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***Abstract*— This paper investigates the stabilization of Linear Time Invariant (LTI) systems by relay feedback. Generally the states measurements are subject to perturbations and noises. However, their effect is often ignored in the design of relay feedback. Here a state-dependent switching law is designed in a robust manner, while taking into account the perturbed states. The stability analysis of the closed loop system leads to qualitative conditions. Then, an LMI reformulation of the stability conditions is proposed to allow a numerical implementation of the results. Computer simulations show the efficiency of the developed method over a numerical example.**

I. INTRODUCTION

Since the '50s, relay feedback systems have been widely studied in control theory. Due to their simplicity and robustness properties, relays represent an interesting substitute to continuous control laws (see for instance [23], [24], [13], [8], [11]). Relays are used for different targets (control, controllers auto-tuning, identification, etc.) and in different application fields (chemistry, electromechanics, biology, etc.) - see for instance [25], [27], [3], [17], [10], [5], [26]. However, the usefulness of relays for stabilization and control does not exclude difficulties and some undesired phenomena. From a theoretical point of view, systems with relay feedback control can be seen as switched systems [16] with a complex behavior. The design of a relay feedback controller is not an obvious problem even for the case of linear systems. In the works [15], [14], the presence of sliding modes, limit cycles and chattering in relay feedback systems are pointed out. These phenomena must not be neglected and their study is theoretically challenging. In particular, for systems with sliding modes the notion of system's solution must be reviewed to take into account the dynamics obtained by fast switching [7], [6]. Frequency domain methods [4] and LMI approaches [19], [20] have also been used for design relay feedback design. Recently, a convex embedding formalism has been used in order to design relay feedback controllers in [11] and [9]. However, to the best of our knowledge, the existing results about relay feedback systems consider the system's states as perfectly known. In real systems, the system's states are generally affected by perturbations and measurement noises.

In this paper we consider the problem of stabilization by a perturbed relay feedback. The value of the state measurements used in the relay is assumed to be affected by a bounded disturbance. The result is formulated as LMI conditions to allow a simple numerical implementation.

The paper is organized as follows: Section II gives the system description and exposes the problem under study. A qualitative stability result is proposed in Section III. In Section IV, LMI conditions are expressed. The proposed conditions provide estimates of the domain of attraction and of the chattering ball. In Section V, a numerical example is given to illustrate the efficiency and the limits of the presented method. Finally, perspectives are given in the last section together with the conclusion.

A. Notations

In this paper we use the notation $\mathbb{R}^+$ to refer to the interval $[0,\infty)$. The transpose of a matrix M is denoted by M^T and if the matrix is symmetric the symmetric elements are denoted by $*$. The notation $M \succeq 0$ (resp. $M \preceq 0$) means that the matrix M is positive (resp. negative) semi-definite, and the notation $M \succ 0$ (resp. $M \prec 0$) means that it is positive definite (resp. negative definite). The identity matrix is denoted by I and both notations $\text{eig}_{min}(M)$ and $\text{eig}_{max}(M)$ are used to refer to the minimum and maximum eigenvalue respectively of a matrix M. For a positive definite matrix $P \in \mathbb{R}^{n\times n}$ and a positive scalar γ, we denote by $\mathcal{E}(P,\gamma)$ the ellipsoid

$$\mathcal{E}(P,\gamma) = \{x \in \mathbb{R}^n : x^T P x \leq \gamma\}, \tag{1}$$

and for all positive scalar r, we denote by $\mathcal{B}(0,r)$ the ball

$$\mathcal{B}(0,r) = \mathcal{E}(I,r) = \{x \in \mathbb{R}^n : x^T x \leq r\}. \tag{2}$$

For a given set $\mathcal{S}$, the notation $\text{Conv}\{\mathcal{S}\}$ indicates the convex hull of the set, $\text{int}\{\mathcal{S}\}$ its interior and $\overline{\mathcal{S}}$ its closure and finally the closed convex hull of the set $\mathcal{S}$ will be noted by $\overline{\text{Conv}}\{\mathcal{S}\}$. The minimum argument of a given function $f : \mathcal{S} \longrightarrow \mathbb{R}$ such that the set $\mathcal{S} \subset \mathbb{R}$ is a finite set of vectors is noted by

$$\arg\min f = \{y \in \mathcal{S} : f(y) \leq f(z), \forall z \in \mathcal{S}\}. \tag{3}$$

*This work was supported by the Agence Nationale pour la Recherche Project ROCC-SYS under the Grant agreement ANR-14-C27-0008 and by the ARCIR Project ESTIREZ.

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For a positive integer N, we denote by $\mathcal{I}_N$ the set $\{1,\ldots,N\}$. We use $\|.\|$ to denote the Euclidean norm for a vector and the associated norm for a matrix. By Δ_N we denote the unit simplex

$$\Delta_N = \left\{\alpha = (\alpha_1,\ldots,\alpha_N)^T \in \mathbb{R}^n : \sum_{i=1}^{N} \alpha_i = 1, \alpha_i \geq 0, i \in \mathcal{I}_N\right\}. \tag{4}$$

II. System description and problem statement

A. System description

Consider the linear system

$$\dot{x} = Ax + Bu, \tag{5}$$

with $x \in \mathbb{R}^n$ and an input u which takes only values in the set $\mathcal{V} = \{v_1,\ldots,v_N\} \subset \mathbb{R}^m$. $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$ are the matrices describing the system.

In the sequel we assume that:

A-1 The pair (A,B) is stabilizable, which means that there exists a matrix K such that the closed-loop matrix $A_{cl} = A + BK$ is Hurwitz.

A-2 The set $\mathrm{Conv}\{\mathcal{V}\}$ is nonempty, and the null vector is contained in its interior ($0 \in \mathrm{int}\{\mathrm{Conv}\{\mathcal{V}\}\}$).

This paper deals with the stabilization of system (5) in the case of a disturbed switching law. We consider a controller given by

$$u(x+e(t)) \in \arg\min_{v\in\mathcal{V}} (x+e(t))^T \Gamma v, \tag{6}$$

where e is an exogenous unknown disturbance considered as a measurable and bounded function from $\mathbb{R}^+$ to $\mathbb{R}^n$ satisfying

$$e(t)^T e(t) \leq \bar{e}, \tag{7}$$

with $\bar{e}$ its upper bound. The matrix $\Gamma \in \mathbb{R}^{n\times m}$ characterizes the switching hyperplanes. The formulation of the controller (6) encompasses the classical sign function in the classical relay feedback. Therefore, if $\mathcal{V} = \{v_1, v_2\} = \{-v, v\}$ with $v > 0$ then we get

$$\begin{aligned} u(x+e(t)) &= -v\mathrm{sign}(\Gamma^T(x+e(t)) \\ &\in \begin{cases} v & \text{if} \quad \Gamma^T(x+e(t)) < 0, \\ \{-v,v\} & \text{if} \quad \Gamma^T(x+e(t)) = 0, \\ v & \text{if} \quad \Gamma^T(x+e(t)) > 0. \end{cases} \end{aligned} \tag{8}$$

It may equally be interpreted as networked control systems with quantization [18]. The closed loop system is modeled by a differential equation with a discontinuous right hand-side. Consequently, to study the stability of the system we will consider the Filippov solutions of differential inclusions. The definitions of differential inclusions and their solutions are given below and used in the sequel. They can be found in [7], [1], [6], [2].

B. Solution concept

The interconnection (5), (6), (7) is the closed-loop system modeled by a discontinuous differential equation of the form

$$\dot{x} = Ax + B\tilde{u}(t,x) = f(t,x), \tag{9}$$

where $\tilde{u}(t,x) = u(x+e(t))$. Therefore, to the discontinuous closed-loop system (9) we associate the differential inclusion

$$\dot{x} \in \mathcal{F}(t,x), \tag{10}$$

with $\mathcal{F}(t,x)$ the set-valued map which can be computed from the differential equation with a discontinuous right hand side using the construction given in [7], [2], [6], [21] and the references therein such that

$$\mathcal{F}(t,x) = \bigcap_{\delta>0} \bigcap_{\mu(\mathcal{S})=0} \overline{\mathrm{Conv}}\{f(t,\mathcal{B}(x,\delta))\setminus \mathcal{S}\}, x \in \mathbb{R}^n, t \in \mathbb{R}^+, \tag{11}$$

where $\overline{\mathrm{Conv}}$ is the closed convex hull, $\mathcal{B}(x,\delta)$ is the open ball centered on x with radius δ, and $\mathcal{S}$ is a set of measure zero with $\mu(\mathcal{S})$ its measure in the sens of Lebesgue. The closed-loop system is then modeled by a differential inclusion for which the notion of a solution was defined in [7], and remembered hereafter.

Definition 1: (Filippov solution) Consider the closed-loop system (9) and its associated differential inclusion (10). A *Filippov solution* of the discontinuous systems (5), (6) over the interval $[t_a, t_b] \subset [0,\infty)$ is an absolutely continuous mapping $y : [t_a, t_b] \longrightarrow \mathbb{R}^n$ satisfying:

$$\dot{y}(t) \in \mathcal{F}(t, y(t)), \quad \text{for almost all} \quad t \in [t_a, t_b], \tag{12}$$

with $\mathcal{F}(t,x)$ given by (11).

A differential inclusion has at least one solution if the set valued map $\mathcal{F}(t,x)$ is locally bounded and takes nonempty, compact and convex values [6], [7], [2], [1]. We adapt as follows the concept of $R\epsilon$-stability from [20] to the context under study.

Definition 2: ($R\epsilon$-stability) Consider positive scalars R and ϵ. Assume that there exists a matrix $P = P^T \succ 0$ such that for all Filippov solutions $x(.)$ with $x(0) \in \mathcal{E}(P,R)$, the value of the state $x(t)$ converges to $\mathcal{E}(P,\epsilon)$ as t goes to infinity. Then system (5), (6) is said to be $R\epsilon$-stable from $\mathcal{E}(P,R)$ to $\mathcal{E}(P,\epsilon)$.

III. Control design

This section deals with the $R\epsilon$-stabilization of system (5), (6), (7). Assumptions A.1 and A.2 are used to prove that there exists a switching matrix Γ such that the system is $R\epsilon$-stable. The results are given in the following.

Theorem 1: Assume that A.1 and A.2 hold. Then there exist positive scalars R and ϵ and matrices $P = P^T \succ 0$ and $\Gamma = PB$ such that the system (5) with control (6) is $R\epsilon$-stable from $\mathcal{E}(P,R)$ to $\mathcal{E}(P,\epsilon)$ for a perturbation (7) with a sufficiently small bound $\bar{e}$.

Proof: Since the pair (A,B) is stabilizable then there exists a gain K such that $A_{cl} = A + BK$ is Hurwitz. Furthermore, for all $\delta > 0$ there exists a matrix $P = P^T \succ 0$ satisfying

$$A_{cl}^T P + PA_{cl} \preceq -2\delta P. \tag{13}$$

Consider the closed-loop system (5), (6), (7) and the associated differential inclusion (10).

We want to prove that for $\Gamma = PB$ there exists $\bar{e}$ such that if $e^T e \leq \bar{e}$ then

$$\sup_{y\in\mathcal{F}(t,x)} \frac{\partial V}{\partial x} y \leq -2\alpha V(x), \tag{14}$$

for some $\alpha > 0$ in a domain $\mathcal{D} \subset \mathbb{R}^n$ which will be determined.

We define for any $z \in \mathbb{R}^n$ the set of index $\mathcal{I}^*(z)$ such that

$$\mathcal{I}^*(z) = \{i \in \mathcal{I}_N : z^T PB(v_j - v_i) \geq 0, \quad \forall j \in \mathcal{I}_N\}. \tag{15}$$

To $\mathcal{I}^*(z)$ we associate for all $z \in \mathbb{R}^n$ the set $\Delta^*(z)$ of vectors defined by

$$\Delta^*(z) = \{\beta \in \Delta_N : \beta_i = 0, \forall i \in \mathcal{I}_N \setminus \mathcal{I}^*(z)\}. \tag{16}$$

Using (15) and (16), the set valued map $\mathcal{F}(t,x)$ in (11) satisfies

$$\mathcal{F}(t,x) \subseteq \mathcal{F}^*(t,x), \tag{17}$$

with

$$\begin{aligned}\mathcal{F}^*(t,x) = & \operatorname*{Conv}_{i\in\mathcal{I}^*(\bar{x}_e(t))} \{Ax + Bv_i\} \\ = & \{Ax + Bv(\beta) : \beta \in \Delta^*(\bar{x}_e(t))\},\end{aligned} \tag{18}$$

$v(\beta) = \sum_{i=1}^{N} \beta_i v_i$, and $\bar{x}_e(t) = x + e(t)$.

Therefore, in order to show (14), it is sufficient to prove that for some positive scalar α we have

$$\sup_{y\in\mathcal{F}^*(t,x)} \frac{\partial V}{\partial x} y \leq -2\alpha V(x), \tag{19}$$

in some domain $\mathcal{D} \subset \mathbb{R}^n$ to be determined.

From (18), and using the fact that the set $\Delta^*(z)$ is compact for all $z \in \mathbb{R}^n$, we have

$$\begin{aligned}\sup_{y\in\mathcal{F}^*(t,x)} \frac{\partial V}{\partial x} y = & \sup_{\beta\in\Delta^*(\bar{x}_e(t))} \left\{\frac{\partial V}{\partial x}(Ax + Bv(\beta))\right\} \\ = & \max_{\beta\in\Delta^*(\bar{x}_e(t))} \left\{\frac{\partial V}{\partial x}(Ax + Bv(\beta))\right\}.\end{aligned} \tag{20}$$

Then, showing (19) is equivalent to prove that for some $\alpha > 0$

$$\max_{\beta\in\Delta^*(\bar{x}_e(t))} \left\{\frac{\partial V}{\partial x}(Ax + Bv(\beta))\right\} \leq -2\alpha V(x), \tag{21}$$

in a domain $\mathcal{D} \subset \mathbb{R}^n$ to be determined below.

From inequality (13), we obtain

$$\frac{\partial V}{\partial x}(A_{cl} x) \leq -2\delta V(x), \forall x \in \mathbb{R}^n. \tag{22}$$

Note that, since the set $\mathrm{Conv}\{\mathcal{V}\}$ is nonempty and the null vector is contained in its interior ($0 \in \mathrm{int}\{\mathrm{Conv}\{\mathcal{V}\}\}$), then there exists a neighborhood of the origin $\mathcal{E}(P,\gamma)$ with $\gamma > 0$ such that for all $x \in \mathcal{E}(P,\gamma)$ we have

$$Kx \in \mathrm{Conv}\{\mathcal{V}\}. \tag{23}$$

Therefore for all $x \in \mathcal{E}(P,\gamma)$ there exist positive scalars $\alpha_j(x)$, $j \in \mathcal{I}_N$, such that $\sum_{j=1}^{N} \alpha_j(x) = 1$ and

$$Kx = \sum_{j=1}^{N} \alpha_j(x) v_j. \tag{24}$$

In the development that follows, we consider the case where (22) and (24) are verified (i.e. for all $x \in \mathcal{E}(P,\gamma)$). From (15), for all $i \in \mathcal{I}^*(\bar{x}_e(t))$ we have

$$(x + e(t))^T PB(v_j - v_i) \geq 0, \forall j \in \mathcal{I}_N. \tag{25}$$

Then, for any $\beta \in \Delta^*(\bar{x}_e(t))$ we have

$$(x + e(t))^T PB(v_j - v(\beta)) \geq 0, \forall j \in \mathcal{I}_N. \tag{26}$$

Multiplying this last inequality by $\alpha_j(x)$ for $j \in \mathcal{I}_N$ ($\alpha_j(x)$ defined in (24)) and summing the N elements, we obtain

$$(x + e(t))^T PB(Kx - v(\beta)) \geq 0. \tag{27}$$

Adding this to the left part of (21), it comes

$$\begin{aligned}& \max_{\beta\in\Delta^*(\bar{x}_e(t))} \left\{\frac{\partial V}{\partial x}(Ax + Bv(\beta))\right\} \\ & \leq \max_{\beta\in\Delta^*(\bar{x}_e(t))} \{x^T(A_{cl}^T P + PA_{cl})x \\ & +2e^T PBKx - 2e^T PBv(\beta)\}.\end{aligned} \tag{28}$$

Then, using (22), we get

$$\begin{aligned}& \max_{\beta\in\Delta^*(\bar{x}_e(t))} \{x^T(A_{cl}^T P + PA_{cl})x \\ & +2e^T PBKx - 2e^T PBv(\beta)\} \\ & \leq \max_{\beta\in\Delta^*(\bar{x}_e(t))} \left\{-2\delta V(x) + 2e^T PBKx - 2e^T PBv(\beta)\right\}.\end{aligned} \tag{29}$$

Then, in order to prove (21) in some domain $\mathcal{D} \subseteq \mathcal{E}(P,\gamma)$, from (29) it is sufficient to show that for some $\alpha > 0$

$$\begin{aligned}& \max_{\beta\in\Delta^*(\bar{x}_e(t))} \left\{-2\delta V(x) + 2e^T PBKx - 2e^T PBv(\beta)\right\} \\ & \leq -2\alpha V(x).\end{aligned} \tag{30}$$

Recall that for any positive number θ

$$2a^T b \leq \frac{1}{\theta} a^T a + \theta b^T b, \forall a, b \in \mathbb{R}^n. \tag{31}$$

Applying (31) to the terms $2e^T PBKx$ and $-2e^T PBv(\beta)$, with

$$\theta = \eta, \quad a_1 = e, \quad b_1 = PBKx, \tag{32}$$

and

$$\theta = \eta, \quad a_2 = e, \quad b_2 = -PBv(\beta), \tag{33}$$

we obtain the following inequality

$$\begin{aligned}& \max_{\beta\in\Delta^*(\bar{x}_e(t))} \left\{-2\delta V(x) + 2e^T PBKx - 2e^T PBv(\beta)\right\} \\ & \leq -2\delta V(x) + 2\eta^{-1} e^T e \\ & +\eta x^T K^T B^T PPBKx \\ & +\eta \max_{\beta\in\Delta^*(\bar{x}_e(t))} \left\{v^T(\beta) B^T PPBv(\beta)\right\}.\end{aligned} \tag{34}$$

Note that there exist $\zeta_{max} > 0$ and $\xi_{max} > 0$ such that

$$\begin{aligned} &x^T K^T B^T P P B K x \leq \zeta_{max} V(x), \forall x \in \mathbb{R}^n, \\ &\max_{\beta \in \Delta^*(\bar{x}_e(t))} \left\{ v(\beta)^T B^T P P B v(\beta) \right\} \leq \xi_{max}. \end{aligned} \tag{35}$$

Also, with $\bar{e}$ a positive scalar satisfying (7), thus from (34) we obtain

$$\begin{aligned} &\max_{\beta \in \Delta^*(\bar{x}_e(t))} \left\{ -2\delta V(x) + 2e^T P B K x - 2e^T P B v(\beta) \right\} \\ &\leq -2\delta V(x) + 2\eta^{-1}\bar{e} + \eta(\zeta_{max} V(x) + \xi_{max}). \end{aligned} \tag{36}$$

Then, (36) is verified (and consequently (14)) if there exists $\alpha > 0$ such that

$$-2\delta V(x) + 2\eta^{-1}\bar{e} + \eta(\zeta_{max} V(x) + \xi_{max}) \leq -2\alpha V(x), \tag{37}$$

which is satisfied if

$$\begin{cases} -\delta V(x) + 2\eta^{-1}\bar{e} + \eta\xi_{max} \leq 0, \\ -\delta + \eta\zeta_{max} + 2\alpha \leq 0. \end{cases} \tag{38}$$

Therefore, if we take $0 < \alpha < \frac{\delta}{2}$ and

$$0 < \eta \leq \frac{-2\alpha + \delta}{\zeta_{max}}, \tag{39}$$

then for a sufficiently small $\bar{e}$ and η, we have (37) (and thus (14)) is satisfied for all $x \in \mathcal{D} := \mathcal{E}(P, \gamma) \setminus \mathcal{E}(P, c(\bar{e}))$, with

$$x^T P x \leq c(\bar{e}) := \frac{2\eta^{-1}\bar{e} + \eta\xi_{max}}{\delta} < \gamma. \tag{40}$$

Therefore, system (5), (6), (7) is $R\epsilon$-stable from $\mathcal{E}(P, \gamma)$ to $\mathcal{E}(P, c)$ with $R = \gamma$ and $\epsilon = c(\bar{e})$. ■

Remark 1: From the proof of Theorem 1, equation (40), we can note that the size of the level set $\mathcal{E}(P, \epsilon)$ depends on the upper bound of the disturbance. Furthermore, it depends also on the upper bound of the control value. Then, the size of the chattering ball $\mathcal{E}(P, \epsilon)$ increases with the amplitude of the control vector (see Figure 1 in Section V).

Remark 2: In Theorem 1, the upper bound $\bar{e}$ of the disturbance is assumed to be sufficiently small in order to ensure the solvability of the stabilization problem. The existence of solutions may be preserved despite the large value of $\bar{e}$.

In this section, qualitative conditions for a robust stabilization of the closed-loop system are given. In order to allow a numerical implementation of the proposed relay controller, an LMI approach is developed hereafter.

IV. LMIs Solution

The first result (Theorem 1) has a qualitative nature. In practice, it is useful to find a constructive procedure which for desired domain of attraction $\mathcal{B}(0, r_\gamma)$ and chattering ball $\mathcal{B}(0, r_c)$ provides a switching law which ensures the $R\epsilon$-stability. In this section a numerical approach to deal with this problem is given. The main idea is to use the existence of a stabilizing linear static feedback gain K in order to re-design a relay feedback control of the form (6). An LMI solution is proposed hereafter. In order to express the result note that for any set $\mathcal{V}$ there exists a finite number n_h of vectors $h_i \in \mathbb{R}^{1\times m}$, $i \in \mathcal{I}_{n_h}$ such that

$$\text{Conv}\{\mathcal{V}\} = \{u \in \mathbb{R}^m : h_i u \leq 1, i \in \mathcal{I}_{n_h}\}. \tag{41}$$

Theorem 2: Assume that A.1 and A.2 hold. Consider the linear closed-loop system (5), (6), (7) with $\Gamma = PB$, P is a design parameter, and positive scalars c, r_c, r_γ, γ such that $r_c < r_\gamma$ and $\alpha > 0$. The matrix $A + BK$ is Hurwitz. If there exist $\epsilon_1 > 0, \epsilon_2 > 0, \epsilon_3 > 0$ and $P = P^T \succ 0$ such that the LMIs

1)

$$\mathcal{M}(v_i) = \begin{bmatrix} M_i & PBK & 0 \\ * & -\epsilon_3 I & -PBv_i \\ * & * & -\psi \end{bmatrix} \preceq 0, \forall i \in \mathcal{I}_N, \tag{42}$$

with

$$\begin{aligned} M_i = &A_{cl}^T P + P A_{cl} \\ &+ (2\alpha + \epsilon_1 - \epsilon_2) P, \\ \psi = &-\epsilon_1 c + \epsilon_3 \bar{e} + \epsilon_2 \gamma, \end{aligned}$$

2)

$$P - K^T h_i^T \gamma h_i K \succ 0, \forall i \in \mathcal{I}_{n_h}, \tag{43}$$

3)

$$P \preceq \frac{\gamma}{r_\gamma} I, \tag{44}$$

4)

$$P \succeq \frac{c}{r_c} I, \tag{45}$$

5)

$$c < \gamma, \tag{46}$$

are feasible, then the system (5), (6) is $R\epsilon$-stable from $\mathcal{E}(P, \gamma)$ to $\mathcal{E}(P, c)$ for a perturbation e satisfying (7). Furthermore, $\mathcal{B}(0, r_\gamma) \subseteq \mathcal{E}(P, \gamma)$ and $\mathcal{E}(P, c) \subseteq \mathcal{B}(0, r_c)$.

Proof: We want to prove that if the set of LMIs (42)-(46) are feasible then the closed-loop system (5), (6), (7) is $R\epsilon$-stable from $\mathcal{E}(P, \gamma)$ to $\mathcal{E}(P, c)$. It is sufficient to prove that

$$\sup_{y \in \mathcal{F}^*(t,x)} \frac{\partial V}{\partial x} y \leq -2\alpha V(x), \forall x \in \mathcal{E}(P, \gamma) \setminus \mathcal{E}(P, c), \tag{47}$$

for some $\alpha > 0$, with $\mathcal{F}^*(t, x)$ is defined in (11).

The static gain K is such that the matrix $A + BK$ is Hurwitz and for x the controller satisfies $Kx \in \text{Conv}\{\mathcal{V}\}$, hence let us define the set $\mathcal{C}_v$ as

$$\mathcal{C}_v = \{x \in \mathbb{R}^n : h_i K x < 1, i \in \mathcal{I}_N\}. \tag{48}$$

Note that, from (43) we have

$$x^T K^T h_i^T h_i K^T x < x^T \frac{P}{\gamma} x. \tag{49}$$

This last inequality means that the level set $\mathcal{E}(P, \gamma)$ satisfies

$$\mathcal{E}(P, \gamma) \subset \mathcal{C}_v. \tag{50}$$

The LMI (42) is equivalent to

$$z^T \mathcal{M}(v_i) z \leq 0, \forall z \in \mathbb{R}^{2n+1}. \tag{51}$$

Considering the vector $z^T = (x, e, 1)^T$, this leads to

$$\begin{aligned}
&x^T(A_{cl}^T P + PA_{cl})x + (2\alpha - \epsilon_2 + \epsilon_1)x^T Px\\
&+2e^T PBKx - 2e^T PBv_i\\
&-\epsilon_3 e^T e - \epsilon_1 c + \epsilon_2\gamma + \epsilon_3\bar{e}\\
&\leq 0, \forall x \in \mathbb{R}^n, \forall e \in \mathbb{R}^n, \forall i \in \mathcal{I}_N.
\end{aligned} \tag{52}$$

Note that for all $x \in \mathcal{C}_v$ there exist N positive scalars $\alpha_j(x)$, $j \in \mathcal{I}_N$, $\sum_{j=1}^N \alpha_j(x) = 1$ such that

$$Kx = \sum_{j=1}^N \alpha_j(x)v_j. \tag{53}$$

Since the constraint (50) is satisfied, then using (52) and (53), we obtain

$$\begin{aligned}
&x^T(A^T P + PA)x + (2\alpha - \epsilon_2 + \epsilon_1)x^T Px\\
&+2e^T PB\sum_{j=1}^N \alpha_j(x)v_j - \epsilon_3 e^T e - 2e^T PBv_i\\
&+2x^T PB\sum_{j=1}^N \alpha_j(x)v_j - \epsilon_1 c + \epsilon_2\gamma + \epsilon_3\bar{e}\\
&\leq 0, \forall x \in \mathcal{E}(P,\gamma), \forall e \in \mathbb{R}^n, \forall i \in \mathcal{I}_N.
\end{aligned} \tag{54}$$

which leads to

$$\begin{aligned}
&\sum_{j=1}^N \alpha_j(x)\{2x^T P(Ax + Bv_j) + (2\alpha - \epsilon_2 + \epsilon_1)x^T Px\\
&+2e^T PB(v_j - v_i) - \epsilon_3 e^T e - \epsilon_1 c + \epsilon_2\gamma\\
&+\epsilon_3\bar{e}\} \leq 0, \forall x \in \mathcal{E}(P,\gamma), \forall e \in \mathbb{R}^n, \forall i \in \mathcal{I}_N.
\end{aligned} \tag{55}$$

By adding and subtracting the term $2\sum_{j=1}^N \alpha_j(x)x^T PB(v_j - v_i)$, we get

$$\begin{aligned}
&\sum_{j=1}^N \alpha_j(x)\{2x^T P(Ax + Bv_i) + (2\alpha - \epsilon_2 + \epsilon_1)x^T Px\\
&+2(x + e(t))^T PB(v_j - v_i) - \epsilon_3 e^T e - \epsilon_1 c + \epsilon_2\gamma\\
&+ \epsilon_3\bar{e}\} \leq 0, \forall x \in \mathcal{E}(P,\gamma), \forall e \in \mathbb{R}^n, \forall i \in \mathcal{I}_N.
\end{aligned} \tag{56}$$

In addition, the LMI (44) is equivalent to the constraint

$$\mathcal{B}(0, r_\gamma) \subset \mathcal{E}(P,\gamma), \tag{57}$$

where $\mathcal{B}(0, r_\gamma)$ is the ball of radius r_γ centered on 0.

Moreover, the LMI (45) is equivalent to the constraint

$$\mathcal{E}(P, c) \subset \mathcal{B}(0, r_c), \tag{58}$$

where r_c is the radius of the ball $\mathcal{B}(0, r_c)$. Inequality (46) guarantees the fact that

$$\mathcal{E}(P, c) \subset \mathcal{E}(P,\gamma). \tag{59}$$

Recall that if $i \in \mathcal{I}^*(x + e(t))$ then

$$(x + e(t))^T PB(v_j - v_i) \geq 0, j \in \mathcal{I}_N. \tag{60}$$

If in addition $x \in \mathcal{E}(P,\gamma) \setminus \mathcal{E}(P, c)$ and the disturbance is such that $e^T e \leq \bar{e}$ then, Then (56) is a result of an S-procedure of these constraints with

$$\frac{\partial V}{\partial x}(Ax + Bv_i) \leq -2\alpha V(x). \tag{61}$$

Then, using the same arguments as in Theorem 1 we may show that

$$\begin{aligned}
\sup_{y\in\mathcal{F}(t,x)} \frac{\partial V}{\partial x}y &= \max_{\beta\in\Delta^*(\bar{x}_e(t))} \frac{\partial V}{\partial x}(Ax + Bv(\beta))\\
&\leq -2\alpha V(x),
\end{aligned} \tag{62}$$

with $v(\beta) = \sum_{i=1}^N \beta_i v_i$, for all $x \in \mathcal{E}(P,\gamma) \setminus \mathcal{E}(P, c)$ and e satisfying (7), which ends the proof. ■

Remark 3: To compute the LMI solution, for a given gain K such that (A, B) is stabilizable, P, c, and γ are taken as LMIs variables. A line search can be used to find the radius r_c and r_γ and a gridding to find the parameters ϵ_1, ϵ_2, and ϵ_3. Optimization algorithm can then be used to maximize r_γ or minimize r_c.

Remark 4: In Theorem 2 the controller gain K is supposed to be given such that the closed-loop continuous system matrix $A + BK$ is Hurwitz. The choice of the controller gain K has an influence on the size of the domain of attraction $\mathcal{E}(P,\gamma)$ and the chattering domain $\mathcal{E}(P, c)$. Although the main contribution here is the switching law design, one can use the results in Theorem 2 to co-design the controller gain K and the switching hyperplane characterizing matrix Γ (and domains $\mathcal{E}(P, c)$ and $\mathcal{E}(P,\gamma)$) by using recursive LMI optimization algorithms for example the one used in [22].

In the sections III and IV the stability conditions of the closed-loop system are given. In order to show the efficiency of the developed method numerical implementations are done and the results are reported in the next section.

V. Numerical example

Consider the linear system (5) with $u \in \mathcal{V} = \{-v, v\} = \{-3, 3\}$, and matrices $A = \begin{bmatrix}-1 & 0.3\\ 0.5 & 1\end{bmatrix}$ and $B = \begin{bmatrix}1\\ 1\end{bmatrix}$. Consider the static gain $K = \begin{bmatrix}0.2295, & -2.7897\end{bmatrix}$ computed using non robust approach given in [12]. The eigenvalues of A are -1.0724, and 1.0724 hence the open-loop linear system is unstable. Applying the method developed above, a relay feedback controller is designed to stabilize the system in the presence of a bounded perturbation $e(t) = \sqrt{\bar{e}} \times \begin{bmatrix}\sin(5t), & \cos(5t)\end{bmatrix}^T$ with $\bar{e} = 0.5 \times 10^{-4}$ and considering a decay rate $\alpha = 0.4$. An algorithm of minimization of c with a line search to find the parameters ϵ_1, ϵ_2, ϵ_3, r_γ and r_c is implemented.

The LMIs are feasible for $r_\gamma = 1.1$, $r_c = 0.1$, $\gamma = 3.8112$, $c = 0.0922$, $P = \begin{bmatrix}1.0397 & -0.5053\\ * & 3.3555\end{bmatrix}$, and with parameters $\epsilon_1 = 1.4216$, $\epsilon_2 = 0$, and $\epsilon_3 = 1.1980 \times 10^3$.

The computer simulations are realized for an initial condition $x(0) = \begin{bmatrix}1.5, & 0.4\end{bmatrix}^T$, and are reported in Figures 1-2.

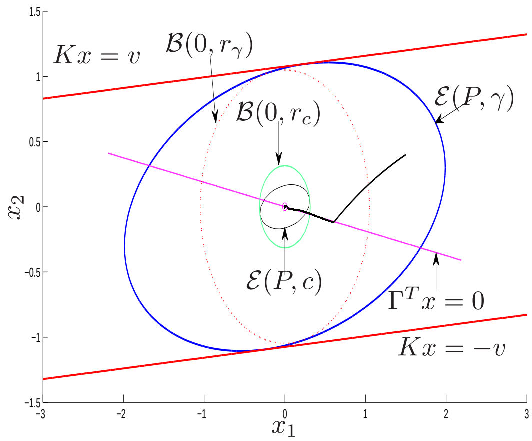

Fig. 1. Phase plant

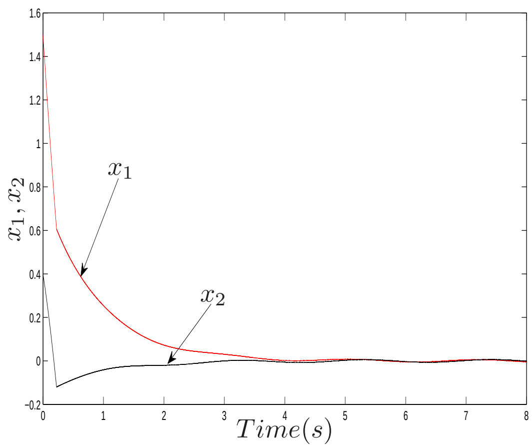

Fig. 2. State variables

As we can see from Figures 1 and 2, the states starting in the domain of attraction converge to a neighborhood of the origin and remain in it. In other words, from Figure 1 we can see that the states starting in the largest level set $\mathcal{E}(P, \gamma)$ contained in the convex $\mathcal{C}_v$ evolve until reaching the smallest level set $\mathcal{E}(P, c)$ surrounding the origin. The states stay bounded and oscillate around the origin indefinitely as it can be seen from Figure 2 and this confirms the provided results.

VI. Conclusion

In this paper a method for relay control design is provided for LTI systems stabilization. The article takes into account the fact that the states measurements used in the relay are affected by perturbations. Qualitative and numerical conditions of stability are given and the domain of attraction is estimated. The efficiency of the method is shown using a numerical example and computer simulations. New approaches will be developed in future works in order to reduce the conservatism of the current method. The studies of output dependent switching law design for linear system stabilization and observer based controller will be equally considered.

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