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Abstract

In this article we are interested in the semi-group stability for finite difference schemes approximations of hyperbolic systems of equations in corner domains. We give generalizations of the results of [CG11] and [Cou15] from the half space geometry to the quarter space geometry. The most interesting fact is that the proofs of [CG11] and [Cou15] can be adaptated with minor changes to apply in the quarter space geometry. This is due to the fact that both methods in [CG11] and [Cou15] are based on energy methods and the construction of auxiliary problems with strictly dissipative boundary conditions which are known to be suitable for the strong well-posed for initial boundary value problems in the quarter space.

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Contents

1 Introduction

In this article we are interested in finite difference schemes approximation for linear hyperbolic problems in the quarter space. Such problems read:

$$\begin{cases} L(\partial)u := \partial_t u + A_1\partial_1 u + A_2\partial_2 u + \sum_{j=3}^{d} A_j\partial_j u = f, \text{ in } [0,\infty[\times\Omega\times\mathbb{R}^{d-2}, \\ B_1 u_{|x_1=0} = g_1, \text{ on } [0,\infty[\times\partial\Omega_1\times\mathbb{R}^{d-2}, \\ B_2 u_{|x_2=0} = g_2, \text{ on } [0,\infty[\times\partial\Omega_2\times\mathbb{R}^{d-2}, \\ u_{|t=0} = u_0, \text{ on } \Omega\times\mathbb{R}^{d-2}, \end{cases} \tag{1}$$

where Ω denotes the quarter space $\mathbb{R}_+^2$ and $\partial\Omega_1$ (resp. $\partial\Omega_2$) is the component of the boundary associated to $\{x_1 = 0\}$ (resp. $\{x_2 = 0\}$). In (1) the coefficients in the interior, the A_j are matrices in $\mathbf{M}_{n\times n}(\mathbb{R})$ while the coefficient on the boundary B_1 (resp. B_2) is an element of $\mathbf{M}_{p_1\times n}(\mathbb{R})$ (resp.$\mathbf{M}_{p_2\times 2}(\mathbb{R})$) where p_1 (resp. p_2) denotes the number of strictly positive eigenvalues of A_1 (resp. A_2).

Finite difference schemes approximations in the quarter space are thus just discretizations of (1) and have pratical motivations in scientific computations. Indeed, due to the impossibility to modeling the full space $\mathbb{R}^d$ during a numerical simulation, all the schemes implemented in a computer lie in a large rectangle and thus numerically boundary conditions have to be specified even for the numerical approximation of a Cauchy problem. Thus the theoretical study of such schemes set in a domain with corners also have more pratical views. About these pratical views we can be more specific and describe, for example the question of absorbing boundary conditions for wave propagation (see for example [EM77]-[Hig86] and [Ehr10]). These conditions are non physical ones and aim to minimize, as much as possible, the "parasite" reflections which occur when the wave hits the artificial boundaries implemented in the simulation of the Cauchy problem. Consequently these conditions are choosen in such a way that the reflections against the boundaries modify or influence as little as possible the approximation in the interior of the box. A similar method is the study of perfectly matched layer (see for example [Ber94]) which are boundary conditions which will only modify the approximation in a small neighborhood of the boundary.

In this article we are interested in the stability of difference schemes approximation set in a space with corner. But before to turn to a more precise description of the notion of stability for schemes with corner let us recall some elements of comparison with the notion of strong well-posedness for continous problems.

Strong well-posedness means existence and uniqueness of the solution of (1) and that this solution is as regular (in the L^2-norm) as the datas of the problem. Such a control of the solution by the data is refered as an energy estimate for (1). In the author knowledge, even for homogeneous initial conditions (that is to say $u_0 \equiv 0$) the strong well-posedness of (1), under suitable conditions, has not been established yet. The main contribution about this question is due to [Osh73], in which the author obtains, thanks to the introduction of a new inversibility condition (we refer to [Osh73] or to [[Ben15], Chapitre 5] for more details), an energy estimate for the L^2-norm of the solution. However the regularity of the source terms of (1) asked to control the L^2-norm of the solution is not explicit. As a consequence, there is a non explicit number of losses of derivatives in the energy estimate and we can not conclude to the strong well-posedness.

However in a particular framework, more precisely for strictly dissipative boundary conditions, that is to say boundary conditions which make the energy decrease, the strong well-posedness (with homogeneous initial datas) is established see [[Ben15], Chapitres 4 and 5]. We also refer to [HR] for a result dealing with three dimensional corners in which, thanks to the strict dissipativity and under an ellipticity assumption on the spatial symbol of the hyperbolic operator, the authors obtain a result of strong well-posedness for corners problems with inhomogeneous initial conditions.

We give some more details about the mentioned previous energy estimates. By analogy with the natural energy estimate in the half space geometry [Kre70], the expected energy estimate for u the solution of (1) is:

$$\begin{aligned}\sup_{t\geq 0} e^{-2\gamma t}\|u(t,\cdot)\|^2_{L^2(\Omega)} \quad &+ \quad \gamma\int_0^\infty e^{-2\gamma t}\|u(t,\cdot)\|^2_{L^2(\Omega)}dt + \sum_{k=1}^{2}\int_0^\infty e^{-2\gamma t}\|u_{|x_k=0}(t,\cdot)\|^2_{L^2(\partial\Omega_k)}dt \qquad (2)\\ &\leq \quad C\left(\|u_0\|^2_{L^2(\Omega)} + \frac{1}{\gamma}\int_0^\infty e^{-2\gamma t}\|f(t,\cdot)\|^2_{L^2(\Omega)}dt + \sum_{k=1}^{2}\int_0^\infty e^{-2\gamma t}\|g_k(t,\cdot)\|^2_{L^2(\partial\Omega_k)}dt\right),\end{aligned}$$

and, in our definition stability for difference schemes approximations of (1) means that some discretized version of (2) (see Definition 2.1 for a precise definition) holds for the solution of the scheme.

1.1 Some results about strong well-posed and stability for the half space geometry

Before to describe the obtained stability results, it is interesting to give a brief overview of the known results on this subject in the simpler geometry of the half space $\widetilde{\Omega} := \{x = (x_1, x')|x_1 > 0, x' \in \mathbb{R}^{d-1}\}$. The associated version of (1) reads:

$$\begin{cases} L(\partial)u = f, \text{ in } [0,\infty[\times\widetilde{\Omega}, \\ Bu_{|x_1=0} = g_1, \text{ on } [0,\infty[\times\partial\widetilde{\Omega}, \\ u_{|t=0} = u_0, \text{ on } \widetilde{\Omega}, \end{cases} \qquad (3)$$

and a finite difference scheme approximation of (3) is (for example for a one time step approximation) given by:

$$\begin{cases} U_j^{n+1} + QU_j^n = \Delta t f_j^{n+1}, \text{ for } n\geq 0,\ j_1\geq 1, j'\in\mathbb{Z}^{d-1}, \\ U_j^{n+1} + B^{0,j_1}U_j^n + B^{1,j_1}U_j^{n+1} = g_j^{n+1}, \text{ for } n\geq 0, \geq 1-\ell_1\leq j_1\leq 0, j'\in\mathbb{Z}^{d-1}, \\ U_j^0 = u_{0,j}, \text{ for } j_1\geq 1-\ell_1, j'\in\mathbb{Z}^{d-1}, \end{cases} \qquad (4)$$

where Q is a dicretization of the spatial differetiation in the interior, B^{0,j_1} and B^{1,j_1} are discretizations of the boundary condition B and finally where $\ell_1 \in \mathbb{N}$ is the stencil of the operator Q in the $(-x_1)$-direction.

Compared to the corner geometry, the theory of semi-group well-posedness for (3) is much more elaborated. Semi-group well-posedness for (3) means existence and uniqueness of a solution u which satisfies the energy estimate: there exists $C > 0$ such that for all $\gamma > 0$ we have

$$\begin{aligned}\sup_{t\geq 0} e^{-2\gamma t}\|u(t,\cdot)\|^2_{L^2(\widetilde{\Omega})} \quad &+ \quad \gamma\int_0^\infty e^{-2\gamma t}\|u(t,\cdot)\|^2_{L^2(\widetilde{\Omega})}dt + \int_0^\infty e^{-2\gamma t}\|u_{|x_1=0}(t,\cdot)\|^2_{L^2(\partial\widetilde{\Omega})}dt \qquad (5)\\ &\leq \quad C\left(\|u_0\|^2_{L^2(\widetilde{\Omega})} + \frac{1}{\gamma}\int_0^\infty e^{-2\gamma t}\|f(t,\cdot)\|^2_{L^2(\widetilde{\Omega})}dt + \int_0^\infty e^{-2\gamma t}\|g_1(t,\cdot)\|^2_{L^2(\partial\widetilde{\Omega})}dt\right).\end{aligned}$$

And, from [Kre70] and [Rau72], we know that the initial boundary value problem in the half space (3) is semi-group well-posed if and only if the so-called uniform Kreiss-Lopatinskii condition is satisfied. This conditions means that in the normal modes analysis no stable mode satisfies the homogeneous boundary condition.

Semi-group stability for the finite difference scheme approximation (4) means (for example) that the solution of (4) satisfies the estimate: there exists C such that for all $\gamma > 0$, for all $\Delta t \in]0,1]$

$$\sup_{n\geq 0} \Delta x_1 e^{-2\gamma n\Delta t} \sum_{j_1=1-\ell_1}^{\infty} \|U^n_{j_1,\cdot}\|^2 + \frac{\gamma}{\gamma\Delta t+1} \sum_{n\geq 0} \Delta t\Delta x_1 e^{-2\gamma n\Delta t} \sum_{j_1=1-\ell_1}^{\infty} \|U^n_{j_1,\cdot}\|^2 + \sum_{n\geq 0} \Delta t e^{-2\gamma n\Delta t} \sum_{j_1=1-\ell_1}^{0} \|U^n_{j_1,\cdot}\|^2$$
$$\leq C\Big(\sum_{j_1=1-\ell_1}^{\infty} \Delta x_1 \|u_0\|^2 + \frac{\gamma\Delta t+1}{\gamma} \sum_{n\geq 1} \Delta t\Delta x_1 e^{-2\gamma n\Delta t} \sum_{j_1=1-\ell_1}^{\infty} \|f^n_{j_1,\cdot}\|^2$$
$$+ \sum_{n\geq 1} \Delta t e^{-2\gamma n\Delta t} \sum_{j_1=1-\ell_1}^{0} \|g^n_{j_1,\cdot}\|^2\Big), \quad (6)$$

where the $\|\cdot\|$-norm is defined by: $for j' \in \mathbb{Z}^{d-1}$,

$$\|U_{\cdot}\|^2 := \prod_{k=2}^{d} \Delta x_k \sum_{j'\in\mathbb{Z}^{d-1}} |U_{j'}|^2,$$

and where the parameters Δt , Δx_k, $k = 1, ..., d$ are the parameters of the cartesian discretization of $[0,\infty[\times\widetilde{\Omega}$. These parameters are assumed to satisfy some CFL (COURANT-FRIEDRICHS-LEWY) condition (that is to say that the ratios $\lambda_k := \frac{\Delta t}{\Delta x_k}$ are constant while $\Delta t \downarrow 0$).

Note that if ones formally takes the limit $\Delta t \downarrow 0$ in the stability estimate for (U^n_j) then he recovers the energy estimate for u that is (5). As a consequence the stability estimate for (U^n_j) is just a discretized version of (5). Once again we have a full characterization of the difference schemes approximations that are strongly stable : the scheme (4) is strongly stable if and only it satisfies the so-called GKS (Gustafsson-Kreiss-Sundström) condition (see [BGS72]). This condition is in some sense a discrete version of the uniform Kreiss-Lopatinskii condition.

The sketch of proof to establish the semi-group stability or the semi-group well-posed in the same and is based in two distinct substeps. In a first time the study is restricted to homogeneous initial conditions and we show the estimate (5) but without the control of the supremum in the left hand side (and also without the term u_0 in the right hand side). This estimate characterized all the problems which are call strongly well-posed (resp. strongly stable) in the setting of continous (resp. discrete) problems. More precisely this estimate in the continous setting reads: there exists $C > 0$ such that for all $\gamma > 0$

$$\gamma \int_0^\infty e^{-2\gamma t} \|u(t,\cdot)\|^2_{L^2(\widetilde{\Omega})} dt + \int_0^\infty e^{-2\gamma t} \|u_{|x_1=0}(t,\cdot)\|^2_{L^2(\partial\widetilde{\Omega})} dt$$
$$\leq C\left(\frac{1}{\gamma} \int_0^\infty e^{-2\gamma t} \|f(t,\cdot)\|^2_{L^2(\widetilde{\Omega})} dt + \int_0^\infty e^{-2\gamma t} \|g_1(t,\cdot)\|^2_{L^2(\partial\widetilde{\Omega})} dt \right),$$

and: there exists $C > 0$ such that for all $\gamma > 0$, $\Delta t \in]0,1]$

$$\frac{\gamma}{\gamma\Delta t+1} \sum_{n\geq 0} \Delta t\Delta x_1 e^{-2\gamma n\Delta t} \sum_{j_1=1-\ell_1}^{\infty} \|U^n_{j_1,\cdot}\|^2 + \sum_{n\geq 0} \Delta t e^{-2\gamma n\Delta t} \sum_{j_1=1-\ell_1}^{0} \|U^n_{j_1,\cdot}\|^2$$
$$\leq C\Big(+ \frac{\gamma\Delta t+1}{\gamma} \sum_{n\geq 1} \Delta t\Delta x_1 e^{-2\gamma n\Delta t} \sum_{j_1=1-\ell_1}^{\infty} \|f^n_{j_1,\cdot}\|^2 + \sum_{n\geq 1} \Delta t e^{-2\gamma n\Delta t} \sum_{j_1=1-\ell_1}^{0} \|g^n_{j_1,\cdot}\|^2\Big),$$

in the discrete framework.

Then in a second time, this estimate characterizing the strong well-posedness (resp. strong stability) is extended to non homogeneous initial conditions and the supremum is added in the left hand side in view to obtain (resp. the discretized version of) (5). Once the estimate is demonstrated we tell that the problem is semi-group well-posed in the continous setting and semi-group stable in the discrete one. Let us note the important fact that in this second step, the main assumption is to assume that the considered problem is strongly well-posed or strongly stable (up to some possibly technical extra assumptions in the discrete framework).

In the continous setting the full characterization of strongly well-posed problems has been established in [Kre70] and its extension to non homogeneous initial conditions establishing the semi-group well-posedness is due to [Rau72] (we also refer to [BG07] or [CP81] for an overview/review of the proofs).

In the discrete setting, the first result showing strong stability for a finite difference scheme is due to [BGS72] and has then been extended to more general finite difference schemes, for example, in [Cou09]-[Cou11]. About the semi-group stability of schemes with non zero initial conditions, the first result applies to one step in time finite difference schemes and is due to [Wu95]. The method of [Wu95] has then been generelized in [CG11]. An other result in the theory of semi-group stability for finite difference schemes with several time steps this time (but restricted to scalar equations) is obtained in [Cou15].

1.2 Generalization of semi-group stability results to corner domains

In this article we give generalizations of the results contained in [CG11] and [Cou15] from the half space to the quarter space geometry. More precisely we show that if we assume that the finite difference scheme approximation is strongly stable (see Definition 2.1 for a precise definition) then a discretized version of (2) (see (22)-(24)) can be obtained for all the finite difference schemes that we are able to deal with in the half space geometry.

As a consequence, the geometry in which the finite difference scheme is set does not prevent to go from strong stability to semi-group stability.[1]

As the reader will see, our proofs follow the main steps of the proofs in [CG11] and [Cou15]. This fact may seem to be surprising but it should not. Indeed the proofs in [CG11] and [Cou15] both relies on the introduction of an auxiliary problem. More precisely in [CG11], the authors first treate the case of one dimensional schemes. Then to generalize their result to multidimensional schemes they use partial Fourier transform in the tangential variables to recover the one dimensional case.

In the one dimensional setting, the auxiliary problem used in [CG11] is the finite difference scheme (4) but with Dirichlet boundary conditions instead of the discretized boundary conditions involving B^{0,j_1} and B^{1,j_1}. The auxiliary problem used in [Cou15] in based on two discrete multipliers coming from the Leray-Gårding method to obtain *a priori* estimates for hyperbolic PDE (see [?]-[?]). Note that the existence of such multipliers was the starting point in the analysis of [Rau72] to go from the strong well-posed to the semi-group stability for initial boundary value problems. Compared with the auxiliary problem used in [CG11], the auxiliary problem of [Cou15] is defined on the full space $\{j \in \mathbb{Z}^d\}$ and thus it permits to use Laplace- partial Fourier transform (without any extension) to translate some energy estimates for the solution in terms of the symbol of the discretization operator.

Then the authors use the fact that each auxiliary finite difference scheme admits *strictly dissipative* boundary condition to show the semi-group stability from the strong stability.

In the continous setting it is known (see for example [[Ben15], Chapitre 4]) that strictly dissipative boundary conditions are suitable for corner problems as well as for problems in the half space. As a consequence, as far as strict dissipativity is concerned, the proofs for finite difference schemes in the half space should also operate for the quarter space geometry and it is effectively the case with sometimes really minor changes. An other important point in the generalization of [Cou15] to the quarter space geometry is that the auxiliary problem is set in the full space. Consequently the use of the Laplace- partial Fourier transform (which is prohibited, without preliminary extension, in the quarter space geometry because there are two "normal" directions) also operates because we are in the full space.

This point will not be true anymore for the generalization of [CG11] for quarter spaces because the auxiliary problem will not be set in the full space. However we show that in that case it is possible to do the analysis of [CG11] and specifically the energy method directly for multidimensional schemes. So we will not

[1] In all the article to make the notations as simple as possible we restricted our subject to domains with only one two dimensional corner. However all the results extended to multi-dimensional corners and/or to domains with several corners.

have to perform any partial Fourier transform and the result extend to the quarter space geometry.

Of course the main assumption in both of the generalization is that the finite difference scheme for the corner problem is strongly stable. In the author knowledge there is no result concerning the full characterization of strongly stable schemes in corner domains in the litterature. Moreover in the author opinion this question could be a challenging one. What is clear is that imposing that each finite difference scheme in the half spaces $\left\{j_1 \geq 1-\ell_1, (j_2, j') \in \mathbb{Z}^{d-1}\right\}$ and $\left\{j_1 \in \mathbb{Z}, j_2 \geq 1-\ell_2, j' \in \mathbb{Z}^{d-2}\right\}$ satisfies the GKS condition will be necessary. However the study of [Osh73] for continous problems tells us that a new condition will be needed. In analogy with the half space geometry, that seems to be a reasonable conjecture is that to characterize strong stability in corner domains a discretized version of this condition will also be necessary.

1.3 Organization of the article

The paper is organized as follows. In Section 2 we introduce the notations and some definitions, in particular we give some new definitions needed to deal with the corner geometry. Then in Section 3 we state the assumptions and the main results. At last Sections 4 and 5 are devoted to the proofs of each generalization.

2 Finite difference schemes and corner

2.1 General notations and definitions

In all what follows we use the short hand notation $[\![\cdot,\cdot]\!]$ for the "intervals of integers", more precisely for $a, b \in \mathbb{R}$ we define $[\![a, b]\!] := [a, b] \cap \mathbb{Z}$.

To describe the finite difference scheme that we will consider we define the following subsets of $\mathbb{Z}^2$, for $j = (j_1, j_2) \in \mathbb{Z}^2$ let:

$$\mathscr{I} := \left\{j \in \mathbb{Z}^2 | j_1, j_2 \geq 1\right\}, \quad \mathscr{C} := \left\{j \in \mathbb{Z}^2 | j_1 \in [\![1-\ell_1, 0]\!], j_2 \in [\![1-\ell_2, 0]\!]\right\} \tag{7}$$

$$\mathscr{B}_1 := \left\{j \in \mathbb{Z}^2 | j_1 \in [\![1-\ell_1, 0]\!], 1 \leq j_2\right\}, \quad \mathscr{B}_2 := \left\{j \in \mathbb{Z}^2 | 1 \leq j_1, j_2 \in [\![1-\ell_2, 0]\!]\right\}, \tag{8}$$

where ℓ_1 (resp. ℓ_2) is a fixed positive integer (that will correspond to the number of space steps of the scheme towards the "left" (resp. "bottom")).

The set $\mathscr{I}$ has to be understood as the discretization of the interior of Ω, $\mathscr{B}_1$ (resp. $\mathscr{B}_2$) as the discretization of the boundary $\partial\Omega_1$ (resp. $\partial\Omega_2$) and finally $\mathscr{C}$ is a discretization of the corner of Ω. Finally, the full set of resolution $\mathscr{R}$ is defined by

$$\mathscr{R} := \mathscr{I} \cup \mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{C}.$$

To state our definition of strong stability we introduce the extended discretizations of the traces $\mathscr{B}_1$ and $\mathscr{B}_2$ defined by:

$$\overline{\mathscr{B}}_1 := \left\{j \in \mathbb{Z}^2 | j_1 \in [\![1-\ell_1, r_1]\!], 1-\ell_2 \leq j_2\right\}, \ \overline{\mathscr{B}}_2 := \left\{j \in \mathbb{Z}^2 | 1-\ell_1 \leq j_1, j_2 \in [\![1-\ell_2, r_2]\!]\right\}, \tag{9}$$

where once again r_1 (resp. r_2) is a fixed positive integer (that will correspond to the number of space steps of the scheme towards the "right" (resp. "top")).

Let $\Delta x_1, \Delta x_2, ...\Delta x_d > 0$ be the space steps of discretization, we define $\Delta x := \Delta x_1 \Delta x_2$, and let Δt be the time step discretization. In a classical setting let us assume that Δt, $\Delta x_1, ..., \Delta x_d$ are related by the CFL numbers which are defined by $\lambda_k := \frac{\Delta t}{\Delta x_k}$ for $k \in [\![1, d]\!]$. Let us recall that the λ_k are kept constant as $\Delta t \downarrow 0$. Note that it implies, in particular, that for all $k_1, k_2 \in [\![1, d]\!]$ we have $\Delta x_{k_1} \sim \Delta x_{k_2}$.

We introduce the following weighted norm on $\ell^2(\mathbb{Z}^d)$. Let $\mathfrak{I} \subseteq \mathbb{Z}^2$ and $u \in \ell^2(\mathfrak{I} \times \mathbb{Z}^{d-2})$ we define:

$$\|u\|^2_{\mathfrak{I}} := \left(\prod_{k=3}^{d} \Delta x_k\right) \|u\|^2_{\ell^2(\mathfrak{I} \times \mathbb{Z}^{d-2})}, \text{ and } |\!|\!|u|\!|\!|^2_{\mathfrak{I}} := \Delta x_1 \Delta x_2 \|u\|^2_{\mathfrak{I}}.$$

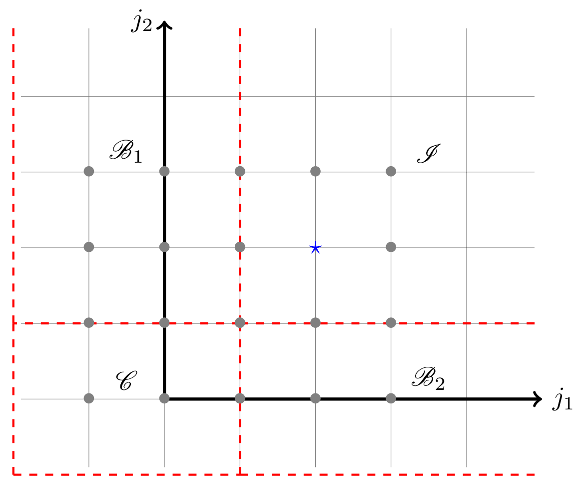

Figure 1: The set of resolution and the dependency set of $U_{2,2}$ for $\ell_1 = 3$, $\ell_2 = 2$ and $r_1 = r_2 = 1$.

We also denote $\langle \cdot, \cdot \rangle_{\mathfrak{I}}$ (resp. $\langle\langle \cdot, \cdot \rangle\rangle_{\mathfrak{I}}$) the scalar product associated to $\| \cdot \|_{\mathfrak{I}}$ (resp. $|\!|\!| \cdot |\!|\!|_{\mathfrak{I}}^2$).

The finite difference scheme approximation of (1) that we are considering reads:

$$\begin{cases} \sum_{\sigma=0}^{s+1} Q^\sigma U_j^{n+\sigma} = \Delta t f_j^{n+s+1}, & \text{for } j \in \mathscr{I} \times \mathbb{Z}^{d-2}, n \geq 0, \\ U_j^{n+s+1} + \sum_{\sigma=0}^{s+1} B_1^{\sigma,j_1} U_j^{n+\sigma} = g_{1,j}^{n+s+1}, & \text{for } j \in \mathscr{B}_1 \times \mathbb{Z}^{d-2}, n \geq 0, \\ U_j^{n+s+1} + \sum_{\sigma=0}^{s+1} B_2^{\sigma,j_2} U_j^{n+\sigma} = g_{2,j}^{n+s+1}, & \text{for } j \in \mathscr{B}_2 \times \mathbb{Z}^{d-2}, n \geq 0, \\ U_j^{n+s+1} + \sum_{\sigma=0}^{s+1} C^{\sigma,j} U_j^{n+\sigma} = h_j^{n+s+1}, & \text{for } j \in \mathscr{C} \times \mathbb{Z}^{d-2}, n \geq 0, \\ U_j^n = u_{n,j}, & \text{for } j \in \mathscr{R} \times \mathbb{Z}^{d-2}, n \in [\![0, s]\!]. \end{cases} \tag{10}$$

Note that (10) has $s+1$ time steps. The operator Q^σ appearing in the first equation of (10) is defined by:

$$Q^\sigma := \sum_{\mu_1=-\ell_1}^{r_1} \sum_{\mu_2=-\ell_2}^{r_2} \sum_{\mu'=-\ell'}^{r'} A^{\sigma,\mu} \mathbf{T}_1^{\mu_1} \mathbf{T}_2^{\mu_2} \mathbf{T}'^{\mu'}, \tag{11}$$

where $\mu := (\mu_1, \mu_2, \mu') \in \mathbb{N}^d$, the coefficients $A^\mu \in \mathbf{M}_{n\times n}(\mathbb{R})$ and where for $k \in [\![1, 2]\!]$, $\mathbf{T}_k^{\mu_k}$ (resp. $\mathbf{T}'^{\mu'}$) denotes the μ_k (resp. μ')-shift operator, that is:

$$\forall u \in \ell^2(\mathbb{Z}^d),\ (\mathbf{T}_1^{\mu_1} u)_j := u_{j_1+\mu_1, j_2, j'},\ (\mathbf{T}_2^{\mu_2} u)_j := u_{j_1, j_2+\mu_2, j'} \text{ and } (\mathbf{T}'^{\mu'} u)_j := u_{j_1, j_2, j'+\mu'}.$$

Also note that in (11) we used the short hand notation: for $\ell', r' \in \mathbb{N}^{d-2}$,

$$\sum_{\mu'=-\ell'}^{r'} := \sum_{k=3}^{d-2} \sum_{\mu_k=-\ell_k}^{r_k}.$$

Thus, in view of its definition, the scheme (10) has stencil $\ell_1 + r_1$ in the j_1-direction and $\ell_2 + r_2$ in the j_2-direction. So to compute the sequence $(U_j^n)_{j\in\mathscr{I}}$ it is needed to know the boundary values $(U_j^n)_{j\in\mathscr{B}_1\cup\mathscr{B}_2}$. This was expected from the analysis of finite difference schemes in the half space. But, and it is a new fact induced by the quarter space geometry, we also need the corner values $(U_j^n)_{j\in\mathscr{C}}$ (see (8)-(7) for a definition of this sets). Also note that in this formulation, the finite difference scheme (10) can be explicit or implicit in time.

A new feature for finite difference scheme in corner domains is that, if we have computed the solution $(U_j^n)_{j\in\mathscr{R}}$ at some time n then, the order of computation of the U_j^{n+1} is not as canonical as in the half space geometry. Indeed for finite difference schemes in the half space the only possible way to compute (U_j^{n+1})

from (U_j^n) is to determine the U_j^{n+1} for j in the interior and then to compute the U_j^{n+1} for j in the discretization of the boundary. This determines (U_j^{n+1}) and the order also of resolution of the scheme in a unique way.

In corner domains we always have to determine first the U_j^{n+1} for $j \in \mathscr{I}$. But we have some degrees of freedom in the order of determination of the U_j^{n+1} for $j \in \mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{C}$. These degrees of freedom lead to different expressions of the boundary and corner operators B_1^{σ,j_1}, B_2^{σ,j_2} and C^{σ,j_1,j_2}. Some of this several possible expressions are described in paragraph 2.2.

We conclude this section by the definition of strongly stable finite difference schemes in the quarter space:

Definition 2.1 (Strong stability) *We say that the difference scheme* (10) *is strongly stable for homogeneous initial conditions if there exists* $C > 0$ *such that for all* $\gamma > 0$ *and* $\Delta t \in]0, 1]$*, the solution* (U_j^n) *of* (10) *with* $u_{0,j} \equiv 0$ *satisfies the estimate*[2]*:*

$$\frac{\gamma}{\gamma\Delta t + 1}\sum_{n\geq s+1}\Delta t e^{-2\gamma n\Delta t}|||U^n|||_{\mathscr{R}}^2 + \sum_{k=1}^{2}\sum_{n\geq s+1}\Delta t\Delta x_{3-k}e^{-2\gamma n\Delta t}\|U^n\|_{\mathscr{B}_k}^2 \leq \tag{12}$$

$$C\left(\frac{\gamma\Delta t+1}{\gamma}\sum_{n\geq s+1}\Delta t e^{-2\gamma n\Delta t}|||f^n|||_{\mathscr{I}}^2 + \sum_{k=1}^{2}\sum_{n\geq s+1}\Delta t\Delta x_{3-k}e^{-2\gamma n\Delta t}\|g_k^n\|_{\mathscr{B}_k}^2 + \sum_{n\geq s+1}\Delta t\Delta x_2 e^{-2\gamma n\Delta t}\|h^n\|_{\mathscr{C}}^2\right).$$

Before we turn to the statement of our main result it may be interesting to give some comments about the notion of strong stability given in Definition 2.1. Indeed many definitions of stability are possible, and to the author knowledge, any definition of strong stability has been proposed for difference schemes for a boundary value problem in the quarter space. Remark that when one takes the limit $\Delta t \downarrow 0$ in (12) then he (formally) recovers the expected energy estimate for initial boundary value problems in the quarter space (2).

2.2 Boundary and corner conditions

In this paragraph we give several possible expressions for the boundary and corner operators B_1^{σ,j_1}, B_2^{σ,j_2} and C^{σ,j_1,j_2} appearing in (10) and we then describe the influence of these expressions on the order of determination of the U_j^{n+1} for $j \in \mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{C}$.

The first possible choice is the simplest one. It is also the one that most looks like the boundary conditions in the half space. We first define for the boundary operators:

$$\text{for } j \in \mathscr{B}_1,\ B_1^{\sigma,j_1} := \sum_{\mu_1=0}^{q_{11}}\sum_{\mu_2=0}^{q_{12}}\sum_{\mu'=-q_1'}^{q_1'} B_1^{\sigma,\mu,j_1}\mathbf{T}_1^{\mu_1}\mathbf{T}_2^{\mu_2}\mathbf{T}'^{\mu'}, \tag{13}$$

$$\text{for } j \in \mathscr{B}_2,\ B_2^{\sigma,j_2} := \sum_{\mu_1=0}^{q_{21}}\sum_{\mu_2=0}^{q_{22}}\sum_{\mu'=-q_2'}^{q_2'} B_2^{\sigma,\mu,j_2}\mathbf{T}_1^{\mu_1}\mathbf{T}_2^{\mu_2}\mathbf{T}'^{\mu'}, \tag{14}$$

where q_{11}, q_{12}, q_{21} and q_{22} are fixed positive integers, $q_1', q_2' \in \mathbb{N}^{d-2}$ and where the B_1^{σ,μ,j_1}, B_2^{σ,μ,j_2} are fixed matrices in $\mathbf{M}_{n\times n}(\mathbb{R})$. And we then define the corner operator by:

$$\text{for } j \in \mathscr{C},\ C^{\sigma,j_1,j_2} := \sum_{\mu_1=0}^{c_1}\sum_{\mu_2=0}^{c_2}\sum_{\mu'=-c'}^{c'} C^{\sigma,\mu,j_1,j_2}\mathbf{T}_1^{\mu_1}\mathbf{T}_2^{\mu_2}\mathbf{T}'^{\mu'}, \tag{15}$$

for c_1, c_2 two positive fixed integers and $c' \in \mathbb{N}^{d-2}$. The coefficients C^{σ,μ,j_1,j_2} are fixed matrices in $\mathbf{M}_{n\times n}(\mathbb{R})$.

[2]Let us remark that by definition of the CFL numbers λ_1 and λ_2 we have $\Delta x_1 \sim \Delta x_2$ and as a consequence one can equivalently use (12) with the last term in the right hand side changed by $\sum_{n\geq s+1}\Delta t\Delta x_1 e^{-2\gamma n\Delta t}\|h^n\|_{\mathscr{C}}^2$ (or even $\sum_{n\geq s+1}\Delta t^2 e^{-2\gamma n\Delta t}\|h^n\|_{\mathscr{C}}^2$) as an estimate for strongly stable finite difference schemes.

With these definitions the terms U_j^{n+1} for $j \in \mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{C}$ only depend on the U_j^n for $j \in \mathscr{R}$ and the U_j^{n+1} for $j \in \mathscr{I}$. So the U_j^{n+1} for $j \in \mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{C}$ can be determined in any order.

A second possibility is to keep (13) and (14) for the equations defining the boundary operators but to change (15) by:

$$\text{for } j \in \mathscr{C},\ C^{\sigma,j_1,j_2} := \sum_{\mu'=-c'}^{c'} \left(\sum_{\mu_1=0}^{c_1} \sum_{\mu_2=0}^{c_2} C^{\sigma,\mu,j_1,j_2} \mathbf{T}_1^{\mu_1} \mathbf{T}_2^{\mu_2} + \sum_{\mu_1=-\ell_1}^{-1} \sum_{\mu_2=0}^{c_{12}} C_{\mathscr{B}_1}^{\sigma,\mu,j_1,j_2} \mathbf{T}_1^{\mu_1} \mathbf{T}_2^{\mu_2} + \sum_{\mu_1=0}^{c_{21}} \sum_{\mu_2=-\ell_2}^{-1} C_{\mathscr{B}_2}^{\sigma,\mu,j_1,j_2} \mathbf{T}_1^{\mu_1} \mathbf{T}_2^{\mu_2} \right) \mathbf{T}'^{\mu'}, \tag{16}$$

where c_{12} and c_{21} are fixed integers and $C_{\mathscr{B}_1}^{\sigma,\mu,j_1,j_2}$, $C_{\mathscr{B}_2}^{\sigma,\mu,j_1,j_2}$ are fixed matrices in $\mathbf{M}_{n\times n}(\mathbb{R})$. With this new definition of C^{s+1,j_1,j_2}, $C^{s+1,j_1,j_2} U_{1,1}^{n+1}$ now involves some terms of the discretized boundaries $\mathscr{B}_1$ and $\mathscr{B}_2$ (more precisely the U_j^{n+1} for $j \in (\llbracket 1-\ell_1, 0\rrbracket \times \llbracket 1, 1+c_{12}\rrbracket) \cup (\llbracket 1, 1+c_{21}\rrbracket \times \llbracket 1-\ell_2, 0\rrbracket)$) and thus it is needed to determine the U_j^{n+1} for $j \in \mathscr{B}_1 \cup \mathscr{B}_2$ before to determine the U_j^{n+1} for $j \in \mathscr{C}$.

The last possibility that we will describe here is to go back to (15) for the equation defining the corner operator and to change the equations defining the boundary operators by:

$$\text{for } j_1 \in \llbracket 1-\ell_1, 0\rrbracket,\ B_1^{\sigma,j_1} := \sum_{\mu'=-q_1'}^{q_1'} \left(\sum_{\mu_1=0}^{q_{11}} \sum_{\mu_2=0}^{q_{12}} B_1^{\sigma,\mu,j_1} \mathbf{T}_1^{\mu_1} \mathbf{T}_2^{\mu_2} + \sum_{\mu_1=-\ell_1}^{-1} \sum_{\mu_2=-\ell_2}^{-1} B_{1,\mathscr{C}}^{\sigma,\mu,j_1} \mathbf{T}_1^{\mu_1} \mathbf{T}_2^{\mu_2} \right) \mathbf{T}'^{\mu'}, \tag{17}$$

$$\text{for } j_2 \in \llbracket 1-\ell_2, 0\rrbracket,\ B_2^{\sigma,j_2} := \sum_{\mu'=-q_2'}^{q_2'} \left(\sum_{\mu_1=0}^{q_{21}} \sum_{\mu_2=0}^{q_{22}} B_2^{\sigma,\mu,j_2} \mathbf{T}_1^{\mu_1} \mathbf{T}_2^{\mu_2} + + \sum_{\mu_1=-\ell_1}^{-1} \sum_{\mu_2=-\ell_2}^{-1} B_{2,\mathscr{C}}^{\sigma,\mu,j_2} \mathbf{T}_1^{\mu_1} \mathbf{T}_2^{\mu_2} \right) \mathbf{T}'^{\mu'}, \tag{18}$$

where $B_{1,\mathscr{C}}^{\sigma,\mu,j_1}, B_{2,\mathscr{C}}^{\sigma,\mu,j_2} \in \mathbf{M}_{n\times n}(\mathbb{R})$. In that case, the boundary terms depend of the corner terms and thus the boundary terms have to be computed after the corner ones.

We summarize the previous discussion in the following definition:

Definition 2.2 *We say that the finite difference scheme approximation* (10)*:*

⋄ *admits decoupled boundary and corner conditions if the boundary operators are given by* (13) *and* (14) *and if the corner operator is given by* (15)*;*

⋄ *is traces to corner if the boundary operators are given by* (13) *and* (14) *and if the corner operator is given by* (16)*;*

⋄ *is corner to traces if the boundary operators are given by* (17) *and* (18) *and if the corner operator is given by* (15).

3 Main results

In all this article we will assume that the boundaries $\mathscr{B}_1$ and $\mathscr{B}_2$ are non characteristic for the scheme (10). In view to state this assumption, let us introduce the following "tangential" operators from $\ell^2(\mathbb{Z})$ to $\ell^2(\mathbb{Z})$: for $z \in \mathbb{C}$ and

$$\text{for } \mu_1 \in \llbracket -\ell_1, r_1\rrbracket,\ \mathbb{A}_1^{\mu_1}(z) := \sum_{\sigma=0}^{s+1} z^\sigma \sum_{\mu_2=-\ell_2}^{r_2} \sum_{\mu'=-\ell'}^{r'} A^{\sigma,\mu} \mathbf{T}_2^{\mu_2} \mathbf{T}'^{\mu'}, \tag{19}$$

$$\text{for } \mu_2 \in \llbracket -\ell_2, r_2\rrbracket,\ \mathbb{A}_2^{\mu_2}(z) := \sum_{\sigma=0}^{s+1} z^\sigma \sum_{\mu_1=-\ell_1}^{r_1} \sum_{\mu'=-\ell'}^{r'} A^{\sigma,\mu} \mathbf{T}_1^{\mu_1} \mathbf{T}'^{\mu'}. \tag{20}$$

As already mentioned in the introduction, our first semi-group stability result holds for explicit with only one time step finite differences schemes (but with an arbitrary number of equations). In (10) we thus set

$s = 0$ and $Q^1 = I$ to obtain (setting also $Q^0 := Q$):

$$\begin{cases} U_j^{n+1} + QU_j^n = \Delta t f_j^{n+1}, & \text{for } j \in \mathscr{I} \times \mathbb{Z}^{d-2}, n \geq 0, \\ U_j^{n+1} + B_1^{0,j_1} U_j^n + B_1^{1,j_1} U_j^{n+1} = g_{1,j}^{n+1}, & \text{for } j \in \mathscr{B}_1 \times \mathbb{Z}^{d-2}, n \geq 0, \\ U_j^{n+1} + B_2^{0,j_2} U_j^n + B_2^{1,j_2} U_j^{n+1} = g_{2,j}^{n+1}, & \text{for } j \in \mathscr{B}_2 \times \mathbb{Z}^{d-2}, n \geq 0, \\ U_j^{n+1} + C^{0,j_1,j_2} U_j^n + C^{0,j_1,j_2} U_j^n = h_j^{n+1}, & \text{for } j \in \mathscr{C} \times \mathbb{Z}^{d-2}, n \geq 0, \\ U_j^0 = u_{0,j}, & \text{for } j \in \mathscr{R} \times \mathbb{Z}^{d-2}. \end{cases} \tag{21}$$

In this framework the operators defined in (19) and (20) becomes:

$$\mathbb{A}_1^{\mu_1}(z) := z\delta_{\mu_1=0} + \sum_{\mu_2=-\ell_2}^{r_2} \sum_{\mu'=-\ell'}^{r'} A^{0,\mu} \mathbf{T}_2^{\mu_2} \mathbf{T}'^{\mu'}, \text{ and } \mathbb{A}_2^{\mu_2}(z) := z\delta_{\mu_2=0} + \sum_{\mu_1=-\ell_1}^{r_1} \sum_{\mu'=-\ell'}^{r'} A^{0,\mu} \mathbf{T}_1^{\mu_1} \mathbf{T}'^{\mu'},$$

And, to save some notations, we also define $\mathbb{A}_k^{\mu_k} := \mathbb{A}_k^{\mu_k}(1)$. The first non characteristicity assumption then reads:

Assumption 3.1 *We assume that there exist two constants $c_1, c_2 > 0$ such that:*

$$\forall u \in \ell^2(\mathbb{Z}^d), \|\mathbb{A}_1^{r_1} u\| \geq c_1 \|u\|, \text{ and } \|\mathbb{A}_2^{r_2} u\| \geq c_1 \|u\|.$$

As in [CG11] we assume that the operator of discretization in the interior Q does not increase the ℓ^2-norm of the solution.

Assumption 3.2 *We assume that for all $u \in \ell^2(\mathbb{Z}^d)$, we have $\|Qu\| \leq \|u\|$.*

Under these assumptions the generalization of [CG11] to the corner space geometry is the following:

Theorem 3.1 *Under Assumptions 3.1 and 3.2, assume that the difference scheme approximation* (21) *is strongly stable in the sense of Definition 2.1 and finally*[3] *assume that $r_1, r_2 \geq 1$, then* (21) *is also semi-group stable. More precisely, there exists $C > 0$ such that for all $\gamma > 0$ and $\Delta t \in]0, 1]$ the solution of* (21) *satisfies the estimate:*

$$\begin{aligned} \sup_{n\geq 0} e^{-2\gamma n\Delta t} |||U^n|||_{\mathscr{R}}^2 \quad &+ \quad \frac{\gamma}{\gamma\Delta t + 1} \sum_{n\geq 0} \Delta t e^{-2\gamma n \Delta t} |||U^n|||_{\mathscr{R}}^2 + \sum_{k=1}^{2} \sum_{n\geq 0} \Delta t \Delta x_{3-k} e^{-2\gamma n\Delta t} \|U^n\|_{\mathscr{B}_k}^2 \leq \\ &C \quad \left(|||u_0|||_{\mathscr{R}}^2 + \frac{\gamma \Delta t + 1}{\gamma} \sum_{n\geq 1} \Delta t \Delta x e^{-2\gamma n\Delta t} |||f^n|||_{\mathscr{I}}^2 \right. \\ &+ \quad \left. \sum_{k=1}^{2} \sum_{n\geq 1} \Delta t \Delta x_{3-k} e^{-2\gamma n\Delta t} \|g_1^n\|_{\mathscr{B}_k}^2 + \sum_{n\geq 1} \Delta t e^{-2\gamma n\Delta t} \|h^n\|_{\mathscr{C}}^2 \right). \end{aligned} \tag{22}$$

Our second main result is the generalization of [Cou15] for corner domains. This result is thus, on the one hand, restricted to scalar equations (that is $n = 1$) but, on the other hand, it can be applied to finite difference schemes with several time steps. To stress that in this framework we are dealing with scalar equations we rewrite the coefficients defining (10), that is $Q^{\sigma,\mu}$, B_1^{σ,μ,j_1}, B_2^{σ,μ,j_2} and $C^{\sigma,\mu,j}$, and the solution (U_j^n) of (10) with lowercase letters. We thus write:

$$\begin{cases} \sum_{\sigma=0}^{s+1} Q^\sigma u_j^{n+\sigma} = \Delta t f_j^{n+s+1}, & \text{for } j \in \mathscr{I} \times \mathbb{Z}^{d-2}, n \geq 0, \\ u_j^{n+s+1} + \sum_{\sigma=0}^{s+1} B_1^{\sigma,j_1} u_j^{n+\sigma} = g_{1,j}^{n+s+1}, & \text{for } j \in \mathscr{B}_1 \times \mathbb{Z}^{d-2}, n \geq 0, \\ u_j^{n+s+1} + \sum_{\sigma=0}^{s+1} B_2^{\sigma,j_2} u_j^{n+\sigma} = g_{2,j}^{n+s+1}, & \text{for } j \in \mathscr{B}_2 \times \mathbb{Z}^{d-2}, n \geq 0, \\ u_j^{n+s+1} + \sum_{\sigma=0}^{s+1} C^{\sigma,j_1,j_2} u_j^{n+\sigma} = h_j^{n+s+1}, & \text{for } j \in \mathscr{C} \times \mathbb{Z}^{d-2}, n \geq 0, \\ u_j^n = u_{n,j}, & \text{for } j \in \mathscr{R} \times \mathbb{Z}^{d-2}, n \in [\![0, s]\!]. \end{cases} \tag{23}$$

[3] If r_1 or r_2 equals zero then Theorem 3.1 remains true. This fact is straightforward direct use of the arguments of [CG11] to treat the case $r_1 = 0$ that we will not reproduce here.

The semi-group stability result then needs extra (or just some modifications) of Assumptions 3.1 and 3.2.

The first assumption is made to ensure the solvability of (23) in the case that it defines an implicit (in time) scheme.

Assumption 3.3 *The operator Q^{s+1} appearing in (23) is an isomorphism on $\ell^2(\mathbb{Z}^d)$. Moreover, for all source terms $f_j \in \ell^2(\mathscr{I} \times \mathbb{Z}^{d-2})$, $g_{1,j} \in \ell^2(\mathscr{B}_1 \times \mathbb{Z}^{d-2})$, $g_{2,j} \in \ell^2(\mathscr{B}_2 \times \mathbb{Z}^{d-2})$ and $h \in \ell^2(\mathscr{C} \times \mathbb{Z}^{d-2})$, the finite difference scheme:*

$$\begin{cases} Q^{s+1}u_j = f_j, & \text{for } j \in \mathscr{I} \times \mathbb{Z}^{d-2}, \\ u_j + B_1^{s+1,j_1}u_j = g_{1,j}, & \text{for } j \in \mathscr{B}_1 \times \mathbb{Z}^{d-2}, \\ u_j + B_2^{s+1,j_2}u_j = g_{2,j}, & \text{for } j \in \mathscr{B}_2 \times \mathbb{Z}^{d-2}, \\ u_j + C^{s+1,j_1,j_2}u_j = h_j, & \text{for } j \in \mathscr{C} \times \mathbb{Z}^{d-2}, \end{cases}$$

admits a unique solution $(u_j) \in \ell^2(\mathscr{R} \times \mathbb{Z}^{d-2})$.

We also assume the following modifications of Assumptions 3.2 and 3.1:

Assumption 3.4 *For all $\eta := (\eta_1, \eta_2, \eta') \in \mathbb{R}^d$, the equation*

$$\sum_{\sigma=0}^{s+1} \widehat{Q^\sigma}(e^{i\eta_1}, e^{i\eta_2}, ..., e^{i\eta_d})z^\sigma = 0, \ \text{where } \widehat{Q^\sigma}(\kappa) := \sum_{k=1}^{d} \sum_{\mu_k=-\ell_k}^{r_k} a^{\sigma,\mu}\kappa_k^{\mu_k},$$

admits $s+1$ simple roots $z_0, ..., z_d$ satisfying that for all $k \in [\![0, d]\!]$, $|z_k| \leq 1$.

Let us recall that Assumption 3.4 implies that the finite difference scheme associated to the Cauchy problem is strongly stable (see ()).

Our last assumption is a modification of Assumption 3.1.

Assumption 3.5 *For $z \in \mathbb{C}$, $\eta' \in \mathbb{R}^{d-2}$; $k \in [\![1, 2]\!]$, $\mu_k \in [\![-\ell_k, r_k]\!]$ and $\eta_{3-k} \in \mathbb{R}$ we define:*

$$a_k^{\mu_k}(z, \eta', \eta_{3-k}) := \sum_{\sigma=0} z^\sigma \sum_{\mu_{3-k}=-\ell_{3-k}}^{r_{3-k}} a^{\sigma,\mu} e^{i\eta_{3-k}\mu_{3-k}} e^{i\eta'\cdot\mu'}.$$

Then $a_1^{-\ell_1}$, $a_1^{r_1}$, $a_2^{-\ell_2}$ and $a_2^{r_2}$ are nonzero on $\{z \in \mathbb{C} | \, |z| \geq 1\} \times \mathbb{R}^{d-1}$ and have nonzero degree compared with z for all (η', η_{3-k}).

Then the result generalizing [Cou15] from the half space to corner spaces is the following:

Theorem 3.2 *Under Assumptions 3.3-3.4 and 3.5. Assume that the difference scheme approximation (23) is strongly stable in the sense of Definition 2.1 then (23) is also semi-group stable. More precisely, there exists $C > 0$ such that for all $\gamma > 0$ and $\Delta t \in]0, 1]$ the solution of (10) satisfies the estimate:*

$$\begin{aligned} \sup_{n\geq 0} e^{-2\gamma n\Delta t} |\!|\!| u^n |\!|\!|_{\mathscr{R}}^2 \quad &+ \quad \frac{\gamma}{\gamma\Delta t + 1} \sum_{n\geq 0} \Delta t e^{-2\gamma n\Delta t} |\!|\!| u^n |\!|\!|_{\mathscr{R}}^2 + \sum_{k=1}^{2} \sum_{n\geq 0} \Delta t \Delta x_{3-k} e^{-2\gamma n\Delta t} \| u^n \|_{\overline{\mathscr{B}_k}}^2 \leq \\ &C \quad \left(\sum_{n=0}^{s} |\!|\!| u_n |\!|\!|_{\mathscr{R}}^2 + \frac{\gamma\Delta t + 1}{\gamma} \sum_{n\geq s+1} \Delta t e^{-2\gamma n\Delta t} |\!|\!| f^n |\!|\!|_{\mathscr{I}}^2 \right. \\ &+ \left. \sum_{k=1}^{2} \sum_{n\geq s+1} \Delta t \Delta x_{3-k} e^{-2\gamma n\Delta t} \| g_k^n \|_{\mathscr{B}_k}^2 + \sum_{n\geq s+1} \Delta t \Delta x_2 e^{-2\gamma n\Delta t} \| h^n \|_{\mathscr{C}}^2 \right). \end{aligned} \tag{24}$$

Remark The proof of Theorem 3.1 in fact needs a weaker definition of strong stability. More precisely, Theorem 3.1 remains true if one assumes that the scheme is strongly stable in the sense of Definition 2.1 with $\mathscr{B}_1$ and $\mathscr{B}_2$ instead of $\overline{\mathscr{B}_1}$ and $\overline{\mathscr{B}_2}$. However, this is not the case for the proof of Theorem 3.2.

We now turn to the proofs of Theorems 3.1 and 3.2.

4 Proof of Theorem 3.1

Following [CG11] the finite difference scheme with discrete Dirichlet conditions on each boundary and at the corner will be a suitable (in the sense that we can demonstrate that the semi-group, interior and extended traces norms of its solutions are controlable by the source terms) auxiliary problem.

4.1 Finite difference schemes with discrete Dirichlet boundary and corner conditions

Let us introduce the auxiliary finite difference scheme of (21) in which we just substitute the boundary and corner conditions by Dirichlet conditions:

$$\begin{cases} U_j^{n+1} + QU_j^n = \Delta t f_j^{n+1}, & \text{for } j \in \mathscr{I} \times \mathbb{Z}^{d-2}, n \geq 0, \\ U_j^{n+1} = g_{1,j}^{n+1}, & \text{for } j \in \mathscr{B}_1 \times \mathbb{Z}^{d-2}, n \geq 0, \\ U_j^{n+1} = g_{2,j}^{n+1}, & \text{for } j \in \mathscr{B}_2 \times \mathbb{Z}^{d-2}, n \geq 0, \\ U_j^{n+1} = h_j^{n+1}, & \text{for } j \in \mathscr{C} \times \mathbb{Z}^{d-2}, n \geq 0, \\ U_j^0 = u_{0,j}, & \text{for } j \in \mathscr{R} \times \mathbb{Z}^{d-2}. \end{cases} \tag{25}$$

The aim of this paragraph is to show that the solution of (25) satisfies the same estimate as in Theorem 3.1 equation (22).

Theorem 4.1 *Under Assumptions 3.1 and 3.2, there exists $C > 0$ such that for all $\gamma > 0$ and $\Delta t \in]0, 1]$ the solution U of (25) satisfies:*

$$\begin{aligned} \sup_{n\geq 0} e^{-2\gamma n\Delta t} |||U^n|||_{\mathscr{R}}^2 \quad &+ \quad \frac{\gamma}{\gamma\Delta t + 1} \sum_{n\geq 0} \Delta t e^{-2\gamma n\Delta t} |||U^n|||_{\mathscr{R}}^2 + \sum_{k=1}^{2} \sum_{n\geq 0} \Delta t \Delta x_{3-k} e^{-2\gamma n\Delta t} \|U^n\|_{\mathscr{B}_k}^2 \\ &\leq \quad C \quad \Bigg(|||u_0|||_{\mathscr{R}}^2 + \frac{\gamma\Delta t + 1}{\gamma} \sum_{n\geq 1} \Delta t e^{-2\gamma n\Delta t} |||f^n|||_{\mathscr{I}}^2 \\ &+ \quad \sum_{k=1}^{2} \sum_{n\geq 1} \Delta t \Delta x_{3-k} e^{-2\gamma n\Delta t} \|g_k^n\|_{\mathscr{B}_k}^2 + \sum_{n\geq 1} \Delta t \Delta x_2 e^{-2\gamma n\Delta t} \|h^n\|_{\mathscr{C}}^2 \Bigg). \end{aligned} \tag{26}$$

The proof of Theorem 4.1 is splitted in two parts. In a first time we show that (26) holds for homogeneous schemes in the interior and in a second time, we show that (26) holds for inhomogeneous schemes in the interior but homogeneous for all the others conditions.

Remark Theorem 4.1 is a key step in the proof of Theorem 3.1. However this result is also interesting for itself. Indeed, it shows that as far as the semi-group stability is concerned, Dirichlet boundary conditions wich are (with Neumann boundary conditions) the simplest ones to use are suitable. This fact is interesting because Dirichlet boundary conditions can lead to severe consistency issues for the finite scheme approximation (21) for example when they are imposed while there exists an outgoing modes (that is a mode which transports the information from the interior of the domain to the boundaries or to the corner).

4.1.1 Homogeneous schemes in the interior

In this paragraph we show that the solution of the scheme (25) with homogeneous source term in the interior (that is $f_j^n \equiv 0$) admits a suitable estimate to show that (25) is strongly stable. More precisely we will show the following lemma:

Lemma 4.1 *Under Assumptions 3.1 and 3.2 there exists $C > 0$ such that for all $\gamma > 0$ and $\Delta t \in]0,1]$ the solution of* (25) *(with $f_j^n \equiv 0$) satisfies the estimate:*

$$\sup_{n\geq 0} e^{-2\gamma n\Delta t} |||U^n|||_{\mathscr{R}}^2 + \frac{\gamma}{\gamma\Delta t+1}\sum_{n\geq 0}\Delta t e^{-2\gamma n\Delta t}|||U^n|||_{\mathscr{R}}^2 + \sum_{k=1}^{2}\sum_{n\geq 0}\Delta t\Delta x_{3-k}e^{-2\gamma n\Delta t}\|U^n\|_{\mathscr{B}_k}^2 \tag{27}$$
$$\leq C\left(|||u_0|||_{\mathscr{R}}^2 + \sum_{k=1}^{2}\sum_{n\geq 1}\Delta t\Delta x_{3-k}e^{-2\gamma n\Delta t}\|g_{3-k}^n\|_{\mathscr{B}_k}^2 + \sum_{n\geq 1}\Delta t\Delta x_2 e^{-2\gamma n\Delta t}\|h^n\|_{\mathscr{C}}^2\right).$$

Proof : Following [CG11], we introduce $Q := -(I + \widetilde{Q})$. Note that from the definition of Q (see (11)) we have:

$$\widetilde{Q} = -\sum_{\mu_1=-\ell_1}^{r_1}\mathbb{A}_1^{\mu_1}\mathbf{T}_1^{\mu_1} = -\sum_{\mu_2=-\ell_2}^{r_2}\mathbb{A}_2^{\mu_2}\mathbf{T}_2^{\mu_2}. \tag{28}$$

From Assumption 3.2 we deduce that:

$$\|\widetilde{Q}U^n\|_{\mathbb{Z}^2}^2 + 2\left\langle \widetilde{Q}U^n, U^n\right\rangle_{\mathbb{Z}^2} \leq 0. \tag{29}$$

We then introduce (W_j^n) the extension of (U_j^n) by zero for $j_1 \leq -\ell_1$ or $j_2 \leq -\ell_2$. In particular in view of (28) we have $\widetilde{Q}W_j^n = \widetilde{Q}U_j^n$ on $\mathscr{I}$ and $\widetilde{Q}W_j^n = 0$ for $j_1 \leq -r_1-\ell_1$ or $j_2 \leq -r_2-\ell_2$. We also define the following subsets of $\mathbb{Z}^2$:

$$\begin{aligned} \mathscr{E}_{\mathscr{C}} &:= \left\{j\in\mathbb{Z}^2 | j_1 \in [\![1-\ell_1-r_1, -\ell_1]\!], j_2\in[\![1-\ell_2-r_2,-\ell_2]\!]\right\}, \\ \mathscr{E}_{\mathscr{B}_1} &:= \left\{j\in\mathbb{Z}^2 | j_1 \in [\![1-\ell_1-r_1, -\ell_1]\!], j_2\in[\![1-\ell_2,\infty[\![\right\}, \\ \mathscr{E}_{\mathscr{B}_2} &:= \left\{j\in\mathbb{Z}^2 | j_1 \in [\![1-\ell_1, \infty[\![, j_2\in[\![1-\ell_2-r_2,-\ell_2]\!]\right\}. \end{aligned} \tag{30}$$

Some computations give:

$$\begin{aligned} \|\widetilde{Q}W^n\|_{\mathbb{Z}^2}^2 &= \|\widetilde{Q}W^n\|_{j_1\geq 1-r_1-\ell_1 \text{ and } j_2\geq 1-r_2-\ell_2}^2 \\ &= \|\widetilde{Q}W^n\|_{\mathscr{E}_{\mathscr{C}}}^2 + \|\widetilde{Q}W^n\|_{\mathscr{C}}^2 + \|\widetilde{Q}W^n\|_{\mathscr{E}_{\mathscr{B}_1}}^2 + \|\widetilde{Q}W^n\|_{\mathscr{B}_1}^2 + \|\widetilde{Q}W^n\|_{\mathscr{E}_{\mathscr{B}_2}}^2 + \|\widetilde{Q}W^n\|_{\mathscr{B}_2}^2 + \|\widetilde{Q}U^n\|_{\mathscr{I}}^2, \end{aligned} \tag{31}$$

and

$$\begin{aligned} \left\langle \widetilde{Q}W^n, W^n\right\rangle_{\mathbb{Z}^2} &= \left\langle \widetilde{Q}W^n, W^n\right\rangle_{j_1\geq 1-\ell_1 \text{ and } j_2\geq 1-\ell_2} \\ &= \left\langle \widetilde{Q}W^n, W^n\right\rangle_{\mathscr{C}} + \left\langle \widetilde{Q}W^n, W^n\right\rangle_{\mathscr{B}_1} + \left\langle \widetilde{Q}W^n, W^n\right\rangle_{\mathscr{B}_2} + \left\langle \widetilde{Q}U^n, U^n\right\rangle_{\mathscr{I}}. \end{aligned}$$

Then for $\mathfrak{I} \in \{\mathscr{C}, \mathscr{B}_1, \mathscr{B}_2, \mathscr{I}\}$ we write:

$$\|\widetilde{Q}W^n\|_{\mathfrak{I}}^2 + 2\left\langle \widetilde{Q}W^n, W^n\right\rangle_{\mathfrak{I}} = \|\widetilde{Q}W^n + W^n\|_{\mathfrak{I}}^2 - \|U^n\|_{\mathfrak{I}}^2.$$

In particular, for $\mathfrak{I} = \mathscr{I}$, using the definition of the scheme (25), we can simplify the previous expression in the following way:

$$\|\widetilde{Q}U^n\|_{\mathscr{I}}^2 + 2\left\langle \widetilde{Q}U^n, U^n\right\rangle_{\mathscr{I}} = \|QU^n\|_{\mathscr{I}}^2 - \|U^n\|_{\mathscr{I}}^2 = \|U^{n+1}\|_{\mathscr{I}}^2 - \|U^n\|_{\mathscr{I}}^2.$$

This term will permit to obtain the supremum in time in the estimate of Lemma 4.1.

From equations (29),(31),(30) and the two previous relations we obtain, thanks to the fact that we have Dirichlet boundary and corner conditions, that:

$$\|U^{n+1}\|_{\mathscr{I}}^2 - \|U^n\|_{\mathscr{I}}^2 + \sum_{\mathfrak{I}\in\{\mathscr{C},\mathscr{B}_1,\mathscr{B}_2\}} \|\widetilde{Q}W^n\|_{\mathscr{E}_{\mathfrak{I}}}^2 + \|\widetilde{Q}W^n + W^n\|_{\mathfrak{I}}^2 \leq \|g_1^n\|_{\mathscr{B}_1}^2 + \|g_2^n\|_{\mathscr{B}_2}^2 + \|h^n\|_{\mathscr{C}}^2.$$

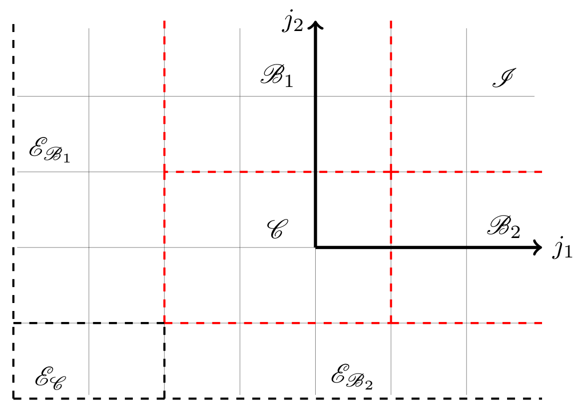

Figure 2: Error terms induced by the extension by zero for $\ell_1 = 3$, $\ell_2 = 2$, $r_1 = 2$ and $r_2 = 1$.

In particular we have:

$$\begin{cases} \|U^{n+1}\|^2_{\mathscr{I}} - \|U^n\|^2_{\mathscr{I}} + \|\widetilde{Q}W^n\|^2_{\mathscr{E}_{\mathscr{B}_1}} + \|\widetilde{Q}W^n + W^n\|^2_{\mathscr{B}_1} + \|\widetilde{Q}W^n + W^n\|^2_{\mathscr{C}} \leq \|g_1^n\|^2_{\mathscr{B}_1} + \|g_2^n\|^2_{\mathscr{B}_2} + \|h^n\|^2_{\mathscr{C}}, \\ \|U^{n+1}\|^2_{\mathscr{I}} - \|U^n\|^2_{\mathscr{I}} + \|\widetilde{Q}W^n\|^2_{\mathscr{E}_{\mathscr{B}_2}} + \|\widetilde{Q}W^n + W^n\|^2_{\mathscr{B}_2} + \|\widetilde{Q}W^n + W^n\|^2_{\mathscr{C}} \leq \|g_1^n\|^2_{\mathscr{B}_1} + \|g_2^n\|^2_{\mathscr{B}_2} + \|h^n\|^2_{\mathscr{C}}. \end{cases} \tag{32}$$

The following lemma is a straightforward generalization of [[CG11]- Lemma 2.2] from finite to infinite dimension. It is however the keystone of the analysis.

Lemma 4.2 *There exist two constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$ we have:*

$$\begin{aligned} \|\widetilde{Q}W^n\|^2_{\mathscr{E}_{\mathscr{B}_1}} + \|\widetilde{Q}W^n + W^n\|^2_{\mathscr{B}_1} + \|\widetilde{Q}W^n + W^n\|^2_{\mathscr{C}} &\geq c_1 \|W^n\|^2_{\overline{\mathscr{B}_1}}, \\ \|\widetilde{Q}W^n\|^2_{\mathscr{E}_{\mathscr{B}_2}} + \|\widetilde{Q}W^n + W^n\|^2_{\mathscr{B}_2} + \|\widetilde{Q}W^n + W^n\|^2_{\mathscr{C}} &\geq c_2 \|W^n\|^2_{\overline{\mathscr{B}_2}}. \end{aligned}$$

The proof of this Lemma is given in paragraph 4.4.

We apply Lemma 4.2 to (32), by definition of (W_j^n) we obtain:

$$\|U^{n+1}\|^2_{\mathscr{I}} - \|U^n\|^2_{\mathscr{I}} + \|U^n\|^2_{\overline{\mathscr{B}_1}} + \|U^n\|^2_{\overline{\mathscr{B}_2}} \leq C \left(\|g_1^n\|^2_{\mathscr{B}_1} + \|g_2^n\|^2_{\mathscr{B}_2} + \|h^n\|^2_{\mathscr{C}} \right),$$

we multiply this equation by $\Delta x e^{-2\gamma n \Delta t}$ and sum for $n \in [\![0, N]\!]$. It follows, from the definition of the CFL numbers that:

$$\begin{aligned} e^{-2\gamma N \Delta t} |\!|\!| U^{N+1} |\!|\!|^2_{\mathscr{I}} \quad &+ \quad \frac{e^{2\gamma \Delta t} - 1}{\Delta t} \sum_{n=1}^{N} \Delta t e^{-2\gamma n \Delta t} |\!|\!| U^n |\!|\!|^2_{\mathscr{I}} + \sum_{k=1}^{2} \frac{1}{\lambda_k} \sum_{n=0}^{N} \Delta t \Delta x_{3-k} \|U^n\|^2_{\overline{\mathscr{B}_k}} \\ &\leq \quad C \left(|\!|\!| u_0 |\!|\!|^2_{\mathscr{R}} + \sum_{k=1}^{2} \frac{1}{\lambda_k} \sum_{n \geq 0} \Delta t \Delta x_{3-k} e^{-2\gamma n \Delta t} \|g_k^n\|^2_{\mathscr{B}_k} + \sum_{n \geq 0} \Delta t \Delta x_2 e^{-2\gamma n \Delta t} \|h^n\|^2_{\mathscr{C}} \right). \end{aligned}$$

We then take the supremum in N in the previous equation to obtain that for $\Delta t \leq 1$:

$$\begin{aligned} & e^{2\gamma \Delta t} \sup_{n \geq 1} e^{-2\gamma n \Delta t} |\!|\!| U^n |\!|\!|^2_{\mathscr{I}} + \gamma \sum_{n \geq 1} \Delta t e^{-2\gamma n \Delta t} |\!|\!| U^n |\!|\!|^2_{\mathscr{I}} + \sum_{k=1}^{2} \sum_{n \geq 0} \Delta t \Delta x_{3-k} \|U^n\|^2_{\overline{\mathscr{B}_k}} \\ & \leq C \left(|\!|\!| u_0 |\!|\!|^2_{\mathscr{R}} + \sum_{k=1}^{2} \sum_{n \geq 0} \Delta t \Delta x_{3-k} e^{-2\gamma n \Delta t} \|g_k^n\|^2_{\mathscr{B}_k} + \sum_{n \geq 0} \Delta t \Delta x_2 e^{-2\gamma n \Delta t} \|h^n\|^2_{\mathscr{C}} \right). \end{aligned} \tag{33}$$

Then we remark that $\frac{\gamma}{\gamma\Delta t+1}\leq\gamma$, $\frac{\gamma\Delta t}{\gamma\Delta t+1}\leq 1$ and use the initial condition to obtain:

$$\begin{aligned}\sup_{n\geq 0} e^{-2\gamma n\Delta t}|||U^n|||_{\mathscr{I}}^2 &+ \frac{\gamma}{\gamma\Delta t+1}\sum_{n\geq 0}\Delta t e^{-2\gamma n\Delta t}|||U^n|||_{\mathscr{I}}^2+\sum_{k=1}^{2}\sum_{n\geq 0}\Delta t\Delta x_{3-k}\|U^n\|_{\overline{\mathscr{B}_k}}^2 \\ &\leq C\left(|||u_0|||_{\mathscr{R}}^2+\sum_{k=1}^{2}\left(\Delta t\Delta x_{3-k}\|g_k^0\|_{\mathscr{B}_k}^2+\sum_{n\geq 1}\Delta t\Delta x_{3-k}e^{-2\gamma n\Delta t}\|g_k^n\|_{\mathscr{B}_k}^2\right)\right. \\ &+ \left.\Delta t\Delta x_2\|h^0\|_{\mathscr{C}}^2+\sum_{n\geq 1}\Delta t\Delta x_2 e^{-2\gamma n\Delta t}\|h^n\|_{\mathscr{C}}^2\right).\end{aligned} \tag{34}$$

At last using the fact that we have Dirichlet boundary conditions we can bound the right hand side by:

$$\begin{aligned}\sup_{n\geq 0} e^{-2\gamma n\Delta t}\Delta t|||U^n|||_{\mathscr{I}}^2+\frac{\gamma}{\gamma\Delta t+1}\sum_{n\geq 0}\Delta t e^{-2\gamma n\Delta t}|||U^n|||_{\mathscr{I}}^2+\sum_{k=1}^{2}\sum_{n\geq 0}\Delta t\Delta x_{3-k}\|U^n\|_{\overline{\mathscr{B}_k}}^2 \\ \leq C\,().\end{aligned} \tag{35}$$

To conclude the proof of Lemma 4.1 we add in (35) the term:

$$\sum_{\mathfrak{J}\in\{\mathscr{C},\mathscr{B}_1,\mathscr{B}_2\}}\sup_{n\geq 0} e^{-2\gamma n\Delta t}|||U^n|||_{\mathfrak{J}}^2+\frac{\gamma\Delta t}{\gamma\Delta t+1}\sum_{n\geq 0}e^{-2\gamma n\Delta t}|||U^n|||_{\mathfrak{J}}^2$$

so that the left hand side of (33) is greater (because $\frac{\gamma\Delta t}{\gamma\Delta t+1}<1$) than the left hand side of (27). Then we remark, thanks to the fact that we have Dirichlet conditions at the boundaries and the corner, that:

$$|||U^n|||_{\mathscr{C}}^2=\Delta x_1\Delta x_2\sum_{j\in\mathscr{C}}|U_j^n|^2=\frac{1}{\lambda_1}\sum_{j\in\mathscr{C}}\Delta t\Delta x_2|h_j^n|^2,$$
$$|||U^n|||_{\mathscr{B}_1}^2=\frac{1}{\lambda_1}\sum_{j\in\mathscr{C}}\Delta t\Delta x_2|g_{1,j}^n|^2, \text{ and } |||U^n|||_{\mathscr{B}_2}^2=\frac{1}{\lambda_2}\sum_{j\in\mathscr{C}}\Delta t\Delta x_1|g_{2,j}^n|^2,$$

which are exactly the weighted norms of the source terms appearing in the right hand side of (27). We then conclude by using the injection $\ell^1\subset\ell^\infty$.

□

4.1.2 Homogeneous Dirichlet conditions

We now turn to schemes which only have a nonzero source term in the interior. The result is the following:

Lemma 4.3 *Under Assumptions 3.1 and 3.2 there exists $C>0$ such that for all $\gamma>0$ and $\Delta t\in]0,1]$ the solution of* (10) *(with $g_1^n, g_2^n, h^n, u_0\equiv 0$) satisfies the estimate:*

$$\begin{aligned}\sup_{n\geq 0} e^{-2\gamma n\Delta t}\|U_j^n\|_{\mathscr{R}}^2+\frac{\gamma}{\gamma\Delta t+1}\sum_{n\geq 0}\Delta t e^{-2\gamma n\Delta t}|||U^n|||_{\mathscr{R}}^2+\sum_{k=1}^{2}\sum_{n\geq 0}\Delta t\Delta x_{3-k}e^{-2\gamma n\Delta t}\|U_j^n\|_{\overline{\mathscr{B}_k}}^2 \\ \leq C\frac{\gamma\Delta t+1}{\gamma}\sum_{n\geq 1}\Delta t e^{-2\gamma n\Delta t}|||f^n|||_{\mathscr{I}}^2\end{aligned}$$

Proof : First we compute:

$$\begin{aligned}\|U^{n+1}\|_{\mathscr{I}}^2-\|U^n\|_{\mathscr{I}}^2 &:= \|(I+\widetilde{Q})U^n+\Delta t f^n\|_{\mathscr{I}}^2-\|U^n\|_{\mathscr{I}}^2, \\ &= \|\widetilde{Q}U^n\|_{\mathscr{I}}^2+2\left\langle\widetilde{Q}U^n,U^n\right\rangle_{\mathscr{I}}+2\Delta t\left\langle QU^n,f^n\right\rangle_{\mathscr{I}}+\Delta t^2\|f^n\|_{\mathscr{I}}^2.\end{aligned}$$

Proceeding as in the proof of Lemma 4.1, we define (W_j^n) the extension of (U_j^n) by zero for $j_1 \leq -\ell_1$ or $j_2 \leq -\ell_2$. If r_1 and r_2 are nonzero then 4.2 holds and we can repeat exactly the same computations as those made in the proof of Lemma 4.1. This leads us to the following inequality (where we strongly used the fact that the conditions at the boundaries and at the corner are homogeneous Dirichlet):

$$\|U^{n+1}\|_{\mathscr{I}}^2 - \|U^n\|_{\mathscr{I}}^2 + \|U^n\|_{\mathscr{B}_1}^2 + \|U^n\|_{\mathscr{B}_2}^2 \leq C\left(\Delta t\,\langle QU^n, f^n\rangle_{\mathscr{I}} + \Delta t^2\|f^n\|_{\mathscr{I}}^2\right).$$

As in the proof of Lemma 4.1, we multiply the previous inequality by $\Delta x e^{-2\gamma n\Delta t}$ and sum over $n \in [\![0, N]\!]$. This gives from the definition of the CFL numbers and by Cauchy-Schwartz inequality:

$$\begin{aligned} e^{-2\gamma N\Delta t}|||U^{N+1}|||_{\mathscr{I}}^2 \quad &+ \quad \frac{e^{2\gamma\Delta t}-1}{\Delta t}\sum_{n=1}^{N}\Delta t e^{-2\gamma n\Delta t}|||U^n|||_{\mathscr{I}}^2 + \sum_{k=1}^{2}\sum_{n=0}^{N}\Delta t\Delta x_{3-k}e^{-2\gamma n\Delta t}\|U^n\|_{\mathscr{B}_k}^2 \\ &\leq \quad C\left(\Delta t\sum_{n=0}^{N}\Delta t e^{-2\gamma n\Delta t}|||f^n|||_{\mathscr{I}}^2 + \sum_{n=0}^{N}e^{-2\gamma n\Delta t}\Delta t\|\sqrt{\Delta x}U^n\|_{\mathscr{I}}\|\sqrt{\Delta x}f^n\|_{\mathscr{I}}\right). \end{aligned}$$

To conclude we use Young's inequality (with parameter $\frac{e^{2\gamma\Delta t}-1}{2\Delta t}$) in the last term of the right hand side and it follows that:

$$\begin{aligned} e^{-2\gamma N\Delta t}|||U^{N+1}|||_{\mathscr{I}}^2 \quad &+ \quad \frac{e^{2\gamma\Delta t}-1}{2\Delta t}\sum_{n=1}^{N}\Delta t e^{-2\gamma n\Delta t}|||U^n|||_{\mathscr{I}}^2 + \sum_{k=1}^{2}\sum_{n=0}^{N}\Delta t\Delta x_{3-k}e^{-2\gamma n\Delta t}\|U^n\|_{\mathscr{B}_k}^2 \\ &\leq \quad C\frac{2\Delta t}{e^{2\gamma\Delta t}+1}\sum_{n=0}^{N}e^{-2\gamma(n+1)\Delta t}\Delta t|||f^n|||_{\mathscr{I}}^2. \end{aligned}$$

The result is obtained from the inequality $\frac{e^{2\gamma\Delta t}-1}{2\Delta t} \leq \gamma \leq \frac{\gamma}{\gamma\Delta t+1}$ and by taking the supremum in N (recall once again that the initial condition is zero).

4.2 Reinforcement of traces estimates

The solution of the auxiliary problem (25), (U_j^n), and more especifically its traces will act like an error term in the final error analysis in the proof of Theorem 3.1. More precisely we will have to control the terms $B_1U_{1,j_2,j'}^{\cdot}$, $B_2U_{j_1,1,j'}^{\cdot}$ and finally $CU_{1,1,j'}^{\cdot}$ which may involve more terms than those controled by (26). It is typically the case if some of the parameters (for example q_{11} or q_{22}) are larger than r_1 and r_2. As a consequence to perform the final error analysis, we need to obtain a better control of the traces in (26). The result is the following:

Theorem 4.2 *Under Assumptions 3.1 and 3.2, let P_1, P_2 be two fixed integers, then there exists $C > 0$ such that for all $\gamma > 0$ the solution (U_j^n) of (25) satisfies:*

$$\begin{aligned} \sup_{n\geq 0} e^{-2\gamma n\Delta t}|||U^n|||_{\mathscr{R}}^2 \quad &+ \quad \frac{\gamma}{\gamma\Delta t+1}\sum_{n\geq 0}\Delta t e^{-2\gamma n\Delta t}|||U^n|||_{\mathscr{R}}^2 + \sum_{k=1}^{2}\sum_{n\geq 0}\Delta t\Delta x_{3-k}e^{-2\gamma n\Delta t}\sum_{j_k=1-\ell_k}^{P_k}\|U_{j_k,\cdot}^n\|_{\ell^2(\mathbb{Z}^{d-1})}^2 \\ &\leq \quad C\left(|||u_0|||_{\mathscr{R}}^2 + \frac{\gamma\Delta t+1}{\gamma}\sum_{n\geq 1}\Delta t e^{-2\gamma(n+1)\Delta t}|||f^n|||_{\mathscr{I}}^2\right. \\ &+ \quad \left.\sum_{k=1}^{2}\sum_{n\geq 1}\Delta t\Delta x_{3-k}e^{-2\gamma n\Delta t}\|g_k^n\|_{\mathscr{B}_k}^2 + \sum_{n\geq 1}\Delta t\Delta x_2 e^{-2\gamma n\Delta t}\|h^n\|_{\mathscr{C}}^2\right). \end{aligned} \tag{36}$$

The proof follows, once again, the proof given in [CG11] up to some little difficulties induced by the corner geometry.

Proof : Let us define the shifted sequence $V_j^n := U_{j_1+1,j_2,j'}^n$. Then, for all $n \geq 0$, (V_j^n) solves the equation:

$$\begin{cases} V_j^{n+1} + QV_j^n = \Delta t f_{j_1+1,j_2,j'}^{n+1} & , j \in \mathscr{I}, \\ V_j^{n+1} = g_{2,j_1+1,j_2,j'}^{n+1} & , j_1 \in [\![1,\infty[\![\text{ and } (j_2,j') \in [\![1-\ell_2,0]\!] \times \mathbb{Z}^{d-2}, \\ V_j^{n+1} = g_{1,j_1+1,j_2,j'}^{n+1} & , j_1 \in [\![1-\ell_1,-1]\!] \text{ and } (j_2,j') \in [\![1,\infty[\![\times \mathbb{Z}^{d-2}, \\ V_j^{n+1} = h_{1,j_1+1,j_2,j'}^{n+1} & , j_1 \in [\![1-\ell_1,-1]\!] \text{ and } (j_2,j') \in [\![1-\ell_2,0]\!] \times \mathbb{Z}^{d-2}, \\ V_j^{n+1} = U_{1,j_2,j'}^{n+1}, & , j_1 = 0 \text{ and } (j_2,j') \in [\![1,\infty[\![\times \mathbb{Z}^{d-2}, \\ V_j^{n+1} = g_{2,1,j_2,j'}^{n+1} & , j_1 = 0 \text{ and } (j_2,j') \in [\![1-\ell_2,0]\!] \times \mathbb{Z}^{d-2}, \\ V_j^0 = u_{0,j_1+1,j_2,j'} & , j \in \mathscr{R}. \end{cases}$$

So we can apply Theorem 4.1 to (V_j^n) to obtain the estimate[4]:

$$\begin{aligned} \Delta t \Delta x_2 e^{-2\gamma n \Delta t} \|V_{r_1,\cdot}^n\|_{\ell^2_{j_2,j'}(\mathbb{Z}^{d-2})}^2 \leq \; C & \Bigg(|\!|\!| u_0 |\!|\!|_{\mathscr{R}}^2 + \frac{\gamma \Delta t + 1}{\gamma} \sum_{n \geq 1} \Delta t e^{-2\gamma(n+1)\Delta t} |\!|\!| f^n |\!|\!|_{\mathscr{I}}^2 \\ & + \sum_{k=1}^{2} \sum_{n \geq 1} \Delta t \Delta x_{3-k} e^{-2\gamma n \Delta t} \|g_k^n\|_{\mathscr{B}_k}^2 \\ & + \sum_{n \geq 1} \Delta t \Delta x_2 e^{-2\gamma n \Delta t} \|h^n\|_{\mathscr{C}}^2 + \sum_{n \geq 1} \Delta t \Delta x_2 \sum_{j_2=1}^{\infty} \|U_{1,j_2,\cdot}^n\|_{\ell^2(\mathbb{Z}^{d-3}}^2 \Bigg). \end{aligned} \tag{37}$$

We then apply again Theorem 4.1 to estimate the last term in the right hand side of (37) (this is effectively possible because $r_1 \geq 1$) and we obtain the following control of $U_{r_1+1,\cdot}^n$:

$$\begin{aligned} \Delta t \Delta x_2 e^{-2\gamma n \Delta t} \|U_{r_1+1,\cdot}^n\|_{\ell^2_{j_2,j'}(\mathbb{Z}^{d-2})}^2 \leq \; C & \Bigg(|\!|\!| u_0 |\!|\!|_{\mathscr{R}}^2 + \frac{\gamma \Delta t + 1}{\gamma} \sum_{n \geq 1} \Delta t e^{-2\gamma(n+1)\Delta t} |\!|\!| f^n |\!|\!|_{\mathscr{I}}^2 \\ & + \sum_{k=1}^{2} \sum_{n \geq 1} \Delta t \Delta x_{3-k} e^{-2\gamma n \Delta t} \|g_k^n\|_{\mathscr{B}_k}^2 + \sum_{n \geq 1} \Delta t \Delta x_2 e^{-2\gamma n \Delta t} \|h^n\|_{\mathscr{C}}^2 \Bigg). \end{aligned} \tag{38}$$

We can then repeat exactly the same reasoning to the sequence $\widetilde{V}_j^n := V_{j_1+1,j_2,j'}^n$ to obtain the analogous of (39) but for $U_{r_1+2,\cdot}^n$. By induction we can then show that for all $P_1 \geq r_1 + 1$ (note that the case $P_1 \leq r_1$ is already included in Theorem 4.1). We thus obtain that:

$$\begin{aligned} \Delta t \Delta x_2 e^{-2\gamma n \Delta t} \sum_{j_1=r_1+1}^{P_1} \|U_{j_1,\cdot}^n\|_{\ell^2_{j_2,j'}(\mathbb{Z}^{d-2})}^2 \leq \; C & \Bigg(|\!|\!| u_0 |\!|\!|_{\mathscr{R}}^2 + \frac{\gamma \Delta t + 1}{\gamma} \sum_{n \geq 1} \Delta t e^{-2\gamma(n+1)\Delta t} |\!|\!| f^n |\!|\!|_{\mathscr{I}}^2 \\ & + \sum_{k=1}^{2} \sum_{n \geq 1} \Delta t \Delta x_{3-k} e^{-2\gamma n \Delta t} \|g_k^n\|_{\mathscr{B}_k}^2 + \sum_{n \geq 1} \Delta t \Delta x_2 e^{-2\gamma n \Delta t} \|h^n\|_{\mathscr{C}}^2 \Bigg), \end{aligned} \tag{39}$$

which combined with Theorem 4.1 give the desired version of (36) for $P_2 \leq r_2$.

To obtain (36) for arbitrary P_2 it is sufficient to reiterate the same arguments but with the shifted sequence $W_j^n := U_{j_1,j_2+1,j'}^n$ and this completes the proof of Theorem 4.2.

□

[4]Note that the term in the right hand side is not sharp because of the shift but it will be sufficient for our discussion.

4.3 End of the proof by error estimate

With Theorem 4.2 in hand the proof of Theorem 3.1 is just an error analysis. More precisely, let (U_j^n) be the solution of (10), we decompose $U_j^n := V_j^n + W_j^n$ where (V_j^n) and (W_j^n) solve respectively:

$$\begin{cases} V_j^{n+1} + QV_j^n = \Delta t f_j^{n+1} & , \text{ for } n \geq 0, j \in \mathscr{I} \times \mathbb{Z}^{d-2}, \\ V_j^{n+1} = g_{1,j}^{n+1} & , \text{ for } n \geq 0, j \in \mathscr{B}_1 \times \mathbb{Z}^{d-2}, \\ V_j^{n+1} = g_{2,j}^{n+1} & , \text{ for } n \geq 0, j \in \mathscr{B}_2 \times \mathbb{Z}^{d-2}, \\ V_j^{n+1} = h_j^{n+1} & , \text{ for } n \geq 0, j \in \mathscr{C} \times \mathbb{Z}^{d-2}, \\ V_j^0 = u_{0,j} & , \text{ for } j \in \mathscr{R} \times \mathbb{Z}^{d-2}, \end{cases}$$

and

$$\begin{cases} W_j^{n+1} + QW_j^n = 0 & , \text{ for } n \geq 0, j \in \mathscr{I} \times \mathbb{Z}^{d-2}, \\ W_j^{n+1} + \sum_{\sigma=0}^1 B_1^{\sigma,j_1} W_{1,j_2}^{n+\sigma} = \widetilde{g}_{1,j}^{n+1} & , \text{ for } n \geq 0, j \in \mathscr{B}_1 \times \mathbb{Z}^{d-2}, \\ W_j^{n+1} + \sum_{\sigma=0}^1 B_2^{\sigma,j_2} W_{j_1,1}^{n+\sigma} = \widetilde{g}_{2,j}^{n+1} & , \text{ for } n \geq 0, j \in \mathscr{B}_2 \times \mathbb{Z}^{d-2}, \\ W_j^{n+1} + \sum_{\sigma=0}^1 C^{\sigma,j_1,j_2} W_{1,1}^{n+\sigma} = \widetilde{h}_j^{n+1} & , \text{ for } n \geq 0, j \in \mathscr{C} \times \mathbb{Z}^{d-2}, \\ W_j^0 = 0 & , \text{ for } j \in \mathscr{R} \times \mathbb{Z}^{d-2}, \end{cases}$$

where the sequences $\widetilde{g}_1$, $\widetilde{g}_2$ and $\widetilde{h}$ are defined by: for all $n \leq 0$,

$$\forall j \in \mathscr{B}_1 \times \mathbb{Z}^{d-2},\ \widetilde{g}_{1,j}^n := -\sum_{\sigma=0}^1 B_1^{\sigma,j_1} V_{1,j_2,j'}^{n+\sigma}, \tag{40}$$

$$\forall j \in \mathscr{B}_2 \times \mathbb{Z}^{d-2},\ \widetilde{g}_{2,j}^n := -\sum_{\sigma=0}^1 B_2^{\sigma,j_2} V_{j_1,1,j'}^{n+\sigma},$$

$$\text{and } \forall j \in \mathscr{C} \times \mathbb{Z}^{d-2},\ \widetilde{h}_j^n := -\sum_{\sigma=0}^1 C^{\sigma,j_1,j_2} V_{1,1,j'}^{n+\sigma}.$$

By construction, (V_j^n) satisfies the estimate (36) so we only have to estimate (W_j^n). We use the fact that (21) is assumed to be strongly stable is the sense of Definition 2.1 to obtain the estimate:

$$\frac{\gamma}{\gamma\Delta t + 1} \sum_{n\geq 0} \Delta t e^{-2\gamma n\Delta t} |||W^n|||_{\mathscr{R}}^2 + \sum_{k=1}^2 \sum_{n\geq 0} \Delta t \Delta x_{3-k} e^{-2\gamma n \Delta t} \|W^n\|_{\mathscr{B}_k}^2 \leq \tag{41}$$

$$C \left(\sum_{k=1}^2 \sum_{n\geq 1} \Delta t \Delta x_{3-k} e^{-2\gamma n\Delta t} \|\widetilde{g}_k^n\|_{\mathscr{B}_k}^2 + \sum_{n\geq 1} \Delta t \Delta x_2 e^{-2\gamma n \Delta t} \|\widetilde{h}^n\|_{\mathscr{C}}^2 \right).$$

We now turn to the estimate of the right hand side of (41), thanks to the reinforced traces estimate for (V_j^n), that is to say (36). From (40) we have to estimate $|B_1^{\sigma,j_1} V_{1,j_2}^{n+\sigma}|^2$, $|B_2^{\sigma,j_2} V_{j_1,1}^{n+\sigma}|^2$ and $|C^{\sigma,j_1,j_2} V_{1,1}^{n+\sigma}|^2$ and we have to distinguish three cases depending of the definition of the operators B_1^{σ,j_1}, B_2^{σ,j_2} and C^{σ,j_1,j_2} (see paragraph 2.2):

$\diamond$ (10) admits decoupled boundary and corner conditions:
In this framework, independently of σ, we have that:

- for $j \in \mathscr{B}_1$, $B_1^{\sigma,j_1} V_{1,j_2}^{n+\sigma}$ involves the $|V_{\widetilde{j}}^n|^2$ for $\widetilde{j}_1 \in [\![1, 1+q_{11}]\!]$, $\widetilde{j}_2 \in [\![j_2, j_2+q_{12}]\!]$;

  for $j \in \mathscr{B}_2$, $B_1^{\sigma,j_2} V_{j_1,1}^{n+\sigma}$ involves the $|V_{\widetilde{j}}^n|^2$ for $\widetilde{j}_1 \in [\![j_1, j_1+q_{21}]\!]$, $\widetilde{j}_2 \in [\![1, 1+q_{22}]\!]$;

- finally for $j \in \mathscr{C}$, $C^{\sigma,j_1,j_2} V_{1,1}^{n+\sigma}$ involves the $|V_{\widetilde{j}}^n|^2$ for $\widetilde{j}_1 \in [\![1, 1+c_1]\!]$, $\widetilde{j}_2 \in [\![1, 1+c_2]\!]$.

Consequently:

- the error term $\|\widetilde{g}_1^n\|_{\mathscr{B}_1}^2$ involves the $|V_j^n|^2$ for $j_1 \in [\![1, 1+q_{11}]\!]$, $j_2 \geq 1$;
  the error term $\|\widetilde{g}_2^n\|_{\mathscr{B}_2}^2$ involves the $|V_j^n|^2$ for $j_1 \geq 1$, $j_2 \in [\![1, 1+q_{22}]\!]$;

- and finally the error term $\|\widetilde{h}_1^n\|_{\mathscr{C}}^2$ involves the $|V_j^n|^2$ for $j_1 \in [\![1, 1+c_1]\!]$, $j_2 \in [\![1, 1+c_2]\!]$.

As first noticed in [CG11] if $r_1 \geq q_{11}$ then we use (26), while if $r_1 \leq q_{11}+1$ we use (36) with $P_1 = q_{11}+1$. In both cases we obtain:

$$\sum_{n\geq 0} \Delta t \Delta x_2 e^{-2\gamma n\Delta t} \|\widetilde{g}_1^n\|_{\mathscr{B}_1}^2 \leq C\Big(|\!|\!| u_0 |\!|\!|_{\mathscr{R}}^2 + \frac{\gamma\Delta t+1}{\gamma} \sum_{n\geq 1} \Delta t e^{-2\gamma(n+1)\Delta t} |\!|\!| f^n |\!|\!|_{\mathscr{I}}^2 + \sum_{k=1}^{2}\sum_{n\geq 1} \Delta t \Delta x_{3-k} e^{-2\gamma n\Delta t} \|g_k^n\|_{\mathscr{B}_k}^2 + \sum_{n\geq 1} \Delta t\Delta x_2 e^{-2\gamma n\Delta t}\|h^n\|_{\mathscr{C}}^2 \Big). \tag{42}$$

The estimate for the term depending on $\|\widetilde{g}_2^n\|_{\mathscr{B}_2}^2$ in the right hand side of (41) follows exactly the same discussion upon r_2 and q_{22}. This permits to show that $\sum_{n\geq 0} \Delta t \Delta x_1 e^{-2\gamma n\Delta t}\|\widetilde{g}_2^n\|_{\mathscr{B}_2}^2$ is bounded by the right hand side of (42).

Finally to estimate the term depending on $\|\widetilde{h}^n\|_{\mathscr{C}}^2$ in the right hand side of (41) we use the fact that in (41), $\|\widetilde{h}^n\|_{\mathscr{C}}^2$ and $\|\widetilde{g}_1^n\|_{\mathscr{B}_1}^2$ have the same weight in terms of Δt and Δx_2. As a consequence (26) gives the desired bound (that is the right hand side of (42)) if $r_1 \geq c_1$, while if $r_1 \leq c_1+1$ we use (36) with $P_1 = c_1+1$.

We thus have shown that:

$$\frac{\gamma}{\gamma\Delta t+1}\sum_{n\geq 0}\Delta t e^{-2\gamma n\Delta t} |\!|\!| W^n |\!|\!|_{\mathscr{R}}^2 + \sum_{k=1}^{2}\sum_{n\geq 0}\Delta t\Delta x_{3-k} e^{-2\gamma n\Delta t}\|W^n\|_{\mathscr{B}_k}^2 \leq C \left(\|u_0\|_{\mathscr{I}}^2 + \frac{\gamma\Delta t+1}{\gamma}\sum_{n\geq 1}\Delta t e^{-2\gamma(n+1)\Delta t} |\!|\!| f^n |\!|\!|_{\mathscr{I}}^2 + \sum_{k=1}^{2}\sum_{n\geq 1}\Delta t\Delta x_{3-k}e^{-2\gamma n\Delta t}\|g_k^n\|_{\mathscr{B}_k}^2 + \sum_{n\geq 1}\Delta t\Delta x_2 e^{-2\gamma n\Delta t}\|h^n\|_{\mathscr{C}}^2 \right). \tag{43}$$

$\diamond$ (10) is of type traces to corner: From Definition 2.2, the terms depending on $\|\widetilde{g}_k^n\|_{\mathscr{B}_k}^2$ in the right hand side of (41) are bounded as in the previous case.

In view of the definition of C^{σ,j_1,j_2}, the norm $\|\widetilde{h}_1^n\|_{\mathscr{C}}^2$ involves the $|V_j^n|^2$ for

$$(j_1,j_2) \in ([\![1,1+c_1]\!]\times[\![1,1+c_2]\!]) \cup ([\![1,1+c_{11}]\!]\times[\![1-\ell_2,0]\!]) \cup ([\![1-\ell_1,0]\!]\times[\![1,1+c_{22}]\!])\,.$$

To estimate the terms $|V_j^n|^2$ for $(j_1,j_2)\in[\![1,1+c_1]\!]\times[\![1,1+c_2]\!]$ (resp.$[\![1,1+c_{11}]\!]\times[\![1-\ell_2,0]\!]$) we use (26) or (36) applied to $P_1 = 1+c_1$ (resp. $P_1 = 1+c_{11}$) depending of the sign of $r_1 - c_1$ (resp. $r_1 - c_{11}$). Finally to estimate $|V_j^n|^2$ for $(j_1,j_2)\in[\![1-\ell_1,0]\!]\times[\![1,1+c_{22}]\!]$ we remark that CFL condition $\frac{x_2}{\lambda_1} = \frac{x_1}{\lambda_2}$ and we then use (26) or (36) applied to $P_2 = 1+c_{22}$, estimate (43) follows.

$\diamond$ (10) is of type corner to traces: In the case, the proof of the estimate (43) follows exactly the same arguments as when (10) admits decoupled boundary and corner conditions so it will not be treated here.

In view of (10) to conclude the proof of Theorem 3.1 we just have to show that $\sup_{n\leq 0} e^{-2\gamma n\Delta t} |\!|\!| W^n |\!|\!|_{\mathscr{R}}^2$ can be bounded by the left hand side of (43). The proof follows exactly the same arguments as in [[CG11], paragraph 2.3 and Appendix A] so we will not give the details here.

□

4.4 Proof of Lemma 4.2

We will only show here the first equality in Lemma 4.2, the proof of the second one is totally equivalent. Let us rewrite the first equality in Lemma 4.2 as:

$$\sum_{j_1=1-r_1-\ell_1}^{-r_1} \|\widetilde{Q}W_{j_1,\cdot}^n\|_{\ell^2(\mathbb{Z}^{d-1})}^2 + \sum_{j_1=1-\ell_1}^{0}\|(\widetilde{Q}W^n+W^n)_{j_1,\cdot}\|_{\ell^2(\mathbb{Z}^{d-1})}^2 \geq c\sum_{j_1=1-\ell_1}^{r_1}\|W_{j_1,\cdot}^n\|_{\ell^2(\mathbb{Z}^{d-1})}^2. \tag{44}$$

In view of the definitions of $\widetilde{Q}$ and (W_j^n), we have that:

$$\sum_{j_1=1-r_1-\ell_1}^{-r_1} \|\widetilde{Q}W_{j_1,\cdot}^n\|_{\ell^2(\mathbb{Z}^{d-1})}^2 = \sum_{j_1=1-r_1-\ell_1}^{-r_1} \left\| \mathbb{A}_1^{r_1} W_{j_1+r_1,\cdot}^n + \sum_{k<r_1} \mathbb{A}^k W_{j_1+k,\cdot}^n \right\|_{\ell^2(\mathbb{Z}^{d-1})}^2 ,$$

$$= \|\mathbb{A}_1^{r_1} W_{1-\ell_1,\cdot}^n\|_{\ell^2(\mathbb{Z}^{d-1})}^2 + \sum_{j_1=2-r_1-\ell_1}^{-r_1} \left\| \mathbb{A}_1^{r_1} W_{1-\ell_1,\cdot}^n + \sum_{k<r_1} \mathbb{A}_1^k W_{j_1+k,\cdot}^n \right\|_{\ell^2(\mathbb{Z}^{d-1})}^2 .$$

As a consequence, equation (44) can be rewritten under the form:

$$\|\mathscr{L} W_{1-\ell_1,\cdot}^n\|_{\ell^2(\mathbb{Z}^{d-1})}^2 + \|\mathscr{L} W_{2-\ell_1,\cdot}^n + \mathscr{L}_1 W_{1-\ell_1,\cdot}^n\|^2 + ... + \|\mathscr{L} W_{r_1,\cdot}^n + \mathscr{L}_{\ell_1+r_1-1}(W_{1-\ell_1,\cdot}^n, ..., W_{r_1-1,\cdot}^n)\|^2$$
$$\geq \sum_{j_1=1-\ell_1}^{r_1} \|W_{j_1,\cdot}^n\|_{\ell^2(\mathbb{Z}^{d-1})}^2 ,$$

where $\mathscr{L} := \mathbb{A}_1^{r_1}$ and where the $\mathscr{L}_k$ are linear and bounded on $\ell^2(\mathbb{Z}^{d-1})^k$ (the precise expression of these operators is not usefull for what follows).

Then we proceed by indution on r_1 and contradiction. Firstly we assume that for all $k \in \mathbb{N}$, there exist two sequences $X_1^k, X_2^k \in \ell^2(\mathbb{Z}^{d-1})$ such that $\|X_1^k\|_{\ell^2(\mathbb{Z}^{d-1})}^2 + \|X_2^k\|_{\ell^2(\mathbb{Z}^{d-1})}^2 = 1$ and satisfying:

$$\forall k \in \mathbb{N}, \quad \|\mathscr{L} X_1^k\|_{\ell^2(\mathbb{Z}^{d-1})}^2 + \|\mathscr{L} X_2^k + \mathscr{L}_1 X_1^k\|_{\ell^2(\mathbb{Z}^{d-1})}^2 \leq \frac{1}{k}.$$

Thus we have that $\|\mathscr{L} X_1^k\|_{\ell^2(\mathbb{Z}^{d-1})} \downarrow 0$ which implies, by Assumption 3.1, that $\|X_1^k\|_{\ell^2(\mathbb{Z}^{d-1})} \downarrow 0$ and thus $\|\mathscr{L}_1 X_1^k\|_{\ell^2(\mathbb{Z}^{d-1})} \downarrow 0$ because $\mathscr{L}_1$ is bounded.

We thus deduce that $\|\mathscr{L} X_2^k\|_{\ell^2(\mathbb{Z}^{d-1})} \downarrow 0$ and finally that $\|X_2^k\|_{\ell^2(\mathbb{Z}^{d-1})} \downarrow 0$. This is a contradiction with the fact that $\|X_1^k\|_{\ell^2(\mathbb{Z}^{d-1})}^2 + \|X_2^k\|_{\ell^2(\mathbb{Z}^{d-1})}^2 = 1$. So (44) holds for $r_1 = 1$. The induction step follows exactly the same proof and will be omitted.

5 Proof of Theorem 3.2

Following [Cou15] the proof of Theorem 3.2 is based on an energy-dissipation balance law which is obtained from the introduction of a multiplier inspired of the multiplier of the Leray-Gårding method [?]-[?] (see also [Rau72] for an analogous approach in the continous framework). We will use exactly the same multiplier as in [Cou15]. Using the fact that this multiplier only depends on the discretization in the interior of the domain, we will show that this multiplier gives "strictly dissipative" boundaries and corner conditions and thus permits to introduce an auxiliary problem (posed in the full space, and as a consequence differing from the one used in Section 4) whose solution admits suitable (for the final error analysis) control of the traces to show the semi-group stability of (21).

Before we turn to a precise statement of the auxiliary problem, let us recall the definition of the multiplier used in [Cou15] and the energy-dissipation balance law that it induces for finite difference schemes in the full space.

We define:

$$L := \sum_{\sigma=0}^{s+1} \mathbf{T}_0^\sigma Q^\sigma, \text{ and } M := \sum_{\sigma=0}^{s+1} \sigma \mathbf{T}_0^\sigma Q^\sigma, \tag{45}$$

where $\mathbf{T}_0^\sigma$ is the time-shifting operator.

Then we have the following balance law:

Lemma 5.1 ([Cou15] Proposition 2) *Under Assumptions 3.3 and 3.4 then there exist a continous coercive quadratic form E and a continous nonnegative quadratic form D on $\ell^2(\mathbb{Z}^d, \mathbb{R})^{s+1}$ such that for all $(v^n)_{n\geq 0}$ with values in $\ell^2(\mathbb{Z}^d, \mathbb{R})$ and all $n \in \mathbb{N}$ we have:*

$$2\,\langle\langle Mv^n, Lv^n \rangle\rangle_{\mathbb{Z}^2} = (s+1)|||Lv^n|||_{\mathbb{Z}^2}^2 + (\mathbf{T}_0 - I)E(v^n, ..., v^{n+s}) + D(v^n, ..., v^{n+s}).$$

Note that we do not require that the sequence (v_j^n) solves any finite difference scheme. Indeed Lemma 5.1 only depends on the coefficient of the discretization in the interior. This observation will be required to extend the proof of [Cou15] to corner domains.

5.1 Homogeneous initial conditions and auxiliary problem with strictly dissipative boundary and corner conditions

As the proof of Theorem 3.1, the proof of Theorem 3.2 uses the linearity of (21) to treat separately the case of homogeneous initial conditions and the case of nonzero initial conditions. For homogeneous initial conditions the proof is a straightforward generalization of the proof in the half space. This proof is given in the following paragraph for a sake of completness. For nonzero initial conditions the proof needs to introduce an auxiliary problem which is posed in the full space and admits strictly dissipative boundary and corner conditions (compared with the proof in the half space where only one strictly dissipative boundary condition is needed).

5.1.1 Proof of Theorem 3.2 for homogeneous initial conditions

We first show Theorem 3.2 for homogeneous initial conditions that is:

Lemma 5.2 *Under Assumptions 3.3-3.4 and 3.5, assume that the difference scheme approximation* (21) *is strongly stable in the sense of Definition 2.1 then u the solution of* (23) *with homogeneous initial conditions satisfies that there exists $C>0$ such that for all $\gamma>0$ and $\Delta t\in]0,1]$ we have the following estimate:*

$$\begin{aligned}\sup_{n\geq 0} e^{-2\gamma n\Delta t}|||u^n|||_{\mathscr{R}}^2 &+ \frac{\gamma}{\gamma\Delta t+1}\sum_{n\geq 0}\Delta t e^{-2\gamma n\Delta t}|||u^n|||_{\mathscr{R}}^2+\sum_{k=1}^{2}\sum_{n\geq 0}\Delta t\Delta x_{3-k}e^{-2\gamma n\Delta t}\|u^n\|_{\mathscr{B}_k}^2\leq \\ C &\left(\frac{\gamma\Delta t+1}{\gamma}\sum_{n\geq s+1}\Delta t e^{-2\gamma n\Delta t}|||f^n|||_{\mathscr{I}}^2+\sum_{k=1}^{2}\sum_{n\geq s+1}\Delta t\Delta x_{3-k}e^{-2\gamma n\Delta t}\|g_k^n\|_{\mathscr{B}_k}^2\right. \\ &+ \left.\sum_{n\geq s+1}\Delta t\Delta x_2 e^{-2\gamma n\Delta t}\|h^n\|_{\mathscr{C}}^2\right).\end{aligned} \tag{46}$$

Proof : By strong stability of (23) (recall that we assumed $u_n\equiv 0$ for $n\in[\![0,s]\!]$) it is sufficient to show that $\sup_{n\geq 0}e^{-2\gamma n\Delta t}|||u^n|||_{\mathscr{R}}^2$ can be bounded by the right hand side of (46). As in the proof of Theorem 4.1 we introduce (w_j^n) the extension of (u_j^n) by zero for $j\in\mathbb{Z}^2\setminus\mathscr{R}$ (recall that we have $Lw_j^n=0$ for $j_1\leq -\ell_1-r_1$ or $j_2\leq-\ell_2-r_2$ and $Lw_j^n=Lu_j^n$ for $j\in\mathscr{I}$, and so do for M). Applying Lemma 5.1, using the nonnegativity of D it follows that:

$$(\mathbf{T}_0-I)E(w^n,...,w^{n+s})\leq 2\,\langle\langle Mw^n,Lw^n\rangle\rangle_{\overline{\mathscr{R}}\setminus\mathscr{I}}-(s+1)|||Lw^n|||_{\overline{\mathscr{R}}\setminus\mathscr{I}}^2+2\Delta t\,\langle\langle Mu^n,f^n\rangle\rangle_{\mathscr{I}}-(s+1)\Delta t^2|||f^n|||_{\mathscr{I}}^2,$$

where $\overline{\mathscr{R}}:=\mathscr{R}\cup\mathscr{E}_{\mathscr{B}_1}\cup\mathscr{E}_{\mathscr{B}_2}\cup\mathscr{E}_{\mathscr{C}}$ (see (30) for the definition of the $\mathscr{E}_{\mathfrak{J}}$). Multiplying by $e^{-2\gamma(n+s+1)\Delta t}$ and summing over $n\in[\![0,N]\!]$ gives:

$$e^{-2\gamma(N+s+1)\Delta t}E(w^{N+1},...,w^{N+s+1})+(1-e^{-2\gamma\Delta t})\sum_{n=1}^{N}e^{-2\gamma(n+s)\Delta t}E(v^n,...v^{n+s})\leq I_{\overline{\mathscr{R}}\setminus\mathscr{I},N}+I_{\mathscr{I},N}, \tag{47}$$

with

$$\begin{aligned}I_{\overline{\mathscr{R}}\setminus\mathscr{I},N} &:= \sum_{n=0}^{N}e^{-2\gamma(n+s+1)\Delta t}\left(2\,\langle\langle Mw^n,Lw^n\rangle\rangle_{\overline{\mathscr{R}}\setminus\mathscr{I}}-(s+1)|||Lw^n|||_{\overline{\mathscr{R}}\setminus\mathscr{I}}^2\right),\\ I_{\mathscr{I},N} &:= \sum_{n=0}^{N}\Delta t e^{-2\gamma(n+s+1)\Delta t}\left(2\,\langle\langle Mu^n,f^n\rangle\rangle_{\mathscr{I}}-(s+1)\Delta t|||f^n|||_{\mathscr{I}}^2\right),\end{aligned}$$

and we will estimate these terms separately. First remark that by definition of L, M and (w_j^n) the terms Lw^n and Mw^n only involve the $u_j^{n+\sigma}$ for $j \in \overline{\mathscr{B}_1} \cup \overline{\mathscr{B}_2}$ and $\sigma \in [\![0, s+1]\!]$. By CFL condition we can always exchange in the equation defining $I_{\overline{\mathscr{R}}\setminus\mathscr{I},N}$, the factor Δx_1 or Δx_2 by Δt (see Section 4 for similar arguments) and we can thus use the trace estimate given by the strong stability of (21) and the trivial bound $I_{\overline{\mathscr{R}}\setminus\mathscr{I},N} \leq I_{\overline{\mathscr{R}}\setminus\mathscr{I},\infty}$ to show that $I_{\overline{\mathscr{R}}\setminus\mathscr{I},N}$ is bounded by the right hand side of (46).

We now turn to $I_{\mathscr{I},N}$. For $j \in \mathscr{I}$ we remark that Mw_j^n only involves the $u_j^{n+\sigma}$ for $j \in \mathscr{R}$ and $\sigma \in [\![0, s+1]\!]$. As a consequence $\Delta t |||Mw^n|||_{\mathscr{I}}$ only involves the $\Delta t |||u^{n+\sigma}|||_{\mathscr{R}}$ for $\sigma \in [\![0, s+1]\!]$. We use this time the interior estimate given by the strong stability assumption on (21) to bound these terms. Applying exactly the same computations as in [Cou15] we show that $I_{\mathscr{I},N}$ is also bounded by the right hand side of (46).

To conclude, we go back to the left hand side of (47) and from the coercivity of E we have that:

$$e^{-2\gamma(N+s+1)\Delta t} |||v^{N+s+1}||| \leq I_{\overline{\mathscr{R}}\setminus\mathscr{I},N} + I_{\mathscr{I},N},$$

and (46) follows by taking the supremum in N.

□

Remark Note that compared to the proof of Theorem 3.1 it is here crucial (in view to obtain a suitable estimate for $I_{\overline{\mathscr{R}}\setminus\mathscr{I},N}$) that the strong stability provides a control of the extended traces on $\overline{\mathscr{B}_1}$ and $\overline{\mathscr{B}_2}$ and not only a control of the traces on $\mathscr{B}_1$ and $\mathscr{B}_2$.

5.1.2 Auxiliary problem

In this paragraph we use the multiplier M to introduce an auxiliary problem defined in the full space for which we can show a semi-group estimate and an extended traces estimate (which will be used in the end of the proof as it has been done in Section 4 to control the error terms).

More precisely the result is the following:

Theorem 5.1 ([Cou15], Theorem 2) *Under Assumptions 3.3-3.4 and 3.5, for all $P_1, P_2 \in \mathbb{N}$, there exists $C > 0$ such that the solution (u_j^n) of:*

$$\begin{cases} Lu_j^n = 0, & \text{for } n \geq 0, j \in \mathscr{I} \times \mathbb{Z}^{d-2}, \\ Mu_j^n = g_{1,j}^{n+s+1}, & \text{for } n \geq 0, j_1 \leq 0, j_2 \geq 1, j' \in \mathbb{Z}^{d-2}, \\ Mu_j^n = g_{2,j}^{n+s+1}, & \text{for } n \geq 0, j_1 \geq 1, j_2 \leq 1, j' \in \mathbb{Z}^{d-2}, \\ Mu_j^n = h_j^{n+s+1}, & \text{for } n \geq 0, j_1 \leq 0, j_2 \leq 0, j' \in \mathbb{Z}^{d-2} \\ u_j^n = u_{n,j}, & \text{for } n \in [\![0, s]\!], j \in \mathbb{Z}^d, \end{cases} \tag{48}$$

satisfies that for all $\gamma > 0$, $\Delta t \in]0, 1]$:

$$\begin{aligned} \sup_{n\geq 0} e^{-2\gamma n\Delta t} |||u^n|||_{\mathbb{Z}^2}^2 \quad &+ \quad \frac{\gamma}{\gamma\Delta t + 1} \sum_{n\geq 0} \Delta t e^{-2\gamma n\Delta t} |||u^n|||_{\mathbb{Z}^2}^2 + \sum_{k=1}^{2} \sum_{n\geq 0} \Delta t \Delta x_{3-k} e^{-2\gamma n\Delta t} \sum_{j_k=1-\ell_k}^{P_k} \|u_{j_k,\cdot}^n\|_{\ell^2(\mathbb{Z}^{d-1})}^2 \\ &\leq \quad C \left(\sum_{n=0}^{s} |||u_n|||_{\mathbb{Z}^2}^2 + \sum_{k=1}^{2} \sum_{n\geq s+1} \Delta t \Delta x_{3-k} e^{-2\gamma n\Delta t} \sum_{j_k\leq 0, j_{3-k}\geq 1} \|g_{k,j_k,j_{3-k},\cdot}^n\|_{\ell^2(\mathbb{Z}^{d-2})}^2 \right. \\ &+ \quad \left. \sum_{n\geq s+1} \Delta t \Delta x_2 e^{-2\gamma n\Delta t} \sum_{j_1\leq 0, j_2\leq 0} \|h_{j_k,j_{3-k},\cdot}^n\|_{\ell^2(\mathbb{Z}^{d-2})}^2 \right). \end{aligned} \tag{49}$$

Before we turn to the proof of Theorem 5.1 let us give some arguments explaining why essentially the same as the proof of Theorem 2 in [Cou15] will effectively operate even if the auxiliary problems differ. Indeed compared to the auxiliary problem introduced in [Cou15], the auxiliary problem (48) is defined by $Mu_j^n = g_{2,j}^{n+s+1}$ and not $Lu_j^n = 0$ in the quarter space $j_1 \geq 1, j_2 \leq 1, j' \in \mathbb{Z}^{d-2}$. Moreover the proof of (49) given in [Cou15] relies on partial Fourier transform which are not *a priori* suitable in the geometry of the quarter space due to the fact that "there are too many normal directions".

However, using the fact that (u_j^n) the solution of (48) is defined in the full space $\mathbb{Z}^d$ any extension will be needed to perform partial Fourier transform and consequently, in this particular setting, we will be able to perform two partial Fourier transforms to obtain the estimate (49) (one transform by trace that have to be controled). The fact that (u_j^n) does not solve $Lu_j^n = 0$ in the quarter space $j_1 \geq 1, j_2 \leq 1, j' \in \mathbb{Z}^{d-2}$ will not be an issue neither. Indeed, in [Cou15] the main part of the proof leading to (49) does not strongly use the equation solved by (u_j^n). Indeed here are the main points in the proof of [Cou15].

Firstly one uses Lemma 5.1 to obtain an estimate of $\sup_{n\geq 0} e^{-2\gamma n\Delta t} |||u^n|||_{\mathbb{Z}^2}^2$, $\sum_{n\geq 0} \Delta t e^{-2\gamma n\Delta t} |||u^n|||_{\mathbb{Z}^2}^2$ and $\sum_{n\geq 0} e^{-2\gamma(n+s+1)\Delta t} |||Lu^n|||_{\mathbb{Z}^2}^2$.

Then to remplace $\sum_{n\geq 0} e^{-2\gamma(n+s+1)\Delta t} |||Lu^n|||_{\mathbb{Z}^2}^2$ by $\sum_{k=1}^2 \sum_{n\geq 0} \Delta t \Delta x_{3-k} e^{-2\gamma n\Delta t} \sum_{j_k=1-\ell_k}^{P_k} \|u_{j_k,\cdot}^n\|_{\ell^2(\mathbb{Z}^{d-1})}^2$ in this estimate, the arguments are the following. In a first time one works in the regime $\gamma\Delta t$ large and only needs algebraic properties of L (but does not use the fact that (u_j^n) solves a precise difference scheme) and thus in this regime we will be able to apply the same proof as in [Cou15].

Then, in the regime $\gamma\Delta t$ small, the first step is to use the estimate to show that Laplace-Fourier transform of (u_j^n) is well-defined. Then by algebraic computations and using the fact that (u_j^n) solves the boundary conditions (but not the fact that it solves $Lu_j^n = 0$ on any quarter space) one obtains that some algebraic quantity involving the Laplace-Fourier transform of (u_j^n) and the symbol of L is bounded by the right hand side of (49). To conclude its remains to show that the considered algebraic quantity controls the extended traces for any considered sequence. This last point is independent of the defintion of (u_j^n) and so it will also works if (u_j^n) solves (48).

Proof : Firstly we apply Lemma 5.1 to (48), in view to demonstrate the analogous estimate than in [Cou15]. From the nonnegativity of D it follows that:

$$\begin{aligned}(\mathbf{T}_0 - I)E(u^n, ..., u^{n+s}) + (s+1)|||Lu^n|||_{\mathbb{Z}^2}^2 \quad &\leq \quad 2\,\langle\langle g_1^n, Lu^n\rangle\rangle_{]\!]-\infty,0]\!]\times[\![1,\infty[\![} + 2\,\langle\langle g_2^n, Lu^n\rangle\rangle_{[\![1,\infty[\![\times]\!]-\infty,0]\!]} \\ &+ \quad 2\,\langle\langle h^n, Lu^n\rangle\rangle_{]\!]-\infty,0]\!]^2}\,,\end{aligned}$$

we then use three times Cauchy-Schwartz combined with Young inequality (with parameter $\frac{(s+1)}{6}$) to obtain:

$$(\mathbf{T}_0 - I)E(u^n, ..., u^{n+s}) + \frac{(s+1)}{2}|||Lu^n|||_{\mathbb{Z}^2}^2 \leq 6\left(\|g_1^n\|_{]\!]-\infty,0]\!]\times[\![1,\infty[\![}^2 + \|g_2^n\|_{[\![1,\infty[\![\times]\!]-\infty,0]\!]}^2 + \|h^n\|_{]\!]-\infty,0]\!]^2}^2\right).$$

We then multiply the latter inequality by $e^{-2\gamma(n+s+1)\Delta t}$ and sum from $n = 0$ to N. Reiterating the same kind of computations as in Section 4 we obtain, from the coercivity of E the following estimate:

$$\begin{aligned}\sup_{n\geq 0} e^{-2\gamma n\Delta t}|||u^n|||_{\mathbb{Z}^2}^2 \quad &+ \quad \frac{\gamma}{\gamma\Delta t+1}\sum_{n\geq 0}\Delta t e^{-2\gamma n\Delta t}|||u^n|||_{\mathbb{Z}^2}^2 + \sum_{n\geq 0} e^{-2\gamma(n+s+1)\Delta t}|||Lu^n|||_{\mathbb{Z}^2}^2 \leq \\ &C \quad \left(\sum_{n=0}^{s}|||u_n|||_{\mathbb{Z}^2}^2 + \sum_{k=1}^{2}\sum_{n\geq s+1}\Delta t\Delta x_{3-k}e^{-2\gamma n\Delta t}\sum_{j_k\leq 0, j_{3-k}\geq 1}\|g_{k,j_k,j_{3-k}\cdot}^n\|_{\ell^2(\mathbb{Z}^{d-2})}^2\right. \\ &+ \quad \left.\sum_{n\geq s+1}\Delta t\Delta x_2 e^{-2\gamma n\Delta t}\sum_{j_1\leq 0, j_2\leq 0}\|h_{j_k,j_{3-k},\cdot}^n\|_{\ell^2(\mathbb{Z}^{d-2})}^2\right).\end{aligned} \tag{50}$$

Using the definition of the CFL numbers we can rewrite the weight $\Delta x_1\Delta x_2$ as $\Delta t\Delta x_2$ or $\Delta t\Delta x_1$. So (50) can also be rewritten under the form:

$$\sup_{n\geq 0} e^{-2\gamma n\Delta t}|||u^n|||^2_{\mathbb{Z}^2} + \frac{\gamma}{\gamma\Delta t+1}\sum_{n\geq 0}\Delta t e^{-2\gamma n\Delta t}|||u^n|||^2_{\mathbb{Z}^2} + \sum_{k=1}^{2}\sum_{n\geq 0}\Delta t\Delta x_{3-k}\sum_{j_k\in\mathbb{Z}} e^{-2\gamma(n+s+1)\Delta t}\|Lu^n_{j_k,\cdot}\|^2_{\ell^2(\mathbb{Z}^{d-1})} \leq$$
$$C\left(\sum_{n=0}^{s}|||u_n|||^2_{\mathbb{Z}^2} + \sum_{k=1}^{2}\sum_{n\geq s+1}\Delta t\Delta x_{3-k}e^{-2\gamma n\Delta t}\sum_{j_k\leq 0, j_{3-k}\geq 1}\|g^n_{k,j_k,j_{3-k}\cdot}\|^2_{\ell^2(\mathbb{Z}^{d-2})}\right.$$
$$\left. + \sum_{n\geq s+1}\Delta t\Delta x_2 e^{-2\gamma n\Delta t}\sum_{j_1\leq 0, j_2\leq 0}\|h^n_{j_k,j_{3-k},\cdot}\|^2_{\ell^2(\mathbb{Z}^{d-2})}\right). \tag{51}$$

To conclude the proof of Theorem 5.1 it is sufficient to explain how (51) implies a good control of the extended traces values $\sum_{k=1}^{2}\sum_{n\geq 0}\Delta t\Delta x_{3-k}e^{-2\gamma n\Delta t}\sum_{j_k=1-\ell_k}^{P_k}\|u^n_{j_k,\cdot}\|^2_{\ell^2(\mathbb{Z}^{d-1})}$.

Following [Cou15] in view to do this it is sufficient to distinguish two cases depending on the value of $\gamma\Delta t$. The proof in the framework $\gamma\Delta t$ large is totally analogous (because as already mentioned it only uses algebraic properties of the operator L) to the one given in [Cou15] and will not be repeated here. Let us recall that when $\gamma\Delta t$ is large we, in fact, obtain the control of infinitely many traces:

$$\sum_{k=1}^{2}\sum_{n\geq 0}\Delta t\Delta x_{3-k}e^{-2\gamma n\Delta t}\sum_{j_k\in\mathbb{Z}}\|u^n_{j_k,\cdot}\|^2_{\ell^2(\mathbb{Z}^{d-1})} \leq$$
$$C\left(\sum_{n=0}^{s}|||u_n|||^2_{\mathbb{Z}^2} + \sum_{k=1}^{2}\sum_{n\geq s+1}\Delta t\Delta x_{3-k}e^{-2\gamma n\Delta t}\sum_{j_k\leq 0, j_{3-k}\geq 1}\|g^n_{k,j_k,j_{3-k}\cdot}\|^2_{\ell^2(\mathbb{Z}^{d-2})}\right.$$
$$\left. + \sum_{n\geq s+1}\Delta t\Delta x_2 e^{-2\gamma n\Delta t}\sum_{j_1\leq 0, j_2\leq 0}\|h^n_{j_k,j_{3-k},\cdot}\|^2_{\ell^2(\mathbb{Z}^{d-2})}\right),$$

and not only a control of finitely many traces.

We now turn to the case $\gamma\Delta t$ small. As the reader will see the proof in this framework will not need any adaptation of the proof given in [Cou15]. However as the proof of [Cou15] relies on partial Fourier transform (which was the only point that we needed to adapt in the proof of Theorem 3.1) it seems important, in the author opinion, to give some comments about the previous claim.

Firstly the estimate (51) shows that the Laplace in time and partial Fourier transforms in spaces (j_1, j') and (j_2, j') of (u^n_j) are well-defined. We denote these transforms by $\widehat{u_{j_2}}^1$ and $\widehat{u_{j_1}}^2$ respectively. We introduce $\tau := \gamma + i\theta$ the dual variable of time for the Laplace transform and $\eta_k := (j_k, j')$ the dual variable of space for the partial Fourier transform in space (j_k, j'). To save some notations we also denote $\eta_{k,\Delta} := (j_k, j')\cdot(\Delta x_k, \Delta x')$.

The following lemma gives a control of $\widehat{u_{j_2}}^1$ and $\widehat{u_{j_1}}^2$ up to some multiplication by the associated symbols of L and M in term of the right hand side of (49) if $\gamma\Delta t < \ln R_0$ for some fixed $R_0 > 1$:

Lemma 5.3 *There exists $C > 0$ such that for all $\gamma > 0$, $\Delta t \in]0, 1]$ satisfying $\gamma\Delta t < \ln R_0$, we have that for*

$k \in [\![1,2]\!]$:

$$\sum_{j_k\in\mathbb{Z}} \int_{\mathbb{R}\times\mathbb{R}^{d-1}} \left| \sum_{\mu_k=-\ell_k}^{r_k} a^{\mu_k}(e^{\tau\Delta t}, \eta_{\Delta,3-k}) \widehat{u_{j_k}}^{3-k}(\tau,\eta_{3-k}) \right|^2 d\theta d\eta_{3-k}$$

$$+ \sum_{j_k\leq 0} \int_{\mathbb{R}\times\mathbb{R}^{d-1}} \left| \sum_{\mu_k=-\ell_k}^{r_k} e^{\tau\Delta t}\partial_z a^{\mu_k}(e^{\tau\Delta t}, \eta_{\Delta,3-k}) \widehat{u_{j_k}}^{3-k}(\tau,\eta_{3-k}) \right|^2 d\theta d\eta_{3-k}$$

$$\leq C \left(\sum_{n=0}^{s} |\!|\!| u_n |\!|\!|_{\mathbb{Z}^2}^2 + \sum_{k=1}^{2} \sum_{n\geq s+1} \Delta t \Delta x_{3-k} e^{-2\gamma n\Delta t} \sum_{j_k\leq 0, j_{3-k}\geq 1} \|g^n_{k,j_k,j_{3-k}\cdot}\|^2_{\ell^2(\mathbb{Z}^{d-2})} \right.$$

$$\left. + \sum_{n\geq s+1} \Delta t \Delta x_2 e^{-2\gamma n\Delta t} \sum_{j_1\leq 0, j_2\leq 0} \|h^n_{j_k,j_{3-k},\cdot}\|^2_{\ell^2(\mathbb{Z}^{d-2})} \right).$$

Then the following lemma gives the control of the trace of any sequence in term of the left hand side of the inequality appearing in Lemma 5.3. More precisely, we have:

Lemma 5.4 *Under Assumptions 3.3-3.4 and 3.5, for $k \in [\![1,2]\!]$ and for any fixed $P_k \in \mathbb{N}$, there exists $C_{P_k} > 0$ such that for all $z \in \mathbb{C}$ such that $1 \leq |z| \leq R_0$, for all $\eta_{3-k} \in \mathbb{R}^{d-1}$ and **for all** sequence $(w_{j_k})_{j_k\in\mathbb{Z}} \in \ell^2(\mathbb{Z},\mathbb{C})$ we have:*

$$\sum_{j_k=-\ell_k-r_k}^{P_k} |w_{j_k}|^2 \leq C_{P_k} \left(\sum_{j_k\in\mathbb{Z}} \int_{\mathbb{R}\times\mathbb{R}^{d-1}} \left| \sum_{\mu_k=-\ell_k}^{r_k} a^{\mu_k}(e^{\tau\Delta t}, \eta_{\Delta,3-k}) w_{j_k+\mu_k} \right|^2 d\theta d\eta_{3-k} \right.$$

$$\left. + \sum_{j_k\leq 0} \int_{\mathbb{R}\times\mathbb{R}^{d-1}} \left| \sum_{\mu_k=-\ell_k}^{r_k} e^{\tau\Delta t}\partial_z a^{\mu_k}(e^{\tau\Delta t}, \eta_{\Delta,3-k}) w_{j_k+\mu_k} \right|^2 d\theta d\eta_{3-k} \right).$$

With Lemmas 5.3 and 5.4 in hand let us describe how to conclude the proof of Theorem 5.1. For $k \in [\![1,2]\!]$, we apply Lemma 5.4 with $z := e^{\tau\Delta t}$ and to the sequences $(\widehat{u_{j_k}}^{3-k}(\tau,\eta_{3-k}))_{j_k\in\mathbb{Z}}$. We then integrate the estimate of Lemma 5.4 with respect to (θ,η_{3-k}) and choose the real part of τ small enougt to apply Lemma 5.3. We thus have:

$$\sum_{j_k=-\ell_k-r_k}^{P_k} \int_{\mathbb{R}\times\mathbb{R}^{d-1}} |w_{j_k}|^2 d\theta d\eta_{3-k} \leq C \left(\sum_{n=0}^{s} |\!|\!| u_n |\!|\!|_{\mathbb{Z}^2}^2 + \sum_{k=1}^{2} \sum_{n\geq s+1} \Delta t \Delta x_{3-k} e^{-2\gamma n\Delta t} \sum_{j_k\leq 0, j_{3-k}\geq 1} \|g^n_{k,j_k,j_{3-k}\cdot}\|^2_{\ell^2(\mathbb{Z}^{d-2})} \right.$$

$$\left. + \sum_{n\geq s+1} \Delta t \Delta x_2 e^{-2\gamma n\Delta t} \sum_{j_1\leq 0, j_2\leq 0} \|h^n_{j_k,j_{3-k},\cdot}\|^2_{\ell^2(\mathbb{Z}^{d-2})} \right).$$

To come back in terms of the sequence $(u^n_{j_k,\cdot})$ we apply Plancherel formula in the left hand side and use the fact that $\gamma\Delta t$ is small to obtain that:

$$\begin{aligned} \sum_{j_k=-\ell_k-r_k}^{P_k} \sum_{n\geq 0} \Delta t e^{-2\gamma n\Delta t} \|u_{j_k,\cdot}\|^2_{\ell^2(\mathbb{Z}^{d-1})} &\leq \sum_{j_k=-\ell_k-r_k}^{P_k} \sum_{n\geq 0} \frac{1-e^{-2\gamma\Delta t}}{2\gamma\Delta t} \Delta t e^{-2\gamma n\Delta t} \|u_{j_k,\cdot}\|^2_{\ell^2(\mathbb{Z}^{d-1})} \\ &= \sum_{j_k=-\ell_k-r_k}^{P_k} \int_{\mathbb{R}\times\mathbb{R}^{d-1}} |w_{j_k}|^2 d\theta d\eta_{3-k}, \end{aligned}$$

which concludes the proof of Theorem 5.1.

□

Remark We conclude this paragraph by giving some elements of proof for Lemmas 5.3 and 5.4. The proof of 5.4 is the most technical one but since it only depends on the tangential operators a^{μ_k}, we will not reiterate it here. The proof of Lemma 5.3 uses the fact that the considered sequence solves an explicit scheme. However this point is only used to establish the estimate (51). As we have already seen this estimate also holds for our choice of auxiliary scheme and consequently we can reiterate exactly the proof given in [Cou15] to show Lemma 5.3.

5.2 End of the proof by error estimate

The case of finite difference schemes with homogeneous initial conditions has already been treated in Paragraph 5.1. So without loss of generality we can assume that in (23) the sequences $(f_j^n), (g_{1,j}^n), (g_{2,j}^n)$ and (h_j^n) are zero. We denote the associated solution by (u_j^n).By linearity of (23) we decompose (u_j^n) into $u_j^n := v_j^n + w_j^n$ where (v_j^n) is the solution of the auxiliary problem with strictly dissipative boundary and corner conditions:

$$\begin{cases} Lv_j^n = 0, & \text{for } n \geq 0, j \in \mathscr{I} \times \mathbb{Z}^{d-2}, \\ Mv_j^n = 0, & \text{for } n \geq 0, j_1 \leq 0, j_2 \geq 1, j' \in \mathbb{Z}^{d-2}, \\ Mv_j^n = 0, & \text{for } n \geq 0, j_1 \geq 1, j_2 \leq 1, j' \in \mathbb{Z}^{d-2}, \\ Mv_j^n = 0, & \text{for } n \geq 0, j_1 \leq 0, j_2 \leq 0, j' \in \mathbb{Z}^{d-2}, \\ v_j^n = \widetilde{u}_{n,j}, & \text{for } n \in [\![0, s]\!], j \in \mathbb{Z}^d, \end{cases} \tag{52}$$

where $(\widetilde{u}_{n,j})$ is the extension of $(u_{n,j})$ by zero for $j \neq \mathscr{R}$; and where (w_j^n) is the solution of (23) with homogeneous initial conditions (but inhomogeneous boundary and corner conditions):

$$\begin{cases} Lw_j^n = 0, & \text{for } n \geq 0, j \in \mathscr{I} \times \mathbb{Z}^{d-2}, \\ w_j^{n+s+1} + \sum_{\sigma=0}^{s+1} B_1^{\sigma,j_1} w_j^{n+\sigma} = \widetilde{g}_{1,j}^{n+s+1}, & \text{for } j \in \mathscr{B}_1 \times \mathbb{Z}^{d-2}, n \geq 0, \\ w_j^{n+s+1} + \sum_{\sigma=0}^{s+1} B_2^{\sigma,j_2} w_j^{n+\sigma} = \widetilde{g}_{2,j}^{n+s+1}, & \text{for } j \in \mathscr{B}_2 \times \mathbb{Z}^{d-2}, n \geq 0, \\ w_j^{n+s+1} + \sum_{\sigma=0}^{s+1} C^{\sigma,j_1,j_2} w_j^{n+\sigma} = \widetilde{h}_j^{n+s+1}, & \text{for } j \in \mathscr{C} \times \mathbb{Z}^{d-2}, n \geq 0, \\ w_j^n = 0, & \text{for } j \in \mathscr{R} \times \mathbb{Z}^{d-2}, n \in [\![0, s]\!]. \end{cases} \tag{53}$$

where $\widetilde{g}_{k,j}^{n+s+1}$ and $\widetilde{h}_j^{n+s+1}$ are the errors at the boundaries and at the corner induced by the sequence (v_j^n). More precisely they are defined by:

$$\text{for } k \in [\![1, 2]\!], \quad \widetilde{g}_{k,j}^{n+s+1} := -v_j^{n+s+1} - \sum_{\sigma=0}^{s+1} B_k^{\sigma,j_k} v_j^{n+\sigma}, \text{ for } j \in \mathscr{B}_k \times \mathbb{Z}^{d-2}, \tag{54}$$
$$\text{and } \widetilde{h}_j^{n+s+1} := -v_j^{n+s+1} - \sum_{\sigma=0}^{s+1} C^{\sigma,j_1,j_2} v_j^{n+\sigma}, \text{ for } j \in \mathscr{C} \times \mathbb{Z}^{d-2}.$$

Using the fact that (53) admits homogeneous initial conditions we can apply the estimate obtained in Paragraph 5.1.1. We thus have:

$$\begin{aligned} \sup_{n\geq 0} e^{-2\gamma n\Delta t} |\!|\!| w^n |\!|\!|_{\mathscr{R}}^2 \quad + \quad & \frac{\gamma}{\gamma\Delta t + 1} \sum_{n\geq 0} \Delta t e^{-2\gamma n\Delta t} |\!|\!| w^n |\!|\!|_{\mathscr{R}}^2 + \sum_{k=1}^{2} \sum_{n\geq 0} \Delta t \Delta x_{3-k} e^{-2\gamma n\Delta t} \| w^n \|_{\mathscr{B}_k}^2 \leq \\ C \quad & \left(\sum_{k=1}^{2} \sum_{n\geq s+1} \Delta t \Delta x_{3-k} e^{-2\gamma n\Delta t} \| \widetilde{g}_k^n \|_{\mathscr{B}_k}^2 + \sum_{n\geq s+1} \Delta t \Delta x_2 e^{-2\gamma n\Delta t} \| \widetilde{h}^n \|_{\mathscr{C}}^2 \right), \end{aligned} \tag{55}$$

and we want to estimate the errors terms $\widetilde{g}_{k,j}^{n+s+1}$ and $\widetilde{h}_j^{n+s+1}$ in terms of the initial datas. In view of (54) from the triangle inequality we deduce that to conclude it is sufficient to control the norms of the terms $-v_j^{n+s+1} + \sum_{\sigma=0}^{s+1} B_k^{\sigma,j_k} v_j^{n+\sigma}$ and $-v_j^{n+s+1} + \sum_{\sigma=0}^{s+1} C^{\sigma,j_1,j_2} v_j^{n+\sigma}$ by the initial datas. To do this we used the strengthened traces and corner estimates obtained for (v_j^n) in Theorem 5.1.

As it has been done in the proof of Theorem 3.1 (see Paragraph 4.3) a discussion depending on the kind of the boundaries and corner conditions in (10) is needed. However as the arguments are totally similar to these described in the proof of Theorem 3.1 we will only here describe the proof when (10) admits decoupled traces and corner conditions (the proofs in the others cases are the same up to different values of P_1 and P_2 (see again Paragraph 4.3)).

Note that in view of the definition of B_1^{σ,j_1} when (10) admits decoupled traces and corner conditions, for fixed n and $j \in \mathscr{B}_1 \times \mathbb{Z}^{d-2}$, the term $-v_j^{n+s+1} + \sum_{\sigma=0}^{s+1} B_1^{\sigma,j_1} v_j^{n+\sigma}$ only involve the $v_{\widetilde{j}}^{n+\sigma}$ for $\widetilde{j} \in [\![1-\ell_1, q_{11}]\!] \times [\![1,\infty[\![\times\mathbb{Z}^{d-2}$ and $\sigma \in [\![0, s+1]\!]$. The triangle inequality and the estimate (49) applied to $P_1 := \max\{r_1, 1+q_{11}\}$ then give (recall that (52) is homogeneous at the boundary):

$$\begin{aligned}\sum_{n\geq s+1} \Delta t \Delta x_2 e^{-2\gamma n\Delta t} \|\widetilde{g}_1^n\|_{\mathscr{B}_1}^2 &\leq C \sum_{n\geq 0} \Delta t \Delta x_2 e^{-2\gamma n \Delta t} \sum_{j_1=1-\ell_1}^{P_1} \|v_{j_1,\cdot}^n\|_{\ell^2(\mathbb{Z}^{d-1})}^2 \\ &\leq C \sum_{n=0}^{s} |||\widetilde{u}_{n,\cdot}|||_{\mathbb{Z}^2}^2.\end{aligned}$$

We then apply exactly the same arguments to estimate $\widetilde{g}_{2,j}^{n+s+1}$ (choosing $P_2 = \max\{r_2, 1+q_{22}\}$ in (49)) and we to obtain that:

$$\sum_{n\geq s+1} \Delta t \Delta x_1 e^{-2\gamma n\Delta t} \|\widetilde{g}_2^n\|_{\mathscr{B}_2}^2 \leq C \sum_{n=0}^{s} |||\widetilde{u}_{n,\cdot}|||_{\mathbb{Z}^2}^2.$$

Finally to deal with $\widetilde{h}_j^{n+s+1}$ we reiterate a last time the previous reasoning with $P_1 = \max\{r_1, 1+c_{11}\}$ in the estimate (49) to obtain:

$$\sum_{n\geq s+1} \Delta t \Delta x_2 e^{-2\gamma n\Delta t} \|\widetilde{h}^n\|_{\mathscr{C}}^2 \leq C \sum_{n=0}^{s} |||\widetilde{u}_{n,\cdot}|||_{\mathbb{Z}^2}^2.$$

As a consequence we have that

$$\sup_{n\geq 0} e^{-2\gamma n\Delta t} |||w^n|||_{\mathscr{R}}^2 + \frac{\gamma}{\gamma\Delta t + 1} \sum_{n\geq 0} \Delta t e^{-2\gamma n\Delta t} |||w^n|||_{\mathscr{R}}^2 + \sum_{k=1}^{2} \sum_{n\geq 0} \Delta t \Delta x_{3-k} e^{-2\gamma n\Delta t} \|w^n\|_{\mathscr{B}_k}^2 \leq C \sum_{n=0}^{s} |||\widetilde{u}_{n,\cdot}|||_{\mathbb{Z}^2}^2,$$

and the same estimate holds for (v_j^n) (by (49) for $P_1 = r_1$ and $P_2 = r_2$). We thus obtain (24) by the triangle inequality and from the fact that, in view of its definition, $|||\widetilde{u}_{n,\cdot}|||_{\mathbb{Z}^2}^2 = |||\widetilde{u}_{n,\cdot}|||_{\mathscr{R}}^2$.

□

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