the proof of the main Theorem 1. Some hints about the proof of the consistency result contained in Proposition 1 are given in Subsection 5.3, and Corollary 1 is proved in Subsection 5.4.

We adopt a strategy developed by Stute in Stute (1995) in order to prove his Theorem 1.1, a well-known result which states that a Kaplan-Meier integral of the form $\int \phi \, dF_n$ can be approximated by a sum of independent terms. This idea is used in Suzukawa (2002) in the context of competing risks. We thus intend to approximate $\widehat{\gamma}_{n,k}$ by the integral $\widetilde{\gamma}_{n,k} = \int \phi_n \, dF_n^{(k)}$ of some deterministic function $\phi_n$, with respect to the Aalen-Johansen estimator, and approximate this integral by the mean $\breve{\gamma}_{n,k}$ of independent variables $U_{i,n}$ (defined a few lines below). The passage from $\widehat{\gamma}_{n,k}$ to $\widetilde{\gamma}_{n,k}$ (which amounts to replacing $\bar{F}_n^{(k)}(t_n)$ by $\bar{F}^{(k)}(t_n)$ in the denominator of $\widehat{\gamma}_{n,k}$) will imply an additional sum of independent variables $V_{i,n}$, which will participate to the asymptotic variance of our estimator.

However, a major difference with Stute (1995) or Suzukawa (2002) is that the function we integrate here, $\phi_n(u) = \frac{1}{\bar{F}^{(k)}(t_n)} \log(u/t_n) \mathbb{I}_{u>t_n}$, is not only an unbounded function, depending on $n$, but it also has a "sliding" support $[t_n, +\infty[$, which is therefore always close to the endpoint $+\infty$ of the distribution $H$. In Stute (1995), a crucial point of the proof consists in temporarily considering that the integrated function $\phi$ has a support which is bounded away from the endpoint of $H$ (condition (2.3) there). Considering the kind of function $\phi_n$ we have to deal with here, we cannot follow the same strategy : dealing with the remainder terms will thus be a particularly challenging part of our work. Finally note that, in order to deal with the ratio $\bar{F}_n^{(k)}(t_n)/\bar{F}^{(k)}(t_n)$ (and somehow approximate $\widehat{\gamma}_{n,k}$ by $\widetilde{\gamma}_{n,k}$) we will have to consider simultaneously integrals (with respect to $F_n^{(k)}$) of $\phi_n$ and of another function $g_n$, defined below, which basically shares the same flaws as $\phi_n$.

Let us first recall or define the following objects :

$$\begin{aligned}
\widehat{\phi}_n(u) &= \frac{1}{\bar{F}_n^{(k)}(t_n)} \log\left(\frac{u}{t_n}\right) \mathbb{I}_{u>t_n} \\
\phi_n(u) &= \frac{1}{\bar{F}^{(k)}(t_n)} \log\left(\frac{u}{t_n}\right) \mathbb{I}_{u>t_n} \\
\gamma_{n,k} &= \int \phi_n(u) dF^{(k)}(u) \stackrel{n\to\infty}{\longrightarrow} \gamma_k \\
\widetilde{\gamma}_{n,k} &= \int \phi_n(u) dF_n^{(k)}(u) \\
\widehat{\gamma}_{n,k} &= \int \widehat{\phi}_n(u) dF_n^{(k)}(u).
\end{aligned}$$

We thus have $\widehat{\gamma}_{n,k} = \Delta_n^{-1}\widetilde{\gamma}_{n,k}$, where

$$\Delta_n = \bar{F}_n^{(k)}(t_n)/\bar{F}^{(k)}(t_n) = \int g_n(u) dF_n^{(k)}(u) \quad \text{and} \quad g_n(u) = \frac{1}{\bar{F}^{(k)}(t_n)} \mathbb{I}_{u>t_n},$$

and we now introduce the following new quantities, related to the Stute-like decomposition of $\widetilde{\gamma}_{n,k}$ and $\Delta_n$ :

$$\begin{aligned}
U_{i,n}^{(1)} &= \frac{\phi_n(Z_i)}{\bar{G}(Z_i)} \delta_i \mathbb{I}_{\mathscr{C}_i=k} \quad \text{and} \quad V_{i,n}^{(1)} = \frac{g_n(Z_i)}{\bar{G}(Z_i)} \delta_i \mathbb{I}_{\mathscr{C}_i=k} \\
U_{i,n}^{(2)} &= \frac{1-\delta_i}{\bar{H}(Z_i)} \psi(\phi_n, Z_i) \quad \text{and} \quad V_{i,n}^{(2)} = \frac{1-\delta_i}{\bar{H}(Z_i)} \psi(g_n, Z_i) \\
U_{i,n}^{(3)} &= \int_0^{Z_i} \psi(\phi_n, u)\, dC(u) \quad \text{and} \quad V_{i,n}^{(3)} = \int_0^{Z_i} \psi(g_n, u)\, dC(u) \\
U_{i,n} &= U_{i,n}^{(1)} + U_{i,n}^{(2)} - U_{i,n}^{(3)} \quad \text{and} \quad V_{i,n} = V_{i,n}^{(1)} + V_{i,n}^{(2)} - V_{i,n}^{(3)}
\end{aligned}$$

where, for any function $f : \mathbb{R}_+ \to \mathbb{R}$, we note (for any given $z \geqslant 0$)

$$\psi(f,z) = \int_z^{+\infty} f(t) dF^{(k)}(t) \quad \text{and} \quad C(z) = \int_0^z \frac{dG(t)}{\bar{H}(t)\bar{G}(t)}.$$

This enables us to finally define the important objects

$$\breve{\gamma}_{n,k} = \frac{1}{n}\sum_{i=1}^n U_{i,n} \quad \text{and} \quad \widehat{\Delta}_n = \frac{1}{n}\sum_{i=1}^n V_{i,n} \tag{7}$$

which are the triangular sums of independent terms which will respectively approximate $\widetilde{\gamma}_{n,k}$ and $\Delta_n$. At the beginning of section 5.1, it will be proved that $\mathbb{E}(U_{i,n}^{(1)}) = \gamma_{n,k}$ and $\mathbb{E}(V_{i,n}^{(1)}) = 1$, while $\mathbb{E}(U_{i,n}^{(2)}) = \mathbb{E}(U_{i,n}^{(3)})$ and $\mathbb{E}(V_{i,n}^{(2)}) = \mathbb{E}(V_{i,n}^{(3)})$, yielding $\mathbb{E}(\breve{\gamma}_{n,k}) = \gamma_{n,k}$ and $\mathbb{E}(\widehat{\Delta}_n) = 1$ ; the terms $U_{i,n}^{(2)}$, $U_{i,n}^{(3)}$, $V_{i,n}^{(2)}$ and $V_{i,n}^{(3)}$ only participate to the variance component of the estimator. The relation between all these quantities is made clearer in the following Lemma :

**Lemma 1.** *We have*

$$\sqrt{v_n}(\widehat{\gamma}_{n,k} - \gamma_k) = \Delta_n^{-1} \left( Z_n + \sqrt{v_n} R_n + \sqrt{v_n}(\gamma_{n,k} - \gamma_k) \right) \tag{8}$$

*where*

$$Z_n = \sqrt{v_n}\Big( (\breve{\gamma}_{n,k} - \gamma_{n,k}) - \gamma_k(\widehat{\Delta}_n - 1) \Big) \quad \textit{and} \quad R_n = (\widetilde{\gamma}_{n,k} - \breve{\gamma}_{n,k}) - \gamma_k(\Delta_n - \widehat{\Delta}_n)$$

The proof of Lemma 1 is simple :

$$\begin{aligned}
\sqrt{v_n}(\widehat{\gamma}_{n,k} - \gamma_k) &= \Delta_n^{-1}\sqrt{v_n}\,(\widetilde{\gamma}_{n,k} - \Delta_n\gamma_k) \\
&= \Delta_n^{-1}\sqrt{v_n}\,\{(\breve{\gamma}_{n,k} - \gamma_{n,k}) + \gamma_k(1-\Delta_n) + (\widetilde{\gamma}_{n,k} - \breve{\gamma}_{n,k}) + (\gamma_{n,k} - \gamma_k)\}
\end{aligned}$$

which leads to the desired relation (8).

The main theorem thus becomes an immediate consequence of the following four results, the second one being the most difficult to establish.

**Proposition 2.** *Under condition* (1) *and assuming that*

$$v_n \overset{n\to\infty}{\longrightarrow} +\infty, \tag{9}$$

*if* $\gamma_k < \gamma_C$*, then*

$$Z_n \overset{d}{\longrightarrow} \mathscr{N}(0,\sigma^2) \qquad \text{as } n \to \infty$$

*where* $\sigma^2$ *is defined in the statement of Theorem 1.*

**Proposition 3.** *Under conditions* (1) *and* (2) *, if* $\gamma_k < \gamma_C$*, then*

$$\widetilde{\gamma}_{n,k} = \check{\gamma}_{n,k} + R_{n,\phi} \qquad \text{and} \qquad \Delta_n = \widehat{\Delta}_n + R_{n,g} \tag{10}$$

*where* $R_{n,\phi}$*,* $R_{n,g}$ *(and consequently* $R_n$ *too) are* $o_{\mathbb{P}}(v_n^{-1/2})$*.*

**Corollary 2.** *Under the conditions of Proposition 3,* $\sqrt{v_n}(\Delta_n - 1)$ *converges in distribution to* $\mathscr{N}(0, 1/(1-r))$ *where* $r = \gamma_k/\gamma_C \in ]0,1[$*.*

**Lemma 2.** *Under conditions* (1), (3) *and* $\sqrt{v_n} g(t_n) \to \lambda \geqslant 0$*, the bias term* $\sqrt{v_n}(\gamma_{n,k} - \gamma_k)$ *in (8) converges to* $\lambda m$ *as* $n \to \infty$*, where* $m$ *is defined in Theorem 1.*

Propositions 2 and 3 will be proved in Sections 5.1 and 5.2 respectively, sometimes with the help of other results stated and established in the Appendix. The proofs of Corollary 2 and Lemma 2 are short, we state them below.

Concerning Corollary 2, once the proof of Proposition 2 has been gone through, it will become clear to the reader that $\sqrt{v_n}(\hat{\Delta}_n - 1)$ converges in distribution to the centred gaussian distribution of variance $1/(1-r)$, because $\sqrt{v_n}(\hat{\Delta}_n - 1) = \sum_{i=1}^n \tilde{W}_{i,n}$ where $\tilde{W}_{i,n} = \frac{\sqrt{v_n}}{n}\tilde{V}_{i,n} = \frac{\sqrt{v_n}}{n}(V_{i,n} - 1)$ are centred, and $\mathbb{V}ar(\tilde{W}_{i,n}) = \frac{1}{n}\frac{1}{1-r} + o(1/n)$ (this is proved similarly as (11) and (15)). Since Proposition 3 states that $\Delta_n = \hat{\Delta}_n + o_{\mathbb{P}}(v_n^{1/2})$, the same central limit theorem holds for $\Delta_n$ and the corollary is proved.

Concerning now Lemma 2, remind that $\gamma_{n,k} = \int \phi_n(u) dF_n^{(k)}(u)$. An integration by parts and the fact that $\bar{F}^{(k)}(x) = x^{-1/\gamma_k} l_k(x)$ yield

$$\sqrt{v_n}(\gamma_{n,k} - \gamma_k) = \sqrt{v_n} \int_1^{+\infty} y^{-1/\gamma_k - 1} \left( \frac{l_k(yt_n)}{l_k(t_n)} - 1 \right) dy,$$

and, using assumption (3) and Proposition 3.1 in de Haan and Ferreira (2006), we can write

$$\int_1^{+\infty} y^{-1/\gamma_k - 1} \left( \frac{l_k(yt_n)}{l_k(t_n)} - 1 \right) dy = g(t_n) \int_1^{+\infty} y^{-1/\gamma_k - 1} h_{\rho_k}(y) dy + o(g(t_n)).$$

The result then follows from assumption $\sqrt{v_n} g(t_n) \to \lambda \geqslant 0$ and the fact that $\int_1^{+\infty} y^{-1/\gamma_k - 1} h_{\rho_k}(y) dy = m$.

In the rest of the paper, we will very often handle the well-known sub-distributions functions $H^{(0)}$ and $H^{(1,k)}$ defined, for all $t \geqslant 0$, by

$$H^{(0)}(t) = \mathbb{P}(Z \leqslant t, \delta = 0) \qquad \text{and} \qquad H^{(1,k)}(t) = \mathbb{P}(Z \leqslant t, \xi = k).$$

Note that we have

$$dH^{(0)} = \bar{F} dG \qquad \text{and} \qquad dH^{(1,k)} = \bar{G} dF^{(k)}.$$

*5.1. Proof of Proposition 2*

We first write

$$Z_n = \sum_{i=1}^n W_{i,n} \quad \text{where} \quad W_{i,n} = \frac{\sqrt{v_n}}{n}\left(\tilde{U}_{i,n} - \gamma_k \tilde{V}_{i,n}\right) \quad \text{and} \quad \begin{cases} \tilde{U}_{i,n} = U_{i,n} - \gamma_{n,k} \\ \tilde{V}_{i,n} = V_{i,n} - 1 \end{cases}$$

where $W_{i,n}$, $\tilde{U}_{i,n}$ and $\tilde{V}_{i,n}$ are centred, because the random variables $U_{i,n}$ and $V_{i,n}$ have expectations respectively equal to $\gamma_{n,k}$ and 1. Indeed, we have

$$\mathbb{E}\left(U_{i,n}^{(1)}\right) = \mathbb{E}\left(\tfrac{\phi_n(Z_i)}{\bar{G}(Z_i)} \delta_i \mathbb{I}_{\mathscr{C}_i = k}\right) = \int \tfrac{\phi_n(u)}{\bar{G}(u)} dH^{(1,k)}(u) = \int \phi_n(u) dF^{(k)}(u) = \gamma_{n,k}$$

and

$$\mathbb{E}\left(U_{i,n}^{(2)}\right) = \mathbb{E}\left(\tfrac{1-\delta_i}{\bar{H}(Z_i)} \psi(\phi_n, Z_i)\right) = \iint \tfrac{1}{\bar{H}(u)} \mathbb{I}_{t>u} \phi_n(t)\, dF^{(k)}(t)\, dH^{(0)}(u) = \iint \tfrac{1}{\bar{G}(u)} \mathbb{I}_{t>u} \phi_n(t)\, dF^{(k)}(t)\, dG(u)$$

as well as

$$\mathbb{E}\left(U_{i,n}^{(3)}\right) \;=\; \mathbb{E}\left(\int_0^{Z_i} \psi(\phi_n, u)\, dC(u)\right) \;=\; \iiint \mathbb{I}_{z>u}\mathbb{I}_{t>u}\phi_n(t)\, dH(z)\, dF^{(k)}(t)\, \tfrac{dG(u)}{\bar{G}(u)\bar{H}(u)} \;=\; \mathbb{E}\left(U_{i,n}^{(2)}\right).$$

The proof for $\mathbb{E}(V_{i,n}) = 1$ is similar.

We will now prove the asymptotic normality of $Z_n$ by using the Lyapunov criteria.

**Lemma 3.** *Under the conditions* (1) *and* (9) *, if* $\gamma_k < \gamma_C$ *:*

(*i*) *we have*

$$\mathbb{V}ar(\tilde{U}_{1,n} - \gamma_k \tilde{V}_{1,n}) \;=\; \mathbb{E}(U_{1,n}^2) + \gamma_k^2 \mathbb{E}(V_{1,n}^2) - 2\gamma_k \mathbb{E}(U_{1,n}V_{1,n}) + o(1) \tag{11}$$

(*ii*) *we have*

$$\mathbb{E}(U_{1,n}^2) \;=\; \int \frac{\phi_n^2(u)}{\bar{G}(u)} dF^{(k)}(u) \;-\; \int (\psi(\phi_n, u))^2\, dC(u) \tag{12}$$

$$\mathbb{E}(V_{1,n}^2) \;=\; \int \frac{g_n^2(u)}{\bar{G}(u)} dF^{(k)}(u) \;-\; \int (\psi(g_n, u))^2\, dC(u) \tag{13}$$

$$\mathbb{E}(U_{1,n}V_{1,n}) \;=\; \int \frac{\phi_n(u) g_n(u)}{\bar{G}(u)} dF^{(k)}(u) \;-\; \int \psi(\phi_n, u)\psi(g_n, u)\, dC(u) \tag{14}$$

(*iii*) *we have, noting* $r = \gamma_k/\gamma_C$ *(which belongs to* $]0,1[$ *under our conditions) as well as* $p = \gamma/\gamma_F = \gamma_C/(\gamma_F + \gamma_C) \in ]0,1[$,

$$\mathbb{V}ar(\tilde{U}_{1,n} - \gamma_k \tilde{V}_{1,n}) \;=\; \frac{1}{\bar{F}_n^{(k)}(t_n)\bar{G}(t_n)} \left(\frac{\gamma_k^2(1+r^2)}{(1-r)^3} + o(1)\right) \;-\; \frac{1-p}{\bar{H}(t_n)} \left(\frac{2\gamma_k^3}{\gamma(2-\gamma_k/\gamma)^3} + o(1)\right) + o(1) \tag{15}$$

**Lemma 4.** *Under the conditions* (1) *and* (9) *, if* $\gamma_k < \gamma_C$*, then*

$\sum_{i=1}^n \mathbb{E}\,|W_{i,n}|^{2+\delta} \to 0$*, as* $n$ *tends to infinity, for some* $\delta > 0$.

We can then immediately prove Proposition 2. Indeed, since $Z_n = \sum_{i=1}^n W_{i,n}$, Lemma 3 yields

$$\mathbb{V}ar(Z_n) = n\mathbb{V}ar(W_{1,n}) = \frac{v_n}{n}\mathbb{V}ar(\tilde{U}_{1,n} - \gamma_k \tilde{V}_{1,n}).$$

which, since $v_n = n\bar{F}_n^{(k)}(t_n)\bar{G}(t_n)$, becomes

$$\mathbb{V}ar(Z_n) \;=\; \gamma_k^2(1+r^2)(1-r)^{-3} \;-\; \left(2(1-p)\gamma_k^3\gamma^{-1}(2-\gamma_k/\gamma)^{-3}\right)\left(\bar{F}^{(k)}(t_n)/\bar{F}(t_n)\right) \;+\; o(1).$$

Therefore, depending on the limit $c$ of the ratio $\bar{F}^{(k)}(t_n)/\bar{F}(t_n)$ when $n \to \infty$ (for instance, it converges to 0 when $\gamma_k < \gamma_F$), it is simple to check that the variance of $Z_n$ converges to the value $\sigma^2$ described in the statement of Theorem 1. Thanks to Lemma 4, the Lyapunov CLT applies and Proposition 2 is proved.

The two subsections 5.1.1 and 5.1.2 are now respectively devoted to the proofs of Lemmas 3 and 4.

*5.1.1. Proof of Lemma 3*

Part (*i*) of the lemma is straightforward : since $\tilde{U}_{1,n}$ and $\tilde{V}_{1,n}$ are centred, we have indeed

$$\begin{aligned}\mathbb{V}ar(\tilde{U}_{1,n} - \gamma_k \tilde{V}_{1,n}) \;&=\; \mathbb{E}((\tilde{U}_{1,n} - \gamma_k \tilde{V}_{1,n})^2) \;=\; \mathbb{E}(\tilde{U}_{1,n}^2) + \gamma_k^2\mathbb{E}(\tilde{V}_{1,n}^2) - 2\gamma_k\mathbb{E}(\tilde{U}_{1,n}\tilde{V}_{1,n}) \\ &=\; (\mathbb{E}(U_{1,n}^2) - \gamma_{n,k}^2) \;+\; \gamma_k^2(\mathbb{E}(V_{1,n}^2) - 1) \;-\; 2\gamma_k(\mathbb{E}(U_{1,n}V_{1,n}) - \gamma_{n,k})\end{aligned}$$

and the result comes by using the fact that $\gamma_{n,k}$ converges to $\gamma_k$ as $n \to \infty$.

Now we proceed to the proof of part (*ii*), and will only prove (12) because, by definition of $\phi_n$ and $g_n$, the proofs for (13) and (14) will be completely similar. First of all, we obviously have

$$(U_{1,n})^2 = (U_{1,n}^{(1)})^2 + (U_{1,n}^{(2)})^2 + (U_{1,n}^{(3)})^2 + 2U_{1,n}^{(1)}U_{1,n}^{(2)} - 2U_{1,n}^{(1)}U_{1,n}^{(3)} - 2U_{1,n}^{(2)}U_{1,n}^{(3)} \tag{16}$$

The first term in the right-hand side of (12) is equal to $\mathbb{E}((U_{1,n}^{(1)})^2)$, and the second one (without the minus sign) is equal to $\mathbb{E}((U_{1,n}^{(2)})^2)$ and to $\mathbb{E}(U_{1,n}^{(1)}U_{1,n}^{(3)})$ because

$$\mathbb{E}((U_{1,n}^{(2)})^2) = \mathbb{E}\left(\frac{1-\delta_1}{\bar{H}^2(Z_1)}(\psi(\phi_n, Z_1))^2\right) = \int \frac{1}{\bar{H}^2(z)}(\psi(\phi_n, z))^2\, dH^{(0)}(z) = \int (\psi(\phi_n, z))^2\, dC(z)$$

and

$$\begin{aligned}
\mathbb{E}(U_{1,n}^{(1)}U_{1,n}^{(3)}) &= \int \frac{\phi_n(z)}{\bar{G}(z)}\left(\int_0^z \psi(\phi_n,u)dC(u)\right)dH^{(1,k)}(z) \\
&= \int \psi(\phi_n,u)\left(\int_u^\infty \phi_n(z)dF^{(k)}(z)\right)dC(u) \;=\; \int (\psi(\phi_n,z))^2\,dC(z)
\end{aligned}$$

The expectation $\mathbb{E}(U_{1,n}^{(1)}U_{1,n}^{(2)})$ equals 0 because $\delta_1(1-\delta_1)$ is constantly 0, and we are now going to prove that $\mathbb{E}((U_{1,n}^{(3)})^2) = 2\mathbb{E}(U_{1,n}^{(2)}U_{1,n}^{(3)})$, which ends the proof of (12) in view of (16). Indeed, noting $h(z) = \int_0^z \psi(\phi_n,u)dC(u)$ and using the simple fact that $h(z) = h(y) + \int_y^z \psi(\phi_n,u)dC(u)$ for every $y < z$, we have

$$\begin{aligned}
\mathbb{E}((U_{1,n}^{(3)})^2) &= \int_0^\infty \left(\int_0^z \psi(\phi_n,y)h(z)dC(y)\right)dH(z) \\
&= \int_0^\infty \left(\int_0^z \psi(\phi_n,y)h(y)dC(y)\right)dH(z) \;+\; \int_0^\infty \left\{\int_0^z \left(\int_y^z \psi(\phi_n,u)dC(u)\right)\psi(\phi_n,y)dC(y)\right\}dH(z) \\
&= \int_0^\infty \left(\int_0^z \psi(\phi_n,y)h(y)dC(y)\right)dH(z) \;+\; \int_0^\infty \left\{\int_0^z \left(\int_0^u \psi(\phi_n,y)dC(y)\right)\psi(\phi_n,u)dC(u)\right\}dH(z) \\
&= 2\int_0^\infty \left(\int_0^z \psi(\phi_n,y)h(y)dC(y)\right)dH(z)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}(U_{1,n}^{(2)}U_{1,n}^{(3)}) &= \mathbb{E}\left(\frac{1-\delta_1}{\bar{H}(Z_1)}\psi(\phi_n,Z_1)\,h(Z_1)\right) \;=\; \int \bar{H}(y)h(y)\psi(\phi_n,y)\frac{dG(y)}{\bar{H}(y)\bar{G}(y)} \\
&= \int_0^\infty \left(\int_y^\infty dH(z)\right)h(y)\psi(\phi_n,y)\,dC(y) \\
&= \int_0^\infty \left(\int_0^z \psi(\phi_n,y)h(z)dC(y)\right)dH(z) \;=\; \frac{1}{2}\mathbb{E}((U_{1,n}^{(3)})^2)
\end{aligned}$$

as announced.

We can now start proving part $(iii)$ of the lemma, in which the exact nature of the function $\phi_n$ matters. First remind that functions $\bar{F}^{(k)}$, $\bar{G}$ and $C$ are regularly varying of respective orders $-1/\gamma_k$, $-1/\gamma_C$ and $1/\gamma$ (for $C$, this is proved in Lemma 8 with $\delta = 1$). Let us define the constants $c_j$ and $d_j$ $(j = 0,1,2)$ by

$$c_j = \frac{j!\gamma_k^j}{(1-q)^{j+1}} \quad \text{and} \quad d_j = \frac{j!\gamma_k^{j+1}}{\gamma(2-\gamma_k/\gamma)^{j+1}}.$$

Since $\gamma_k < \gamma_C$ was assumed, then according to Lemma 7 part $(ii)$ (applied first with $a+b = 1/\gamma_C - 1/\gamma_k < 0$ for $c_j$, and then with $a+b = -2/\gamma_k + 1/\gamma = (1/\gamma_C - 1/\gamma_k) + (1/\gamma_F - 1/\gamma_k) < 0$ for $d_j$) , we have

$$\int_{t_n}^\infty \log^j\left(\frac{u}{t_n}\right)\frac{dF^{(k)}(u)}{\bar{G}(u)} \;\sim\; c_j\frac{\bar{F}^{(k)}(t_n)}{\bar{G}(t_n)} \quad \text{and} \quad \int_{t_n}^\infty \log^j\left(\frac{u}{t_n}\right)\left(\frac{\bar{F}^{(k)}(u)}{\bar{F}^{(k)}(t_n)}\right)^2\frac{dC(u)}{C(t_n)} \;\stackrel{n\to\infty}{\longrightarrow}\; d_j. \tag{17}$$

Hence, by definition of $\phi_n$, $g_n$, the first terms of $\mathbb{E}(U_{1,n}^2)$, $\mathbb{E}(V_{1,n}^2)$ and $\mathbb{E}(U_{1,n}V_{1,n})$ in relations (12), (13) and (14) are respectively equivalent (as $n\to\infty$) to $c_2D(t_n)$, $c_0D(t_n)$ and $c_1D(t_n)$ where $D(t_n)$ denotes

$$D(t_n) = \frac{1}{\bar{F}^{(k)}(t_n)\bar{G}(t_n)}.$$

Since $c_2 + \gamma_k^2c_0 - 2\gamma_kc_1$ is found to be equal to $\gamma_k^2(1+r^2)/(1-r)^3$, then in view of (11) this proves the first term in relation (15). We now need to obtain equivalent expressions for the quantities $\int(\psi(\phi_n,u))^2\,dC(u)$, $\int(\psi(g_n,u))^2\,dC(u)$ and $\int\psi(\phi_n,u)\psi(g_n,u)\,dC(u)$ in order to prove the second part of relation (15) and therefore finish the proof of Lemma 3.

For saving space, we will use temporarily the following notations :

$$l_n(u) = \log(u/t_n) \quad , \quad R_n(u) = \bar{F}^{(k)}(u)/\bar{F}^{(k)}(t_n)$$

According to the technical Lemma 9 of the Appendix and, after splitting the integral into $\int_0^{+\infty}$ and $\int_{t_n}^{+\infty}$, we can write

$$\int (\psi(\phi_n,u))^2\,dC(u) \;=\; \gamma_{n,k}^2 \int_0^{t_n} dC(u)$$

$$+\int_{t_n}^{+\infty} \Big( l_n^2(u) R_n^2(u) + \gamma_k^2\,(u/t_n)^{-2/\gamma_k} + 2\gamma_k l_n(u) R_n(u)\,(u/t_n)^{-1/\gamma_k}\Big) dC(u) + o(C(t_n)), \quad (18)$$

where $o(C(t_n))$ in (18) is due to part $(ii)$ of Lemma 7 and to the fact that $\epsilon_n(u)$ in Lemma 9 converges to 0 uniformly in $u$. According to the second part of relation (17), we thus have

$$\int (\psi(\phi_n,u))^2\,dC(u) \;=\; (\gamma_k^2 + d_2 + \gamma_k^2 d_0 + 2\gamma_k d_1) C(t_n) + o(C(t_n)) \quad (19)$$

The other terms are treated similarly (using the fact that $\psi(g_n,u)=1$ when $u \leqslant t_n$, and $= \bar F^{(k)}(u)/\bar F^{(k)}(t_n)$ when $u > t_n$) and we obtain

$$\int (\psi(g_n,u))^2\,dC(u) \;=\; (1+d_0)\,C(t_n) + o(C(t_n)), \quad (20)$$

$$\int \psi(\phi_n,u)\psi(g_n,u)\,dC(u) \;=\; (\gamma_k + d_1 + \gamma_k d_0)\,C(t_n) + o(C(t_n)). \quad (21)$$

In view of (11), combining (19), (20) and (21) and using Remark 3 (following Lemma 8) to write that $C(t_n) \sim (1-p)/\bar H(t_n)$ (as $n\to\infty$), this proves the second term in relation (15).

*5.1.2. Proof of Lemma 4*

We have to prove that, for some $\delta>0$ small enough, $n\mathbb{E}\,|W_{1,n}|^{2+\delta}$ tends to 0, as $n\to\infty$. In the sequel, *cst* denotes an unspecified absolute positive constant. According to the definition of $W_{1,n}$, it is clear that

$$n\,|W_{1,n}|^{2+\delta} \;\leqslant\; cst\,\frac{v_n^{1+\delta/2}}{n^{1+\delta}}\left(\sum_{j=1}^{3}\left|U_{1,n}^{(j)}\right|^{2+\delta} + \gamma_k^{2+\delta}\sum_{j=1}^{3}\left|V_{1,n}^{(j)}\right|^{2+\delta} + |\gamma_{n,k}-\gamma_k|^{2+\delta}\right)$$

First, we clearly have $n^{-1-\delta}\,v_n^{1+\delta/2}\,|\gamma_{n,k}-\gamma_k|^{2+\delta} \longrightarrow 0$ as $n\to\infty$. Secondly, since $V_{1,n}^{(j)}$ has the same form as $U_{1,n}^{(j)}$, with $g_n$ instead of $\phi_n$ (*i.e.* without the log factor), we will only prove that there exists some $\delta>0$ such that, as $n\to\infty$,

$$n^{-1-\delta}\,v_n^{1+\delta/2}\mathbb{E}\left|U_{1,n}^{(j)}\right|^{2+\delta} = n^{-\delta/2}\left(\bar F^{(k)}(t_n)\bar G(t_n)\right)^{1+\delta/2}\mathbb{E}\left|U_{1,n}^{(j)}\right|^{2+\delta} \overset{n\to\infty}{\longrightarrow} 0 \qquad \text{for } j\in\{1,2,3\} \quad (22)$$

For $j=1$, we have

$$\mathbb{E}\left|U_{1,n}^{(1)}\right|^{2+\delta} \;=\; \int_0^{+\infty}\left|\frac{\phi_n(z)}{\bar G(z)}\right|^{2+\delta} dH^{(1,k)}(z) \;=\; (\bar F^{(k)}(t_n))^{-2-\delta}\int_{t_n}^{+\infty}(\bar G(z))^{-2-\delta}\,(\log(z/t_n))^{2+\delta}\,\bar G(z)dF^{(k)}(z)$$

$$=\; \left(\bar F^{(k)}(t_n)\bar G(t_n)\right)^{-1-\delta}\int_{t_n}^{+\infty}(\log(z/t_n))^{2+\delta}\left(\frac{\bar G(t_n)}{\bar G(z)}\right)^{1+\delta}\frac{dF^{(k)}(z)}{\bar F^{(k)}(t_n)}.$$

Applying part $(ii)$ of Lemma 7 for $\alpha = 2+\delta$, $a=(1+\delta)/\gamma_C$ and $b=-1/\gamma_k$ (with $\delta$ sufficiently small so that $a+b = 1/\gamma_C - 1/\gamma_k + \delta/\gamma_C$ is kept $<0$), and using the fact that $v_n = n\bar F^{(k)}(t_n)\bar G(t_n)\to\infty$, this ends the proof of (22) for $j=1$.

For $j=2$, we have

$$\mathbb{E}\left|U_{1,n}^{(2)}\right|^{2+\delta} = \int_0^{+\infty}\left|\frac{\psi(\phi_n,z)}{\bar H(z)}\right|^{2+\delta}\bar F(z)dG(z).$$

By definition of $\psi$, $\phi_n$, and $\gamma_{n,k}$, we have $\psi(\phi_n,z)=\gamma_{n,k}$ when $z\leqslant t_n$. Therefore, splitting the integral above into two integrals $\int_0^{t_n}$ and $\int_{t_n}^{+\infty}$ we obtain

$$\mathbb{E}\left|U_{1,n}^{(2)}\right|^{2+\delta} = I_1(t_n) + I_2(t_n),$$

where, on one hand,

$$I_1(t_n) \;=\; (\gamma_{n,k})^{2+\delta}\int_0^{t_n}\frac{\bar F(z)dG(y)}{(\bar H(z))^{2+\delta}} \;=\; (\gamma_{n,k})^{2+\delta}\int_0^{t_n}\frac{dC(z)}{(\bar H(z))^{\delta}} \;\leqslant\; (\gamma_{n,k})^{2+\delta}\frac{C(t_n)}{(\bar H(t_n))^{\delta}}$$

and, on the other hand, using the technical Lemma 9, for some $\delta' > 0$,

$$\begin{aligned}
I_2(t_n) &\leqslant \int_{t_n}^{+\infty} \left| \log\left(\frac{z}{t_n}\right) \frac{\bar{F}^{(k)}(z)}{\bar{F}^{(k)}(t_n)} + \gamma_k \left(\frac{z}{t_n}\right)^{-1/\gamma_k} + \epsilon_n(u) \left(\frac{z}{t_n}\right)^{-1/\gamma_k+\delta'} \right|^{2+\delta} \frac{\bar{F}(z)dG(z)}{(\bar{H}(z))^{2+\delta}} \\
&\leqslant cst \left\{ \int_{t_n}^{+\infty} \log^{2+\delta}\left(\frac{z}{t_n}\right) \left(\frac{\bar{F}^{(k)}(z)}{\bar{F}^{(k)}(t_n)}\right)^{2+\delta} \frac{dC(z)}{(\bar{H}(z))^{\delta}} \right. \\
&\quad \left. + \ \gamma_k^{2+\delta} \int_{t_n}^{+\infty} \left(\frac{z}{t_n}\right)^{-(2+\delta)/\gamma_k} \frac{dC(z)}{(\bar{H}(z))^{\delta}} + \sup_{u>t_n} |\epsilon_n(u)|^{2+\delta} \int_{t_n}^{+\infty} \left(\frac{z}{t_n}\right)^{(-1/\gamma_k+\delta')(2+\delta)} \frac{dC(z)}{(\bar{H}(z))^{\delta}} \right\} \\
&= cst \frac{C(t_n)}{(\bar{H}(t_n))^{\delta}} \left\{ \int_{t_n}^{+\infty} \log^{2+\delta}\left(\frac{z}{t_n}\right) \left(\frac{\bar{F}^{(k)}(z)}{\bar{F}^{(k)}(t_n)}\right)^{2+\delta} \left(\frac{\bar{H}(t_n)}{\bar{H}(z)}\right)^{\delta} \frac{dC(z)}{C(t_n)} \right. \\
&\quad \left. + \ \gamma_k^{2+\delta} \int_{t_n}^{+\infty} \left(\frac{z}{t_n}\right)^{-(2+\delta)/\gamma_k} \left(\frac{\bar{H}(t_n)}{\bar{H}(z)}\right)^{\delta} \frac{dC(z)}{C(t_n)} + o(1) \int_{t_n}^{+\infty} \left(\frac{z}{t_n}\right)^{(-1/\gamma_k+\delta')(2+\delta)} \left(\frac{\bar{H}(t_n)}{\bar{H}(z)}\right)^{\delta} \frac{dC(z)}{C(t_n)} \right\}
\end{aligned}$$

Applying Lemma 8 to $\delta = 1$, we have $C(t_n) = O\left(1/\bar{H}(t_n)\right)$, therefore $I_1(t_n) = O\left((\bar{H}(t_n))^{-1-\delta}\right)$. It is then easy to check that $n^{-\delta/2}\left(\bar{F}^{(k)}(t_n)\bar{G}(t_n)\right)^{1+\delta/2} I_1(t_n)$ tends to 0, because $\bar{F}^{(k)} \leqslant \bar{F}$ and $n\bar{H}(t_n) \to \infty$, since $\bar{H}(t_n) \geqslant \bar{F}^{(k)}(t_n)\bar{G}(t_n)$.

For $I_2(t_n)$, since by Lemma 8 the function $C$ is regularly varying with index $1/\gamma$, the application of part (*ii*) of Lemma 7 to $\alpha = 0$ or $2+\delta$ and to various couples of values of $a$ and $b$ finally yields $I_2(t_n) = O\left((\bar{H}(t_n))^{-1-\delta}\right)$, and consequently $n^{-\delta/2}\left(\bar{F}^{(k)}(t_n)\bar{G}(t_n)\right)^{1+\delta/2} I_2(t_n)$ tends to 0.

We now come to the study of relation (22) for $j = 3$. We have

$$\mathbb{E}\left|U_{1,n}^{(3)}\right|^{2+\delta} = \int_0^{+\infty} \left(\int_0^z \psi(\phi_n, u)dC(u)\right)^{2+\delta} dH(z).$$

Proceeding as above by splitting the integral into two integrals $\int_0^{t_n}$ and $\int_{t_n}^{+\infty}$, we obtain

$$\mathbb{E}\left|U_{1,n}^{(3)}\right|^{2+\delta} = J_1(t_n) + J_2(t_n),$$

where

$$J_1(t_n) = (\gamma_{n,k})^{2+\delta} \int_0^{t_n} (C(z))^{2+\delta} dH(z) = -(\gamma_{n,k})^{2+\delta} \bar{H}(t_n) \left(C(t_n)\right)^{2+\delta} \int_0^{t_n} \left(\frac{C(z)}{C(t_n)}\right)^{2+\delta} \frac{d\bar{H}(z)}{\bar{H}(t_n)}$$

and

$$J_2(t_n) \leqslant cst(J_2^{(1)}(t_n) + J_2^{(2)}(t_n)),$$

where

$$J_2^{(1)}(t_n) = \int_{t_n}^{+\infty} \left(\int_0^{t_n} \psi(\phi_n, u)dC(u)\right)^{2+\delta} dH(z) = \gamma_{n,k}^{2+\delta} \int_{t_n}^{+\infty} (C(t_n))^{2+\delta} dH(z) = \gamma_{n,k}^{2+\delta} \left(C(t_n)\right)^{2+\delta} \bar{H}(t_n)$$

and, using the technical Lemma 9 as we did some lines above,

$$\begin{aligned}
J_2^{(2)}(t_n) &= \int_{t_n}^{+\infty} \left(\int_{t_n}^z \psi(\phi_n, u)dC(u)\right)^{2+\delta} dH(z) \\
&\leqslant cst \left( \int_{t_n}^{+\infty} \left( \int_{t_n}^z \log\left(\frac{u}{t_n}\right) \frac{\bar{F}^{(k)}(u)}{\bar{F}^{(k)}(t_n)} dC(u) \right)^{2+\delta} dH(z) + \gamma_k \int_{t_n}^{+\infty} \left( \int_{t_n}^z \left(\frac{u}{t_n}\right)^{-1/\gamma_k} dC(u) \right)^{2+\delta} dH(z) \right. \\
&\quad \left. + \sup_{u>t_n} |\epsilon_n(u)|^{2+\delta} \int_{t_n}^{+\infty} \left( \int_{t_n}^z \left(\frac{u}{t_n}\right)^{-1/\gamma_k+\delta'} dC(u) \right)^{2+\delta} dH(z) \right).
\end{aligned} \tag{23}$$

Using Lemma 8 and part (*iii*) of Lemma 7, we find that both $J_1(t_n)$ and $J_2^{(1)}(t_n)$ are $O\left((\bar{H}(t_n))^{-1-\delta}\right)$ and, though the term $J_2^{(2)}(t_n)$ is more involved, we are also going to prove below that the same property holds for $J_2^{(2)}(t_n)$ : this will finish the proof of Lemma 4 because $n^{-\delta/2}\left(\bar{F}^{(k)}(t_n)\bar{G}(t_n)\right)^{1+\delta/2} (\bar{H}(t_n))^{-1-\delta}$ tends to

0, as already seen in the proof for $j = 2$.

We only treat the first integral in the right-hand side of (23), since the two others are very similar, *i.e.* we need to prove that

$$\int_{t_n}^{+\infty} \left( \int_{t_n}^{z} \log\left(\frac{u}{t_n}\right) \frac{\bar{F}^{(k)}(u)}{\bar{F}^{(k)}(t_n)} dC(u) \right)^{2+\delta} dH(z) = O\left( \frac{1}{(\bar{H}(t_n))^{1+\delta}} \right). \tag{24}$$

Now,

$$\int_{t_n}^{+\infty} \left( \int_{t_n}^{z} \log\left(\frac{u}{t_n}\right) \frac{\bar{F}^{(k)}(u)}{\bar{F}^{(k)}(t_n)} dC(u) \right)^{2+\delta} dH(z) \;=\; (C(t_n))^{2+\delta} \int_{t_n}^{+\infty} \left( \int_{1}^{z/t_n} \log(y) \frac{\bar{F}^{(k)}(yt_n)}{\bar{F}^{(k)}(t_n)} \frac{dC(yt_n)}{C(t_n)} \right)^{2+\delta} dH(z)$$

Using Potter-bounds (41) for $\bar{F}^{(k)} \in RV_{-1/\gamma_k}$, integration by parts and then Potter-bounds (41) for $C \in RV_{1/\gamma}$, it is easy to see that for $n$ sufficiently large and $\epsilon > 0$, there exists some positive constants $c$, $c'$, $c''$ such that

$$\left( \int_{1}^{z/t_n} \log(y) \frac{\bar{F}^{(k)}(yt_n)}{\bar{F}^{(k)}(t_n)} \frac{dC(yt_n)}{C(t_n)} \right)^{2+\delta} \;\leqslant\; c \log^{2+\delta}\left(\frac{z}{t_n}\right) \left(\frac{z}{t_n}\right)^{a} + c'\left(\frac{z}{t_n}\right)^{a} + c''.$$

where $a = (2+\delta)(\frac{1}{\gamma} - \frac{1}{\gamma_k} + 2\epsilon)$. Consequently

$$\begin{aligned}
&\int_{t_n}^{+\infty} \left( \int_{t_n}^{z} \log\left(\frac{u}{t_n}\right) \frac{\bar{F}^{(k)}(u)}{\bar{F}^{(k)}(t_n)} dC(u) \right)^{2+\delta} dH(z) \\
\leqslant\;& (C(t_n))^{2+\delta} \bar{H}(t_n) \left( -c \int_{t_n}^{+\infty} \log^{2+\delta}\left(\frac{z}{t_n}\right) \left(\frac{z}{t_n}\right)^{a} \frac{d\bar{H}(z)}{\bar{H}(t_n)} - c' \int_{t_n}^{+\infty} \left(\frac{z}{t_n}\right)^{a} \frac{d\bar{H}(z)}{\bar{H}(t_n)} - c'' \right).
\end{aligned}$$

This yields (24), by using part $(ii)$ of Lemma 7 to this value of $a$, to $b = -1/\gamma$ (and to $\alpha = 2+\delta$ or $\alpha = 0$), as well as Lemma 8.

### *5.2. Proof of Proposition 3*

Let us start with an important note. In Proposition 3, the main result is that the remainder terms $R_{n,\phi}$ and $R_{n,g}$ are $o_{\mathbb{P}}(v_n^{-1/2})$. Proving this will be conducted in a similar way as proving that $R_n$ is $o_{\mathbb{P}}(n^{-1/2})$ in Theorem 1.1 of Stute (1995). But, recall that in our situation, the function that we integrate here is $\phi_n$, which is depending on $n$, with a "sliding" support $[t_n, +\infty[$. We will need to be particularly cautious with integrability issues, especially when dealing with U-statistics for the terms $R_{n,2}$ and $R_{n,3}$ in the remainder $R_{n,C}$, defined below.

Before we proceed with the proof, let us define the following empirical (sub)-distribution functions : for $t \geqslant 0$,

$$H_n(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_{Z_i \leqslant t}, \quad H_n^{(0)}(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_{Z_i \leqslant t}\ \delta_i, \quad H_n^{(1,k)}(t) = \sum_{i=1}^{n} \mathbb{I}_{Z_i \leqslant t}\ \delta_i\ \mathbb{I}_{\mathscr{C}_i = k}.$$

First note that, since $g_n$ is the function $\phi_n$ without the log factor, it should be clear to the reader that proving that $\Delta_n = \hat{\Delta}_n + R_{n,g}$ and $\sqrt{v_n} R_{n,g} = o_{\mathbb{P}}(1)$ will be simpler than proving that $\widetilde{\gamma}_{n,k} = \check{\gamma}_{n,k} + R_{n,\phi}$ and $\sqrt{v_n} R_{n,\phi} = o_{\mathbb{P}}(1)$. We will thus only prove the latter two relations.

Let us start with the first one, in other words let us define the remainder term $R_{n,\phi}$. Remind that the definitions of $\widetilde{\gamma}_{n,k}$ and $\check{\gamma}_{n,k}$ are $\widetilde{\gamma}_{n,k} = \int \phi_n(u) dF_n^{(k)}(u)$ and $\check{\gamma}_{n,k} = \bar{U}_n^{(1)} + \bar{U}_n^{(2)} - \bar{U}_n^{(3)}$, where $\bar{U}_n^{(j)}$ denotes the mean of the $n$ variables $U_{i,n}^{(j)}$. We need to decompose the integral of $\phi_n$ with respect to $F_n^{(k)}$, which is a stepwise subdistribution function which jumps at the (ordered) observations $Z_{(i)}$ are equal to $\mathbb{I}_{\xi_{(i)}=k}/(n\bar{G}_n(Z_{(i-1)}))$. But it is known that (see Lemma 2.1 in Stute (1995))

$$\frac{1}{\bar{G}_n(Z_{(i-1)})} = \exp\left\{ n \int_0^{Z_{(i)}^-} \log(1 + (n\bar{H}_n(x))^{-1})\, dH_n^{(0)}(x) \right\}$$

Therefore, using the fact that $\bar{G}(z) = \exp\left(-\int_0^z \bar{H}^{-1} dH^{(0)}\right)$, we have

$$\begin{aligned}\widetilde{\gamma}_{n,k} &= \frac{1}{n}\sum_{i=1}^n \frac{\phi_n(Z_{(i)})}{\bar{G}(Z_{(i)}^-)} \mathbb{I}_{\xi_{(i)}=k} \frac{\bar{G}(Z_{(i)}^-)}{\bar{G}_n(Z_{(i-1)})} \\ &= \frac{1}{n}\sum_{i=1}^n \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)} \mathbb{I}_{\xi_i=k} \exp\left( n\int_0^{Z_i^-} \log(1+(n\bar{H}_n(x))^{-1})\, dH_n^{(0)}(x) \;-\; \int_0^{Z_i^-} \frac{dH^{(0)}}{\bar{H}} \right).\end{aligned}$$

Consequently, using the mean value theorem for exp, and introducing the important notations

$$\begin{aligned}B_{i,n} &= n\int_0^{Z_i^-} \log(1+(n\bar{H}_n(x))^{-1})\, dH_n^{(0)}(x) \;-\; \int_0^{Z_i^-} \frac{dH_n^{(0)}}{\bar{H}_n} \\ C_{i,n} &= \int_0^{Z_i^-} \frac{dH_n^{(0)}}{\bar{H}_n} \;-\; \int_0^{Z_i^-} \frac{dH^{(0)}}{\bar{H}},\end{aligned}$$

it is easy to see that

$$\begin{aligned}\widetilde{\gamma}_{n,k} &= \frac{1}{n}\sum_{i=1}^n \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)} \mathbb{I}_{\xi_i=k} \;+\; \frac{1}{n}\sum_{i=1}^n \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)} \mathbb{I}_{\xi_i=k} B_{i,n} \;+\; \frac{1}{n}\sum_{i=1}^n \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)} \mathbb{I}_{\xi_i=k} C_{i,n} \\ &\qquad + \; \frac{1}{2n}\sum_{i=1}^n \phi_n(Z_i)\mathbb{I}_{\xi_i=k}(B_{i,n}+C_{i,n})^2 e^{\Delta_{i,n}} \\ &= \bar{U}_n^{(1)} \;+\; R_{n,B} \;+\; \frac{1}{n}\sum_{i=1}^n \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)} \mathbb{I}_{\xi_i=k} C_{i,n} \;+\; R_{n,\Delta}, \qquad (25)\end{aligned}$$

where $\bar{U}_n^{(1)}$ is the first term in the definition of $\check{\gamma}_{n,k}$, and $\Delta_{i,n}$ is a random quantity lying between $\int_0^{Z_i^-} \bar{H}^{-1} dH^{(0)}$ and $n\int_0^{Z_i^-} \log(1+(n\bar{H}_n(x))^{-1})\, dH_n^{(0)}(x)$.

What we now need to do is to show that the term involving the quantity $C_{i,n}$ in relation (25) above can be written as $\bar{U}_n^{(2)} - \bar{U}_n^{(3)}$ plus a remainder term $R_{n,C}$, and therefore we have $\widetilde{\gamma}_{n,k} = \check{\gamma}_{n,k} + R_{n,\phi}$, where

$$R_{n,\phi} \;=\; R_{n,B} + R_{n,C} + R_{n,\Delta}. \qquad (26)$$

The rest of the proof will, afterwards, be devoted to showing that each term of $R_{n,\phi}$ is $o_{\mathbb{P}}(v_n^{-1/2})$.

Proceeding as in Stute (1995) or Suzukawa (2002), and using the fact that for any given function $f$ we have $\int f\, dH_n^{(1,k)} = \frac{1}{n}\sum_{i=1}^n f(Z_i)\mathbb{I}_{\xi_i=k}$, we can write

$$\frac{1}{n}\sum_{i=1}^n \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)} \mathbb{I}_{\xi_i=k} C_{i,n} \;=\; -C_n^{(1)} + 2C_n^{(2)} - C_n^{(3)} + R_{n,1} \qquad (27)$$

where

$$\begin{aligned}C_n^{(1)} &= \iiint \frac{\phi_n(z)}{\bar{G}(z)\bar{H}^2(v)} \mathbb{I}_{z>v}\mathbb{I}_{u>v}\, dH_n(u) dH_n^{(0)}(v) dH_n^{(1,k)}(z), \\ C_n^{(2)} &= \iint \frac{\phi_n(z)}{\bar{G}(z)\bar{H}(v)} \mathbb{I}_{z>v}\, dH_n^{(0)}(v) dH_n^{(1,k)}(z), \\ C_n^{(3)} &= \iint \frac{\phi_n(z)}{\bar{G}(z)\bar{H}(v)} \mathbb{I}_{z>v}\, dH^{(0)}(v) dH_n^{(1,k)}(z), \\ R_{n,1} &= \iint \frac{\phi_n(z)}{\bar{G}(z)} \mathbb{I}_{z>v} \frac{(\bar{H}_n - \bar{H})^2(v)}{\bar{H}^2(v)\bar{H}_n(v)} dH_n^{(0)}(v) dH_n^{(1,k)}(z).\end{aligned}$$

Note that $C_n^{(1)}$ and $C_n^{(2)}$ are a kind of $U$-statistics, which need to be approximated by sums of independent variables called Hoeffding decompositions : more precisely, if we introduce the functions (important in the sequel)

$$h(v,w) \;=\; \frac{\phi_n(w)}{\bar{G}(w^-)\bar{H}(v)} \mathbb{I}_{w>v}\mathbb{I}_{v<\infty}\mathbb{I}_{w<\infty} \quad \text{and} \quad \underline{h}(u,v,w) \;=\; h(v,w)\frac{\mathbb{I}_{u>v}}{\bar{H}(v)} \qquad (28)$$

for $u \in \mathbb{R}$, $v \in \mathbb{R} \cup \{+\infty\}$ and $w \in \mathbb{R} \cup \{+\infty\}$, then these decompositions are defined by

$$\begin{aligned}
\widehat{C}_n^{(1)} &= \iiint \underline{h}(u,v,w)dH_n(u)dH^{(0)}(v)dH^{(1,k)}(w) + \\
&\quad \iiint \underline{h}(u,v,w)dH(u)dH_n^{(0)}(v)dH^{(1,k)}(w) + \\
&\quad \iiint \underline{h}(u,v,w)dH(u)dH^{(0)}(v)dH_n^{(1,k)}(w) \\
&\quad - 2\iiint \underline{h}(u,v,w)dH(u)dH^{(0)}(v)dH^{(1,k)}(w) \qquad (29)\\
\widehat{C}_n^{(2)} &= \iint h(v,w)dH_n^{(0)}(v)dH^{(1,k)}(w) + \iint h(v,w)dH^{(0)}(v)dH_n^{(1,k)}(w) \\
&\quad - \iint h(v,w)dH^{(0)}(v)dH^{(1,k)}(w). \qquad (30)
\end{aligned}$$

Therefore, if we introduce the remainder terms

$$R_{n,2} = C_n^{(1)} - \widehat{C}_n^{(1)} \quad \text{and} \quad R_{n,3} = C_n^{(2)} - \widehat{C}_n^{(2)} \qquad (31)$$

then (27) becomes

$$\frac{1}{n}\sum_{i=1}^n \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)}\mathbb{I}_{\xi_i=k}C_{i,n} = -\widehat{C}_n^{(1)} + 2\widehat{C}_n^{(2)} - C_n^{(3)} + R_{n,C} \quad \text{where} \quad R_{n,C} = R_{n,1} - R_{n,2} + 2R_{n,3}.$$

We are thus left to prove that $-\widehat{C}_n^{(1)} + 2\widehat{C}_n^{(2)} - C_n^{(3)} = \bar{U}_n^{(2)} - \bar{U}_n^{(3)}$. This is indeed the case because, if we note

$$\theta_n = \iint h(v,w)dH^{(0)}(v)dH^{(1,k)}(w), \qquad (32)$$

then, by definition of $\underline{h}$, the last (fourth) term in $\widehat{C}_n^{(1)}$ equals $-2\theta_n$, the third one equals $C_n^{(3)}$, the second one is (because $dH^{(1,k)}(w) = \bar{G}(w^-)dF^{(k)}(w)$)

$$\begin{aligned}
\iiint \underline{h}(u,v,w)dH(u)dH_n^{(0)}(v)dH^{(1,k)}(w) &= \iint h(v,w)dH_n^{(0)}(v)dH^{(1,k)}(w) \\
&= \frac{1}{n}\sum_{i=1}^n (1-\delta_i)\int h(Z_i,w)\bar{G}(w^-)\,dF^{(k)}(w) \\
&= \frac{1}{n}\sum_{i=1}^n \frac{1-\delta_i}{\bar{H}(Z_i)}\int_{Z_i}^{\infty} \phi_n(w)\,dF^{(k)}(w) \\
&= \frac{1}{n}\sum_{i=1}^n \frac{1-\delta_i}{\bar{H}(Z_i)}\psi(\phi_n, Z_i) = \bar{U}_n^{(2)}
\end{aligned}$$

and the first one is (because $dH^{(1,k)}(w) = \bar{G}(w^-)dF^{(k)}(w)$ and $dH^{(0)}(v) = \bar{F}(v)dG(v)$)

$$\begin{aligned}
\iiint \underline{h}(u,v,w)dH_n(u)dH^{(0)}(v)dH^{(1,k)}(w) &= \frac{1}{n}\sum_{i=1}^n \iint \underline{h}(Z_i,v,w)dH^{(0)}(v)dH^{(1,k)}(w) \\
&= \frac{1}{n}\sum_{i=1}^n \iint \frac{\phi_n(w)}{\bar{G}(v)\bar{H}(v)}\mathbb{I}_{v<w}\mathbb{I}_{v<Z_i}\,dG(v)\,dF^{(k)}(w) \\
&= \frac{1}{n}\sum_{i=1}^n \int_0^{Z_i} \psi(\phi_n, v)\,dC(v) = \bar{U}_n^{(3)}.
\end{aligned}$$

Likewise, the first term of $\widehat{C}_n^{(2)}$ equals $\bar{U}_n^{(2)}$, the second one equals $C_n^{(3)}$, and the last one equals $-\theta_n$. After straightforward simplifications, we obtain the desired equality $-\widehat{C}_n^{(1)} + 2\widehat{C}_n^{(2)} - C_n^{(3)} = \bar{U}_n^{(2)} - \bar{U}_n^{(3)}$, and the proof of $\widetilde{\gamma}_{n,k} = \check{\gamma}_{n,k} + R_{n,\phi}$ is over.

The proof of Proposition 3 is now based on the following two lemmas : Lemma 5 is proved in subsection 5.2.1, and Lemma 6 is the longest to establish, its proof will be split across subsections 5.2.2 to 5.2.5.

**Lemma 5.** *If conditions* (1) *and (2) hold with* $\gamma_k < \gamma_C$*, then we have*

$$\sqrt{v_n}R_{n,B} = o_{\mathbb{P}}(1), \ \sqrt{v_n}R_{n,1} = o_{\mathbb{P}}(1), \ \text{and} \ \sqrt{v_n}R_{n,\Delta} = o_{\mathbb{P}}(1).$$

**Lemma 6.** *If conditions* (1) *and (2) hold with* $\gamma_k < \gamma_C$*, then we have*

$$\sqrt{v_n} R_{n,j} = o_{\mathbb{P}}(1) \qquad \text{for } j = 2 \text{ and for } j = 3.$$

*5.2.1. Proof of Lemma 5*

• We start with the remainder term $R_{n,B}$, which is defined as

$$R_{n,B} = \frac{1}{n} \sum_{i=1}^{n} \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)} \mathbb{I}_{\xi_i = k} B_{i,n},$$

where $B_{i,n} = n \int_0^{Z_i^-} \log(1 + (n\bar{H}_n(x))^{-1})\, dH_n^{(0)}(x) - \int_0^{Z_i^-} \frac{dH_n^{(0)}}{\bar{H}_n}$. Since, for all $x \geqslant 0$, $x - \frac{x^2}{2} \leqslant \log(1+x) \leqslant x$, we obtain

$$-\frac{1}{2n} \int_0^{Z_i^-} \frac{dH_n^{(0)}(x)}{(\bar{H}_n(x))^2} \leqslant B_{i,n} \leqslant 0$$

and then

$$\begin{aligned} |R_{n,B}| &\leqslant \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)} \mathbb{I}_{\xi_i = k} \frac{1}{2n} \left( \int_0^{Z_i^-} \frac{dH_n^{(0)}(x)}{(\bar{H}_n(x))^2} \right) \right\} \\ &\leqslant \sup_{0 \leqslant x < Z_{(n)}} \left( \frac{\bar{H}(x)}{\bar{H}_n(x)} \right)^2 \frac{1}{2n^2} \sum_{i=1}^{n} \left\{ \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)} \mathbb{I}_{\xi_i = k} \left( \int_0^{Z_i^-} \frac{dH_n^{(0)}(x)}{(\bar{H}(x))^2} \right) \right\}. \qquad (33) \end{aligned}$$

But

$$\int_0^{Z_i^-} \frac{dH_n^{(0)}(x)}{(\bar{H}(x))^2} = \int_0^{Z_i^-} \frac{dH^{(0)}(x)}{(\bar{H}(x))^2} + \int_0^{Z_i^-} \frac{d(H_n^{(0)} - H^{(0)})(x)}{(\bar{H}(x))^2},$$

so, if we define

$$T_n^{(1)} = \frac{1}{n} \sum_{i=1}^{n} T_{i,n}^{(1)} \text{ and } T_n^{(2)} = \frac{1}{n} \sum_{i=1}^{n} T_{i,n}^{(2)}$$

where

$$\begin{aligned} T_{i,n}^{(1)} &= \frac{1}{n} \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)} \mathbb{I}_{\xi_i = k} \int_0^{Z_i^-} \frac{dH^{(0)}(x)}{(\bar{H}(x))^2} \\ T_{i,n}^{(2)} &= \frac{1}{n} \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)} \mathbb{I}_{\xi_i = k} \int_0^{Z_i^-} \frac{d(H_n^{(0)} - H^{(0)})(x)}{(\bar{H}(x))^2}, \end{aligned}$$

then it remains to prove (thanks to part $(i)$ of Lemma 10) that $\sqrt{v_n} T_n^{(1)} = o_{\mathbb{P}}(1)$ and $\sqrt{v_n} T_n^{(2)} = o_{\mathbb{P}}(1)$.

Concerning $T_n^{(1)}$, since $\bar{H} \geqslant \bar{H}^{(0)}$ implies that $T_{i,n}^{(1)} \leqslant \frac{1}{n} \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)\bar{H}^{(0)}(Z_i^-)} \mathbb{I}_{\xi_i = k}$, then $\sqrt{v_n} T_n^{(1)} = o_{\mathbb{P}}(1)$ is a consequence of Lemma 11, used with $\alpha = 0$ and $d = 1$.

Concerning $T_n^{(2)}$, an integration by parts yields

$$\begin{aligned} \left| \int_0^{Z_i^-} \frac{d(H_n^{(0)} - H^{(0)})(x)}{(\bar{H}(x))^2} \right| &\leqslant \frac{|\bar{H}_n^{(0)} - \bar{H}^{(0)}|(Z_i^-)}{\bar{H}^2(Z_i^-)} + 2 \int_0^{Z_i^-} \frac{|\bar{H}_n^{(0)} - \bar{H}^{(0)}|(x)}{\bar{H}^3(x)}\, dH(x) + |\bar{H}_n^{(0)}(0) - \bar{H}^{(0)}(0)| \\ &\leqslant \sup_{0 \leqslant x < Z_{(n)}} \sqrt{n} \frac{|\bar{H}_n^{(0)} - \bar{H}^{(0)}|(x)}{(\bar{H}^{(0)}(x))^{\frac{1}{2} - \alpha}} \left( \frac{\bar{H}^{(0)}(x)}{\bar{H}(x)} \right)^{\frac{1}{2} - \alpha} \times \\ &\quad \left( \frac{1}{\sqrt{n} \left( \bar{H}(Z_i^-) \right)^{\frac{3}{2} + \alpha}} + \frac{2}{\sqrt{n}} \int_0^{Z_i^-} \frac{dH(x)}{\left( \bar{H}(x) \right)^{\frac{5}{2} + \alpha}} \right) + |\bar{H}_n^{(0)}(0) - \bar{H}^{(0)}(0)|, \end{aligned}$$

for any given $0 < \alpha < \frac{1}{2}$. Lemma 10 (applied with $a = 1/2 - \alpha < 1/2$) and the fact $\bar{H}^{(0)} \leqslant \bar{H}$ thus imply that

$$\left| \int_0^{Z_i^-} \frac{d(H_n^{(0)} - H^{(0)})(x)}{(\bar{H}(x))^2} \right| \leqslant O_{\mathbb{P}}(1) \frac{1}{\sqrt{n} \left( \bar{H}(Z_i^-) \right)^{\frac{3}{2} + \alpha}} + |\bar{H}_n^{(0)}(0) - \bar{H}^{(0)}(0)|,$$

so that, by definition of $T_{i,n}^{(2)}$, the desired statement $\sqrt{v_n}T_n^{(2)} = o_{\mathbb{P}}(1)$ is a consequence of Lemma 11, applied with $\alpha > 0$ sufficiently small and $d = \frac{3}{2}$, and of

$$\frac{\sqrt{v_n}}{n}\sum_{i=1}^{n}\frac{1}{n}\frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)}\mathbb{I}_{\xi_i=k}|\bar{H}_n^{(0)}(0)-\bar{H}^{(0)}(0)| = \sqrt{n}|\bar{H}_n^{(0)}(0)-\bar{H}^{(0)}(0)| \times \frac{\sqrt{\bar{F}^{(k)}(t_n)\bar{G}(t_n)}}{n} \times \bar{U}_n^{(1)} = o_{\mathbb{P}}(1).$$

Indeed $\bar{U}_n^{(1)}$ converges to $\gamma_k$ and $\bar{H}_n^{(0)}(0) - \bar{H}^{(0)}(0)$ equals $\frac{1}{n}\sum_{i=1}^n \mathbb{I}_{\delta_i=0} - \mathbb{P}(\delta = 0)$, which is $O_{\mathbb{P}}(n^{-1/2})$ by the standard central limit theorem.

- Let us now turn to the remainder term $R_{n,1}$, which is defined as

$$R_{n,1} = \iint \frac{\phi_n(z)}{\bar{G}(z)}\mathbb{I}_{z>v}\frac{(\bar{H}_n-\bar{H})^2(v)}{\bar{H}^2(v)\bar{H}_n(v)}dH_n^{(0)}(v)dH_n^{(1,k)}(z).$$

A simple calculation leads to

$$R_{n,1} \leqslant \sup_{0\leqslant x<Z_{(n)}}\left(\sqrt{n}\frac{|H_n-H|(x)}{(\bar{H}(x))^{\frac{1}{2}-\alpha}}\right)^2 \sup_{0\leqslant x<Z_{(n)}}\frac{\bar{H}(x)}{\bar{H}_n(x)} \times \frac{1}{n^2}\sum_{i=1}^{n}\left\{\frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)}\mathbb{I}_{\xi_i=k}\left(\int_0^{Z_i^-}\frac{dH_n^{(0)}(v)}{(\bar{H}(v))^{2+2\alpha}}\right)\right\}$$

for any $0 < \alpha < \frac{1}{2}$. Taking $\alpha$ sufficiently small, the rest of the proof is very similar to the one for $R_{n,B}$ (compare to (33)) and relies on Lemma 10 and Lemma 11.

- We can finally deal with the last remainder term $R_{n,\Delta}$, defined as

$$R_{n,\Delta} = \frac{1}{2n}\sum_{i=1}^{n}\phi_n(Z_i)\mathbb{I}_{\xi_i=k}(B_{i,n}+C_{i,n})^2e^{\Delta_{i,n}},$$

where $\Delta_{i,n}$ is a random quantity lying between $a_n := \int_0^{Z_i^-}\bar{H}^{-1}dH^{(0)}$ and $b_n := \int_0^{Z_i^-}\log(1+(n\bar{H}_n(x))^{-1})\,dH_n^{(0)}(x)$. Since $e^a = 1/\bar{G}(Z_i^-)$, we have

$$R_{n,\Delta} = \frac{1}{2n}\sum_{i=1}^{n}\frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)}\mathbb{I}_{\xi_i=k}(B_{i,n}+C_{i,n})^2e^{\Delta_{i,n}-a_n}.$$

Since $b_n - a_n = B_{i,n} + C_{i,n}$, where $B_{i,n} < 0$ and $C_{i,n} = \int_0^{Z_i^-}\frac{dH_n^{(0)}}{\bar{H}_n} - \int_0^{Z_i^-}\frac{dH^{(0)}}{\bar{H}}$, we clearly have

$$e^{(\Delta_{i,n}-a)} \leqslant \max(1, e^{C_{i,n}}) \leqslant e^{|C_{i,n}|}.$$

But $C_{i,n} = \hat{\Lambda}_{n,G}(Z_i) - \Lambda_G(Z_i)$, where $\Lambda_G$ is the cumulative hazard function associated to $G$, and $\hat{\Lambda}_{n,G}$ its Nelson-Alen estimator. Relying on Zhou (1991) Theorem 2.1, we can deduce that $\sup_{1\leqslant i\leqslant n}|C_{i,n}| = O_{\mathbb{P}}(1)$. Hence, $e^{(\Delta_{i,n}-a)} = O_{\mathbb{P}}(1)$.
Now,

$$C_{i,n} - \frac{1}{2n}\int_0^{Z_i^-}\frac{dH_n^{(0)}}{(\bar{H}_n)^2} \leqslant B_{i,n} + C_{i,n} \leqslant C_{i,n}.$$

By writing

$$C_{i,n} = \int_0^{Z_i^-}\frac{d(H_n^{(0)}-H^{(0)})}{\bar{H}_n} + \int_0^{Z_i^-}\left(\frac{1}{\bar{H}_n}-\frac{1}{\bar{H}}\right)dH^{(0)},$$

we prove (using Lemma 10 and simple integrations as for the previous treatment of $T_n^{(2)}$ above) that $|C_{i,n}| \leqslant O_{\mathbb{P}}(1)\ 1/(\sqrt{n}(\bar{H}(Z_i))^{1/2+\alpha}) + |\bar{H}_n^{(0)}(0) - \bar{H}^{(0)}(0)|$ for $0 < \alpha < 1/2$.

Hence, on one hand $(C_{i,n})^2 \leqslant O_{\mathbb{P}}(1)(n(\bar{H}(Z_i))^{1+2\alpha})^{-1} + O_{\mathbb{P}}(n^{-1}) \leqslant O_{\mathbb{P}}(1)(n(\bar{H}^{(0)}(Z_i))^{1+2\alpha})^{-1} + O_{\mathbb{P}}(n^{-1})$, and on the other hand

$$\left(C_{i,n} - \frac{1}{2n}\int_0^{Z_i^-}\frac{dH_n^{(0)}}{(\bar{H}_n)^2}\right)^2 \leqslant O_{\mathbb{P}}(1)\left(\frac{1}{n(\bar{H}^{(0)}(Z_i))^{1+2\alpha}} + \frac{1}{n^2(\bar{H}^{(0)}(Z_i))^2}\right) + O_{\mathbb{P}}(n^{-1}),$$

for any given $0 < \alpha < 1/2$ (where the $O_{\mathbb{P}}(n^{-1})$ comes from $|\bar{H}_n^{(0)}(0) - \bar{H}^{(0)}(0)|^2$, which does not depend on $i$). Therefore, it is sufficient to prove that

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\sqrt{v_n}\log(Z_i/t_n)\mathbb{I}_{\xi_i=k}\mathbb{I}_{Z_i>t_n}}{\bar{F}^{(k)}(t_n)\bar{G}(Z_i^-)n(\bar{H}^{(0)}(Z_i))^{1+2\alpha}} \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n}\frac{\sqrt{v_n}\log(Z_i/t_n)\mathbb{I}_{\xi_i=k}\mathbb{I}_{Z_i>t_n}}{\bar{F}^{(k)}(t_n)\bar{G}(Z_i^-)n^2(\bar{H}^{(0)}(Z_i))^2}$$

are $o_{\mathbb{P}}(1)$, and that

$$n^{-1/2}\left(\bar{F}^{(k)}(t_n)\bar{G}(t_n)\right)^{1/2}\times\frac{1}{n}\sum_{i=1}^{n}\frac{\log(Z_i/t_n)\mathbb{I}_{\xi_i=k}\mathbb{I}_{Z_i>t_n}}{\bar{F}^{(k)}(t_n)\bar{G}(Z_i^-)}$$

is $o_{\mathbb{P}}(1)$ as well. But the first two statements are consequences of Lemma 11 with $\alpha>0$ sufficiently close to 0 and, respectively, $d=1$ and $d=2$. And for the third statement, the expectation of the expression turns out (thanks to Lemma 7 part (*ii*)) to be equivalent to a constant times $n^{-1/2}(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{1/2}$, which tends to 0.

*5.2.2. Preliminaries to the proof of Lemma 6*

We start this section by introducing important objects, issued from an idea appearing (to the best of our knowledge) in Stute (1994). We define the improper variables $(V_i)_{1\leqslant i\leqslant n}$ and $(W_j)_{1\leqslant j\leqslant n}$ by

$$V_i=\begin{cases}Z_i & \text{if } \delta_i=0\\ +\infty & \text{if } \delta_i=1\end{cases}\qquad\text{and}\qquad W_j=\begin{cases}+\infty & \text{if } \delta_j=0 \text{ or } \mathscr{C}_j\neq k\\ Z_j & \text{if } \delta_j=1 \text{ and } \mathscr{C}_j=k\end{cases}$$

which have $H^{(0)}$ and $H^{(1,k)}$ for respective subdistribution functions. We thus have $1-\delta_i=\mathbb{I}_{V_i<\infty}$ and $\mathbb{I}_{\mathscr{C}_j=k}=\mathbb{I}_{W_j<\infty}$, which, according to the definitions of $C_n^{(1)}$ and $C_n^{(2)}$ on one hand, and of functions $h$ and $\underline{h}$ (in (28)) on the other hand, leads to

$$C_n^{(2)}\;=\;\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\phi_n(Z_j)}{\bar{G}(Z_j^-)\bar{H}(Z_i)}\mathbb{I}_{Z_j>Z_i}(1-\delta_i)\delta_j\mathbb{I}_{\mathscr{C}_j=k}\;=\;\frac{1}{n^2}\sum\sum_{i\neq j}h(V_i,W_j)$$

and

$$C_n^{(1)}\;=\;\frac{1}{n^3}\sum_{l=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\phi_n(Z_j)}{\bar{G}(Z_j^-)\bar{H}^2(Z_i)}\mathbb{I}_{Z_j>Z_i}\mathbb{I}_{Z_l>Z_i}(1-\delta_i)\delta_j\mathbb{I}_{\mathscr{C}_j=k}\;=\;\frac{1}{n^3}\sum\sum\sum_{i\neq j,i\neq l}\underline{h}(Z_l,V_i,W_j).$$

Since the latter triple sum is not convenient, we also define

$$C_n^{(1)}\;=\;\widetilde{C}_n^{(1)}+\widetilde{\widetilde{C}}_n^{(1)}\quad\text{where}\quad\widetilde{C}_n^{(1)}=\frac{1}{n^3}\sum\sum_{i,j,l}\sum_{distincts}\underline{h}(Z_l,V_i,W_j)\quad\text{and}\quad\widetilde{\widetilde{C}}_n^{(1)}=\frac{1}{n^3}\sum\sum_{i\neq j}h(V_i,W_j)/\bar{H}(V_i),$$

where $\widetilde{C}_n^{(1)}$ will be the quantity approximated by $\widehat{C}_n^{(1)}$, and $\widetilde{\widetilde{C}}_n^{(1)}$ will be a remainder. We can indeed rewrite (31) as

$$R_{n,3}\;=\;\tfrac{n(n-1)}{n^2}\left(\tfrac{n^2}{n(n-1)}C_n^{(2)}-\widehat{C}_n^{(2)}\right)-\widehat{C}_n^{(2)}/n,\qquad(34)$$

$$R_{n,2}\;=\;\tfrac{n(n-1)(n-2)}{n^3}\left(\tfrac{n^3}{n(n-1)(n-2)}\widetilde{C}_n^{(1)}-\widehat{C}_n^{(1)}\right)-\tfrac{3n-2}{n^2}\widehat{C}_n^{(1)}+\widetilde{\widetilde{C}}_n^{(1)}.\qquad(35)$$

The terms in parentheses in (34) and (35) turn out to be genuine U-statistics of 2 and 3 variables, denoted by

$$\mathscr{U}_n=\tfrac{1}{n(n-1)}\sum\sum_{i\neq j}\mathscr{H}(V_i,W_j)\quad\text{and}\quad\mathscr{V}_n=\tfrac{1}{n(n-1)(n-2)}\sum\sum_{i,j,l}\sum_{distincts}\underline{\mathscr{H}}(Z_l,V_i,W_j)\qquad(36)$$

where functions $\mathscr{H}$ and $\underline{\mathscr{H}}$ will be defined in a few lines (relation (37)) after some preliminaries, certainly well-known in the U-statistics literature, but which we include here to make our proof self-contained (and since we are dealing with improper variables).

If $V$ and $W$ denote *independent* improper random variables with subdistribution functions $H^{(0)}$ and $H^{(1,k)}$ (*i.e.* $V=Z\mathbb{I}_{\delta=0}+\infty\mathbb{I}_{\delta=1}$ and $W=Z'\delta'\mathbb{I}_{\mathscr{C}'=k}+\infty(1-\delta'+\mathbb{I}_{\mathscr{C}'\neq k})$ where $(Z,\delta,\mathscr{C})$ and $(Z',\delta',\mathscr{C}')$ are independent copies of $(Z_1,\delta_1,\mathscr{C}_1)$), we introduce the following notations : for any function $g:[0,\infty]\times[0,\infty]\to\mathbb{R}$,

$$g_{1\bullet}(v)=\mathbb{E}(g(v,W))\quad\text{and}\quad g_{\bullet 1}(w)=\mathbb{E}(g(V,w))\ ,$$

as well as, for any function $g:[0,\infty[\times[0,\infty]\times[0,\infty]\to\mathbb{R}$, with $Z$ (of distribution function $H$) independent of $V$ and $W$,

$$g_{1\bullet\bullet}(u)=\mathbb{E}(g(u,V,W))\quad,\quad g_{\bullet 1\bullet}(v)=\mathbb{E}(g(Z,v,W))\quad\text{and}\quad g_{\bullet\bullet 1}(w)=\mathbb{E}(g(Z,V,w)).$$

Since $h(v,w)=0$ whenever $v$ or $w$ equals $\infty$, we then have (the proof is simple)

$$\theta_n=\iint h(v,w)dH^{(0)}(v)dH^{(1,k)}(w)=\mathbb{E}(h(V,W))=\mathbb{E}(\underline{h}(Z,V,W)).$$

Therefore, setting (for $z$ in $[0,\infty[$ and $v$ and $w$ in $[0,\infty]$)

$$\begin{array}{rcl} \mathscr{H}(v,w) & = & h(v,w) - h_{1\bullet}(v) - h_{\bullet 1}(w) + \theta_n \\ \underline{\mathscr{H}}(z,v,w) & = & h(z,v,w) - \underline{h}_{1\bullet\bullet}(z) - \underline{h}_{\bullet 1\bullet}(v) - \underline{h}_{\bullet\bullet 1}(w) + 2\theta_n \end{array} \tag{37}$$

it is then not difficult to check (using (29) and (30)) that $\mathscr{U}_n$ and $\mathscr{V}_n$ in relation (36) are indeed equal to the differences in parentheses in relations (34) and (35), respectively. Lemma 6 thus becomes a consequence of the following facts : $\sqrt{v_n}\mathscr{U}_n = o_{\mathbb{P}}(1)$, $\sqrt{v_n}\mathscr{V}_n = o_{\mathbb{P}}(1)$, and

$$\text{the three sequences } \ \widehat{\widehat{\widehat{C}}}{}_n^{(1)}\ ,\ \widehat{C}_n^{(1)}/n \ \text{ and } \ \widehat{C}_n^{(2)}/n \ \text{ are } \ o_{\mathbb{P}}(v_n^{-1/2}). \tag{38}$$

We will prove these statements in the next 3 subsections.

*5.2.3. Proof of $\sqrt{v_n}\mathscr{U}_n = o_{\mathbb{P}}(1)$*

We note $\mathscr{I} = \{I = (i,j)\,;\, 1 \leqslant i < j \leqslant n\}$, $\mathscr{H}_I = \mathscr{H}(V_i, W_j)$ when $I = (i,j) \in \mathscr{I}$, and $N = n(n-1)/2$. It is clear that it suffices to prove that

$$S_N = o_{\mathbb{P}}(1) \quad \text{ where } \quad S_N = \sum_{I\in\mathscr{I}} \tfrac{\sqrt{v_n}}{N}\mathscr{H}_I.$$

The good point is that $S_N$ turns out to be a sum of identically distributed centred and uncorrelated random variables $\mathscr{H}_I$, but unfortunately these variables $\mathscr{H}_I$ are not square-integrable and potentially only have a moment of order slightly larger than 4/3 when $\gamma_k < \gamma_C$. In order to deal with this difficulty, since we cannot handle directly the $L^p$ norm of $S_N$ of order $p = 4/3$, we will follow a strategy similar to that found in Csorgo, Szyszkowicz and Wang (2008), based on truncation. We set

$$\mathscr{H}^*(v,w) \ = \ \mathscr{H}(v,w)\mathbb{I}_{|\mathscr{H}(v,w)|\leqslant M_n} - \mathbb{E}(\mathscr{H}_1\mathbb{I}_{|\mathscr{H}_1|\leqslant M_n}) \ \text{ where } \ \left\{ \begin{array}{rcl} \mathscr{H}_1 & = & \mathscr{H}(V_1,W_2) \\ M_n & = & n^2/\sqrt{v_n} \end{array}\right. \tag{39}$$

The variables $\mathscr{H}_I^* = \mathscr{H}^*(V_i,W_j)$ $(I \in \mathscr{I})$ are centred and bounded, but they lose the non-correlation property of the variables $\mathscr{H}_I$. This is why we define now

$$\mathscr{H}_I^{**} \ = \ \mathscr{H}^{**}(V_i,W_j) \ \text{ where } \ \mathscr{H}^{**}(v,w) = \mathscr{H}^*(v,w) - \mathscr{H}^*_{1\bullet}(v) - \mathscr{H}^*_{\bullet 1}(w)$$

which are centred and bounded but are also uncorrelated (see part $(i)$ of Lemma 13), and we write

$$S_N \ = \ S_N^{(1)} + S_N^{(2)} \ = \ \tfrac{\sqrt{v_n}}{N}\textstyle\sum_{I\in\mathscr{I}}\mathscr{H}_I^{**} \ + \ \tfrac{\sqrt{v_n}}{N}\sum_{I\in\mathscr{I}}(\mathscr{H}_I - \mathscr{H}_I^{**}). \tag{40}$$

We thus need to prove that $S_N^{(1)}$ and $S_N^{(2)}$ both converge to 0 in probability.

Concerning $S_N^{(1)}$, since the $\mathscr{H}_I^{**}$ are centred and uncorrelated, we have

$$\mathbb{E}\left((S_N^{(1)})^2\right) \ = \ (v_n/N^2)\mathbb{E}\left((\textstyle\sum_{I\in\mathscr{I}}\mathscr{H}_I^{**})^2\right) \ = \ (v_n/N)\mathbb{E}\left((\mathscr{H}^{**}(V_1,W_2))^2\right) \ \leqslant \ cst(v_n/n^2)\mathbb{E}(\mathscr{H}_1^2\mathbb{I}_{|\mathscr{H}_1|\leqslant M_n})$$

where $\mathscr{H}_1$ was defined in (39) (the justification of the last inequality is postponed to part $(ii)$ of Lemma 13). Remind that $\mathscr{H}_1$ is not square-integrable and $M_n = n^2/\sqrt{v_n} = n^{3/2}/(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{1/2}$, and introduce $m_n = n^{3/2}/(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{1/2-\epsilon} = o(M_n)$ for some given small $\epsilon > 0$. We then write

$$\mathbb{E}\left((S_N^{(1)})^2\right) \leqslant cst\frac{v_n}{n^2}m_n^{2/3}\mathbb{E}\left(|\mathscr{H}_1|^{4/3}\right) \ + \ cst\frac{v_n}{n^2}M_n^{2/3}\mathbb{E}\left(|\mathscr{H}_1|^{4/3}\mathbb{I}_{|\mathscr{H}_1|>m_n}\right) \ = \ cst(A_n + B_n).$$

Thanks to Lemma 12 (parts $(i)$ and $(ii)$) and to the definition of $m_n$, the term $A_n$ is bounded by a quantity which is equivalent (as $n \to \infty$) to $\frac{v_n}{n^2}m_n^{2/3}\left(\frac{v_n}{n}\right)^{-2/3} = (\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{2\epsilon/3} = o(1)$. We now rely on Hölder's inequality for dealing with the term $B_n$ . Let $p > 1$ and $q > 1$ such that $1/p + 1/q = 1$. Since $\theta_n = \mathbb{E}(\mathscr{H}_1)$, again thanks to Lemma 12 ($(i)$, $(ii)$ and $(v)$), for $p$ sufficiently close to 1 so that $4p/3 < 1 + (1 + 2\gamma_k/\gamma_C)^{-1}$, we have

$$\begin{array}{rcl} B_n & \leqslant & \left(\frac{v_n}{n}\right)^{2/3}\left(\mathbb{E}|\mathscr{H}_1|^{4p/3}\right)^{1/p}(\mathbb{P}(|\mathscr{H}_1| > m_n))^{1/q} \\ & \leqslant & \left(\frac{v_n}{n}\right)^{2/3}\left(O\left(\left(\frac{v_n}{n}\right)^{2(1-4p/3)}\right)\right)^{1/p} m_n^{-1/q}\,(4\theta_n)^{1/q} \\ & \leqslant & O(1)\left(\frac{v_n}{n}\right)^{2/3+2(1-1/q)-8/3} M_n^{-1/q}(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{-\epsilon/q}(-\log\bar{G}(t_n))^{1/q} \\ & \leqslant & O(1)\,v_n^{-3/2q}(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{-2\epsilon/q}\left((\bar{G}(t_n))^{\epsilon}(-\log\bar{G}(t_n))\right)^{1/q} \ = \ o(1)\left(n(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{1+4\epsilon/3}\right)^{-3/2q} \end{array}$$

which converges to 0 thanks to assumption (2), for $\epsilon > 0$ small enough (we used part $(v)$ of Lemma 12 in the third upper bound).

We are thus left to prove that $S_N^{(2)}$ also converges to 0, but this time in $L^1$. We start by writing that

$$\begin{aligned}\mathbb{E}\left(|S_N^{(2)}|\right) &\leqslant \frac{\sqrt{v_n}}{N}\sum_{I\in\mathscr{I}}\mathbb{E}\left(|\mathscr{H}_I - \mathscr{H}_I^{**}|\right) = \sqrt{v_n}\,\mathbb{E}\left(|\mathscr{H}_1 - \mathscr{H}^*(V_1,W_2) + \mathscr{H}_{1\bullet}^*(V_1) + \mathscr{H}_{\bullet 1}^*(W_2)|\right)\\ &\leqslant 4\sqrt{v_n}\,\mathbb{E}\left(|\mathscr{H}_1|\mathbb{I}_{|\mathscr{H}_1|>M_n}\right),\end{aligned}$$

the last inequality being proved in the appendix (part $(iii)$ of Lemma 13). The follow-up is a bit similar to the treatment of $B_n$ above, relying on Lemma 12 (parts $(i)$, $(ii)$ and $(v)$) and on Hölder's inequality : for $p > 1$ close to 1 and a large $q$ such that $1/p + 1/q = 1$, we can write

$$\begin{aligned}\mathbb{E}\left(|S_N^{(2)}|\right) &\leqslant 4\sqrt{v_n}M_n^{-1/3}\left(\mathbb{E}|\mathscr{H}_1|^{4p/3}\right)^{1/p}\left(\mathbb{P}(|\mathscr{H}_1| > M_n)\right)^{1/q}\\ &\leqslant O(1)\,v_n^{-3/2q}(-\log \bar{G}(t_n))^{1/q} \leqslant O(1)\left\{\left(n(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{1+\epsilon}\right)^{-3/2}(\bar{G}(t_n))^{3\epsilon/2}(-\log\bar{G}(t_n))\right\}^{1/q}\end{aligned}$$

which, for $\epsilon > 0$ small enough, is $o(1)$ thanks to assumption (2).

### *5.2.4. Proof of* $\sqrt{v_n}\mathscr{V}_n = o_{\mathbb{P}}(1)$

The proof is very similar to the one contained in the previous subsection. We nonetheless provide a few details to convince the reader of the validity of the result. We note now $\mathscr{I} = \{I = (i,j,l)\,;\, 1 \leqslant i < j < l \leqslant n\}$ and $\underline{\mathscr{H}}_I = \underline{\mathscr{H}}(Z_l, V_i, W_j)$ when $I = (i,j,l) \in \mathscr{I}$, with $N = n(n-1)(n-2)/6$ denoting the cardinal of the index set $\mathscr{I}$. Since the observations $(Z_i)_{i\leqslant n}$ are i.i.d., it should be clear to the reader that it suffices to prove that

$$S_N = o_{\mathbb{P}}(1) \quad \text{where} \quad S_N = \sum_{I\in\mathscr{I}}\frac{\sqrt{v_n}}{N}\underline{\mathscr{H}}_I.$$

As previously, the problem lies with the moments of the centred and uncorrelated variables $\underline{\mathscr{H}}_I$, and now we only have a guaranteed moment of order slightly more than 6/5 instead of 4/3 in the previous situation. Fortunately, the cardinal $N$ is now of order $n^3$, which turns out to be the right compensation.

We thus define, for $(u,v,w) \in [0,\infty[\times[0,\infty]\times[0,\infty]$,

$$\underline{\mathscr{H}}^*(u,v,w) = \underline{\mathscr{H}}(u,v,w)\mathbb{I}_{|\underline{\mathscr{H}}(u,v,w)|\leqslant M_n} - \mathbb{E}(\underline{\mathscr{H}}_1\mathbb{I}_{|\underline{\mathscr{H}}_1|\leqslant M_n}) \quad \text{where} \quad \left\{\begin{array}{lcl}\underline{\mathscr{H}}_1 &=& \underline{\mathscr{H}}(Z_3,V_1,W_2)\\ M_n &=& n^3/\sqrt{v_n}\end{array}\right.$$

as well as

$$\underline{\mathscr{H}}_I^{**} = \underline{\mathscr{H}}^{**}(Z_l,V_i,W_j) \quad \text{where} \quad \underline{\mathscr{H}}^{**}(u,v,w) = \underline{\mathscr{H}}^*(u,v,w) - \underline{\mathscr{H}}_{1\bullet\bullet}^*(u) - \underline{\mathscr{H}}_{\bullet 1\bullet}^*(v) - \underline{\mathscr{H}}_{\bullet\bullet 1}^*(w)$$

which are centred and bounded but are also uncorrelated (see part $(i)$ of Lemma 13 in the Appendix), and we write

$$S_N = S_N^{(1)} + S_N^{(2)} = \frac{\sqrt{v_n}}{N}\sum_{I\in\mathscr{I}}\underline{\mathscr{H}}_I^{**} + \frac{\sqrt{v_n}}{N}\sum_{I\in\mathscr{I}}(\underline{\mathscr{H}}_I - \underline{\mathscr{H}}_I^{**}).$$

Introducing $m_n = M_n(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{\epsilon}$ and skipping details, we assess that

$$\mathbb{E}\left((S_N^{(1)})^2\right) \leqslant cst\frac{v_n}{n^3}m_n^{4/5}\mathbb{E}\left(|\underline{\mathscr{H}}_1|^{6/5}\right) + cst\frac{v_n}{n^3}M_n^{4/5}\mathbb{E}\left(|\underline{\mathscr{H}}_1|^{6/5}\mathbb{I}_{|\mathscr{H}_1|>m_n}\right)$$

and that this quantity converges to 0, as $n \to \infty$, thanks to parts $(i)$ and $(iii)$ of Lemma 12. The same argument is used to prove that $\mathbb{E}(|S_N^{(2)}|) \overset{n\to\infty}{\longrightarrow} 0$.

### *5.2.5. Proof of relation (38)*

Let us first prove that, for some $d \in ]4/5, 1[$, $\mathbb{E}(|\sqrt{v_n}\widetilde{\widetilde{C}}_n^{(1)}|^d)$ tends to 0, as $n$ tends to infinity. Recall that $\widetilde{\widetilde{C}}_n^{(1)} = \frac{1}{n^3}\sum\sum_{i\neq j} h(V_i,W_j)/\bar{H}(V_i)$. Since $d < 1$, we have

$$\mathbb{E}(|\sqrt{v_n}\widetilde{\widetilde{C}}_n^{(1)}|^d) \leqslant n^{d/2-3d}\left(\bar{G}(t_n)\bar{F}^{(k)}(t_n)\right)^{d/2} n(n-1)\mathbb{E}\left(|h(V_1,W_2)/\bar{H}(V_1)|^d\right).$$

According to part $(iv)$ of Lemma 12, the right-hand side of the inequality above is $O(1)\,v_n^{2-5d/2}$, which tends to 0, since $d > 4/5$, and so we are done.

Let us now prove that $\mathbb{E}(|\sqrt{v_n}\widehat{C}_n^{(1)}/n|)$ tends to 0, as $n$ tends to infinity. $\widehat{C}_n^{(1)}$ is defined in (29), where the expectation of each of the four integrals is $\theta_n$ : therefore, we only need to prove that $\frac{\sqrt{v_n}}{n}\theta_n$ tends to 0. This is straightforward using part $(v)$ of Lemma 12.

We can prove in a very similar way that $\mathbb{E}(|\sqrt{v_n}\widehat{C}_n^{(2)}/n|)$ tends to 0, as $n$ tends to infinity.

### *5.3. Proof of Proposition 1*

Using the same notations as in the begining of Section 5, we have,

$$\widehat{\gamma}_{n,k} - \gamma_k \;=\; \Delta_n^{-1}\left(\frac{Z_n}{\sqrt{v_n}} + R_n + (\gamma_{n,k} - \gamma_k)\right).$$

The fact that $\frac{Z_n}{\sqrt{v_n}} \xrightarrow{\mathbb{P}} 0$ is due to the application of a triangular weak law of large numbers (see Chow and Teicher (1997) for example) to $\frac{1}{n}\sum \tilde{U}_{i,n}$ and to $\frac{1}{n}\sum \tilde{V}_{i,n}$. By carrefully following the proof of proposition 3 in Section 5.2, we can see that $R_n = o_{\mathbb{P}}(1)$. The condition $\gamma_k < \gamma_C$ is not used, neither in the treatment of $\frac{Z_n}{\sqrt{v_n}}$ nor in that of $R_n$. Details are omited.

### *5.4. Proof of Corollary 1*

The proof is very similar to the one of Theorem 2 in Worms and Worms (2016), with $\gamma_k$ and $\bar{F}^{(k)}$ here replacing $\gamma_1$ and $\bar{F}$ there. For completeness, we provide some details about it. Reminding the notations $d_n = \bar{F}^{(k)}(t_n)/p_n \to \infty$ and $\Delta_n = \frac{\bar{F}_n^{(k)}(t_n)}{\bar{F}^{(k)}(t_n)}$, we easily write

$$\frac{\hat{x}^{(k)}_{p_n,t_n}}{x^{(k)}_{p_n}} - 1 \;=\; \frac{t_n}{x^{(k)}_{p_n}}(\Delta_n d_n)^{\widehat{\gamma}_{n,k}} - 1 \;=\; \Delta_n^{\widehat{\gamma}_{n,k}}\left(\frac{t_n}{x^{(k)}_{p_n}}\; d_n^{\gamma_k} T_n^1 + T_n^2 + T_n^3\right),$$

where $T_n^1 := d_n^{\widehat{\gamma}_{n,k}-\gamma_k} - 1$, $T_n^2 := \frac{t_n}{x^{(k)}_{p_n}} d_n^{\gamma_k} - 1$ and $T_n^3 := 1 - \Delta_n^{-\widehat{\gamma}_{n,k}}$. We are going to prove that both $T_n^2$ and $T_n^3$ are $o_{\mathbb{P}}(\log d_n/\sqrt{v_n})$, and that $\frac{\sqrt{v_n}}{\log d_n}T_n^1 \longrightarrow \mathscr{N}\left(\lambda m, \sigma^2\right)$ : this will conclude the proof, since both $\Delta_n$ (Corollary 2) and $\frac{t_n}{x^{(k)}_{p_n}}\; d_n^{\gamma_k}$ tend to 1.

Concerning $T_n^1$, the mean value theorem yields

$$\frac{\sqrt{v_n}}{\log d_n}T_n^1 = \frac{\sqrt{v_n}}{\log d_n}\left(e^{(\widehat{\gamma}_{n,k}-\gamma_k)\log(d_n)} - 1\right) = \sqrt{v_n}(\widehat{\gamma}_{n,k} - \gamma_k)\exp(E_n),$$

where $|E_n| \leqslant |\widehat{\gamma}_{n,k} - \gamma_k|\log d_n$ and therefore $E_n$ tends to 0 in probability thanks to Theorem 1 and assumption (4). The desired result for $T_n^1$ is then implied by Theorem 1 again.

Concerning the fact that $T_n^2 = o_{\mathbb{P}}(\log d_n/\sqrt{v_n})$, the proof is completely similar to the evoked one in Worms and Worms (2016), so we omit it here (basically, this is based on some uniform regular variation implied by the assumed negativity of the second order parameter $\rho_k$, and on the assumption that $\sqrt{v_n}g(t_n)$ converges).

Finally, concerning $T_n^3$ we use the mean value theorem to write

$$\frac{\sqrt{v_n}}{\log d_n}T_n^3 = \frac{\widehat{\gamma}_{n,k} D_n^{-\widehat{\gamma}_{n,k}-1}}{\log d_n}.\sqrt{v_n}(\Delta_n - 1),$$

where $D_n$ lies between $\Delta_n$ and 1. But Corollary 2 (and the consistency of $\widehat{\gamma}_{n,k}$) implies that $\widehat{\gamma}_{n,k} D_n^{-\widehat{\gamma}_{n,k}-1} \xrightarrow{\mathbb{P}} \gamma_k$ on one hand, and $\sqrt{v_n}(\Delta_n - 1) = O_{\mathbb{P}}(1)$ on the other hand ; therefore, $\frac{\sqrt{v_n}}{\log d_n}T_n^3 = O(1/\log(d_n)) = o(1)$.

## 6. Appendix

This appendix contains various results : some of them are used repeatedly in the proof of the main result (in particular Proposition 4, Lemmas 7, and 10, and to a lesser extent Lemmas 9 and 8), the other ones concern parts of the main proof which are postponed to the appendix for better clarity of the main flow of the proof (Lemmas 11, 12 and 13).

**Definition 1.** *An ultimately positive function $f : \mathbb{R}^+ \to \mathbb{R}$ is regularly varying (at infinity) with index $\alpha \in \mathbb{R}$, if*

$$\lim_{t\to+\infty} \frac{f(tx)}{f(t)} = x^{\alpha} \quad (\forall x > 0).$$

*This is noted $f \in RV_{\alpha}$. If $\alpha = 0$, $f$ is said to be slowly varying.*

**Proposition 4.** *(See de Haan and Ferreira (2006) Proposition B.1.9)*
*Suppose $f \in RV_{\alpha}$. If $x < 1$ and $\epsilon > 0$, then there exists $t_0 = t_0(\epsilon)$ such that for every $t \geqslant t_0$,*

$$(1-\epsilon)x^{\alpha+\epsilon} < \frac{f(tx)}{f(t)} < (1+\epsilon)x^{\alpha-\epsilon}$$

*and if $x \geqslant 1$ ,*

$$(1-\epsilon)x^{\alpha-\epsilon} < \frac{f(tx)}{f(t)} < (1+\epsilon)x^{\alpha+\epsilon}. \tag{41}$$

**Lemma 7.** *Let $x \in \mathbb{R}_+^*$ , $\alpha \in \mathbb{R}_+$, $\beta > -1$, and for $a$ and $b$ real numbers, $f$ and $g$ are two regular varying functions at infinity, with index, respectively, $a$ and $b$. Then, as $t \to +\infty$,*

*(i)* $J_{\beta}(x) = \int_1^{+\infty} \log^{\beta}(y)\ y^{-x-1}dy = \frac{\Gamma(\beta+1)}{x^{\beta+1}}$.

*(ii)* $I_{\alpha,a,b} = \int_1^{+\infty} \log^{\alpha}(y)\ \frac{f(yt)}{f(t)}\ \frac{dg(yt)}{g(t)} \to \frac{b\Gamma(\alpha+1)}{(-a-b)^{\alpha+1}}$, *if $a+b<0$*

*(iii)* $J_{a,b} = \int_0^1 \frac{f(yt)}{f(t)}\ \frac{dg(yt)}{g(t)} \to \frac{b}{a+b}$, *if $a+b>0$*

Proof :

*(i)* A simple change of variable and the definition of the $\Gamma$ function yields the result.

*(ii)* For the sake of simplicity, we are going to treat the case $a < 0$ and $b < 0$. The only difference for the other cases is the sign in front of the $\epsilon$ or $\epsilon'$ appearing below (coming from the application of (41) several times), which can depend on the sign of $a$, $b$ or another constant, but does not affect the result. Using Potter-bounds (41) for $f$ yields, for $n$ sufficiently large and $\epsilon > 0$ ,

$$(1+\epsilon)\int_1^{+\infty} \log^{\alpha}(y)\ y^{a+\epsilon}\ \frac{dg(yt_n)}{g(t_n)} \leqslant I_{\alpha,a,b} \leqslant (1-\epsilon)\int_1^{+\infty} \log^{\alpha}(y)\ y^{a-\epsilon}\ \frac{dg(yt_n)}{g(t_n)}.$$

Let us treat only the upper bound and the case $\alpha \neq 0$ (the other cases being similar). By integration by parts, with $a+b<0$, we have

$$\int_1^{+\infty} \log^{\alpha}(y)\ y^{a-\epsilon}\ \frac{dg(yt_n)}{g(t_n)} = -\alpha\int_1^{+\infty} \log^{\alpha-1}(y)\ y^{a-1-\epsilon}\ \frac{g(yt_n)}{g(t_n)}dy - (a-\epsilon)\int_1^{+\infty} \log^{\alpha}(y)\ y^{a-1-\epsilon}\ \frac{g(yt_n)}{g(t_n)}dy.$$

Using Potter-bounds (41) for $g$ yields, for $n$ sufficiently large and $\epsilon' > 0$

$$\int_1^{+\infty} \log^{\alpha}(y)\ y^{a-\epsilon}\ \frac{dg(yt_n)}{g(t_n)} \leqslant -\alpha(1-\epsilon')J_{\alpha-1}(-a-b+\epsilon+\epsilon') - (a-\epsilon)(1+\epsilon')J_{\alpha}(-a-b+\epsilon-\epsilon').$$

Doing the same with the lower bound and making $\epsilon$ and $\epsilon'$ tend to 0, yields the result after simplifications.

*(iii)* As in *(ii)*, using Potter-bounds (41) for $f$, integration by parts and then again (41) for $g$ yields the result.

**Lemma 8.** *For any $\delta > 0$, let $C_{\delta}$ denote the function*

$$C_{\delta}(t) = \int_0^t \frac{dG(v)}{\bar{G}(v)\bar{H}^{\delta}(v)}.$$

*Under condition* (1)*, this function is regularly varying of order $\delta/\gamma$ and we have $C_{\delta}(t) \sim (\gamma/\gamma_C)/(\delta\bar{H}^{\delta}(t))$, as $t \to +\infty$.*

Proof : by writing $\bar{H}^{\delta}(t)C_{\delta}(t) = -\int_0^1 \frac{\bar{H}^{\delta}(t)}{\bar{H}^{\delta}(tu)} \frac{\bar{G}(t)}{\bar{G}(tu)} \frac{d\bar{G}(tu)}{\bar{G}(t)}$, the lemma is an immediate consequence of part $(iii)$ of Lemma 7, with $a+b = (\delta/\gamma + 1/\gamma_C) + (-1/\gamma_C) = \delta/\gamma > 0$ and $-b/(a+b) = (\gamma/\gamma_C)/\delta$.

**Remark 3.** *In the Lemma above, $C_1$ is the important function $C$ introduced at the beginning of Section 5, and thus $C(t) \sim (\gamma/\gamma_C)/\bar{H}(t) = (1-\gamma/\gamma_F)/\bar{H}(t)$, as $t \to +\infty$. Hence, $C$ is regularly varying at infinity with index $1/\gamma$, a property which proves useful several times in the main proofs.*

**Lemma 9.** *Let $\psi(\phi_n, u) = \int_u^{+\infty} \phi_n(s) dF^{(k)}(x)$, for $u \geqslant 0$ and $\phi_n(u) = \frac{1}{\bar{F}^{(k)}(t_n)} \log(u/t_n) \mathbb{I}_{u>t_n}$. Under condition* (1), *we have*

$$\begin{aligned}\psi(\phi_n, u) &= \gamma_{n,k}, \text{ if } u \leqslant t_n \\ &= \log\left(\frac{u}{t_n}\right) \frac{\bar{F}^{(k)}(u)}{\bar{F}^{(k)}(t_n)} + \gamma_k \left(\frac{u}{t_n}\right)^{-1/\gamma_k} + \epsilon_n(u) \left(\frac{u}{t_n}\right)^{-1/\gamma_k+\delta} \quad \text{if } u > t_n,\end{aligned}$$

*where $\epsilon_n(u)$ is a sequence tending to 0 uniformly in $u$, as $n \to \infty$, and $\delta$ a positive real number such that $-\frac{1}{\gamma_k} + \delta < 0$.*

Proof : We only consider the second situation where $u > t_n$ (the first one is straightforward) :

$$\int_u^{+\infty} \phi_n(s) dF^{(k)}(x) = -\int_{\frac{u}{t_n}}^{+\infty} \log(y) \frac{d\bar{F}^{(k)}(yt_n)}{\bar{F}^{(k)}(t_n)}$$

An integration by part and the fact that $\bar{F}^{(k)}$ is regularly varying at infinity with index $-1/\gamma_k$, yields

$$\int_u^{+\infty} \phi_n(s) dF^{(k)}(x) = \log\left(\frac{u}{t_n}\right) \frac{\bar{F}^{(k)}(u)}{\bar{F}^{(k)}(t_n)} + \gamma_k \left(\frac{u}{t_n}\right)^{-1/\gamma_k} + \Delta_n(u),$$

where

$$\Delta_n(u) = \int_{\frac{u}{t_n}}^{+\infty} \left( \frac{\bar{F}^{(k)}(yt_n)}{\bar{F}^{(k)}(t_n)} - y^{-1/\gamma_k} \right) \frac{dy}{y}$$

Let $\delta$ be a positive real number. Then

$$\begin{aligned}|\Delta_n(u)| &= \left| \int_{\frac{u}{t_n}}^{+\infty} y^{-1/\gamma_k - 1 + \delta} \left( y^{1/\gamma_k - \delta} \frac{\bar{F}^{(k)}(yt_n)}{\bar{F}^{(k)}(t_n)} - y^{-\delta} \right) dy \right| \\ &\leqslant \sup_{y \geqslant 1} \left| y^{1/\gamma_k - \delta} \frac{\bar{F}^{(k)}(yt_n)}{\bar{F}^{(k)}(t_n)} - y^{-\delta} \right| \int_{\frac{u}{t_n}}^{+\infty} y^{-1/\gamma_k - 1 + \delta} dy,\end{aligned}$$

where the function $y \to y^{1/\gamma_k - \delta} \bar{F}^{(k)}(y)$ is regularly varying with index $-\delta$. Then since

$$\sup_{y \geqslant 1} \left| y^{1/\gamma_k - \delta} \frac{\bar{F}^{(k)}(yt_n)}{\bar{F}^{(k)}(t_n)} - y^{-\delta} \right| \overset{n \to \infty}{\longrightarrow} 0$$

and, when $-\frac{1}{\gamma_k} + \delta < 0$, we have $\int_{\frac{u}{t_n}}^{+\infty} y^{-1/\gamma_k - 1 + \delta} dy = cst\, (u/t_n)^{-1/\gamma_k + \delta}$, this concludes the proof.

**Lemma 10.** *Recalling that $H$ is a distribution function with infinite right endpoint, we have :*

$(i)$ $\sup_{0 \leqslant x < Z^{(n)}} \bar{H}(x)/\bar{H}_n(x) = O_{\mathbb{P}}(1)$

$(ii)$ *for any $a < 1/2$,*

$$\sqrt{n} \sup_{t \geqslant 0} \frac{|\bar{H}_n(t) - \bar{H}(t)|}{(\bar{H}(t))^a} = O_{\mathbb{P}}(1) \quad \text{and} \quad \sqrt{n} \sup_{t \geqslant 0} \frac{|\bar{H}_n^{(0)}(t) - \bar{H}^{(0)}(t)|}{(\bar{H}^{(0)}(t))^a} = O_{\mathbb{P}}(1).$$

Proof : part $(i)$ is well known (see for instance section 3 of chapter 10 of Shorack and Wellner (1986)), while the two statements in $(ii)$ are proved by usual empirical processes techniques, showing that the family of functions $(f_t)_{t<\infty}$ defined in one case by $f_t(z) = \mathbb{I}_{z>t}/(\bar{H}(t))^a$, and in the other case by $f_t(\delta, z) = (1-\delta)\mathbb{I}_{z>t}/(\bar{H}^{(0)}(t))^a$ are Donsker whenever $a < 1/2$ (using respective square integrable envelope functions $f^*(z) = 1/(\bar{H}(z))^a$ and $f^*(\delta, z) = (1-\delta)/(\bar{H}^{(0)}(z))^a$, which bound from above the functions $f_t$ uniformly in $t$) .

**Lemma 11.** *Under conditions (1) and (2), suppose that $\alpha \geqslant 0$ and $d \geqslant 1$ are real numbers. If $\gamma_k < \gamma_C$ and*

$$X_{i,n} = \frac{\sqrt{v_n}}{n^{1+d}} \frac{\phi(Z_i)}{\bar{G}(Z_i)(\bar{H}^{(0)}(Z_i))^{d+\alpha}} \mathbb{I}_{\xi_i=k},$$

*then we have $\sum_{i=1}^n X_{i,n} \overset{\mathbb{P}}{\longrightarrow} 0$, as $n$ tends to infinity, if $\alpha$ is 0 or sufficiently close to it.*

Proof :

According to the LLN for triangular arrays, we need to prove the following three statements :

$$\begin{array}{rcl} (i) & \forall \epsilon > 0, \ \sum_{i=1}^n \mathbb{P}(|X_{i,n}| > \epsilon) & \overset{n\to\infty}{\longrightarrow} 0 \\ (ii) & \sum_{i=1}^n \mathbb{E}((X_{i,n})^2 \mathbb{I}_{|X_{i,n}|\leqslant 1}) & \overset{n\to\infty}{\longrightarrow} 0 \\ (iii) & \sum_{i=1}^n \mathbb{E}(X_{i,n} \mathbb{I}_{|X_{i,n}|\leqslant 1}) & \overset{n\to\infty}{\longrightarrow} 0 \end{array}$$

But, $X_{i,n}$ being positive, $(iii)$ clearly implies $(ii)$. We thus need to prove that $(i)$ and $(iii)$ hold.

Let us start with assertion $(i)$. If $\epsilon > 0$ is given, then

$$X_{i,n} = \frac{v_n^{1/2}}{n^{1+d}} \frac{\log(Z_i/t_n)}{\bar{F}^{(k)}(t_n)\bar{G}(t_n)(\bar{H}^{(0)}(t_n))^{d+\alpha}} \mathbb{I}_{Z_i>t_n} \mathbb{I}_{\xi_i=k} \frac{\bar{G}(t_n)}{\bar{G}(Z_i)} \left( \frac{\bar{H}^{(0)}(t_n)}{\bar{H}^{(0)}(Z_i)} \right)^{d+\alpha}.$$

Now, put $a = \frac{1}{\gamma_C} + \frac{d+\alpha}{\gamma}$ ($> 0$); since, for a given $\epsilon' > 0$, there exists $c > 0$ such that $\forall x \geqslant 1$, $\log(x) \leqslant cx^{\epsilon'}$, and using Potter-bounds (41) for $\bar{G}^{-1}(\bar{H}^{(0)})^{-(d+\alpha)} \in RV_{-a}$, we can write (using the definition of $v_n$)

$$\begin{aligned} \{ |X_{i,n}| > \epsilon \} &\subset \left\{ v_n^{-1/2} n^{-d} \left(\bar{H}^{(0)}(t_n)\right)^{-(d+\alpha)} c(1+\epsilon') \left(\frac{Z_i}{t_n}\right)^{a+2\epsilon'} > \epsilon \right\} \cap \{\xi_i = k \text{ and } Z_i > t_n\} \\ &\subset \left\{ Z_i > c(\epsilon,\epsilon')\, t_n w_n \right\} \cap \{\xi_i = k \text{ and } Z_i > t_n\}, \end{aligned}$$

where $w_n = \left(v_n^{1/2} n^d \left(\bar{H}^{(0)}(t_n)\right)^{d+\alpha}\right)^{1/(a+2\epsilon')}$ and $c(\epsilon,\epsilon')$ is a constant depending on $\epsilon$ and $\epsilon'$ only. Consequently, if $w_n$ tends to infinity,

$$\begin{aligned} \sum_{i=1}^n \mathbb{P}(|X_{i,n}| > \epsilon) &\leqslant n\mathbb{E}(\mathbb{I}_{Z_i > c(\epsilon,\epsilon')\, t_n w_n} \mathbb{I}_{\xi_i=k}) = n \int_{c(\epsilon,\epsilon')\, t_n w_n}^{\infty} \bar{G}(x) dF^k(x) \\ &\leqslant v_n \frac{(\bar{F}^{(k)}\bar{G})(c(\epsilon,\epsilon')\, t_n w_n)}{(\bar{F}^{(k)}\bar{G})(t_n)} \\ &\leqslant cst\ v_n w_n^{-\beta}, \end{aligned}$$

where $\beta = \frac{1}{\gamma_C} + \frac{1}{\gamma_k} - \epsilon'$ and the last inequality is due to Potter-bounds (41) applied to $\bar{F}^{(k)}\bar{G} \in RV_{-\frac{1}{\gamma_C}-\frac{1}{\gamma_k}}$. Then, assertion $(i)$ above will be true as soon as we prove that $w_n \to \infty$ and $v_n w_n^{-\beta} \to 0$, as $n \to \infty$. Since $\bar{H}^{(0)}(t)$ is equivalent to a positive constant times $\bar{H}(t)$ when $t \to +\infty$, and $\bar{H}(t_n) \geqslant v_v/n$, then $w_n^{a+2\epsilon'} \geqslant cst\ (n^{-\eta} v_n)^r$, for $r = \frac{1}{2} + d + \alpha > 0$ and $\eta = \frac{\alpha}{r} \geqslant 0$. Assumption (2) finally yields that $w_n$ tends to $+\infty$, since $0 \leqslant \eta \leqslant \eta_0$ for $\alpha$ sufficiently close to 0.

Now, proving that $v_n w_n^{-\beta}$ tends to 0 is equivalent to proving that $v_n^{-(a+2\epsilon')/\beta} v_n^{1/2} n^d \left(\bar{H}^{(0)}(t_n)\right)^{d+\alpha}$ tends to $+\infty$. The same arguments as in the previous paragraph yield that it is sufficient to prove that $v_n^A n^{-\alpha} = (n^{-\eta} v_n)^A$ tends to $+\infty$, for $A = -(a+2\epsilon')/\beta + 1/2 + d + \alpha$ and $\eta = \frac{\alpha}{A}$. This is a consequence of hypothesis (2), since $A > 0$ and $\alpha \leqslant \eta_0 A$, for $\alpha$ sufficiently close to 0. This ends the proof of $(i)$.

Let us now start the proof of assertion $(iii)$. If $\epsilon > 0$ is given, using Potter-Bounds (41) for $\bar{G}^{-1}(\bar{H}^{(0)})^{-(d+\alpha)}$ which belongs to $RV_{-a}$, and introducing $h(x) = \log(x) x^{a-\epsilon}$, we find that (for some positive constant $c$)

$$\mathbb{I}_{|X_{i,n}|\leqslant 1} \mathbb{I}_{Z_i > t_n} \leqslant \mathbb{I}_{h(Z_i/t_n) \leqslant c w_n} \mathbb{I}_{Z_i > t_n}$$

where we set $w_n = v_n^{1/2} n^d \left(\bar{H}^{(0)}(t_n)\right)^{d+\alpha}$. Hence, denoting by $h^{-1}$ the inverse function of $h$,

$$\mathbb{I}_{|X_{i,n}|\leqslant 1} \mathbb{I}_{Z_i > t_n} \mathbb{I}_{\xi_i=k} \leqslant \mathbb{I}_{t_n < Z_i < t_n h^{-1}(c w_n)} \mathbb{I}_{\xi_i=k}.$$

Consequently, using once again Potter-Bounds (41) and bounding the log with a constant times a power of

$z/t_n$, we get

$$\begin{aligned} n\mathbb{E}(X_{1,n}\mathbb{I}_{|X_{1,n}|\leqslant 1}) &\leqslant \frac{v_n^{1/2}}{n^d}\int_{t_n}^{t_n h^{-1}(cw_n)} \frac{\log(z/t_n)}{\bar F^{(k)}(t_n)\bar G(z)(\bar H^{(0)}(z))^{d+\alpha}} dH^{(1,k)}(z) \\ &\leqslant cst\ \frac{v_n}{w_n}\int_1^{h^{-1}(cw_n)} s^{b+2\epsilon'}\ \frac{dF^{(k)}(st_n)}{\bar F^{(k)}(t_n)}, \end{aligned}$$

where $b=\frac{d+\alpha}{\gamma}$ and $\epsilon'>0$ is some given positive value (the inequality $\log(s)\leqslant cst\, s^{\epsilon'}$, $\forall s\geqslant 1$, was used). But, by integration by parts and (41) applied to $\bar F^{(k)}$, setting $h_n=h^{-1}(cw_n)$, we have

$$\frac{v_n}{w_n}\int_1^{h^{-1}(cw_n)} s^{b+2\epsilon'}\ \frac{dF^{(k)}(st_n)}{\bar F^{(k)}(t_n)} \leqslant cst\frac{v_n}{w_n}\left(1+h_n^{b-1/\gamma_k+3\epsilon'}\right).$$

Proceeding similarly as in the previous paragraphs, we find that $w_n/v_n\to\infty$ (and thus $w_n$ and $h_n$ as well) thanks to assumption (2), for $\alpha$ close to 0. We are thus left to prove that $(v_n/w_n)\times h_n^{b'}$ tends to 0, where $b'=b-1/\gamma_k+3\epsilon'$. If $b-1/\gamma_k$ is negative, this is immediate. We thus suppose that $b-1/\gamma_k\geqslant 0$ and, after some simple computations, we find out that $(v_n/w_n)h_n^{b'}$ tends to 0 if $v_n^{-a+\epsilon'}w_n^{a-b'-\epsilon'}$ tends to $\infty$, a property which holds true thanks to assumption (2), for $\alpha$ close to 0 (we omit the details).

**Lemma 12.** *Suppose that $V_1$ and $W_2$ are independent improper random variables of respective subdistribution functions $H^{(0)}$ and $H^{(1,k)}$, and $Z_3$ is independent of $V_1$ and $W_2$ and has distribution $H$. Consider $h$, $\underline{h}$, $\mathscr{H}$ and $\underline{\mathscr{H}}$ the functions defined in (28) and (37).*

*(i) For any $d\geqslant 1$, there exist some positive constants $c$ and $c'$ such that*

$$\mathbb{E}\left(\,|\mathscr{H}^d(V_1,W_2)|\,\right)\leqslant c\,\mathbb{E}\left(\,h^d(V_1,W_2)\,\right)\quad \text{and}\quad \mathbb{E}\left(\,|\underline{\mathscr{H}}^d(Z_3,V_1,W_2)|\,\right)\leqslant c'\,\mathbb{E}\left(\,\underline{h}^d(Z_3,V_1,W_2)\,\right).$$

*(ii) For any $d\in]1,1+(1+2\gamma_k/\gamma_C)^{-1}[$, we have*

$$\mathbb{E}\left(\,h^d(V_1,W_2)\,\right)=O\left((\bar F^{(k)}(t_n)\bar G(t_n))^{2(1-d)}\right).$$

*In particular, if $\gamma_k<\gamma_C$, then $\mathbb{E}(h^{4/3}(V_1,W_2))$ is of the order of $(\bar F^{(k)}(t_n)\bar G(t_n))^{-2/3}$ and $\mathbb{E}(h^d(V_1,W_2))$ is finite whenever $d$ is (greater than but) sufficiently close to $4/3$.*

*(iii) For any $d\in]1,1+(1+3\gamma_k/\gamma_C)^{-1}[$, we have*

$$\mathbb{E}\left(\,\underline{h}^d(Z_3,V_1,W_2)\,\right)=O\left((\bar F^{(k)}(t_n)\bar G(t_n))^{3(1-d)}\right).$$

*In particular, if $\gamma_k<\gamma_C$, then $\mathbb{E}(\underline{h}^{6/5}(Z_3,V_1,W_2))$ is of the order of $(\bar F^{(k)}(t_n)\bar G(t_n))^{-3/5}$ and $\mathbb{E}(\underline{h}^d(Z_3,V_1,W_2))$ is finite whenever $d$ is (greater than but) sufficiently close to $6/5$.*

*(iv) For any $d\in]1/2,(2\gamma_C^{-1}+\gamma_F^{-1}+\gamma_k^{-1})/(3\gamma_C^{-1}+2\gamma_F^{-1})[$, we have $\mathbb{E}\left(h^d(V_1,W_2)/\bar H^d(V_1)\right)=O\left((\bar F^{(k)}(t_n)\bar G(t_n))^{2-3d}\right)$. In particular, if $\gamma_k<\gamma_C$ then taking $\delta$ (greater than but) sufficiently close to $4/5$ is permitted, otherwise it is $2/3$ instead of $4/5$.*

*(v) The integral $\theta_n=\iint h(v,w)dH^{(0)}(v)dH^{(1,k)}(w)$ is equivalent, as $n\to\infty$, to $\gamma_k(-\log\bar G(t_n))$.*

Proof :

(i) Let $d\geqslant 1$, and remind that $h$ is a non-negative function. Using several times the inequality $|a+b|^d\leqslant 2^{d-1}(|a|^d+|b|^d)$, we can write

$$\mathbb{E}(|\mathscr{H}^d(V_1,W_2)|)\leqslant cst\left\{\mathbb{E}(h^d(V_1,W_2))+\mathbb{E}[(h_{1\bullet}(V_1))^d]+\mathbb{E}[(h_{\bullet 1}(W_2))^d]+(\mathbb{E}(h(V_1,W_2)))^d\right\}.$$

But using the fact that the $L^1$ norm is bounded by the $L^d$ norm whenever $d\geqslant 1$, we have $(\mathbb{E}(h(V_1,W_2)))^d\leqslant\mathbb{E}(h^d(V_1,W_2))$ and it is quite simple to prove (by independency of $V_1$ and $W_2$) that it is also the case of $\mathbb{E}[(h_{1\bullet}(V_1))^d]=\mathbb{E}[(\mathbb{E}(h(V_1,W_2)|V_1))^d]\leqslant\mathbb{E}[\mathbb{E}(h^d(V_1,W_2)|V_1)]=\mathbb{E}(h^d(V_1,W_2))$, as well as for $\mathbb{E}[(h_{\bullet 1}(W_2))^d]$. The inequality is thus proved. The other one (concerning $\underline{\mathscr{H}}$ and $\underline{h}$) is proved similarly.

$(ii)$ Let $d > 1$. Since $h(v, \infty) = h(\infty, w) = 0$ $(\forall v, w)$, we have

$$\begin{aligned}\mathbb{E}(h^d(V_1, W_2)) &= (\bar{F}^{(k)}(t_n))^{-d} \iint \log^d(w/t_n)(\bar{H}(v)\bar{G}(w))^{-d}\mathbb{I}_{w>t_n}\mathbb{I}_{w>v}dH^{(0)}(v)dH^{(1,k)}(w) \\ &= (\bar{F}^{(k)}(t_n))^{1-d}\int_{t_n}^{\infty} \log^d(w/t_n)\left(\int_0^w \frac{dG(v)}{\bar{G}(v)\bar{H}^{d-1}(v)}\right)\bar{G}^{1-d}(w)\frac{dF^{(k)}(w)}{\bar{F}^{(k)}(t_n)} \\ &= \frac{C_{d-1}(t_n)}{(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{d-1}}\int_{t_n}^{\infty} \log^d(w/t_n)\left(\frac{\bar{G}(t_n)}{\bar{G}(w)}\right)^{d-1}\frac{C_{d-1}(w)}{C_{d-1}(t_n)}\frac{dF^{(k)}(w)}{\bar{F}^{(k)}(t_n)}\end{aligned}$$

where the function $C_{d-1}$ was defined in the statement of Lemma 8. This lemma and Lemma 7, applied with $\alpha = d$, $a = (d-1)/\gamma_C + (d-1)/\gamma$ and $b = -1/\gamma_k$ (the constraint specified on $d$ certifies that $a + b < 0$), imply that the integral in the previous line converges to a constant. And Lemma 8 also implies that the ratio in front of this integral is equivalent, as $n \to \infty$, to a positive constant times $\left(\bar{H}(t_n)\bar{F}^{(k)}(t_n)\bar{G}(t_n)\right)^{1-d}$, which is itself lower than $\left(\bar{F}^{(k)}(t_n)\bar{G}(t_n)\right)^{2(1-d)}$, as desired.

$(iii)$ Let $d > 1$. By definition of $\underline{h}$ in (28), and proceeding as in the previous item, $\mathbb{E}(\underline{h}^d(Z_3, V_1, W_2))$ equals

$$\begin{aligned}&(\bar{F}^{(k)}(t_n))^{-d}\iiint \log^d\left(\tfrac{w}{t_n}\right)(\bar{H}(v))^{-2d}(\bar{G}(w))^{-d}\mathbb{I}_{w>t_n}\mathbb{I}_{w>v}\mathbb{I}_{u>v}dH(u)dH^{(0)}(v)dH^{(1,k)}(w) \\ = &\frac{C_{2d-2}(t_n)}{(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{d-1}}\int_{t_n}^{\infty}\log^d(w/t_n)\left(\frac{\bar{G}(t_n)}{\bar{G}(w)}\right)^{d-1}\frac{C_{2d-2}(w)}{C_{2d-2}(t_n)}\frac{dF^{(k)}(w)}{\bar{F}^{(k)}(t_n)},\end{aligned}$$

which is equivalent to $O\left((\bar{H}(t_n))^{2-2d}(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{1-d}\right) = O\left((\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{3(1-d)}\right)$ as soon as, thanks to Lemma 7, the sum $((d-1)/\gamma_C + (2d-2)/\gamma) - 1/\gamma_k$ is negative, which turns out to be true whenever $d < 1 + (2 + 3\gamma_k/\gamma_C)^{-1}$, as specified.

$(iv)$ The proof is very similar to the previous ones, starting from

$$\mathbb{E}\left(h^d(V_1, W_2)/\bar{H}^d(V_1)\right) = (\bar{F}^{(k)}(t_n))^{-d}\iint \log^d(w/t_n)(\bar{H}(v))^{-2d}(\bar{G}(w))^{-d}\mathbb{I}_{w>t_n}\mathbb{I}_{w>v}dH^{(0)}(v)dH^{(1,k)}(w)$$

so we omit the details.

$(v)$ Noting that $-\log\bar{G}$ is slowly varying at infinity null at 0, we have

$$\theta_n = \int_{t_n}^{\infty}\log(w/t_n)\left(\int_0^w dG(v)/\bar{G}(v)\right)\frac{dF^{(k)}(w)}{\bar{F}^{(k)}(t_n)} = (-\log\bar{G}(t_n))\left(-\int_1^{\infty}\log(u)\frac{-\log\bar{G}(ut_n)}{-\log\bar{G}(t_n)}\frac{d\bar{F}^{(k)}(ut_n)}{\bar{F}^{(k)}(t_n)}\right)$$

which can be dealt with using part $(ii)$ of Lemma 7 with $\alpha = 1$, $a = 0$ and $b = -1/\gamma_k$ : the obtained constant is indeed equal to $\gamma_k$.

**Lemma 13.** *In this Lemma, various notations defined in sections 5.2.2 to 5.2.4 are used.*

$(i)$ *The variables $\mathscr{H}_I^{**}$ for $I \in \{(i,j)\,;\, 1 \leqslant i < j \leqslant n\}$ are centred and uncorrelated . This is also true for the variables $\underline{\mathscr{H}}_I^{**}$ for $I \in \{(i,j,l)\,;\, 1 \leqslant i < j < l \leqslant n\}$.*

$(ii)$ *We have $\mathbb{E}\left[(\mathscr{H}^{**}(V_1, W_2))^2\right] \leqslant 48\mathbb{E}[\mathscr{H}_1^2\mathbb{I}_{|\mathscr{H}_1| \leqslant M_n}]$.*

$(iii)$ *We have $\mathbb{E}\left(\,|\mathscr{H}_1 - \mathscr{H}^*(V_1, W_2) + \mathscr{H}_{1\bullet}^*(V_1) + \mathscr{H}_{\bullet 1}^*(W_2)|\,\right) \leqslant 4\mathbb{E}\left(|\mathscr{H}_1|\mathbb{I}_{|\mathscr{H}_1| > M_n}\right)$*

Proof :

$(i)$ Let us consider the first situation, where $\mathscr{I} = \{(i,j)\,;\, 1 \leqslant i < j \leqslant n\}$. First, if $I = (i,j) \in \mathscr{I}$, then $\mathbb{E}(\mathscr{H}_I^{**}) = 0 - \mathbb{E}(\mathscr{H}_{1\bullet}^*(V_i)) - \mathbb{E}(\mathscr{H}_{\bullet 1}^*(W_j))$ ; but, by definition of $\mathscr{H}_{1\bullet}^*$ and independency of $V_i$ and $W_j$, we have $\mathbb{E}(\mathscr{H}_{1\bullet}^*(V_i)) = \mathbb{E}(\mathscr{H}^*(V_i, W_j)) = 0$, and $\mathbb{E}(\mathscr{H}_{\bullet 1}^*(W_j)) = 0$ is obtained similarly, so we proved that $\mathbb{E}(\mathscr{H}_I^{**}) = 0$. Note that we can prove (with similar arguments) that $\mathscr{H}_{1\bullet}^{**}(v) = \mathscr{H}_{\bullet 1}^{**}(w) = 0$ for every $v, w$ in $[0, \infty]$, a property which is repeatedly used below . Let us now deal with the non-correlation of $\mathscr{H}_I^{**}$ and $\mathscr{H}_{I'}^{**}$, by considering the various cases where $I \neq I'$ with $I = (i,j)$ and $I' = (k,l)$ are in $\mathscr{I}$.

If all four indices $i, j, k, l$ are distinct, then non-correlation of $\mathscr{H}_I^{**}$ and $\mathscr{H}_{I'}^{**}$ is immediate by mutual independence of the variables $Z_1, \ldots, Z_n$.

If $i = k$ but $j \neq l$, then $\mathbb{E}(\mathscr{H}_I^{**}\mathscr{H}_{I'}^{**}) = \mathbb{E}(\psi(V_i))$ where $\psi(v) = \mathbb{E}(\mathscr{H}^{**}(v, W_j)\mathscr{H}^{**}(v, W_l)) = (\mathscr{H}_{1\bullet}^{**}(v))^2 = 0$, by independence of $V_i$ with $(W_j, W_l)$, and of $W_j$ and $W_l$.

The case $i \neq k$ and $j = l$ is similar using $\mathscr{H}_{\bullet 1}^{**}(\cdot) \equiv 0$.

If $i = l$ but $j \neq k$, then $\mathbb{E}(\mathscr{H}_I^{**}\mathscr{H}_{I'}^{**}) = \mathbb{E}(\psi(V_i, W_i))$ where $\psi(v, w) = \mathbb{E}(\mathscr{H}^{**}(v, W_j)\mathscr{H}^{**}(V_k, w)) = \mathscr{H}_{1\bullet}^{**}(v)\mathscr{H}_{\bullet 1}^{**}(w) = 0 \times 0 = 0$ ; the case $j = k$ and $i \neq l$ is treated similarly.

Note that the case $i = l$ and $j = k$ (*i.e.* $\mathscr{H}_I^{**} = \mathscr{H}^{**}(V_i, W_j)$, $\mathscr{H}_{I'} = \mathscr{H}^{**}(V_j, W_i)$) is not permitted (it would lead to dependency) since we cannot have simultaneously $i < j$ and $j < i$ ; this is the reason why, in the beginning of section 5.2.3, we restricted the study of the sum $\mathscr{U}_n$ to that of the sum $S_N$ having terms $\mathscr{H}(V_i, W_j)$ satisfying $i < j$.

The second situation, for $\underline{\mathscr{H}}_I^{**}$ and $\underline{\mathscr{H}}_{I'}^{**}$ with $I \neq I'$ in $\mathscr{I} = \{I = (i, j, l)\,;\, 1 \leqslant i < j < l \leqslant n\}$, is a bit more tedious (with more cases to detail) but very similar, so we omit its proof.

(*ii*) We start by the trivial bound

$$\mathbb{E}[(\mathscr{H}^{**}(V_1, W_2))^2] \leqslant 4\left\{\mathbb{E}[(\mathscr{H}^{*}(V_1, W_2))^2] + \mathbb{E}[(\mathscr{H}_{1\bullet}^{*}(V_1))^2] + \mathbb{E}[(\mathscr{H}_{\bullet 1}^{*}(W_2))^2]\right\}.$$

Noting $\mathscr{H}_1^- = \mathscr{H}_1\mathbb{I}_{|\mathscr{H}_1| \leqslant M_n}$, we can write, on one hand, by definition of $\mathscr{H}^*$, $\mathbb{E}[(\mathscr{H}^*(V_1, W_2))^2] \leqslant 2\left\{\mathbb{E}[(\mathscr{H}_1^-)^2] + (\mathbb{E}[\mathscr{H}_1^-])^2\right\} \leqslant 4\mathbb{E}[(\mathscr{H}_1^-)^2]$. On the other hand, if $W$ is independent of $V_1$, we have $\mathbb{E}[(\mathscr{H}_{1\bullet}^*(V_1))^2] = \mathbb{E}[(\mathbb{E}[\mathscr{H}^*(V_1, W)|V_1])^2] \leqslant \mathbb{E}[\mathbb{E}[(\mathscr{H}^*(V_1, W))^2|V_1]] = \mathbb{E}[(\mathscr{H}^*(V_1, W_2))^2]$, which is the same term as the first one, and is thus lower than $4\mathbb{E}[(\mathscr{H}_1^-)^2]$. The same is true of $\mathbb{E}[(\mathscr{H}_{\bullet 1}^*(W_2))^2]$, so the desired inequality is proved.

(*iii*) First recall that $\mathscr{H}_1$ denotes $\mathscr{H}(V_1, W_2)$. Now, since $\mathscr{H}_1$ is centred and we trivially have $\mathscr{H}_1 = \mathscr{H}_1\mathbb{I}_{|\mathscr{H}_1| \leqslant M_n} + \mathscr{H}_1\mathbb{I}_{|\mathscr{H}_1| > M_n}$, noting $\mathscr{H}_1^+ = \mathscr{H}_1\mathbb{I}_{|\mathscr{H}_1| > M_n}$ yields

$$\mathscr{H}_1 - \mathscr{H}^*(V_1, W_2) = \mathscr{H}_1^+ - \mathbb{E}(\mathscr{H}_1^+).$$

Secondly, using the fact that $\mathscr{H}_{1\bullet}(\cdot) \equiv 0$ (simple to prove), we can write

$$\mathscr{H}_{1\bullet}^*(v) = \mathbb{E}(\mathscr{H}(v, W)\mathbb{I}_{|\mathscr{H}(v,W)| \leqslant M_n}) - \mathbb{E}(\mathscr{H}_1\mathbb{I}_{|\mathscr{H}1| \leqslant M_n}) = -\mathscr{H}_{1\bullet}^+(v) + \mathbb{E}(\mathscr{H}_1^+),$$

where $\mathscr{H}_{1\bullet}^+(v)$ denotes $\mathbb{E}(\mathscr{H}(v, W)\mathbb{I}_{|\mathscr{H}(v,W)| > M_n})$ and satisfies $\mathbb{E}(\mathscr{H}_{1\bullet}^+(V_1)) = \mathbb{E}(\mathscr{H}_1^+)$, and similarly

$$\mathscr{H}_{\bullet 1}^*(w) = -\mathscr{H}_{\bullet 1}^+(w) + \mathbb{E}(\mathscr{H}_1^+)$$

with $\mathscr{H}_{\bullet 1}^+(w) = \mathbb{E}(\mathscr{H}(V, w)\mathbb{I}_{|\mathscr{H}(V,w)| > M_n})$ and $\mathbb{E}(\mathscr{H}_{\bullet 1}^+(W_2)) = \mathbb{E}(\mathscr{H}_1^+)$. Summing these three terms finally leads to

$$\mathbb{E}\left(|\mathscr{H}_1 - \mathscr{H}^*(V_1, W_2) + \mathscr{H}_{1\bullet}^*(V_1) + \mathscr{H}_{\bullet 1}^*(W_2)|\right) \;=\; \mathbb{E}\left(|\mathscr{H}_1^+ - \mathscr{H}_{1\bullet}^+(V_1) - \mathscr{H}_{\bullet 1}^+(W_2) + \mathbb{E}(\mathscr{H}_1^+)|\right)$$

which is lower than $4\mathbb{E}(|\mathscr{H}_1^+|)$, as announced.

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