# Conditional expected likelihood technique for compound Gaussian and Gaussian distributed noise mixtures

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Yuri I. Abramovich, *Fellow, IEEE*, Olivier Besson, *Senior Member, IEEE*, and Ben A. Johnson, *Senior Member, IEEE*

***Abstract*—Expected likelihood (EL) technique for quality assessment of parameter estimates of signals embedded in Gaussian noise is extended in this paper over the case where useful signals are immersed in a mixture of compound Gaussian and Gaussian-distributed noises. The main problem here is that analytical expressions for distributions of such mixtures do not exist in most cases. Moreover, in some cases like $K$-distributed noise only, where closed-form expressions for the data distribution are available, the traditional Cramér–Rao bound does not exist. This makes the EL technique even more important for parameter estimation performance assessment. In this paper, for the so-called conditional model, we introduce test statistics whose distribution for the true (actual) parameters does not depend on these parameters and specifics of texture distribution, which makes them applicable for EL applications. We illustrate the utility of this EL technique by studying and predicting the performance breakdown of some direction of arrival estimators in a mixture of $K$-distributed and Gaussian noise.**

***Index Terms*—Direction of arrival estimation, expected likelihood, noise mixture.**

## I. Introduction

Direction of arrival (DoA) estimation of multiple sources is a difficult problem that very rarely can be provided with the globally optimum solution, maximum likelihood (ML) in most cases. Indeed, only an exhaustive global search can guarantee strict optimality of its solution. For this reason, in most practical applications, different ML-proxy techniques are used. Most advanced are the so-called subspace-based techniques such as MUSIC [1] whereby instead of a global search over $P$ DoA, one-dimensional (1D) search for the $P$ maxima of the pseudo-spectrum function is required. In [2]–[4] it was proven that these techniques may be treated as large-sample equivalents of the accurate ML estimator for $T \to \infty$ independent and identically distributed (i.i.d.) Gaussian snapshots. In fact, it was proven in [2]–[4] that asymptotic ($T \to \infty$) DoA mean-square error (MSE) in this case tends to the Cramér-Rao bound. Similar results have been derived for other subspace techniques such as MODE [5] or ESPRIT [6].

However, it was proven in [7] that, when $M \to \infty, T \to \infty$ and $M/T \to c$, where $M$ is the number of array elements, the localization function of MUSIC is not consistent and that MUSIC is not capable of consistently resolving sources within $O(1/M)$ angular distance. This inconsistency manifests itself in the so-called threshold area where, due to small sample support $T$ and/or signal to noise ratio (SNR), the mean-square error of MUSIC is well above the CRB. This departure, which corresponds to the appearance of severely erroneous DoA estimates (outliers), occurs under threshold conditions $(T, \mathrm{SNR})$ where the genuine ML estimator still provides CRB-consistent estimation accuracy [8]. Mestre, based on random matrix theory, proposed in [7], [9], [10] an enhancement of MUSIC, namely G-MUSIC,which provides consistent estimates of the pseudo-spectrum function in the $M \to \infty, T \to \infty, M/T \to c$ regime. Johnson *et al.* [8] investigated in detail the performance of G-MUSIC compared to that of ML, and provided explanations on why it improves over MUSIC without yet filling the gap between G-MUSIC-specific and ML-intrinsic threshold conditions.

For MUSIC, G-MUSIC as well as for other methods, much attention has been devoted on how to detect ("predict") the onset of the threshold regime, and how to possibly rectify ("cure") these method-specific outliers. A pivotal tool for such analysis of performance breakdown prediction is the so-called expected likelihood (EL) technique developed in [11]–[14]. The original EL approach was derived in the Gaussian case to test whether a candidate matrix, say $\mathbf{R}_c$, is a plausible covariance matrix for the observed data $\mathbf{X} \in \mathbb{C}^{M \times T}$. It amounts to testing $H_0 : \mathcal{E}\left\{\mathbf{X}\mathbf{X}^H\right\} = \mathbf{R}_c$ against $H_1 : \mathcal{E}\left\{\mathbf{X}\mathbf{X}^H\right\} = \mathbf{R}$ where $\mathbf{R}$ is unknown, and relies on the likelihood ratio for this test. A fundamental property is that $LR(\mathbf{R}_0)$, where $\mathbf{R}_0$ is the true covariance matrix of $\mathbf{X}$, has a distribution that depends only on $M$ and $T$. Therefore, this distribution can be computed and used to assess the plausibility of $\mathbf{R}_c$. For DoA estimation problems where the covariance matrix depends on some parameter vector $\boldsymbol{\eta}$ (typically $\boldsymbol{\eta}$ includes DoA, sources power and white noise power), this approach enables one to assess the quality of an estimate $\hat{\boldsymbol{\eta}}$ by computing $LR(\mathbf{R}(\hat{\boldsymbol{\eta}}))$ and comparing it to $LR(\mathbf{R}(\boldsymbol{\eta}_0))$ where $\boldsymbol{\eta}_0$ is the true parameter vector. Naturally, the latter is not known a priori, but for the considered conditional [11], [12] and unconditional [14] Gaussian models, the

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introduced $LR(\mathbf{R}(\boldsymbol{\eta}_0))$ is shown to have a distribution that does not depend on $\boldsymbol{\eta}_0$, being specified by $(M, T)$ for unconditional models and by $(M, T, P)$ for conditional models. Specifically, it was demonstrated in [8] that MUSIC-produced sets of DoA estimates that contain an outlier generate LR values that are statistically far smaller than the ones specified by the probability density function (p.d.f.) of $LR(\mathbf{R}(\boldsymbol{\eta}_0))$. Thus the problem of statistically reliable prediction of MUSIC and G-MUSIC specific breakdown has been addressed in [8]. Analysis of LR values generated by G-MUSIC demonstrated that G-MUSIC-specific outliers also produce LR values outside of the $LR(\mathbf{R}(\boldsymbol{\eta}_0))$ distribution support. We also demonstrated that for certain small sample support $T$ and/or SNR values, even genuine ML technique implemented via search for global likelihood function (LF) maximum starts to consistently produce estimation outliers. Yet, in this case, these ML-specific outliers are as likely as the true DoA values, which means the onset of the limits of the ML estimation paradigm, since the ML is no longer associated with the CRB accuracy in this ML-breakdown region. This EL analysis established the gap between the ML-proxy (MUSIC, G-MUSIC) specific and MLE-intrinsic threshold conditions and led to techniques capable of outliers rectification by driving the LR values produced by rectified DoA estimates into support of the $LR(\mathbf{R}(\boldsymbol{\eta}_0))$ distribution. Majority of these results has been derived for Gaussian distributions although the EL approach was recently extended to complex elliptically symmetric distributions in [15]–[17], but mostly for the purpose of covariance matrix estimation.

In some applications, e.g., in HF direction finding applications, external noise is dominated by lightning strikes and being practically white for linear uniform arrays is strongly non-Gaussian. Also, in radar applications, the main source of external disturbance, in this instance clutter, is better described by a non-Gaussian distribution [18]–[21]. In [22], we addressed DoA estimation in pure $K$-distributed noise. The thermal Gaussian noise was neglected, on the ground that the non-Gaussian (external) to Gaussian (internal) noise ratio is generally high. Due to the availability of the p.d.f. of the observations in closed-form, it was possible to derive the CRB (whenever it existed) as well as the ML estimator. However, an outcome of [22] was that internal noise should be taken into account even for high external to thermal noise ratio as, with very spiky clutter, some texture values can fall well below the thermal noise power. Therefore in [23] we addressed the more complicated problem of multiple DoA estimation of useful signals immersed into a mixture of $K$-distributed and Gaussian noises. A practical iterative DoA estimation scheme was introduced and compared to potential competitors such as the recently proposed robust G-MUSIC (RG-MUSIC) technique [24], [25]. In the large sample support or high SNR scenario, most methods were close to a lower bound derived under the (rather unrealistic) assumption of known texture of the compound Gaussian noise. The latter bound was achieved by the corresponding (clairvoyant) ML estimator which assumes all texture values known. In addition to being a not realistic assumption, in [23] the search for ML estimates was conducted in the close vicinity of the true DoA values, and therefore it does not reflect the threshold behavior of the genuine (global search) ML estimator. Moreover, as expected, for small sample support and/or SNR values, we observed strong departure from the clairvoyant lower bound. Yet, the behavior of all methods in this threshold area is not fully understood. Indeed, a major problem here is that the analytical distribution for such a noise mixture does not exist in closed-form, and therefore availability of a CRB or of the exact ML estimator as references is still an open problem in this case. In fact, in absence of an analytical expression for the likelihood function, the standard approach where brute-force global ML search for the optimal DoA estimates in the threshold area enables one to reveal the ML threshold conditions is not applicable. Therefore, an approach, similar to EL, is definitely required that would be capable of evaluating the proximity of the likelihood metric for the derived DoA estimates to the statistical values of this metric calculated at the true DoA values. This is the aim of this paper.

Specifically, the paper is organized as follows. In Section II, we present the model at hand, briefly review the DoA estimators derived in [23] and we propose a new estimator which enables one to relax some assumptions made in [23]. Then, we introduce two test statistics which meet the requirements for EL application, i.e., their distributions, when evaluated at the true parameters, depend only on $(M, T, P)$. Therefore, they allow for assessment of the estimators introduced above, especially in the threshold area. This is the aim of the simulations of Section III where we evaluate the mean-square error of the various estimators and show that outliers can be safely detected by the EL approach. The analysis enables one to reveal the estimators-specific as well as the ML-intrinsic breakdown conditions.

## II. Data Model, DoA Estimators and Expected Likelihood Approach

### A. Data Model

The data model is essentially that of [23] and is given by

$$\mathbf{x}_t = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}_t + \sqrt{\tau_t}\mathbf{n}_t + \sigma_w \mathbf{w}_t \tag{1}$$

where $\boldsymbol{\theta} = \begin{bmatrix} \theta_1 & \cdots & \theta_P \end{bmatrix}^T$ is the vector of the DoA and $\mathbf{A}(\boldsymbol{\theta}) = \begin{bmatrix} \mathbf{a}(\theta_1) & \cdots & \mathbf{a}(\theta_P) \end{bmatrix}$ is the manifold matrix of the $M$-element array. We assume a conditional model for which the emitted waveforms $\mathbf{s}_t$ are treated as deterministic unknowns. $\mathbf{n}_t$ and $\mathbf{w}_t$ are independent and identically distributed vectors drawn from $\mathbf{n}_t \sim \mathbb{C}\mathcal{N}(\mathbf{0}, \mathbf{I})$ and $\mathbf{w}_t \sim \mathbb{C}\mathcal{N}(\mathbf{0}, \mathbf{I})$. $\sigma_w^2$ stands for the thermal noise power. $\tau_t$ is a positive variable which we choose to treat as deterministic unknown, similarly to $\mathbf{S}$. The choice of a conditional model rather than an unconditional model where $\mathbf{S}$ and $\boldsymbol{\tau}$ would be considered random with a prior distribution is due to the fact that these prior distributions may not be completely known and that derivation of maximum likelihood estimators is simpler in the conditional model.

### B. DoA Estimation

For the sake of clarity, we first provide a quick overview of the two main estimators derived in [23] and then we propose a new

TABLE I
APPROXIMATE MAXIMUM LIKELIHOOD ESTIMATION OF $\mathbf{A}$ AND $\boldsymbol{\tau}$ FOR KNOWN $\sigma_w^2$

**Input:** $\mathbf{X}$, initial estimate $\mathbf{A}^{(0)} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_P \end{bmatrix}$
1: **for** $n = 1, \ldots$ **do**
2: Estimate $\tau_t^{(n)} = \max\left(M^{-1}\mathbf{x}_t^H \mathbf{P}_{\mathbf{A}^{(n-1)}}^{\perp} \mathbf{x}_t, \sigma_w^2\right) - \sigma_w^2$
3: Estimate $\mathbf{A}^{(n)} = \mathcal{P}_P\left(\sum_{t=1}^T \left(\tau_t^{(n)} + \sigma_w^2\right)^{-1} \mathbf{x}_t \mathbf{x}_t^H\right)$
4: **end for**

exact ML estimator which relaxes some assumptions made in [23]. Under the stated assumptions, the p.d.f. of the data matrix $\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_T \end{bmatrix}$ is given by

$$p(\mathbf{X}; \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{S}) \propto \prod_{t=1}^T \left[\left(\tau_t + \sigma_w^2\right)^{-M} \exp\left\{-\left(\tau_t + \sigma_w^2\right)^{-1} \|\mathbf{x}_t - \mathbf{A}(\boldsymbol{\theta})\mathbf{s}_t\|^2\right\}\right] \quad (2)$$

where $\propto$ means proportional to, $\boldsymbol{\tau} = \begin{bmatrix} \tau_1 & \cdots & \tau_T \end{bmatrix}^T$ and $\mathbf{S} = \begin{bmatrix} \mathbf{s}_1 & \cdots & \mathbf{s}_T \end{bmatrix}$. In [23] we first investigated a *clairvoyant* maximum likelihood estimator of $\boldsymbol{\theta}$, assuming that $\boldsymbol{\tau}$ and $\sigma_w^2$ are known. This clairvoyant ML estimator of $\boldsymbol{\theta}$ was obtained as

$$\boldsymbol{\theta}_{|\boldsymbol{\tau}, \sigma_w^2}^{\mathrm{ML}} = \arg\min_{\boldsymbol{\theta}} \sum_{t=1}^T \frac{\mathbf{x}_t^H \mathbf{P}_{\mathbf{A}(\boldsymbol{\theta})}^{\perp} \mathbf{x}_t}{\tau_t + \sigma_w^2} \quad (3)$$

where $\mathbf{P}_{\mathbf{A}(\boldsymbol{\theta})}^{\perp} = \mathbf{I} - \mathbf{A}(\boldsymbol{\theta})\left(\mathbf{A}(\boldsymbol{\theta})^H \mathbf{A}(\boldsymbol{\theta})\right)^{-1} \mathbf{A}(\boldsymbol{\theta})^H$. In [23], only a local search for the maximum of the concentrated likelihood function in (3) was implemented, meaning that $\boldsymbol{\theta}_{|\boldsymbol{\tau}, \sigma_w^2}^{\mathrm{ML}}$ was sought in close vicinity of the true DoA $\boldsymbol{\theta}_0$. A lower bound, namely the averaged-over-$\boldsymbol{\tau}$ CRB conditioned on $\boldsymbol{\tau}$, i.e., $\mathcal{E}\left\{CRB(\boldsymbol{\theta}|\boldsymbol{\tau})\right\}$ was also derived. With the local search of the maximum implemented, it was observed that the mean-square error (MSE) was very close to this bound, whatever SNR. However, the performance of the genuine ML estimator, which consists of a global search for the maximum of $\sum_{t=1}^T (\tau_t + \sigma_w^2)^{-1} \mathbf{x}_t^H \mathbf{P}_{\mathbf{A}(\boldsymbol{\theta})}^{\perp} \mathbf{x}_t$ was not studied in [23], which would have revealed the ML-intrinsic performance breakdown conditions. We will fill this gap in the next section.

In order to relax the assumption of known $\tau_t$, we also considered an approximate maximum likelihood estimation of both the unstructured steering matrix $\mathbf{A}$ and $\boldsymbol{\tau}$, assuming only $\sigma_w^2$ is known. This method relies on the fact that, for a given $\mathbf{A}$, the MLE of $\tau_t$ is

$$\tau_{t|\mathbf{A}, \sigma_w^2}^{\mathrm{ML}} = \max\left(M^{-1}\mathbf{x}_t^H \mathbf{P}_{\mathbf{A}}^{\perp} \mathbf{x}_t, \sigma_w^2\right) - \sigma_w^2 \quad (4)$$

while, for known $\boldsymbol{\tau}$, the MLE of $\mathbf{A}$ is

$$\mathbf{A}_{|\boldsymbol{\tau}, \sigma_w^2}^{\mathrm{ML}} = \mathcal{P}_P\left(\sum_{t=1}^T \frac{\mathbf{x}_t \mathbf{x}_t^H}{\tau_t + \sigma_w^2}\right) \quad (5)$$

where $\mathcal{P}_P(.)$ stands for the $P$ principal subspace of the matrix between parentheses. The approximate maximum likelihood (AML) estimator of $\mathbf{A}$ and $\boldsymbol{\tau}$ was thus implemented through the iterative procedure described in Table I. We let $\mathbf{A}_{|\sigma_w^2}^{\mathrm{AML}}$ denote the value obtained at the end of the simulations. The DoA were then estimated as

$$\boldsymbol{\theta}_{|\sigma_w^2}^{\mathrm{AML}} = \mathrm{MUSIC}\left[\mathbf{A}_{|\sigma_w^2}^{\mathrm{AML}}\right] \quad (6)$$

where MUSIC [.] stands for the conventional MUSIC algorithm applied to a $M \times P$ matrix whose columns form a basis for the signal subspace.

In the present paper, we go one step further by assuming that neither $\boldsymbol{\tau}$ nor $\sigma_w^2$ are known and we derive the maximum likelihood estimator of $\boldsymbol{\theta}$ and $\gamma_t = \tau_t + \sigma_w^2$. Indeed, when $\sigma_w^2$ is unknown, one should consider $\gamma_t = \tau_t + \sigma_w^2$ as a whole unknown since the p.d.f. of the data matrix $\mathbf{X}$ is given by

$$p(\mathbf{X}; \boldsymbol{\theta}, \boldsymbol{\gamma}, \mathbf{S}) \propto \prod_{t=1}^T \left[\gamma_t^{-M} \exp\left\{-\gamma_t^{-1} \|\mathbf{x}_t - \mathbf{A}(\boldsymbol{\theta})\mathbf{s}_t\|^2\right\}\right] \quad (7)$$

where $\boldsymbol{\gamma} = \begin{bmatrix} \gamma_1 & \cdots & \gamma_T \end{bmatrix}^T$. It is readily verified that

$$\max_{\mathbf{S}} p(\mathbf{X}|\mathbf{A}(\boldsymbol{\theta}), \boldsymbol{\gamma}, \mathbf{S}) \propto \prod_{t=1}^T \gamma_t^{-M} \exp\left\{-\gamma_t^{-1} \mathbf{x}_t^H \mathbf{P}_{\mathbf{A}(\boldsymbol{\theta})}^{\perp} \mathbf{x}_t\right\}. \quad (8)$$

Moreover,

$$\max_{\gamma_t} \gamma_t^{-M} \exp\left\{-\gamma_t^{-1} \mathbf{x}_t^H \mathbf{P}_{\mathbf{A}(\boldsymbol{\theta})}^{\perp} \mathbf{x}_t\right\} \propto \left(\mathbf{x}_t^H \mathbf{P}_{\mathbf{A}(\boldsymbol{\theta})}^{\perp} \mathbf{x}_t\right)^{-M} \quad (9)$$

so that the ML estimator of $\boldsymbol{\theta}$ is obtained as

$$\begin{aligned} \boldsymbol{\theta}^{\mathrm{ML}} &= \arg\max_{\boldsymbol{\theta}} \prod_{t=1}^T \left(\mathbf{x}_t^H \mathbf{P}_{\mathbf{A}(\boldsymbol{\theta})}^{\perp} \mathbf{x}_t\right)^{-M} \\ &= \arg\max_{\boldsymbol{\theta}} -\sum_{t=1}^T \log\left(\mathbf{x}_t^H \mathbf{P}_{\mathbf{A}(\boldsymbol{\theta})}^{\perp} \mathbf{x}_t\right). \quad (10) \end{aligned}$$

In the simulations section, we will compare the performance of $\boldsymbol{\theta}^{\mathrm{ML}}$ to that of $\boldsymbol{\theta}_{|\boldsymbol{\tau}, \sigma_w^2}^{\mathrm{ML}}$.

### C. Expected Likelihood Approach

In order to assess the various estimators above and others, i.e., in order to detect potential outliers, we suggest to resort to the EL principle and, consequently, we look for a test statistic (actually a likelihood ratio) whose distribution, when evaluated at the true parameters, does not depend on $\mathbf{S}$ or $\boldsymbol{\tau}$ or $\boldsymbol{\theta}_0$ but only on known parameters. Then, to assess the quality of any given estimate, the test statistic evaluated at the estimated parameters will be checked against the test statistic at the true parameters. If a discrepancy is observed, typically when the LR for the estimated parameters fall outside the support of the LR for the true parameters, then the estimates are deemed erroneous. Towards this end, one could think of using the likelihood function in (10), evaluated at some estimate $\hat{\boldsymbol{\theta}}$, for assessment of the latter, by comparison with the LF evaluated at the true DoA $\boldsymbol{\theta}_0$. However, when evaluated at the true steering matrix $\mathbf{A}_0$, one has

$$\begin{aligned} \mathbf{x}_t^H \mathbf{P}_{\mathbf{A}_0}^{\perp} \mathbf{x}_t &= \left[\sqrt{\tau_t}\mathbf{n}_t + \sigma_w \mathbf{w}_t\right]^H \mathbf{P}_{\mathbf{A}_0}^{\perp} \left[\sqrt{\tau_t}\mathbf{n}_t + \sigma_w \mathbf{w}_t\right] \\ &= \left[\sqrt{\tau_t}\mathbf{n}_t + \sigma_w \mathbf{w}_t\right]^H \mathbf{U}_0 \mathbf{U}_0^H \left[\sqrt{\tau_t}\mathbf{n}_t + \sigma_w \mathbf{w}_t\right] \\ &\overset{d}{=} (\tau_t + \sigma_w^2) \tilde{\mathbf{w}}_t^H \mathbf{U}_0 \mathbf{U}_0^H \tilde{\mathbf{w}}_t \quad (11) \end{aligned}$$

where $\overset{d}{=}$ means "distributed as". In the previous equation, we used the fact that, when $\tau_t$ is deterministic, $\sqrt{\tau_t}\mathbf{n}_t + \sigma_w \mathbf{w}_t \sim \mathbb{CN}\left(\mathbf{0}, (\tau_t + \sigma_w^2)\mathbf{I}_M\right)$ and hence it is distributed as $\sqrt{\tau_t + \sigma_w^2}\tilde{\mathbf{w}}_t$ where $\tilde{\mathbf{w}}_t \sim \mathbb{CN}\left(\mathbf{0}, \mathbf{I}_M\right)$. Moreover $\mathbf{U}_0$ is an $M \times (M-P)$ semi-unitary matrix whose columns form an orthonormal basis for $\mathcal{R}(\mathbf{A}_0)^\perp$, the subspace orthogonal to $\mathbf{A}_0$, and hence $\mathbf{U}_0^H \tilde{\mathbf{w}}_t \sim \mathbb{CN}\left(\mathbf{0}, \mathbf{I}_{M-P}\right)$. This likelihood function depends on $\tau_t$, and hence its distribution is dependent on the statistics of $\tau_t$. To get rid of this dependency, one might consider, for any estimate $\hat{\boldsymbol{\theta}}$ and corresponding basis $\mathbf{U}(\hat{\boldsymbol{\theta}})$ for the associated noise subspace, the following data

$$\mathbf{z}_t(\hat{\boldsymbol{\theta}}) = \frac{\mathbf{U}(\hat{\boldsymbol{\theta}})^H \mathbf{x}_t}{\left\|\mathbf{U}(\hat{\boldsymbol{\theta}})^H \mathbf{x}_t\right\|} \tag{12}$$

which, for the true $\boldsymbol{\theta}_0$, follows a complex angular central Gaussian (ACG) distribution with parameter matrix $\mathbf{I}_{M-P}$, since

$$\mathbf{z}_t(\boldsymbol{\theta}_0) = \frac{\mathbf{U}_0^H \mathbf{x}_t}{\left\|\mathbf{U}_0^H \mathbf{x}_t\right\|} \overset{d}{=} \frac{\mathbf{U}_0^H \tilde{\mathbf{w}}_t}{\left\|\mathbf{U}_0^H \tilde{\mathbf{w}}_t\right\|}.$$

In other words, $\mathbf{z}_t(\boldsymbol{\theta}_0)$ is uniformly distributed on the unit sphere. It follows that quality of an estimate $\hat{\boldsymbol{\theta}}$ may be assessed by proximity of the scatter matrix of $\mathbf{z}_t(\hat{\boldsymbol{\theta}})$ to the identity matrix. This suggests the use of the traditional sphericity test [26]

$$ST\left(\hat{\boldsymbol{\theta}}\right) = \frac{\det(\mathbf{R}(\hat{\boldsymbol{\theta}}))}{\left[(M-P)^{-1}\mathrm{Tr}\{\mathbf{R}(\hat{\boldsymbol{\theta}})\}\right]^{M-P}} \tag{13}$$

where

$$\mathbf{R}(\hat{\boldsymbol{\theta}}) = T^{-1}\sum_{t=1}^{T} \mathbf{z}_t(\hat{\boldsymbol{\theta}})\mathbf{z}_t^H(\hat{\boldsymbol{\theta}}). \tag{14}$$

Note that, since $\mathbf{z}_t(\hat{\boldsymbol{\theta}})$ is unit norm, $ST(\hat{\boldsymbol{\theta}}) \propto \det(\mathbf{R}(\hat{\boldsymbol{\theta}}))$. The sphericity test measures the spreading of the eigenvalues of the matrix $\mathbf{R}(\hat{\boldsymbol{\theta}})$, whose sum is equal to one. The maximum is achieved when all eigenvalues are equal, which amounts to $\mathbf{R}(\hat{\boldsymbol{\theta}})$ being the identity matrix. The suggested here "expected likelihood" test is a monotonic function of $\det(\mathbf{R}(\hat{\boldsymbol{\theta}}))$ where the threshold for acceptance or rejection of $\hat{\boldsymbol{\theta}}$ is defined by the distribution of $\det(\mathbf{R}(\boldsymbol{\theta}_0))$, which is that of $\det(T^{-1}\sum_{t=1}^{T}\mathbf{z}_t(\boldsymbol{\theta}_0)\mathbf{z}_t^H(\boldsymbol{\theta}_0))$ with $\mathbf{z}_t(\boldsymbol{\theta}_0)$ uniformly distributed on the complex sphere. Note that, in the absence of an analytical expression for the p.d.f. of this statistic, the threshold is computed from Monte-Carlo simulations. The introduced test checks the "whiteness" of data after projection onto the subspace orthogonal to $\mathbf{A}(\hat{\boldsymbol{\theta}})$. The EL sphericity test meets all requirements: its distribution, when evaluated at the true DoA $\boldsymbol{\theta}_0$, does not depend on the DoA nor on the statistics of the texture distribution, and is only specified by $M$, $P$ and $T$.

The sphericity test, which is meant at testing the spreading of the eigenvalues of $\mathbf{R}(\hat{\boldsymbol{\theta}})$ can also be viewed a generalized likelihood ratio for testing whether the covariance matrix of a Gaussian distributed data is the identity matrix. Since, $\mathbf{z}_t(\boldsymbol{\theta}_0)$ follows a complex angular central Gaussian (ACG) distribution with parameter matrix $\mathbf{I}_{M-P}$, this suggests using the likelihood ratio for testing whether the scatter matrix of $\mathbf{z}_t(\hat{\boldsymbol{\theta}})$ is the identity matrix, considering that $\mathbf{z}_t(\hat{\boldsymbol{\theta}})$ follows an ACG distribution. Such generalized likelihood ratio test for ACG distributions was indeed derived in [16], [17]. Let $\mathbf{Z} = \begin{bmatrix}\mathbf{z}_1 & \cdots & \mathbf{z}_T\end{bmatrix}$ be a $(M-P) \times T$ matrix where each $\mathbf{z}_t$ follows a complex ACG distribution with scatter matrix $\boldsymbol{\Sigma}_0$. Let $\boldsymbol{\Sigma}_c$ be a candidate for this scatter matrix. Then the LR for testing $\boldsymbol{\Sigma}_c$ is

$$LR_{\text{ACG}} = \det(\boldsymbol{\Sigma}_{\text{ML}}\boldsymbol{\Sigma}_c^{-1})^T \prod_{t=1}^{T}\left[\frac{\mathbf{z}_t^H\boldsymbol{\Sigma}_c^{-1}\mathbf{z}_t}{\mathbf{z}_t^H\boldsymbol{\Sigma}_{\text{ML}}^{-1}\mathbf{z}_t}\right]^{-(M-P)} \tag{15}$$

where $\boldsymbol{\Sigma}_{\text{ML}}$ obeys

$$\boldsymbol{\Sigma}_{\text{ML}} = \frac{M-P}{T}\sum_{t=1}^{T}\frac{\mathbf{z}_t\mathbf{z}_t^H}{\mathbf{z}_t^H\boldsymbol{\Sigma}_{\text{ML}}^{-1}\mathbf{z}_t}. \tag{16}$$

As shown in [16], [17], this LR, when evaluated at the true scatter matrix $\boldsymbol{\Sigma}_0$ (equal to $\mathbf{I}_{M-P}$ in our case) has a distribution that depends on $M$, $P$ and $T$ only. Therefore, such a likelihood ratio, applied to $\mathbf{z}_t(\hat{\boldsymbol{\theta}})$, can serve as a means to detect if the data is white after projection. The difference compared to the sphericity test is that these LR are not based on the same assumptions. Note also that $LR_{\text{ACG}}$ is more complicated to compute than the sphericity test.

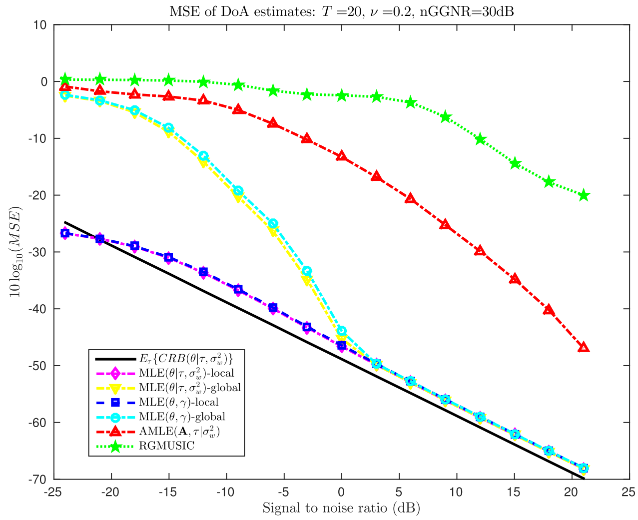
## III. Simulations

We consider a uniform linear array of $M = 20$ elements spaced a half-wavelength apart and a scenario with two sources, whose DoA are $\theta_1 = 16°$ and $\theta_2 = 18°$. The waveforms are generated from a Gaussian distribution with covariance matrix $P_s\mathbf{I}_2$. The compound-Gaussian noise follows a Weibull distribution: the $\tau_t$ are Gamma distributed with shape parameter $\nu$ and scale parameter $\nu^{-1}$ so that $\mathcal{E}\{\tau_t\} = 1$. We set $\nu = 0.2$ in the sequel. The total noise power is thus $1 + \sigma_w^2$ and the non Gaussian to Gaussian noise ratio is simply $\text{nGGNR} = 10\log_{10}\sigma_w^{-2}$. The signal to noise ratio is defined as $\text{SNR} = 10\log_{10}(P_s/(1+\sigma_w^2))$. $10^5$ Monte-Carlo simulations were run to estimate the mean-square error (MSE) of the various estimators.

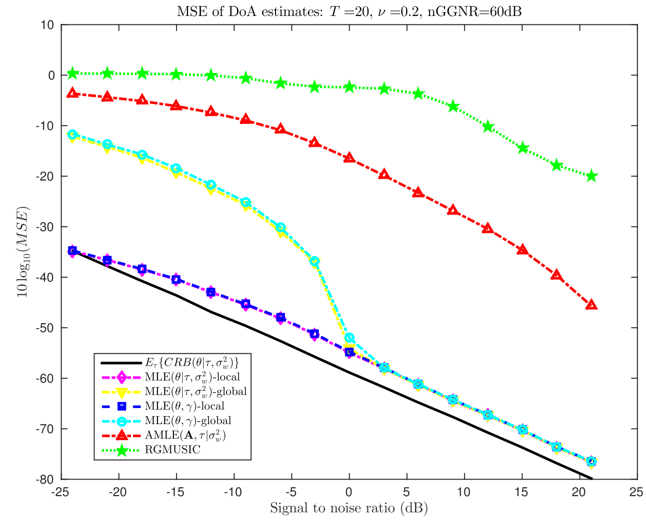
As for $\boldsymbol{\theta}^{\text{ML}}_{|\boldsymbol{\tau},\sigma_w^2}$ and $\boldsymbol{\theta}^{\text{ML}}$, the search for the maximum of the corresponding likelihood function was carried out using two protocols:

1) In a first implementation, the region $\{-\pi/2 \leq \theta_1 < \pi/2; \theta_1 < \theta_2)\}$ was sampled on a grid with step $\Delta\theta = 1°$ and the search for the maximum of the LF was carried out on this grid, yielding an initial estimate $(\theta_1^{(0)}, \theta_2^{(0)})$. Then, the search was refined on $[\theta_1^{(0)} - \Delta\theta/2, \theta_1^{(0)} + \Delta\theta/2] \times [\theta_2^{(0)} - \Delta\theta/2, \theta_2^{(0)} + \Delta\theta/2]$. This method is referred to as "global" in the figures, since it provides the global maximum of the LF.
2) A second implementation consists in searching the maximum of the LF in the vicinity of the true DoA: more precisely, the maximum was constrained to be at most a half beam-width apart from $\boldsymbol{\theta}_0$. This method is referred to as "local" in the figures.

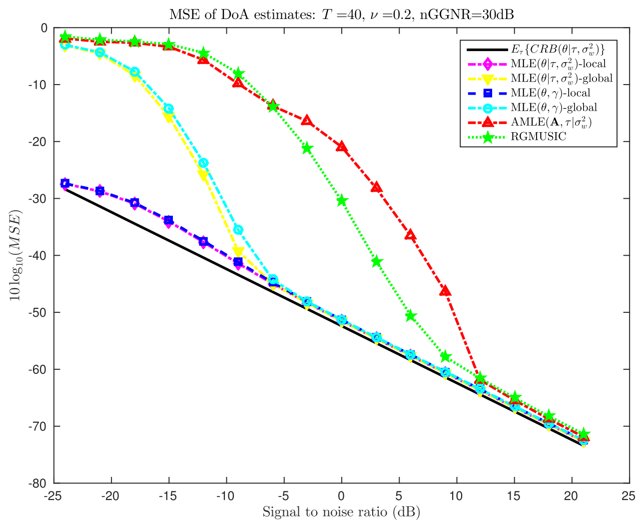
We also consider RG-MUSIC [24], [25] which computes a consistent estimate (in the random matrix theory sense) of the noise projection matrix from the eigenvalue decomposition of



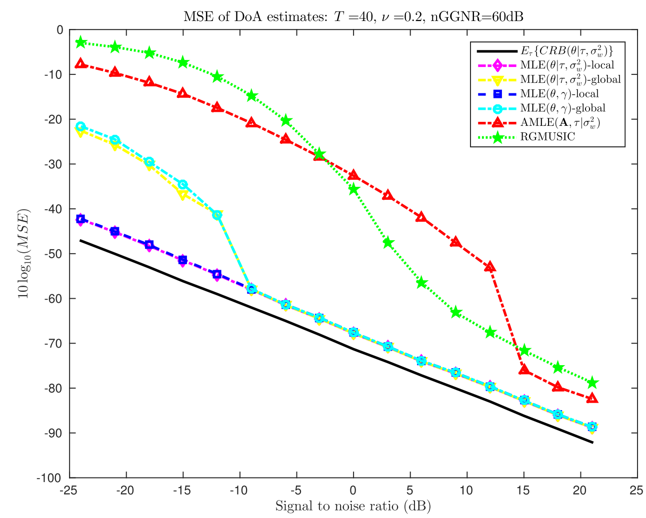

(a) $T = 20$, nGGNR = 30dB



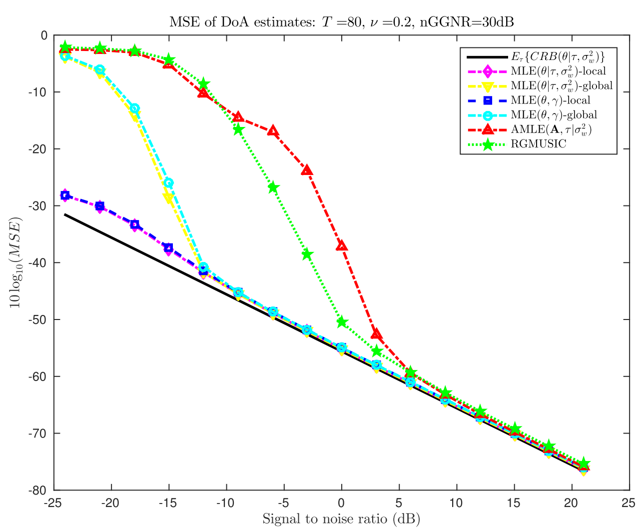

(b) $T = 20$, nGGNR = 60dB



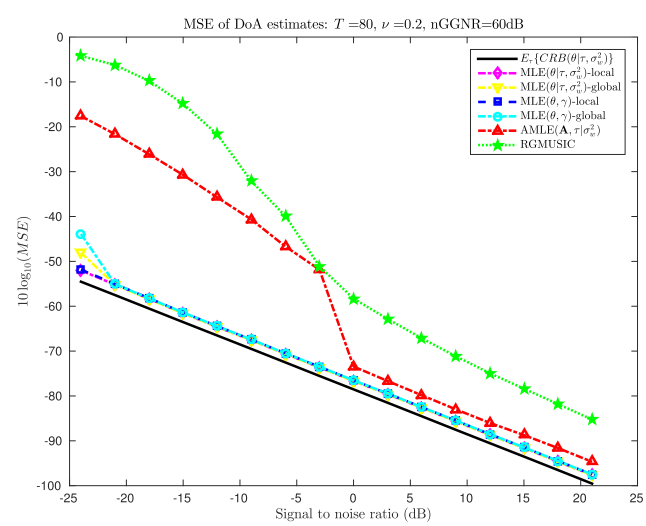

(c) $T = 40$, nGGNR = 30dB




(d) $T = 40$, nGGNR = 60dB




(e) $T = 80$, nGGNR = 30dB




(f) $T = 80$, nGGNR = 60dB

Fig. 1. Mean square error of estimators versus SNR. $\nu = 0.2$.

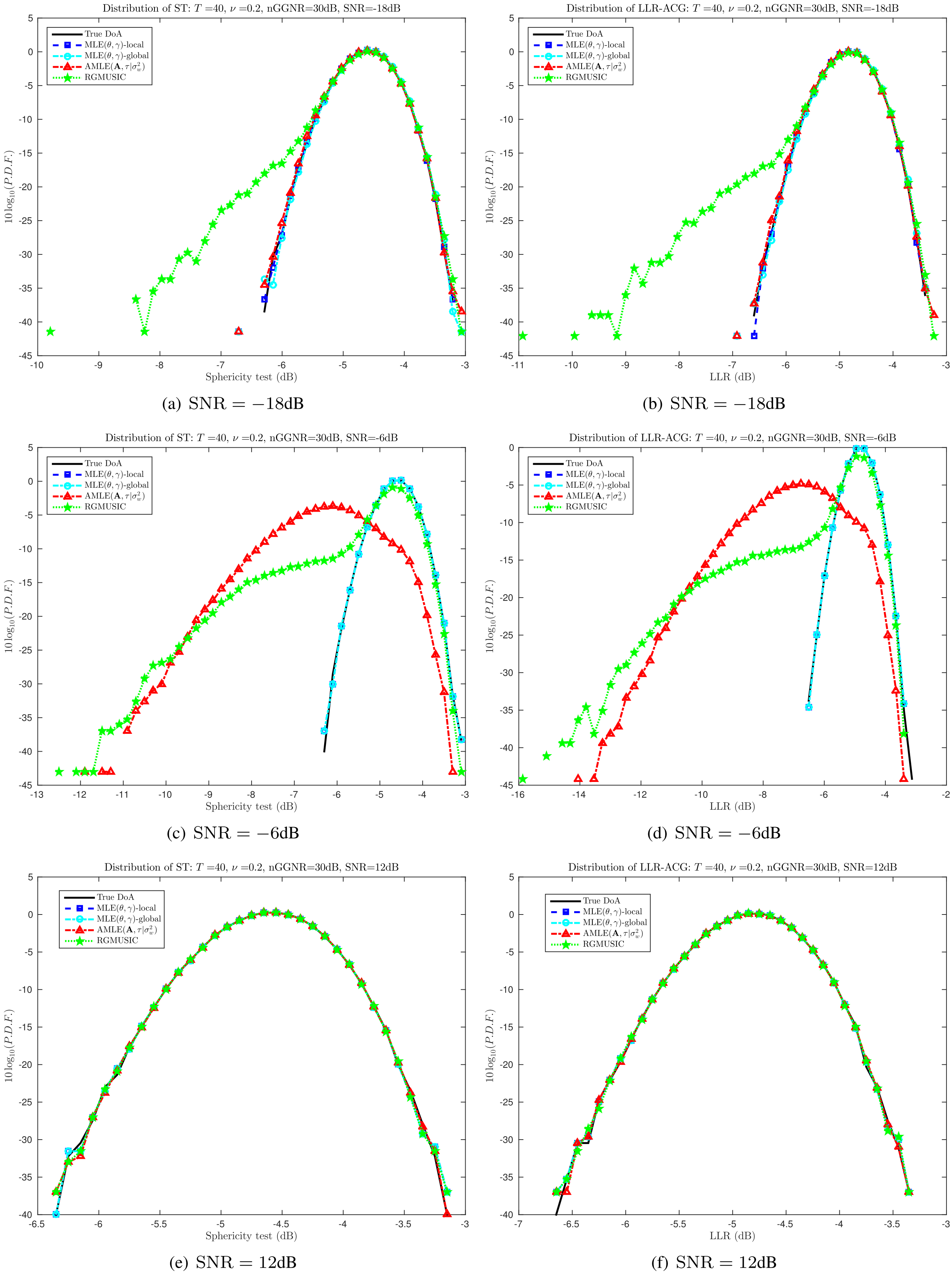
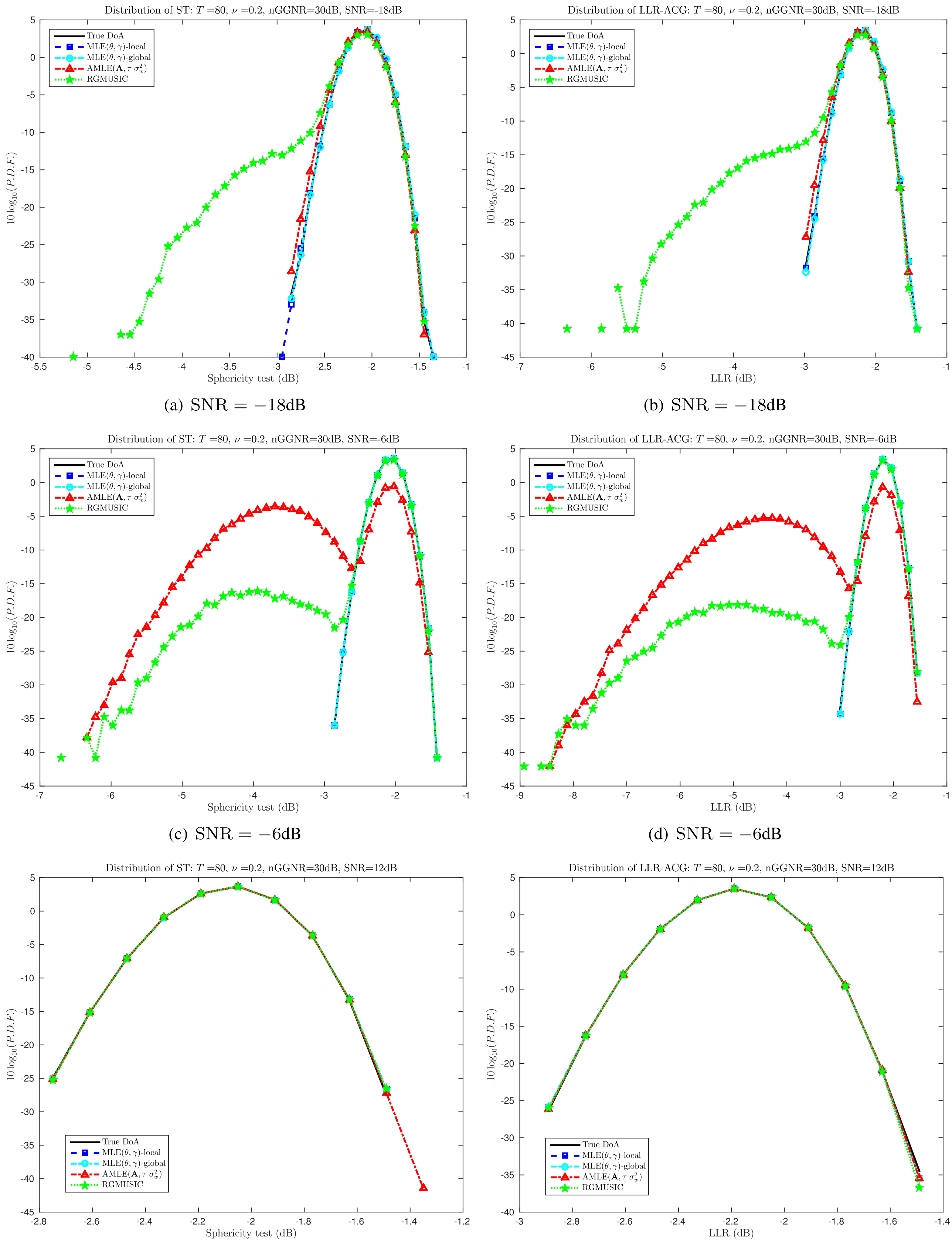



Fig. 2. Probability density function of sphericity test and $LLR_{\text{ACG}}$ for various SNR. $T = 40$, $\nu = 0.2$ and nGGNR $= 30$ dB.




(a) SNR $= -18$dB

(b) SNR $= -18$dB

(c) SNR $= -6$dB

(d) SNR $= -6$dB

(e) SNR $= 12$dB

(f) SNR $= 12$dB

Fig. 3. Probability density function of sphericity test and $LLR_{\text{ACG}}$ for various SNR. $T = 80$, $\nu = 0.2$ and nGGNR $= 30$ dB.

$\hat{\mathbf{R}}_{\text{RG-MUSIC}} = \lim_{k\to\infty} \hat{\mathbf{R}}_k$ with

$$\hat{\mathbf{R}}_{k+1} = \frac{1}{T}\sum_{t=1}^{T} u\left(\frac{1}{M}\mathbf{x}_t^H \hat{\mathbf{R}}_k^{-1}\mathbf{x}_t\right)\mathbf{x}_t\mathbf{x}_t^H .$$

In the simulations $u(x) = (1+\alpha)/(\alpha+x)$ with $\alpha = 0.01$.

### A. Mean-Square Error of DoA Estimates

We first focus on evaluating the mean-square error of the various estimators, with two main objectives: 1) reveal the MLE-intrinsic performance breakdown conditions i.e., the SNR conditions under which the global search for the maximum of the likelihood function completely fails in providing reliable estimates and 2) compare $\boldsymbol{\theta}^{\text{ML}}_{|\boldsymbol{\tau},\sigma_w^2}$ of [23] which assumes both $\tau_t$ and $\sigma_w^2$ are known to the present $\boldsymbol{\theta}^{\text{ML}}$ which does not make this assumption. Fig. 1 displays the MSE versus SNR. Two main observations can be made. First, $\boldsymbol{\theta}^{\text{ML}}_{|\boldsymbol{\tau},\sigma_w^2}$ and $\boldsymbol{\theta}^{\text{ML}}$ provide the same MSE, whatever the type of implementation, local or global search for the maximum. This means that $\boldsymbol{\theta}^{\text{ML}}$ is an interesting option as it does not require as many assumptions as $\boldsymbol{\theta}^{\text{ML}}_{|\boldsymbol{\tau},\sigma_w^2}$. Secondly, a significant difference is seen between the two implementations. When a local search is performed, both $\boldsymbol{\theta}^{\text{ML}}_{|\boldsymbol{\tau},\sigma_w^2}$ and $\boldsymbol{\theta}^{\text{ML}}$ stick to $\mathcal{E}\left\{CRB(\boldsymbol{\theta}|\boldsymbol{\tau})\right\}$ whatever SNR, which confirms that the latter is a local bound. In contrast, with the global search for the maximum, under some SNR threshold, both $\boldsymbol{\theta}^{\text{ML}}_{|\boldsymbol{\tau},\sigma_w^2}$ and $\boldsymbol{\theta}^{\text{ML}}$ significantly depart form the bound revealing the intrinsic-ML performance breakdown conditions. As was already observed for MUSIC and G-MUSIC, RG-MUSIC and $\boldsymbol{\theta}^{\text{AML}}_{|\sigma_w^2}$ begin to produce bad results at SNR where the global MLE is still consistent, and the difference can be significant, e.g., 20 dB at $T = 40$ and nGGNR $= 30$ dB. As will be shown below, the distribution of the sphericity test or the likelihood ratio enables one to capture this phenomenon.

### B. Distribution of the Likelihood Ratios and Detection of Outliers

In order to understand these different breakdown conditions, the sphericity test (ST) and $LLR_{\text{ACG}} = \log LR_{\text{ACG}}$ are instrumental, as illustrated in Figs. 2 and 3 where we plot their distributions, for three values of SNR: SNR $= -18$ dB where even ML breaks, SNR $= -6$ dB where ML produces consistent estimates while RG-MUSIC and $\boldsymbol{\theta}^{\text{AML}}_{|\sigma_w^2}$ do not, and SNR $= 12$ dB where all algorithms achieve the lower bound. It is clear that, when even MLE breaks down, it produces estimates whose ST or LLR is commensurate with that at the true DoA, which is a symptom of complete failure. On the other hand, for SNR $= -6$ dB, it is clearly seen that only RG-MUSIC and $\boldsymbol{\theta}^{\text{AML}}_{|\sigma_w^2}$ produce DoA estimates whose ST or LLR is outside the support of the p.d.f. for $\boldsymbol{\theta}_0$. Finally, at SNR $= 12$ dB, all estimates produce ST or LLR commensurate with those at $\boldsymbol{\theta}_0$.

We now illustrate how the EL principle, through examination of ST and $LLR_{\text{ACG}}$ enable one to *detect outliers*, i.e., severely erroneous DoA estimates. Towards this end, we proceeded in the following way. For each method, among the $10^5$ Monte-Carlo simulations, we decided that an estimate $\hat{\boldsymbol{\theta}}$ is a "true outlier" if $\|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\|^2$ is superior to 100 times the corresponding Cramér-Rao bound. Next, the sphericity test or the log likelihood ratio were computed for this $\hat{\boldsymbol{\theta}}$: whenever they are below a threshold, $\hat{\boldsymbol{\theta}}$ is declared an outlier according to the EL principle. The probability of correct outlier detection is the ratio of the number of declared outliers in the set of true outliers to the number of true outliers. The thresholds of the ST or the LR were set to ensure a probability of false alarm equal to $10^{-2}$ when ST or LLR are calculated with the true scatter matrix $\mathbf{I}_{M-P}$. Figs. 4 and 5 plot the actual percentage of outliers and the probability of detecting them by comparing the ST or LLR to a threshold. As can be observed, for very low SNR where even ML breaks down, there is no way to detect these outliers. However, in the threshold area, where ML is accurate but RG-MUSIC or $\boldsymbol{\theta}^{\text{AML}}_{|\sigma_w^2}$ produce erroneous estimates, the latter can be rather reliably

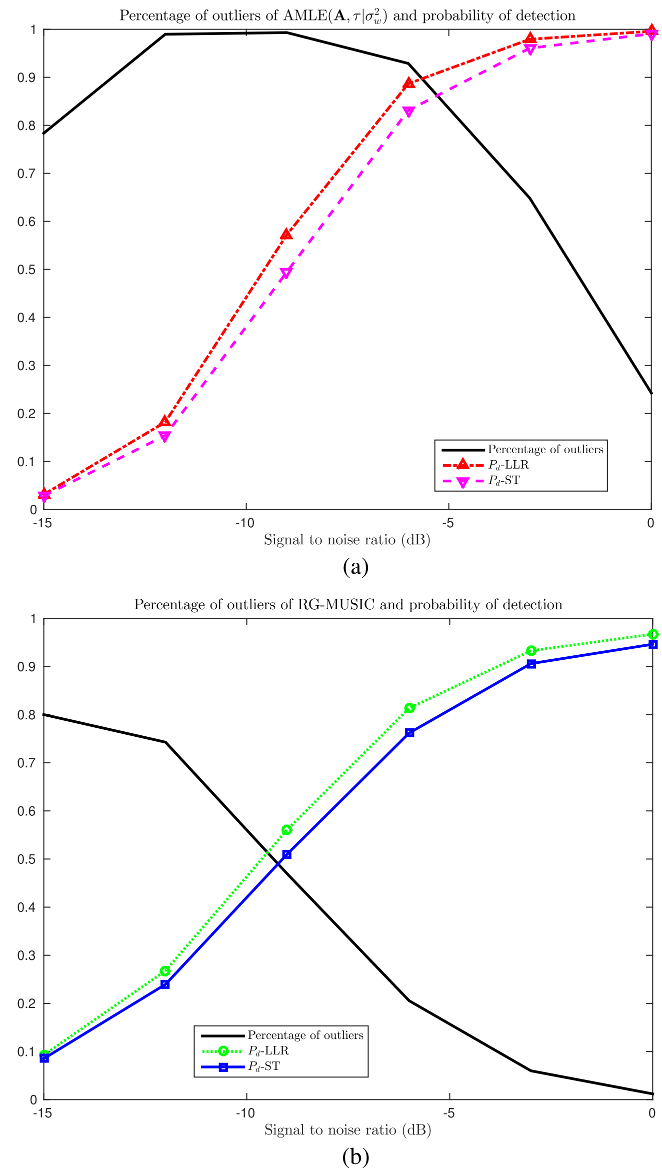



Fig. 4. Probability of detecting outliers using the EL approach for RG-MUSIC and $\boldsymbol{\theta}^{\text{AML}}_{|\sigma_w^2}$. $P_{fa} = 10^{-2}$, $T = 40$, $\nu = 0.2$ and nGGNR $= 30$ dB.

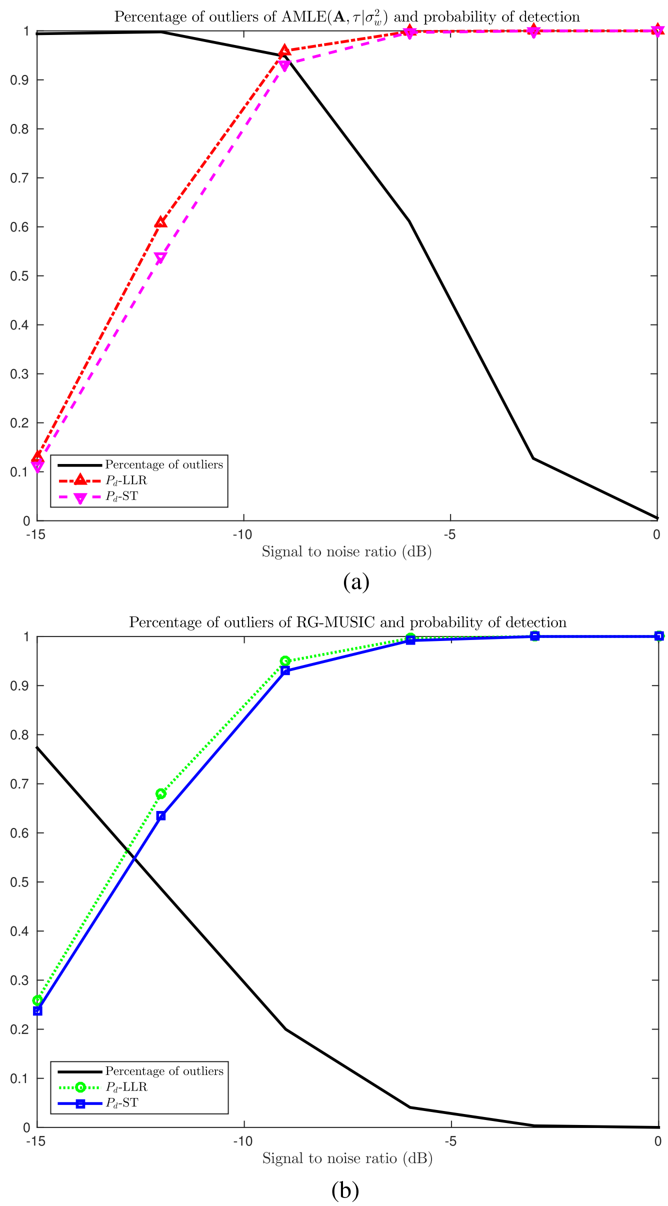



Fig. 5. Probability of detecting outliers using the EL approach for RG-MUSIC and $\boldsymbol{\theta}^{\text{AML}}_{|\sigma_w^2}$. $P_{fa} = 10^{-2}$, $T = 80$, $\nu = 0.2$ and nGGNR = 30 dB.

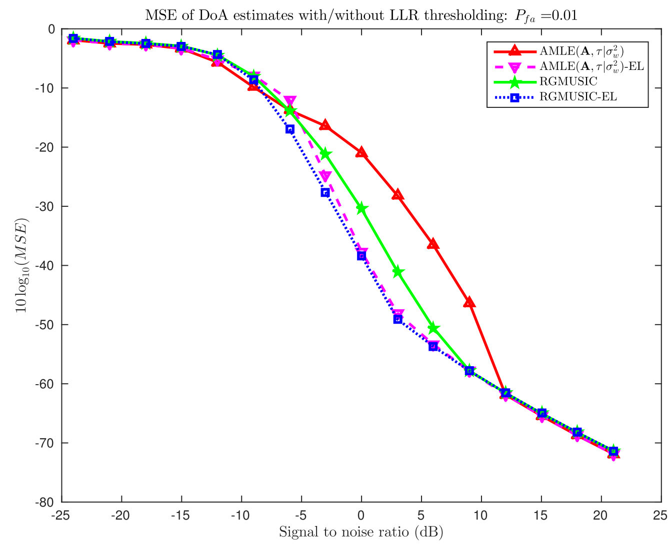



Fig. 6. MSE of RG-MUSIC and $\boldsymbol{\theta}^{\text{AML}}_{|\sigma_w^2}$ with/without thresholding . $P_{fa} = 10^{-2}$, $T = 40$, $\nu = 0.2$ and nGGNR = 30 dB.

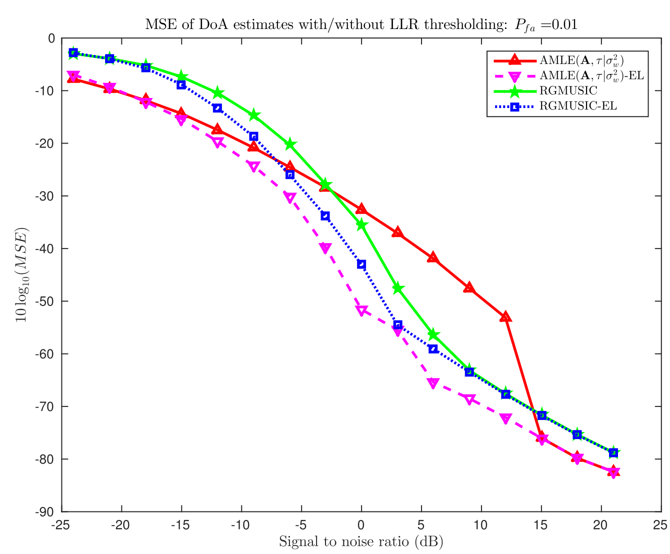



Fig. 7. MSE of RG-MUSIC and $\boldsymbol{\theta}^{\text{AML}}_{|\sigma_w^2}$ with/without thresholding . $P_{fa} = 10^{-2}$, $T = 40$, $\nu = 0.2$ and nGGNR = 60 dB.

detected using the EL principle. We notice that $LLR_{\text{ACG}}$ is slightly better than the sphericity test. However, the difference is not important and ST is easier to obtain than $LLR_{\text{ACG}}$.

To illustrate further how this comparison of ST or LLR to a threshold can help, we plot in Figs. 6 and 7 the MSE obtained when all estimates below the LLR threshold have been removed. Clearly, an improvement is observed, especially in the threshold area where the MSE is noticeably decreased. However, the gap between $\boldsymbol{\theta}^{\text{AML}}_{|\sigma_w^2}$ or RG-MUSIC and $\boldsymbol{\theta}^{\text{ML}}$ is not completely filled with the selected false alarm threshold. By raising this threshold, we may improve MSE of the selected estimates further at expense of discarding more legitimate estimates.

## IV. Conclusions

In this paper we continued our investigation into the DoA estimation problem for signals immersed in a mixture of external compound Gaussian noise and internal Gaussian noise, with the main focus on studying the threshold performance of DoA estimation routines. Since the p.d.f. of the noise mixture is not available, we resorted to the maximum likelihood methodology by treating random texture values as deterministic unknown parameters. The conditional likelihood function and associated likelihood ratio which tests sphericity of the projected out snapshots have been introduced for DoA estimates quality assessment using the expected likelihood principle. Specifically, the introduced sphericity test and likelihood ratio for the true DoA is described by distribution that does not depend on DoA nor on texture distribution, but is fully specified by antenna dimension, number of sources and sample support. Efficiency of estimation-method-specific outliers detection and ML-intrinsic breakdown prediction by the EL approach has been demonstrated.

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