

A uniform cube-free morphism is k-power-free for all integers $k \geq 4$

Francis Wlazinski

▶ **To cite this version:**

Francis Wlazinski. A uniform cube-free morphism is k-power-free for all integers $k \geq 4$. 2016. hal-01417750

HAL Id: hal-01417750
https://hal.science/hal-01417750

Preprint submitted on 15 Dec 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0 International License

A uniform cube-free morphism is k-power-free for all integers $k \geq 4$

Francis Wlazinski *

December 15, 2016

Abstract

The fact that the image of a word by a morphism contains a cube implies some relations. Under some assumptions, we show that these relations lead to more precise results. This allows us to establish that a cube-free uniform morphism is a k-power-free morphism for all integers $k \geq 4$.

1 Introduction

For all integers $k \geq 2$, a k-power is a repetition of k consecutive and identical sequences. A square and a cube respectively correspond to the special cases of $k = 2$ and $k = 3$. An overlap is a word of the form $xuxux$ where x is a letter and u is a word. An infinite word without squares over a three-symbol alphabet and an infinite word without overlaps over a two-symbol alphabet were given by Thue [16, 17] (see also the translation of Berstel [3]). These words are obtained by iterating morphisms.

To find morphisms that generate k-power-free words, a method is to consider k-power-free morphisms. By definition, a k-power-free morphism maps a k-power-free word to a k-power-free word. Starting with a letter (so k-power-free), the word generated by the iteration of a k-power-free morphism is thus k-power-free. But non-k-power-free morphisms can generate k-power-free words. For instance, the Fibonacci morphism $\varphi : \{a, b\} \to \{a, b\}; a \mapsto ab; b \mapsto a$ generates the word $abaababaabaababaababa...$, which is 4-power-free [7]. But φ is not 4-power-free since $\varphi(b^3a) = a^4b$.

Even if we know different ways to verify whether a morphism generates a k-power-free word [2, 6, 7, 14] but also whether a morphism is k-power-free [5, 9, 8, 15, 18], a simple question remains unanswered; is a k-power-free morphism also a $(k + 1)$-power-free morphism?

In [1], Bean, Ehrenfeucht, and McNulty gave a partial result; the morphism h defined by $h(a) = abacbab$, $h(b) = cdabcabd$, $h(c) = cdacabcbd$ and $h(d) = cdacbcacbd$ is square-free but not cube-free. Another partial result exists for the Thue-Morse morphism $\mu : \{a, b\} \to \{a, b\}; a \mapsto ab; b \mapsto ba$. It is k-power-free for every integer $k > 2$ [4].

When $k \geq 4$, we proved in [19] that a k-power-free uniform morphism is $(k + 1)$-power-free. But one of the properties required to prove this result (Lemma 3.1 in [19]) does not hold for $k = 3$. We give a new one (Lemma 3.4) that allows us to conclude for cube-free uniform morphisms (Theorem 4.1 and Proposition 4.2).

2 Preliminaries

We assume the reader is familiar (if not, see for instance [12, 13]) with basic notions on words and morphisms. We use notations that largely come from [19].

*L.A.M.F.A., UMR 7352 du CNRS; Université de Picardie Jules Verne; 33 rue Saint-Leu; 80 039 Amiens Cedex 1; France

2.1 Words

Let w be a non-empty word and let i, j be two integers such that $0 \leq i-1 \leq j \leq |w|$. We denote by $w[i..j]$ the factor of w such that $|w[i..j]| = j-i+1$ and $w = pw[i..j]s$ for two words s and p verifying $|p| = i-1$. When $i = j$, we also denote by $w[i]$ the factor $w[i..i]$, which is the i^{th} letter of w. In particular, $w[1]$ and $w[|w|]$ are respectively the first and the last letter of w. We denote $\text{Fcts}(w)$ the set of all factors of the word w.

Powers of a word u are defined inductively by $u^0 = \varepsilon$ and $u^n = uu^{n-1}$ for all integers $n \geq 1$. Given an integer $k \geq 2$, since the case ε^k is of little interest, we called a k-power any word u^k with $u \neq \varepsilon$. Given an integer $k \geq 2$, a word is *k-power-free* if it does not contain any k-power as factor. A *primitive* word is a word that is not a k-power of an another word whatever the integer $k \geq 2$. A (non-empty) k-power v^k is called *pure* if any proper factor of v^k is k-power-free.

The following proposition gives the well-known solutions (see [11]) to an elementary equation in words and will be widely used in the following sections:

Proposition 2.1 *Let A be an alphabet and u, v, w three words over A.*

1. *If $vu = uw$ and $v \neq \varepsilon$ then there exist two words r and s over A, and an integer n such that $u = r(sr)^n$, $v = rs$ and $w = sr$.*
2. *If $vu = uv$, then there exist a word w over A and two integers n and p such that $u = w^n$ and $v = w^p$.*

We also need a property on words that is an immediate consequence of Proposition 2.1(2).

Lemma 2.2 [8, 10] *If a non-empty word v is an internal factor of vv, i.e., if there exist two non-empty words x and y such that $vv = xvy$ then there exist a non-empty word t and two integers $i, j \geq 1$ such that $x = t^i$, $y = t^j$, and $v = t^{i+j}$.*

We also use the following result which is a corollary of a result of Fine and Wilf [11, 12].

Corollary 2.3 (Keränen) [8] *Let x and y be two words. If a power of x and a power of y have a common factor of length at least equal to $|x| + |y| - gcd(|x|, |y|)$ then there exist two words t_1 and t_2 such that x is a power of t_1t_2 and y is a power of t_2t_1 with t_1t_2 and t_2t_1 primitive words. Furthermore, if $|x| > |y|$ then x is not primitive.*

2.2 Morphisms

Let A and B be two alphabets. A *morphism* f from A^* to B^* is a mapping from A^* to B^* such that $f(uv) = f(u)f(v)$ for all words u, v over A.

Let $k \geq 2$ be an integer. A morphism f on A is *k-power-free* if and only if $f(w)$ is k-power-free for all k-power-free words w over A.

A morphism f on A is called *prefix* (resp. *suffix*) if, for all letters a and b in A, the word $f(a)$ is not a prefix (resp. not a suffix) of $f(b)$. A prefix (resp. suffix) morphism f is non-erasing that is $f(a) \neq \varepsilon$ for all letters a. A morphism is *bifix* if it is prefix and suffix.

A morphism f from A^* to B^* is a *ps-morphism* if and only if the equalities $f(a) = ps$, $f(b) = ps'$, and $f(c) = p's$ with $a, b, c \in A$ (possibly $c = b$) and p, s, p', and $s' \in B^*$ imply $b = a$ or $c = a$.

Let us recall some definitions and properties that will be used in the sequel. Lemma 2.4 derives directly from the definitions of a prefix or a suffix morphism. A detailed proof is left to the reader.

Lemma 2.4 *Let f be a bifix morphism on an alphabet A and let u, v, w, and t be words over A.*
The equality $f(u) = f(v)p$ with p be a prefix of $f(w)$ implies $u = vw'$ for a prefix w' of w such that $f(w') = p$.
And the equality $f(u) = sf(v)$ with s a suffix of $f(t)$ implies $u = t'v$ for a suffix t' of t such that $f(t') = s$.

Lemma 2.5 [8, 10] *If f is not a ps-morphism then f is not a k-power-free morphism for all integers $k \geq 2$.*

Lemma 2.6 [19] *Let f be a ps-morphism from A^* to B^* and let u, v and w be words over A such that $f(u) = \delta\beta$, $f(v) = \alpha\beta$, and $f(w) = \alpha\gamma$ for some non-empty words α, β, γ, and δ over B. Then it implies $v = v_1 a v_2$, $u = u_1 b v_2$, and $w = v_1 c w_2$ for some words v_1, v_2, u_1, and w_2, and some letters a, b, and c. Moreover, we have either $b = a$ or $c = a$.*

Furthermore, if $|\delta| < |f(u[1])|$ then $u_1 = \varepsilon$ and if $|\gamma| < |f(w[|w|])|$ then $w_2 = \varepsilon$.

Assuming that $f(\overline{w}) = pu^k s$ for a factor $\overline{w}$ of a word w and a non-empty word u, and assuming that $\overline{w}$ contains a factor w_0 such that $|f(w_0)| = |u|$, Lemma 2.8 states that $\overline{w}$ necessarily contains a k-power w'^k such that $f(w')$ is a conjugate of u. We will say that $f(w)$ contains a synchronised k-power u^k. More precisely:

Definition 2.7 *Let $k \geq 2$ be an integer. Let f be a morphism from A^* to B^*, w be a word over A, and u be a non-empty word over B such that $f(w)$ contains the k-power u^k. Let $\overline{w}$ be a shortest factor of w whose image by f contains u^k, i.e., $f(\overline{w}) = pu^k s$ with $|p| < |f(\overline{w}[1])|$ and $|s| < |f(\overline{w}[|\overline{w}|])|$.*

We say that $f(w)$ and u^k are synchronised if there exist three words w_0, w_1, and w_2 such that $|f(w_0)| = |u|$ and $\overline{w} = w_1 w_0 w_2$ with $p = \varepsilon$ if $w_1 = \varepsilon$, and $s = \varepsilon$ if $w_2 = \varepsilon$.

The three following results, which will be used in this paper, were proved in [19].

Lemma 2.8 *Let $k \geq 2$ be an integer, let f be a ps-morphism, and let w be a word such that $f(w) = pu^k s$ with $|p| < |f(w[1])|$ and $|p| < |f(w[1])|$. If $f(w)$ contains a synchronised k-power then w starts or ends with a k-power w_0^k such that $f(w_0)$ and u are conjugated.*

Lemma 2.9 *Let $k \geq 3$ be an integer and let $\kappa \in \{k; k+1\}$. Let f be a morphism from A^* to B^* and let ω be a word over A such that $f(\omega) = pU^\kappa S$ for some words p, S and $U \neq \varepsilon$ over B such that $|p| < |f(\omega[1])|$. Moreover, we assume that $|S| < |f(\omega[|\omega|])|$ when $\kappa = k+1$ and v^3 is a chosen factor of a pure k-power v^k.*

When one of the four following situations holds, there exist a word $\breve{\omega}$ such that $f(\breve{\omega}) = p'(U')^\kappa S'$ for some words p', S', and $U' \neq \varepsilon$ over B satisfying $|p'| < |f(\breve{\omega}[1])|$, $0 < |U'| < |U|$, and $f(\breve{\omega})$ and $(U')^\kappa$ are synchronised only if $f(\omega)$ and U^κ are synchronised.

1. *$d_{\mathrm{v}} = 1$, $|D_{\mathrm{v}} f(v)^2| < |U|$, and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for every integer $j \in [2, \kappa]$.*
2. *$d_{\mathrm{v}} = 1$, $L_{j,v} \cup R_{j,v} \neq \emptyset$ for every integer $j \in [2, \kappa-1]$, and there exists a positive integer ϕ such that $\omega[n_{\mathrm{v}}..|\omega|]$ starts with $v^{\phi+2}$ and $\sup\left\{2|f(v)|; |D_{\mathrm{v}} f(v)^\phi|\right\} \leq |U| < |D_{\mathrm{v}} f(v)^{\phi+1}|$.*
3. *$d_{\mathrm{v}} = 0$, $|D_{\mathrm{v}} f(v)^2| \leq |U|$, and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for every integer $j \in [1, \kappa]$.*
4. *$d_{\mathrm{v}} = 0$, $|U| < |D_{\mathrm{v}} f(v)^2| < |D_{\mathrm{v}} U|$, and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for every integer $j \in [1, \kappa - 1]$.*

Proposition 2.10 *Let A and B be two alphabets and let $k \geq 4$ be an integer. A k-power-free uniform morphism is a $(k+1)$-power-free morphism.*

3 About cube-free-morphisms

As mentioned in the introduction, Lemma 3.1 in [19] is no longer valid for $k = 3$. Even if parts of its proof can be extended to this case, some new problems appear. The following lemma is one of the situations we can obtain:

Lemma 3.1 *Let f be a ps-morphism from A^* to B^*. Let us assume that there exist ρ, μ, α, β, and θ words over B and q_1, q_2, and q words over A such that $q \neq \varepsilon$, $\rho = \alpha\beta$ is not the image of a word by f, $\theta = \mu\rho\rho\mu\rho\rho\mu$, $f(q) = \rho\mu\rho$, $f(q_1)$ ends with $\beta\theta$, $f(q_2)$ starts with $\theta\alpha$, and the words $q_1[2..|q_1|]\, q$ and $q\, q_2[1..|q_2|-1]$ are cube-free. Then f is not cube-free.*

Proof of Lemma 3.1 is simply done using iteratively Lemma 3.2. By induction, if f was not cube-free, we could find an infinite sequence $(\chi_i)_{i\geq 0}$ of non-empty words starting with $\chi_0 = q$ such that $|f(\chi_i)| > |f(\chi_{i+1})| > 0$; it is impossible.

Lemma 3.2 *Let f be a ps-morphism from A^* to B^*. Let us assume that there exist ρ, μ, α, β, and θ words over B and q_1, q_2, and q words over A such that $q \neq \varepsilon$, $\rho = \alpha\beta$ is not the image of a word by f, $\theta = \mu\rho\rho\mu\rho\rho\mu$, $f(q) = \rho\mu\rho$, $f(q_1)$ ends with $\beta\theta$, $f(q_2)$ starts with $\theta\alpha$, and the words $q_1[2..|q_1|]\,q$ and $q\,q_2[1..|q_2|-1]$ are cube-free.*

Then either f is not cube-free or there exist ρ', μ', α', β', and θ' words over B and q'_1, q'_2, and q' words over A such that $q' \neq \varepsilon$, $\rho' = \alpha'\beta'$ is not the image of a word by f, $\theta' = \mu'\rho'\rho'\mu'\rho'\rho'\mu'$, $f(q') = \rho'\mu'\rho'$, $f(q'_1)$ ends with $\beta'\theta'$, $f(q'_2)$ starts with $\theta'\alpha'$, and the words $q'_1[2..|q'_1|]\,q'$ and $q'\,q'_2[1..|q'_2|-1]$ are cube-free.

And more particularly, we have $|f(q')| < |f(q)|$.

Proof.

Let us first remark that the condition $q \neq \varepsilon$ is simply a consequence that ρ is not the image of a word by f. Indeed, it implies $\rho \neq f(\varepsilon) = \varepsilon$. Therefore, $f(q) = \rho\mu\rho \neq \varepsilon$ and so $q \neq \varepsilon$. Moreover, the fact that f is a ps-morphism implies that f is bifix and non-erasing.

We can write $f(q_1) = \pi_1\beta\theta$ and $f(q_2) = \theta\alpha\sigma_2$ for two words π_1 and σ_2 over B. Let us denote $\varphi = \mu\rho$ and $\psi = \rho\mu$ so $\theta = \varphi f(q)\psi$.

Let Q_1 be the smallest suffix of q_1 whose image by f ends with ψ and Q_2 be the smallest prefix of q_2 whose image by f starts with φ. Since $\rho \neq \varepsilon$, it implies that Q_1 and Q_2 are not empty. There exist two words Π_1 and Σ_2 such that $f(Q_1) = \Pi_1\psi$, $f(Q_2) = \varphi\Sigma_2$, $|\Pi_1| < |f(Q_1[1])|$ and $|\Sigma_2| < |f(Q_2[|Q_2|])|$.

We have Q_1qQ_2 factor of q_1qq_2 and $f(Q_1qQ_2) = \Pi_1\psi f(q)\varphi\Sigma_2 = \Pi_1(\rho\mu)^3\rho\Sigma_2$.

If Q_1qQ_2 is cube-free, it ends the proof; f is not cube-free. Hence, Q_1qQ_2 contains a shortest cube q'^3; any proper factor of q'^3 is cube-free, i.e., q'^3 is a pure cube. So we can write $Q_1qQ_2 = q'_1q'^3q'_2$ for some words q'_1 and q'_2.

Let us remark that Q_1q is necessarily cube-free. Indeed, if $Q_1 \neq q_1$ then Q_1q is a suffix of $q_1[2..|q_1|]q$ which is cube-free by assumption. If $Q_1 = q_1$, by definition of Q_1, it means that ψ is not a suffix of $f(q_1[2..|q_1|])$ and so $|f(q_1[2..|q_1|])| < |\rho\mu|$. It implies that $|f(Q_1[1])| = |f(q_1[1])| > |\pi_1\beta\varphi f(q)| \geq |f(q_1[2..|q_1|]q)|$. In particular, it means that the first letter of q_1, i.e., $q_1[1] = Q_1[1]$ is not a letter of $q_1[2..|q_1|]q$. Since no cube appears in $q_1[2..|q_1|]q$, it follows that $q_1q = Q_1q$ is cube-free.

In the same way, we obtain that qQ_2 is cube-free.

So q'^3 is neither a factor of Q_1q nor a factor of qQ_2. It follows that $|q'_1q'^3| > |Q_1q|$ and $|q'^3q'_2| > |qQ_2|$ that is $|q'_2| < |Q_2|$ and $|q'_1| < |Q_1|$.

Let φ' be the greatest prefix of $\varphi = \mu\rho$ such that $f(Q_1q)\varphi'$ is a prefix of $f(q'_1q'^3)$. Let ψ' be the greatest suffix of $\psi = \rho\mu$ such that $\psi' f(qQ_2)$ is a suffix of $f(q'^3q'_2)$.

By these definitions, the word $\psi' f(q)\varphi' = \psi'\rho\mu\rho\varphi'$ is a common factor (not necessarily the greatest) of $f(q')^3$ and $(\rho\mu)^4$.

Note that the inequality $|\psi' f(q)\varphi'| \geq |f(q')| + |\rho\mu|$ is equivalent to $|f(q')| \leq |\psi'| + |\varphi'| + |\rho|$.

Case 1: $|f(q')| \leq |\psi'| + |\varphi'| + |\rho|$.

By Corollary 2.3, there exist two words t_1 and t_2, and two integers i and j such that $f(q') = (t_1t_2)^i$ and $\psi = \rho\mu = (t_2t_1)^j$ with t_1t_2 and t_2t_1 primitive words. If $j \geq 2$, $f(Q_1q)$ ends with $\psi f(q) = (\rho\mu)^2\rho = (t_2t_1)^{2j}\rho$ where $2j > 3$. If $i \geq 2$, $f(q'^2) = (t_1t_2)^{2i}$ with q'^2 a proper prefix of q'^3 and $2i > 3$. In this two cases, the image by f of a cube-free word contains a cube; f is not cube-free. So $i = j = 1$.

We have $f(Q_1qQ_2) = \Pi_1(t_2t_1)^3\rho\Sigma_2 = f(q'_1q'^3q'_2) = f(q'_1)(t_1t_2)^3f(q'_2)$. Since t_1t_2 is not an internal factor of $(t_1t_2)^2$ (otherwise, by Lemma 2.2, t_1t_2 would not be primitive), $|f(q'_1)| < |f(Q_1)| = |\Pi_1t_2t_1|$ and $|f(q'_2)| < |f(Q_2)| = |t_2t_1\Sigma_2|$, we have either $\Pi_1 = f(q'_1)t_1$ and $f(q'_2) = t_1\rho\Sigma_2$, or $\Pi_1t_2 = f(q'_1)$ and $t_2f(q'_2) = \rho\Sigma_2$.

Case 1.1: $\Pi_1 = f(q'_1)t_1$.

Since $|\Pi_1| < |f(Q_1[1])| = |f((q'_1q)[1])|$ and f bifix, it means that $q'_1 = \varepsilon$, $\Pi_1 = t_1$, and $f(Q_1) = \Pi_1\psi = t_1t_2t_1 = f(q')t_1$ with t_1 a prefix of $f(q')$. Since f is a ps-morphism so a bifix morphism, by Lemma 2.4, we obtain that there exists a prefix x (possibly empty) of q' such that $f(x) = t_1$. From $f(q') = t_1t_2 = f(x)t_2$ with t_2 a prefix of $f(q)$ and f bifix, we obtain that there exists a prefix y of q such that $f(y) = t_2$.

From $f(q) = \rho\mu\rho = f(yx)\rho$ and f bifix, we obtain that ρ is the image of a word; a contradiction with the definition of ρ in the hypotheses of this lemma.

Case 1.2: $\Pi_1t_2 = f(q'_1)$.

This case is solved in the same way as Case 1.1,

From $f(Q_1) = \Pi_1t_2t_1 = f(q'_1)t_1$, we obtain that t_1 is the image of a word. From $f(q') = t_1t_2$, we obtain that t_2 is the image of a word. It follows that ρ is the image of a word; a contradiction with the definition of ρ.

Case 2: $|f(q')| > |\psi'| + |\varphi'| + |\rho|$.

If $q'_1 = \varepsilon$, i.e., $Q_1qQ_2 = q'^3q'_2$ then, by definition of ψ', it means that $\psi' = \psi = \rho\mu$. It follows that $|f(q')| > |\psi'| + |\varphi'| + |\rho| = |f(q)| + |\varphi'|$. Furthermore, $|f(q')^3| > |\Pi_1| + 2|f(q)| + 2|\varphi'| = |\Pi_1\varphi| + |f(q)| + |\rho| + 2|\varphi'| = |f(Q_1q)| + |\rho| + 2|\varphi'|$. Since $|\rho| \neq 0$, we have $|\rho| + 2|\varphi'| > |\varphi'|$. If $q'_2 \neq \varepsilon$ then $|f(q'_2)| > |\Sigma_2|$ and, by definition of φ', we obtain $|f(q')^3| = |f(Q_1q)\varphi'|$; this is impossible. Hence, $q'_2 = \varepsilon$ and $\varphi' = \varphi = \mu\rho$. It follows that $|f(q')^2| = |f(Q_1qQ_2)| - |f(q')| = |\Pi_1\rho\mu f(q)\mu\rho\Sigma_2| - |f(q')| < |\Pi_1\rho\mu\Sigma_2|$. If $|q'| \geq 2$ then $|f(q')f(q')| > |\Pi_1\Sigma_2| + |f(q')| > |\Pi_1\Sigma_2| + |\psi| + |\varphi| + |\rho| = |\Pi_1\Sigma_2| + |\rho\mu\rho\mu\rho|$ with $|\rho| \neq 0$; this conflicts with previous inequality. Consequently, we have $|q'| = 1$ and $q'^3 = Q_1qQ_2$ with Q_1, q, and Q_2 non-empty words. Therefore, $Q_1 = q = Q_2 = q'$ with $\rho\mu\rho = f(q) = f(Q_1) = \Pi_1\rho\mu$, i.e., $\mu\rho = \rho\mu$. By Proposition 2.1(2), there exist a non-empty word ω over A and two integers n, and p such that $\rho = \omega^n$ and $\mu = \omega^p$. Since $\rho \neq \varepsilon$, we obtain that $n \geq 1$. It follows that $f(Q_1q)$ contains ω^{3n+2p} with Q_1q cube-free and $3n + 2p \geq 3$; f is not cube-free.

In the same way, we obtain that $q'_2 \neq \varepsilon$.

Since $Q_1qQ_2 = q'_1q'^3q'_2$, we have $|f(q'_1[1])| = |f(Q_1[1])| > |\Pi_1|$ and $|f(q'_2[|q'_2|])| = |f(Q_2[|Q_2|])| > |\Sigma_2|$. Hence, $|\psi'| < |\psi|$ and $|\varphi'| < |\varphi|$. By definition of ψ' and φ', it implies that $f(q')^3 = \psi' f(q)\varphi'$.

We have $|\psi'| + 2|\rho| + |\mu| + |\varphi'| = |\psi' f(q)\varphi'| = |f(q')^3| > 3(|\psi'| + |\varphi'| + |\rho|)$ that is $|\mu| > 2|\psi'| + 2|\varphi'| + |\rho|$. It means that μ starts with φ' and ends with ψ'.

The word $f(q')$ starts with $\psi'\rho$ and ends with $\rho\varphi'$. There exist two words X and Y such that $f(q') = \psi'\rho X = Y\rho\varphi'$. Since $|f(q')| > |\psi'| + |\varphi'| + |\rho|$, we have $|X| > |\varphi'|$ and $|Y| > |\psi'|$. Therefore, there exist two non-empty words X' and Y' such that $X = X'\varphi'$, $Y = \psi'Y'$, and $f(q') = \psi'\rho X'\varphi' = \psi'Y'\rho\varphi'$. It follows that $\rho X' = Y'\rho$. By proposition 2.1(1), there exist two words r and s and an integer i such that $\rho = r(sr)^i$, $X' = sr$, and $Y' = rs$. Let us also note that μ ends with Y' and starts with X'.

If $i \geq 1$ then $f(Q_1q)$ contains $\mu\rho\mu$ that itself contains $Y'\rho X' = (rs)^{2+i}r$ with Q_1q cube-free and $2 + i \geq 3$; f is not cube-free.

Let us now consider the case $i = 0$. We have $\rho = r$, $Y' = \rho s$, $X' = s\rho$, and $f(q') = \psi'\rho s\rho\varphi'$. From $f(q')^3 = \psi' f(q)\varphi' = \psi'\rho\mu\rho\varphi'$, we also obtain that $\mu = s\rho\varphi' f(q')\psi'\rho s$. Let us remark that μ starts and ends with s. But the word μ also starts with φ' and also ends with ψ'.

In particular, the word $f(Q_1q)$ contains $\mu(\rho)\mu$ that itself contains $\psi'\rho s(\rho)s\rho\varphi'$.

If $|s| \leq |\varphi'|$ then s is a prefix of φ' and $\psi'\rho s\,\rho\,s\rho\varphi'$ contains the cube $(\rho s)^3$. If $|s| \leq |\psi'|$ then s is a suffix of ψ' and $\psi'\rho s\,\rho\,s\rho\varphi'$ contains the cube $(s\rho)^3$. If $|s| > |\varphi'|$, $|s| > |\psi'|$, and $|s| \leq |\varphi'| + |\psi'|$ then there exist three words a, b, and c such that $s = abc$, $\varphi' = ab$, and $\psi' = bc$. It follows that $\psi'\rho s\,\rho\,s\rho\varphi'$ contains the cube $(bc\rho a)^3$. In these three cases, $f(Q_1q)$ contains a cube with Q_1q cube-free; f is not cube-free.

The remaining case is $|s| > |\varphi'|$, $|s| > |\psi'|$, and $|s| > |\varphi'| + |\psi'|$; there exists a non-empty word μ' such that $s = \varphi'\mu'\psi'$ and we have $f(q') = \psi'\rho\varphi'\mu'\psi'\rho\varphi'$.

Let us denote $\alpha' = \psi'\rho$, $\beta' = \varphi'$, $\rho' = \psi'\rho\varphi' = \alpha'\beta'$, and $\theta' = \mu'\rho'\rho'\mu'\rho'\rho'\mu'$. We have $f(q') = \rho'\mu'\rho'$ and $\mu = s\rho\varphi' f(q')\psi'\rho s = \varphi'\mu'\rho' f(q')\rho'\mu'\psi' = \varphi'\theta'\psi'$.

Since $f(Q_1) = f(q'_1)\psi'$ ends with μ, we obtain that $f(q'_1)$ ends with $\varphi'\theta' = \beta'\theta'$. And, since $f(Q_2) = \varphi' f(q'_2)$ starts with $\mu\rho$, we obtain that $f(q'_2)$ starts with $\theta'\psi'\rho = \theta'\alpha'$. Moreover, by Lemma 2.4, it means that ψ' and φ'

are images of words by f.

Since f is bifix, if $\rho' = \psi'\rho\varphi'$ is the image of a word by f, it is the same for ρ; a contradiction with the hypotheses. So ρ' is not the image of a word by f.

Since q'_1q' is a prefix of Q_1q and $q'q'_2$ is a suffix of qQ_2, the words q'_1q' and $q'q'_2$ are cube-free.

Finally, since $3|f(q')| = |\psi' f(q)\varphi'| < |\psi f(q)\varphi| \leq 3|f(q)|$, we have $|f(q')| < |f(q)|$.

□

Lemma 3.3 *Let f be a ps-morphism from A^* to B^*. Let q_1, q_2, and q be non-empty words over A and let $w = q_1q^3q_2$. Let us assume that $f(w) = \pi_1 f(z)^3\sigma_2$, $|\pi_1| < |f(q_1[1])|$, $|\sigma_2| < |f(q_2[|q_2|])|$, and z is a non-empty word over A such that z^3 is a pure cube.*

If $|q| \leq 2$, if $|z| \geq 2$, or if $2|f(q)| \geq |f(z)|$ then f is not cube-free.

Otherwise, either f is not cube-free or there exist two words X and Y such that $|Y| \leq 1$, $2|f(q)| < |f(X)| = |f(z)| < 3|f(q)|$, and $q_1q^3q_2 = X^3Y$ with q_1 a prefix of X, q_2 a suffix of XY, and $|f(Y)| < |\sigma_2 f(z)|$ or $q_1q^3q_2 = YX^3$ with q_1 a prefix of YX, q_2 a suffix of X, and $|f(Y)| < |\pi_1 f(z)|$.

Proof.

Let us first remark that we can assume that the image by f of any proper factor of z^3 is cube-free. Otherwise, it ends the proof; f is not cube-free.

The hypotheses imply $|f(z)| > |f(q)|$. Furthermore, there exist a suffix σ_1 of $f(q_1)$ and a prefix π_2 of $f(q_2)$ such that $f(q_1) = \pi_1\sigma_1$ and $f(q_2) = \pi_2\sigma_2$. It means that $f(z)^3 = \sigma_1 f(q)^3\pi_2$.

If $|\sigma_1 f(q^3)| \leq |f(z)^2 f(z[1..|z|-1])|$ then $f(z)^2 f(z[1..|z|-1])$ contains the cube $f(q)^3$ with $z^2(z[1..|z|-1])$ a proper factor of z^3 and so cube-free. It ends the proof; f is not cube-free. Identically, if $|f(q^3)\pi_2| \leq |f(z[2..|z|])f(z)^2|$ then f is not cube-free. Consequently, we obtain that $|\pi_2| < |f(z[|z|])|$ and $|\sigma_1| < |f(z[1])|$; there exist two non-empty words α and β such that $f(z) = \sigma_1\alpha = \beta\pi_2$ and $f(q)^3 = \alpha f(z)\beta$.

Let us note that $\alpha \neq \varepsilon$ or $\beta \neq \varepsilon$. Otherwise, $f(z^2)$ contains $f(q)^3$. It ends the proof; f is not cube-free.

If $|f(q^2)| \geq |f(z)|$ then $|\alpha|+|\beta| = |f(q^3)|-|f(z)| \geq |f(q)|$ and the length of $\alpha f(z)\beta \in \mathrm{Fcts}\left(f(z)^3\right) \cap \mathrm{Fcts}\left(f(q)^3\right)$ is at least $|f(z)| + |f(q)|$. Let us note that this situation particularly happens when $|z| \geq 2$ because in this case $|\alpha| + |\beta| \geq |f(z[2..|z|])| + |f(z[1..|z|-1])| \geq |f(z)| > |f(q)|$. By Corollary 2.3, there exist two words z_1 and z_2, and two integers i and j such that $f(z) = (z_1z_2)^i$ and $f(q) = (z_2z_1)^j$. The inequality $|f(z)| > |f(q)|(> 0)$ implies $i > j \geq 1$. It follows that $f(z)^2 = (z_1z_2)^{2i}$ with $2i > 3$ that is $f(z)^2$ contains a cube; f is not cube-free.

From now on, z is a letter and $|f(q^2)| < |f(z)|$, i.e., $0 < |\alpha| + |\beta| < |f(q)|$. We obtain that $f(q)^3 = \alpha f(z)\beta$ starts and ends with $\alpha\beta$, i.e., $\alpha\beta$ is a prefix and a suffix of $f(q)$. There exist two non-empty words φ and ψ such that $f(q) = (\alpha\beta)\varphi = \psi(\alpha\beta)$ and $f(z) = \beta\varphi f(q)\psi\alpha$. By Proposition 2.1(1), there exist two words ρ and μ, and an integer j such that $\alpha\beta = \rho(\mu\rho)^j$, $\varphi = \mu\rho$, and $\psi = \rho\mu$.

Since $f(z)^2$ contains $(\psi\alpha)(\beta\varphi) = \rho\mu\rho(\mu\rho)^j\mu\rho = \rho(\mu\rho)^{j+2}$ and $f(z)^2$ cube-free, we necessarily have $j = 0$.

In this case, if we denote $\theta = \varphi f(q)\psi$, we have $\alpha\beta = \rho$, $f(q) = \rho\mu\rho$, $f(z) = \beta\mu\rho f(q)\rho\mu\alpha$. $\theta = \mu\rho\rho\mu\rho\rho\mu$, $f(z) = \beta\theta\alpha$, $f(q_1) = \pi_1\beta\theta$ and $f(q_2) = \theta\alpha\sigma_2$.

Since f is a *ps*-morphism so is non-erasing and since $q \neq \varepsilon$, we have $\rho\mu \neq \varepsilon$. Since $f(z)$ is cube-free, we necessarily have $\mu \neq \varepsilon$ and $\rho \neq \varepsilon$. Otherwise, the factor $\theta = \mu\rho\rho\mu\rho\rho\mu$ of $f(z)$ would be equal to ρ^4 or μ^3.

We also necessarily have $q_1[2..|q_1|]\,q$ and $q\,q_2[1..|q_2|-1]$ cube-free. Otherwise, $f(q_1[2..|q_1|]\,q)$ or $f(q\,q_2[1..|q_2|-1])$ both factors of $f(z)^2$ would contain a cube.

By Lemma 3.1, if ρ is not the image of a word by f then f is not cube-free; it ends the proof.

Let us now assume that ρ is the image of a word and let $\dot{q}$ be the non-empty word such that $f(\dot{q}) = \rho = \alpha\beta(\neq \varepsilon)$. Since f is bifix, $f(q) = \rho\mu\rho$ and $\mu \neq \varepsilon$, there exist a non-empty word $\overline{q}$ such that $f(\overline{q}) = \mu$. It follows that $q = \dot{q}\overline{q}\dot{q}$ and necessarily $|q| \geq 3$. In particular, $|f(qq\dot{q}\overline{q})| = |f(z)|$ and $qq\dot{q}\overline{q}$ is an internal factor of w; $f(w)$ and $f(z)^3$ are synchronised.

Furthermore, if we denote $x = \overline{q}\dot{q}q\dot{q}\overline{q}$, we obtain that $f(x) = \theta$, $f(q_1) = \pi_1\beta f(x)$, and $f(q_2) = f(x)\alpha\sigma_2$. Since f is bifix and by Lemma 2.4, it follows that $q_1 = W_1x$ for a non-empty word W_1 verifying $f(W_1) = \pi_1\beta(\neq \varepsilon)$ and $q_2 = xW_2$ for a non-empty word W_2 verifying $f(W_2) = \alpha\sigma_2(\neq \varepsilon)$.

Since $f(\dot{q}) = \rho = \alpha\beta$, $|\pi_1| < |f(q_1[1])| = |f(W_1[1])|$, and $|\sigma_2| < |f(q_2[|q_2|])| = |f(W_2[|W_2|])|$, by Lemma 2.6, we obtain that $\dot{q} = a$, $W_1 = b$, $W_2 = c$ for some letters a, b, c. Moreover, we have $b = a$ or $c = a$. It means that $q_1q^3q_2 = bx(\dot{q}\bar{q}\dot{q})^3xc = bxaxaxc$.

If $b = a$, let $X = ax$. It follows that $q_1q^3q_2 = X^3c$ with $|f(X)| = |f(ax)| = |f(\dot{q}x)| = |\beta f(x)\alpha| = |f(z)| = 3|f(q)| - |\rho|$ and $|f(c)| - |\sigma_2| = |\alpha| < |f(q)| < |f(z)|$.

In the same way, if $c = a$, let $X = xa$. We have $q_1q^3q_2 = bX^3$ with $|f(xa)| = |f(z)|$ and $|f(b)| < |\pi_1 f(z)|$. □

Lemma 3.4 *Let f be a ps-morphism from A^* to B^*. Let v and T be non-empty words over A such that v^3 is a pure cube. Let us assume that $f(T) = \pi_1 f(v)^3\sigma_2$ with $|\pi_1| < |f(T[1])|$, $|\sigma_2| < |f(T[|T|])|$, π_1 a suffix of the image by f of a shortest word v_1 and σ_2 a prefix of the image by f of a shortest word v_2.*
Then one of the following holds:

- *(P.1): There exist a cube x^3, a word y over A, and a word Z over B such that*
  - *(P.1.1): $T = x^3y$, $|y| \leq 1$, $f(y) = \pi_1\sigma_2$, $f(x) = \pi_1 Z$, and $f(v) = Z\pi_1$*
  - *(P.1.2): or $T = yx^3$, $|y| = 1$, $f(y) = \pi_1\sigma_2$, $f(x) = Z\sigma_2$, and $f(v) = \sigma_2 Z$.*
- *(P.2): There exist a pure cube x^3 and a non-empty word y over A such that*
  - *(P.2.1): $T = x^3y$ with $|f(x^2)| < |\pi_1 f(v)|$*
  - *(P.2.2): or $T = yx^3$ with $|f(x^2)| < |f(v)\sigma_2|$.*
- *(P.3): $T = t^3$, $|v| \geq 3$, $|t| = 1$ (i.e., t is a letter), $2|f(v)| < |f(t)| < 3|f(v)|$, and there exist two words $x \neq \varepsilon$ and y such that $|f(x)| = |f(t)|$ and*
  - *(P.3.1): $v_1v^3v_2 = x^3y$ with v_1 a prefix of x, v_2 a suffix of xy, and $|f(y)| < |f(t)\sigma_2'|$*
  - *(P.3.2): or $v_1v^3v_2 = yx^3$ with v_1 a prefix of yx, v_2 a suffix of x, and $|f(y)| < |\pi_1' f(t)|$*

  where π_1' and σ_2' are the words such that $f(v_1[1]) = \pi_1'\pi_1$ and $f(v_2[|v_2|]) = \sigma_2\sigma_2'$.
- *(P.4): f is not cube-free.*

Proof.

If T is cube-free, it ends the proof; f is not cube-free.

So T contains at least one cube. Among the cubes of T, we choose one whose image by f is a shortest; we can write $T = t_1t^3t_2$ where $|f(t)| = \min\{|f(t')| \text{ where } t'^3 \in \text{Fcts}(T)\}$. By this definition, since f is bifix (as any ps-morphism) and so non-erasing, t^3 is a pure cube.

If $t_1 \neq \varepsilon$ and $t_2 \neq \varepsilon$ then, by Lemma 3.3, T verify (P.1) or (P.4).

As in the proof of Lemma 3.1 done in [19], if a power of $f(t)$ and a power of $f(v)$ have a common factor of length at least $|f(t)| + |f(v)|$, we obtain that T verify (P.1).

From now, let us assume the converse holds, i.e., any common factor of $f(t)^3$ and $f(v)^3$ is of length at most $|f(t)| + |f(v)|$. It means that $|f(t)^3| - |\sigma_2| < |f(t)| + |f(v)|$ when $t_1 \neq \varepsilon$ and $t_2 = \varepsilon$, and that $|f(t)^3| - |\pi_1| < |f(t)| + |f(v)|$ when $t_1 = \varepsilon$ and $t_2 \neq \varepsilon$. That is T verify (P.2) with $x = t$.

Let us now treat the case where $t_1 = t_2 = \varepsilon$. In this case, $f(v)^3$ is factor of $f(t^3) = f(T)$. Hence, $2|f(v)| < |f(t)|$ and $|\pi_1| + |\sigma_2| = 3|f(t)| - 3|f(v)| > 3|f(v)|$. If $\pi_1 = \varepsilon$ then $f(t^2)$ contains the cube $f(v)^3$ with t^2 cube-free; f is not cube-free. In the same way, if $\sigma_2 = \varepsilon$ then f is not cube-free. It follows that $v_1 \neq \varepsilon$ and $v_2 \neq \varepsilon$; there exist a prefix π_1' of $f(v_1[1])$ and a suffix σ_2' of $f(v_2[|v_2|])$ such that $f(v_1v^3v_2) = \pi_1'\pi_1 f(v)^3\sigma_2\sigma_2' = \pi_1' f(t)^3\sigma_2'$. By Lemma 3.3 with $q = v$, we obtain that either f is not cube-free (for instance if $|v| \leq 2$ or if $|t| \geq 2$) or T verify (P.3). □

By Lemma 2.8, we immediately obtain:

Corollary 3.5 *With hypotheses and notations of Lemma 3.4, if $f(T)$ and $f(v)^3$ are synchronised then either f is not cube-free or T verifies (P.1).*

4 Special case of uniform morphisms

As a consequence of Lemma 3.4 and using Lemma 2.9, we can reduce a word whose image by a uniform morphism contains a cube. We can now state our main result.

Theorem 4.1 *Let A and B be two alphabets. A cube-free uniform morphism is k-power-free for all integers $k \geq 4$.*

It is a consequence of Proposition 2.10 and Proposition 4.2:

Proposition 4.2 *Let A and B be two alphabets. A cube-free uniform morphism is 4-power-free.*

Proof.

We follow the approach of the proof of Lemma 4.1 in [19]. We have to verify that all the steps are checked for $k = 3$.

Let f be uniform morphism from A^* to B^*. We assume that f is not 4-power-free and we want to show that f is not cube-free. The morphism f must be a ps-morphism. Otherwise, by Lemma 2.5, f is not cube-free; it ends the proof.

Let w a shortest 4-power-free word whose image by f contains a 4-power. That is $f(w) = pu^4s$ for a non-empty word u and two words p and s over B. If $f(w)$ and u^4 are synchronised, by Lemma 2.8, w contains a 4-power; a contradiction with the definition of w.

Now, let us assume that f is a ps-morphism and that $f(w)$ and u^4 are not synchronised. In particular, it implies that f is bifix and injective.

The central point of this proof is that, starting with w and u, we use iteratively reduction of Lemma 2.6 in [19] on the word whose image contains a 4-power in such a way that there is no reduction left. That is, no situation of the hypotheses of this lemma can be founded after this procedure; we obtain new words W et U such that $f(W) = pU^4s$ with p a proper prefix of $W[1]$, s a proper suffix of $W[|W|]$, and $f(W)$ and U^4 are not synchronised.

We will show that either f is not cube-free, or $f(W)$ and U can again be reduced using Lemma 2.6 in [19]; a contradiction with their definitions.

Step 1: For any pure cube v^3 of W, the words U^4 and $f(v)^3$ do not have any common factor of length at least $|U| + |f(v)|$.

By contradiction, let us assume that U^4 and $f(v)^3$ have a common factor of length at least $|U| + |f(v)|$. By Corollary 2.3, there exist two words t_1 and t_2, and two integers r and q such that $f(v) = (t_1t_2)^r$ and $U = (t_2t_1)^q$ with t_1t_2 and t_2t_1 primitive words.

If $r \geq 2$ then $f(v^2) = (t_1t_2)^{2r}$ with $2r > 3$ that is $f(v^2)$ contains a cube with v^2 cube-free by definition of v; f is not cube-free.

If $r = 1$ then it implies $q \geq 2$. Otherwise, v^q would be an internal factor of v^3 that is of W with $|f(v)^q| = |U|$; $f(W)$ and U^3 would be synchronised. Consequently, if $W = v_1v^3v_2$ for some words v_1 and v_2 then $f(W) = f(v_1)(t_1t_2)^3f(v_2) = pU^4s = p(t_2t_1)^{4q}s$ with $q \geq 2$.

Let x be the greatest integer such that $p(t_2t_1)^x$ is a prefix of $f(v_1v)$ and let y be the greatest integer such that $(t_2t_1)^ys$ is a suffix of $f(vv_2)$. There exist four words t'_p, t''_p, t'_s, and t''_s such that $t_2t_1 = t'_pt''_p = t'_st''_s$, $f(v_1v) = p(t_2t_1)^xt'_p$, $f(vv_2) = t''_s(t_2t_1)^ys$, and $f(v) = t''_p(t_2t_1)^{4q-x-y-2}t'_s$.

If $x = 0$ then $|f(v_1)| < p$. It implies that $v_1 = \varepsilon$. Therefore, we obtain that $f(v_1v^2) = pt'_pf(v) = (t_1t_2)^2$ starts with a prefix of $p(t_2t_1)^2$. Since t_2t_1 is not a primitive word, by Lemma 2.2, (t_2t_1) is not an internal factor of $(t_2t_1)^2$. It implies that $p = t_1$ and $t'_p = t_2$. In the same way, if $y = 0$, we obtain that $s = t_2$ and $t''_s = t_1$.

Since $f(v_1v)$ ends with t_1t_2, since $f(vv_2)$ starts with t_1t_2, if $x \geq 1$ and $t'_p \neq t_2$, or if $y \geq 1$ and $t''_s \neq t_1$ then (t_2t_1) is an internal factor of $(t_2t_1)^2$. By Lemma 2.2, we obtain that t_2t_1 is not a primitive word; a contradiction with the definition of t_2t_1.

Therefore, $t'_p = t_2 = t'_s$, $t''_p = t_1 = t''_s$, $f(v_1v) = pt_2f(v)^x$, $f(vv_2) = f(v)^yt_1s$, and $x + y + 1 = 4q - 1$. Since f is bifix, it follows that $f(v_1v)$ ends with $f(v)^x$ and $f(vv_2)$ starts with $f(v)^y$. Consequently, we obtain that v^{4q-1}

is an internal factor W with $4q-1 \geq q$. It implies that v^q is an internal factor W with $|f(v)^q| = |U|$, i.e., $f(W)$ and u^3 are synchronised; a contradiction with the hypotheses.

Step 2: $W[2..|W|-1]$ contains a cube and so a pure-cube.

By contradiction, let us assume that $W[2..|W|-1]$ is cube-free. It implies that W starts or ends with a pure cube. Let s_1 and p_5 be the words such that $f(W[1]) = ps_1$ and $f(W[|W|]) = p_5 s$ that is $U^4 = s_1 f(W[2..|W|-1])p_5$.

If $|s_1| \leq |U^3|$ then there exists a word U_c such that $s_1 U_c$ is the prefix of $U^4 = s_1 f(W[2..|W|-1])p_5$ of length $|s_1 U|$. Trivially, the word U_c is a conjugated of U (and $|U_c| = |U|$).

If $|s_1| + |p_5| \leq |U|$, we naturally have $|s_1| \leq |U^3|$. Moreover $|s_1| + |U_c^3| + |p_5| \leq |U^4|$. It means that $f(W[2..|W|-1])$ starts with U_c^3. Since $W[2..|W|-1]$ is a cube-free word, it ends the proof; f is not cube-free.

Let us now study the case where $|s_1| + |p_5| > |U|$.

Let us recall that, since we assume that $W[2..|W|-1]$ is cube-free, any pure cube of $W = W[1..|W|]$ is necessarily a prefix or a suffix of it.

If W starts with a pure cube v^3, let W_{com} be the greatest prefix of $s_1 f(v[2..|v|])f(v^2)$ that is factor of U^4 so a common factor of a power of $f(v)$ and a power of U. Let us remark that, if $W = v^3$ then $W_{com} = U^4$ else $W_{com} = s_1 f(v[2..|v|])f(v^2)$.

By Step 1, we have $|W_{com}| < |U| + |f(v)|$. By definition of W_{com}, if $W = v^3$ then $W_{com} = U^4 = s_1 f(v[2..|v|])f(v)f(v[1..|v|-1])p_5$ would be a common factor of $f(v)^3$ and U^3 with $|W_{com}| \geq |s_1| + |f(v)| + |p_5| > |f(v)| + |U|$; a contradiction. Hence, $W \neq v^3$ and $|W_{com}| = |f(v)| + |s_1 f(v[2..|v|])f(v)| > |f(v)| + 2|s_1|$. It implies that $|s_1| < |U|/2$.

In the case where W ends with a cube v'^3, we similarly obtain that $|p_5| < |U|/2$.

If W starts with a cube then $|s_1| < |U|/2$. Since $|s_1| + |p_5| > |U|$, we obtain that $|p_5| > |U|/2$. It implies that $W[2..|W|]$ is cube-free. But $f(W[1..|W|])$ starts with $ps_1 U_c^3$, i.e., $f(W[2..|W|])$ contains the cube U_c^3; f is not cube-free.

In the same way, if W ends with a cube, we also obtain either a contradiction with the assumptions or that f is not cube-free.

Step 3: For any pure cube $v^3 \in \mathrm{Fcts}\,(W[2..|W|-1])$, the word $f(v)^3$ is an internal factor of U^3 and $|f(v^2)| < |U|$.

For any pure cube $v^3 \in \mathrm{Fcts}\,(W[2..|W|-1])$, the word $f(v)^3$ is an internal factor of U^4. So $|f(v)^3| < |U| + |f(v)|$, i.e., $|f(v)^2| < |U|$ and $|f(v)^3| < \frac{3}{2}|U|$. That is $f(v)^3$ is an internal factor of U^3. It implies that $c_{\mathrm{v}} = 1, 2$ or 3.

Let us recall that, for all integers $j \in [1; 5 - c_{\mathrm{v}}]$, $f(v)^3$ is an internal factor of $p_j U^{c_{\mathrm{v}}} s_{j+c_{\mathrm{v}}}$. Consequently, if $\widehat{v}_j$ is the shortest factor of $W[i_j..i_{j+c_{\mathrm{v}}}]$ such that $f(\widehat{v}_j)$ contains $f(v)^3$ then, by Corollary 3.5, $\widehat{v}_j$ verifies Property $(P.1)$ for all integers $j \in [1; 5 - c_{\mathrm{v}}]$.

We are going to see that it implies that W can be reduced; a final contradiction.

Case 3.1: $c_{\mathrm{v}} = 3$

We necessarily have $d_{\mathrm{v}} = 1$ and $|D_{\mathrm{v}} f(v)| (\leq |f(v^2)|) < |U| \leq |D_{\mathrm{v}} f(v^2)|$. For all integers $j \in [1; 2]$, if $\widehat{v}_j$ verifies $(P.1.1)$ then ${x_j}^3 \in L_{j+1,v}$. And if $\widehat{v}_j$ verifies $(P.1.2)$ then ${x_j}^3 \in R_{j+1,v}$. In other words, we have $L_{j+1,v} \cup R_{j+1,v} \neq \emptyset$ with $j+1 \in [2; 3]$; by Lemma 2.9(2), we can reduce W.

Case 3.2: $c_{\mathrm{v}} \neq 3$ and $d_{\mathrm{v}} = 1$

We necessarily have $c_{\mathrm{v}} = 2$, i.e, $5 - c_{\mathrm{v}} = 3$. For all integers $j \in [1; 3]$, if $\widehat{v}_j$ verifies $(P.1.1)$ then ${x_j}^3 \in L_{j+1,v}$. And if $\widehat{v}_j$ verifies $(P.1.2)$ then ${x_j}^3 \in R_{j+1,v}$. That is $L_{j,v} \cup R_{j,v} \neq \emptyset$ for all integers $j \in [2; 4]$; by Lemma 2.9(1), a reduction can be done.

Case 3.3: $c_{\mathrm{v}} \neq 3$ and $d_{\mathrm{v}} = 0$

If $c_{\mathrm{v}} = 1$ then $|D_{\mathrm{v}} f(v)^2| \leq |U|$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for all integers $j \in [1; 4]$. By Lemma 2.9(3), a reduction can be done.

If $c_{\mathrm{v}} = 2$ then $|U| < |D_{\mathrm{v}} f(v)^2|$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for all integers $j \in [1; 3]$; by Lemma 2.9(4), a reduction can be done.

□

References

[1] D. R. Bean, A. Ehrenfeucht, and G. F. McNulty. Avoidable patterns in strings of symbols. *Pacific J. of Math*, 85(2):261–294, 1979.

[2] J. Berstel. Mots sans carré et morphismes itérés. *Discrete Applied Mathematics*, 29:235–244, 1980.

[3] J. Berstel. Axel Thue's papers on repetition in words: a translation. Technical Report 20, Laboratoire de Combinatoire et d'Informatique Mathématique, Université du Québec, Montréal, 1995.

[4] F.-J. Brandenburg. Uniformly growing k-th power-free homomorphisms. *Theoretical Computer Science*, 23(1):69–82, 1983.

[5] M. Crochemore. Sharp characterizations of squarefree morphisms. *Theoretical Computer Science*, 18:221–226, 1982.

[6] M. Crochemore. Régularités évitables (thèse d'état). Technical Report 83-43, LITP, November 1983.

[7] J. Karhumäki. On cube-free ω-words generated by binary morphisms. *Discrete Applied Mathematics*, 5:279–297, 1983.

[8] V. Keränen. On the k-freeness of morphisms on free monoids. Annales Academiae Scientarium Fennicae 61, Series A, 1986.

[9] M. Leconte. A characterization of power-free morphisms. *Theoretical Computer Science*, 38:117–122, 1985.

[10] M. Leconte. *Codes sans répétition*. PhD thesis, LITP Université Paris 6, october 1985.

[11] M. Lothaire. *Combinatorics on words*, volume 17 of *Encyclopedia of Mathematics*. Addison-Wesley, 1983. Reprinted in 1997 by Cambridge University Press in the Cambridge Mathematical Library, Cambridge, UK, 1997.

[12] M. Lothaire. *Algebraic Combinatorics on words*, volume 90 of *Encyclopedia of Mathematics*. Cambridge University Press, Cambridge, UK, 2002.

[13] M. Lothaire. *Applied Combinatorics on Words*, volume 105 of *Encyclopedia of Mathematics*. Cambridge University Press, Cambridge, UK, 2005.

[14] G. Richomme and P. Séébold. Conjectures and results on morphisms generating k-power-free words. *Int. J. of Foundations of Computer. Science*, 15(2):307–316, 2004.

[15] G. Richomme and F. Wlazinski. About cube-free morphisms. In H. Reichel and S. Tison, editors, *STACS'2000*, volume 1770 of *Lecture Notes in Computer Science*, pages 99–109. Springer-Verlag, 2000.

[16] A. Thue. Uber unendliche zeichenreihen. *Kristiania Videnskapsselskapets Skrifter Klasse I. Mat.-naturv*, 7:1–22, 1906.

[17] A. Thue. Uber die gegenseitige Lage gleigher Teile gewisser Zeichenreihen. *Kristiania Videnskapsselskapets Skrifter Klasse I. Mat.-naturv*, 1:1–67, 1912.

[18] F. Wlazinski. A test-set for k-power-free binary morphisms. *TIA*, 35:437–452, 2001.

[19] F. Wlazinski. Reduction in non-$(k+1)$-power-free morphisms. *RAIRO Theor. Inform. Appl.*, Volume 50, Number 1, January-March 2016, Special issue dedicated to the 15th "Journées montoises d'informatique théorique":3–20, 2016.