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A Comment on Argumentation

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Abstract

We use the theory of defaults and their meaning of [GS16] to develop (the outline of a) new theory of argumentation.

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1 Introduction

1.1 Abstract description

Argumentation is about putting certain objects together. There are three things to consider:
(1) the objects themselves, and their inner structure (if they have any) - this inner structure may be revealed successively, or be immediately present,

(2) rules about how to put them together,

(3) avoid certain results (contradictions) in the resulting pattern.

To help intuition, we picture as result of an argumentation, an inheritance network the agents can agree on. This network may consist of strict and defeasible rules only, with no elements or sets it is applied to. Think of the argumentation going on when writing a book about medical diagnosis. This will not be about particular cases, but about strict and default rules. “Symptom x is usually a sign of illness y, but there are the following exceptions: …” In addition, the network might contain cycles. There is nothing wrong with cycles. Mathematics is full of cycles, equivalences and their proofs. But consider also the following: We work in the set of adult land mammals. “Most elefants weigh more than 1 ton.” “Most elements (i.e. adult land mammals) which weigh more than 1 ton are elefants.” There is nothing in principle wrong with this either - except, in reality, we forgot perhaps about hippopotamuses.

Arguments need not be contradictions to what exists already. They can be confirmations, elaborations, etc. For instance, we might have the default rule that birds fly, and clarify that penguins don’t fly. This is not a contradiction, but an elaboration.

1.2 The structure of the objects

Facts are either so simple that a dispute seems unreasonable. Or, they are a combination of basic facts and (default) rules, like, what I see through my microscope is really there, and not an artifact of some speck of dust on the lenses. For simplicity, facts will be basic, undisputable facts.

Expert opinion may be considered a default rule, where details stay unexplained, perhaps even unexplainable by the expert himself.

Rules (classical or defaults) have three aspects:

(1) the rule itself,

(2) the application of the rule,

(3) the result of the application of the rule.

Classical rules cannot be contested. We can contest their application, i.e. one of their prerequisites, or their result, and, consequently, their application. We can confirm their result by different means, likewise, their application.

Default rules are much more complicated, but not fundamentally different. Again, we can attack their application, by showing that one of the prerequisites does not hold, or, that we are in an (known) exceptional case. We can attack the conclusion, and, consequently, the rule, or its application. In particular, we may attack the conclusion, without attacking the application or the rule itself, by arguing that we are in a surprising exceptional case - and perhaps try to find a new set of exceptions. We can attack the default rule itself, as in the case of “normally, tigers are vegans”. We can confirm a rule by confirming its conclusion, or adding a new rule, which gives the same result. We can elaborate a default rule, by adding an exception set, stating that all exceptions are known, and give the list of exceptions, etc. We can stop homogenousness (downward inheritance) e.g. for Quakers which are Republicans, we stop inheriting pacifism (or its opposite). This is not a contradiction to the default itself, but to the downward inheritance of the default (or to homogenousness) by meta-default, to be precise.

Obviously, the more we add (possible) properties to the objects (here default rules), the more we can attack, elaborate, confirm.

In the following section, we describe our general picture:

(1) there is an arbiter which checks for consistency, and directs the discussion,
(2) how to handle classical arguments and resulting contradictions,
(3) how to handle default arguments.

2 The classical part

We suppose there is an arbiter, whose role is to check consistency, and to authorise participants to speak.
If the arbiter detects an inconsistency, then he points out the “culprits”, i.e. minimal inconsistent sets. As he
detects inconsistencies immediately, the last argument will be in all those sets. The last argument need not be
the problem, it might be one of the earlier arguments.
He asks all participants if they wish to retract one of the arguments involved in at least one minimal inconsistent
set. (They have to agree unanimously on such retraction.) If there is no minimal inconsistent set left, the
argumentation proceeds with the “cleaned” set of arguments, as if the inconsistency did not arise. Of course,
arguments which were based on some of the retracted arguments are now left “hanging in the air”, and may be
open to new attacks.
If not, i.e. at least one minimal inconsistent set is left, the participants can defend (and attack) the arguments
involved in those sets. The arbiter will chose the argument to be attacked/defended. See Example 2.1 (page 3)
below. Suppose $\alpha$ is the argument chosen, then a defense will try to prove or argue for $\alpha$, an attack will try to
prove or argue for $\neg \alpha$, i.e. it is possible or consistent that $\neg \alpha$. In particular, an attacker might try to prove $\bot$,
or some other unlikely consequence of $\alpha$ (and some incontested $\beta$’s), and he need not begin with some $\alpha \rightarrow \gamma$,
it might be a more roundabout attack.
If at least one minimally inconsistent set is left with all elements defended, then there is a deadlock, and the
arbiter declares failure.
Consider

Example 2.1

We argue semantically. Let $A := \{x, a\}$, $B := \{x, b\}$, $C := \{x, c\}$, $Y := \{a, b, c\}$. Let $Y$ be the last set added.
For $A, B, C$, the situation is symmetrical. Let $Z \neq Z'$ be $A, B, or C$, then $Y \cap Z \neq \emptyset$, but $Y \cap Z \cap Z' = \emptyset$, $Z \cap Z' = \{x\}$, etc. Moreover, $A \cap B \subseteq C$, etc. Thus, $A$ and $B$ together are an argument for $C$, etc., so they
argue for each other, and there is no natural way to chose any of $A, B, C$ to be attacked. Thus, it is at the
discretion of the parties involved (or the arbiter) to chose the aim of any attack - apart from $Y$, which is not
supported by any of $A, B, C$. Still, $Y$ might in the end be the strongest argument.
We may add $D, E$, with $D := \{x, d\}$, $Y := \{a, b, c, d\}$ etc., the example may be extended to arbitrarily many
sets.

At any moment, any argument can be attacked, not only if an inconsistency arises. We may continue an
argumentation, even if not all minimally inconsistent subsets are treated as yet, but the arbiter has to keep
track of them, and of the use of their elements. They and their consequences may still be questioned.

3 Defaults

3.1 The classical part of defaults

We see defaults primarily not as rules, but as relatively complicated classical constructions, which we may see
as objects for the moment. The default character is in applying those objects, not in the objects themselves.
We follow here the theory described in Chapter 11 of [GS16], and for convenience of the reader, we repeat in
Section 5 (page 7) (essentially) Section 11.4 of [GS16].
In our view, a (semantical) default $(X : Y)$ says:

(1) “most” elements of $X$ are in $Y$, }
(2) there may be exception sets $X_1, X_2, \text{etc.}$ of $X$, where the elements are “mostly” not in $Y$ (but $X_1 \cup X_2 \cup \ldots$ has to be a “small” subset of $X$),

(3) in addition, there may be a “very small” subset $X' \subseteq X$, which contains “surprise elements” (i.e. not previously known exceptions), which are not in $Y$;

(4) in addition, we may require that subsets of $X$ “normally” behave in a homogenous way.

The notions of “most”, “small” etc. are left open, a numerical interpretation suffices for the intuition. These notions are discussed in depth e.g. in [GS08f] and [GS10].

Introducing a default has to result in a (classically) consistent theory. E.g., it must not be the case that $\forall x \in X \, x \not\in Y$, this contradicts the first requirement about defaults (and any reasonable interpretation of “most”).

3.2 The default part of defaults

This leads to a hierarchy as defined in Section 5.1 (page 10) below. We use the hierarchy to define the use of the defaults.

To use the standard example with birds, penguins, fly, we proceed as follows. Suppose we introduce a bird $x$ into the discussion. We try to put $x$ as low as possible in the hierarchy, i.e. into the set of birds, but not into any known exception set, and much less into any “surprise” set. Only (classical) inconsistency, as checked by the arbiter, may force us to climb higher. Thus, unless there is a contradiction, we let $x$ fly.

3.3 Attacks against defaults and their conclusions

Classical rules are supposed to be always true. Thus, classical rules themselves cannot be attacked, and an attack against a classical conclusion has to be an attack against one of its prerequisites.

Attacks against defaults can be attacks against

(1) the rule itself,

(2) one of the prerequisites,

(3) membership in or not in one of the exception sets,

(4) membership in or not in the surprise set,

(5) perhaps even the notions of size involved,

(6) etc.

Each component of a default rule may be attacked.

4 Comments

We assume that there is no fundamental difference between facts and conclusions: Usually, we were told facts, remember facts, have read facts, observed facts (perhaps with the help of a telescope etc.). These things can go wrong. Situations where things are obvious, and no error seems humanly possible, will not be contradicted.

4.1 Auxiliary elements

We now introduce some auxiliary elements which may help in the argumentation.
(1) “I agree.”
This makes an error in this aspect less likely, as both parties agree - but still possible!

(2) “I confirm.”
I am very certain about this aspect.

(3) Expert knowledge:
Expert knowledge and its conclusions act as “black box defaults”, which the expert himself may be unable
to analyse. Other experts (in the same field) will share the conclusion. (This is simplified, of course.)
(One way to contest an expert’s conclusion is to point out that he neglected an aspect of the situation,
which is outside his expertise. His “language of reasoning” is too poor for the situation.)

(4) The arbiter may ask questions.

4.2 Examples of attacks

(1) Defaults:
Normally, there is a bus line number 1 running every 10 minutes between 10 and 11 in the morning.
Attack: No, the conclusion is wrong.
Question: Why?
Elaboration:

(1.1) No, the default is wrong (e.g.: it is line number 2 running every 10 minutes).
(1.2) Yes, but this is not homogenous, i.e. does not break down to subsets, and we know more. (For
instance, we know that today is Tuesday or Wednesday, and it runs that often only Monday, Thursday,
Friday, Saturday, Sunday - but we do not know this, only that is does not apply to all days of the
week.)
(1.3) Yes, but today is an exception, and we know this. (e.g., we know that today is Tuesday, and we
know that Tuesday is an exception.) (In addition, there might be exceptional Tuesdays, Christmas
market day, etc. . . . )
(1.4) Yes, but I do not know why this is an exception. (This is a surprise case, I know about different
days, but today should not be an exception, still I was just informed that it does not hold today.)
We do not attack the default, nor the applicability - but agree that it fails here.

(2) Classical conclusions:
From $A$ and $B$, $C$ follows classically.
Attack: $C$ does not hold.
Question: Why?
Elaboration:

(2.1) $A$ does not hold or $B$ does not hold, but I do not know which.
(2.2) $A$ does not hold.
(2.3) $B$ does not hold.
(2.4) $A$ does not hold, and $B$ does not hold.

(3) Fact: $A$ holds.
Attacks: No, $A$ does not hold.
Question: Why?
Elaboration:
(3.1) You remember incorrectly.
(3.2) You did not observe well.
(3.3) Your observation tools do not work.
(3.4) You were told something wrong.
(3.5) etc.

(4) Expert knowledge, expert concludes that $A$.
   Attack: $A$ does not hold.
   Question: Why?
   Elaboration:
   The situation involves aspects where you are not an expert. It is beyond your language. (Of course, the expert can ask for elaboration . . . .)
5 Appendix - Section 11.4 in [GS16]

Definition 5.1
If $\alpha \not\sim \phi$ or $\alpha \not\not\sim \phi$, we say that the default $\phi$ is attached to $\alpha$.

Given any fixed default theory, let $\mathcal{A}$ be the set of $\alpha$ (or $M(\alpha)$), to which some default is attached.

We work here in propositional logic, and on the semantic level.

We assume a classical background theory $B$, and a set of classical formulas $\alpha_1, \ldots, \alpha_n$ to which defaults are attached, $\alpha_i \not\sim \phi_i,1, \ldots, \alpha_i \not\sim \phi_i,j_i$, where some or all of the $\not\sim$ may also be $\not\not\sim$ (without being necessarily $\not\sim \neg$).

Condition 5.1
We assume the following consistency conditions:

1. $B$ is classically consistent
2. For each $\alpha_i$, the defaults attached to $\alpha_i$ together with $B$ are jointly consistent.
   
   In particular, the theory of defaults attached to one of the $\alpha_i$ must be consistent. For instance, $\alpha_i \vdash \alpha_j$ and $\alpha_i \not\sim \neg \alpha_j$ together are inconsistent. We thus rule out default theories like $\{\phi/\phi, \neg \phi/\neg \phi\}$.

A negated default like $\alpha \not\not\sim \phi$ needs a model of $\alpha \land \neg \phi$ to be consistent, so we replace $\alpha \not\not\sim \phi$ by $\alpha \land \neg \phi$ for the consistency check.

The $\vdash$-relation between the $\alpha_i$’s gives a specificity relation by strict inclusion. We use this for the inheritance relation, and to solve (some) conflicts.

Definition 5.2
This definition describes how to obtain a consistent default theory at every point in the universe, using a theory revision approach, with specificity solving some conflicts. Of course, modifications are possible, but the general idea seems sound.

We define the set of valid defaults at some point. This influences the relation $\sqsubset$ as defined in Definition 5.4 (page 9), but not the relation $\subset$ as defined in Definition 5.5 (page 10), as the latter depends only on the sets to which defaults are attached, and not which defaults are attached.

We consider now some classical formula $\beta$ - it may be one of the $\alpha_i$’s to which defaults are attached, or not.

1. Visible defaults at $\beta$
   
   The defaults visible at $\beta$: All defaults attached to some $\alpha_i$ such that $\beta \vdash \alpha_i$ are considered visible at $\beta$.

2. Valid defaults at $\beta$
   
   (2.1) The visible defaults are ordered by the $\vdash$ relation between the $\alpha$’s to which they are attached. The more specific ones are considered stronger defaults (for this $\beta$).
   
   Of course, we can plug in here any partial relation, if it seems more suitable, e.g. $\sim$ itself, as is often done in defeasible inheritance networks.

   (2.2) Consider now the set of visible defaults, together with the classical information available at $\beta$. (If $\beta \vdash \beta$, but not conversely, we need not consider $\beta'$, etc.) If this set is inconsistent:
   
   (2.2.1) Consider first in parallel all minimal inconsistent subsets involving classical information. They must contain at least one default, as the classical information was supposed to be consistent. Eliminate simultaneously from each such set the weakest (by the $\vdash$ relation) defaults. (If there are several weakest ones, eliminate all the weakest ones.)
   
   As contradictions involving classical, i.e., strongest information, seem to be more serious, we do these sets first.
(2.2.2) Consider now all remaining minimal inconsistent subsets of default information only. (Note that some might already have been eliminated in the previous step, but for other reasons.) Proceed as in the previous step, i.e. eliminate the weakest information.

(2.2.3) We call the remaining defaults, visible at \( \beta \), the defaults valid at \( \beta \). We will now work with the valid defaults only.

We now define “relevant” sets, sets where some default may change. We work in some fixed default theory, recall from Definition 5.1 (page 7) that \( \mathcal{A} \) is the set of \( \alpha \) or \( M(\alpha) \) to which at least one default is attached.

**Definition 5.3**

Consider the model variant of \( \mathcal{A} \).

Let \( S(\mathcal{A}) := \{ \bigcap A' : A' \subseteq \mathcal{A} \} \) and \( U(\mathcal{A}) := \{ \bigcup A' : A' \subseteq \mathcal{A} \} \),

Let \( \mathcal{R} := \{ X - Y : X - Y \neq \emptyset, X \in S(\mathcal{A}), Y \in U(\mathcal{A}) \} \),

where \( X = U \) (the universe), and \( Y = \emptyset \) are possible.

\( R \in \mathcal{R} \) is called a relevant set (or formula).

Let \( \mathcal{R}_f \) be the \( \subseteq \)-minimal elements of \( \mathcal{R} \) - “f” for finest.

Obviously, the elements of \( \mathcal{R}_f \) are pairwise disjoint. The motivation for \( \mathcal{R}_f \) is the following. Let \( X - Y \in \mathcal{R}_f \), then all elements of \( X - Y \) should satisfy all defaults valid at \( X \), but no other, as they are in no other sets to which defaults are attached.

**Example 5.1**

Let \( A, A' \subseteq U, A'' \subseteq A' \), let defaults be attached to \( A, A', A'' \), let all resulting intersections and set-differences be non-empty, if possible, e.g. \( A'' - A \neq \emptyset \).

See Diagram 5.1 (page 9) for illustration.

So \( \mathcal{A} = \{ A, A', A'' \} \),


and

\( \mathcal{R}_f = \{ U - A - A', A - A', A' - A - A'', A'' - A, A \cap A' - A'', A \cap A'' \} \)

It is useful to code the sets \( A, A', A'' \) by 3 bits, e.g. the left bit codes \( A \), the middle one \( A' \), the right one \( A'' \).

We then have

\( \mathcal{R}_f = \{ U - A - A' = 000, A - A' = 100, A \cap A' - A'' = 110, A \cap A'' = 111, A'' - A = 011, A' - A - A'' = 010 \} \),

where the codes 001, 101 do not exist, as \( A'' \subseteq A' \).

We will order the models by groups, and within groups by their “quality” satisfying defaults. We first define the latter relation, to be denoted \( \sqsubseteq \). It will be used to order the models within the sets \( \omega(A) \), see Construction 5.1 (page 11).
**Definition 5.4**
Suppose we are at some relevant $\beta$, with the valid defaults $\phi_1, \ldots, \phi_n$. We order the models of $\beta$ according to satisfaction of the $\phi_i$. There are different possibilities to define $\sqsubseteq$:

1. By subsets: if the set of $\phi_i$ satisfied by $m$ is a subset of those satisfied by $m'$, then $m'$ is better than $m$, $m' \sqsubseteq m$.

2. By cardinality: if the set of $\phi_i$ satisfied by $m$ is a smaller than the set of those satisfied by $m'$, then $m'$ is better than $m$, $m' \subset m$.

3. Some more complicated order, preferring certain defaults over others. (This might be interesting for contrary-to-duty obligations. Note that we solve here the case of additional information in real-life situations, too.)

4. In particular, we may order the valid defaults by specificity of the $\alpha$ to which they are attached, and satisfy the most specific ones first, then the next, etc., resulting in a lexicographic order.

Note that the construction is more adapted to sets with rare exceptions, than to a classification, like vertebrates into fish, amphibians, reptiles, birds, and mammals, where all are exceptions - and those outside these subsets are the real exceptions.
The dotted line separates $\omega(X)$ from $\mu(X)$

Example, the sets

Diagram 5.1

5.1 The construction

Note that, again, the following definition is independent of the actual defaults, and uses only the fact that defaults are attached to the $A \in \mathcal{A}$. The order to be constructed, $\ll$, will be used as scaffolding in the construction of $\prec$ in Construction 5.1 (page 11), which orders packets of models.

(See Example 5.2 (page 12) and Diagram 5.1 (page 9) and Diagram 5.2 (page 12).)

Recall Definition 5.3 (page 8).

Definition 5.5
We order $R_f$ as follows:

$X \prec Y$ iff $\{A_i : X \subseteq A_i\} \subset \{A_i : Y \subseteq A_i\}$ for $X, Y \in R_f$,
i.e. by the subset relation, of the $A_i$ they are in.

This order expresses exceptionality of the $X \in R_f$. If $X \prec Y$, then $Y$ is a subset of more (by the subset relation) $A_i$ than $X$ is, defaults were attached to the $A_i$, so the $A_i$ are sets of exceptions. For instance, $A_0$ might be the set of birds, $A_1$ the set of penguins, being a penguin (an element of $A_0$ and $A_1$) is more exceptional than being a bird (an element of $A_0$ only).

Recall the coding of the $X \in R_f$ in Example 5.1 (page 8). This example is continued in Example 5.2 (page 12), and $\prec$ is then (the transitive closure of) $U - A - A' - A''$ (000) $\prec A - A'$ (100) $\prec A \cap A' - A''$ (110) $\prec A \cap A''$ (111), $U - A - A' - A''$ (000) $\prec A' - A - A''$ (010) $\prec A'' - A$ (011) $\prec A \cap A''$ (111), $A' - A - A''$ (010) $\prec A \cap A' - A''$ (110).

See Diagram 5.2 (page 12).

We are now ready to construct the preferential relation $\prec$ between model sets, it is implicitly extended to their elements.

Take e.g. in Example 5.2 (page 12) the set $A \cap A' - A''$. It will “see” the defaults valid for $A \cap A'$, but not necessarily those for $A''$. Every element of $A \cap A' - A''$ can satisfy all those defaults, or only a part of them (or none). The best elements are those which satisfy all, the worst those which satisfy none. The precise relation is described by $\sqsubset$, see Definition 5.4 (page 9). Let us call $\mu(X)$ the set of those elements which satisfy all defaults, the set of the others $\omega(X)$, for given $X$. Elements which do not satisfy all defaults are doubly exceptional, as they are not in any subclass where this failure is “explained”. They are the “unexcused” exceptions, they are surprises, and doubly exceptional. So they should sit higher up in the hierarchy.

We do not think that there is a unique reasonable solution. Two ideas come to mind:

(1) Put all sets $\omega(X)$ above all other elements, in their own hierarchy, defined by $\prec$, and ordered internally by $\sqsubset$. This is the radical approach.

(2) A less radical idea is to put them above the immediate $\prec$-successors of $X$. E.g. $\omega(A' - A'' - A')$ will sit above $A \cap A' - A''$ and $A' \cap A'' - A$ - or, rather, above $\mu(A \cap A' - A'')$ and $\mu(A' \cap A'' - A)$. This is what we will do here.

Construction 5.1

(See Example 5.2 (page 12) and Diagram 5.3 (page 13).)

We define a “packetwise” order $\prec$ between model sets, it will then be extended to the elements.

The relation $\prec$ has two parts:

(1) $\mu(X) \prec \mu(Y)$ iff $X \sqsubset Y$

for $X, Y \in R_f$

(2) $\mu(X) \prec \omega(Y)$ iff $X \sqsubset Y$ or $X = Y$ or $X$ is a direct $\prec$-successor of $Y$

for $X, Y \in R_f$.

Note:

(1) We do not continue the order above the $m \in \omega(A)$, they are maximal elements - except for the interior order $\sqsubset$ among themselves.

(2) The $\mu(X)$ are “flat”, all are best possible, There is no interior order in this set $\mu(X)$.

(3) The inside structure of the $\omega(X)$ is given by $\sqsubset$ according to Definition 5.4 (page 9) for this $X$.

(4) If any of the sets is empty, we just omit it and close under transitivity.
More formally, for $X, X' \in \mathcal{R}_f$, and element wise:

$$m \prec m' := \begin{cases} 
\text{there is } X \text{ and } m, m' \in \omega(X) \text{ and } m \sqsubseteq m' \\
\text{or} \\
\text{there are } X, X' \text{ and } m \in \mu(X) \text{ and } m' \in \mu(X') \text{ and } X \vartriangleleft X' \\
\text{or} \\
\text{there are } X, X' \text{ and } m \in \mu(X) \text{ and } m' \in \omega(X') \text{ and } (X \vartriangleleft X' \text{ or } X = X' \text{ or } X \text{ is a direct } \vartriangleleft \text{-successor of } X')
\end{cases}$$

The final order between models is thus basically lexicographic with two parts, first by $\prec$, then by $\sqsubseteq$.

**Example 5.2**

We continue Example 5.1 (page 8), see also Diagram 5.1 (page 9).

Recall that $\mathcal{A} = \{A, A', A''\}$, and $\mathcal{R}_f = \{U - A - A' - A'' = 000, A - A' = 100, A \cap A' - A'' = 110, A \cap A'' = 111, A'' - A = 011, A' - A - A'' = 110\}$.

1. We define $\prec$ by Definition 5.5 (page 10): $U - A - A' - A'' = 000 \prec A - A' = 100 \prec A \cap A' - A'' = 110 \prec A \cap A'' = 111, A'' - A = 011 \prec A' - A - A'' = 110 \prec A \cap A'' = 111$, $A' - A - A'' = 110 \prec A \cap A' - A'' = 110$, closed by transitivity.

See Diagram 5.2 (page 12).

2. We construct the relation between models according to Construction 5.1 (page 11). as follows (considering only the model sets, and neglecting the internal structure of the $\omega(X)$). Constructing the full relation (and closing under transitivity) is then trivial.

$$\mu(000) \prec \mu(100) \prec \mu(110) \prec \mu(111), \mu(000) \prec \mu(010) \prec \mu(011) \prec \mu(111), \mu(010) \prec \mu(110),$$

$$\mu(100) \prec \omega(000), \mu(010) \prec \omega(000), \mu(110) \prec \omega(010), \mu(011) \prec \omega(010), \mu(111) \prec \omega(110),$$

Finally, we close under transitivity.

See Diagram 5.3 (page 13).
Relation between sets

Diagram 5.2
Diagram 5.3

Relation between model sets
References

