Relative Kähler-Einstein metric on Kähler varieties of positive Kodaira dimension

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Abstract. — In this paper we introduce a new notion of canonical metric. The notion of generalized Kähler-Einstein metric on the Kähler varieties with an intermediate Kodaira dimension is not suitable and we need to replace the twisted Kähler-Einstein metric (KE) to new notion of **Relative Kähler-Einstein metric** (RKE) for such varieties. This affirm a crucial error of the canonical metric introduced by Song-Tian[**1**][**2**], Tsuji [**51**],and Zeriahi-Eyssidieux-Guedj[**47**].We highlight that to get C^∞-solution of CMA equation of relative Kähler Einstein metric we need Song-Tian-Tsuji measure (which has minimal singularities with respect to other relative volume forms) be C^∞-smooth and special fiber has canonical singularities. Moreover, we conjecture that if we have relative Kähler-Einstein metric then our family is stable in the sense of Alexeev,and Kollar-Shepherd-Barron. By inspiring the work of Greene-Shapere-Vafa-Yau semi-Ricci flat metric, we introduce fiberwise Calabi-Yau foliation which relies in context of generalized notion of foliation. In final, we give Bogomolov-Miyaoka-Yau inequality for minimal varieties with intermediate Kodaira dimension

Contents

Key words and phrases. — relative Kähler-Einstein metric, Iitaka fibration, canonical metric.

1. Introduction

Let X_0 be a projective variety with canonical line bundle $K \to X_0$ of Kodaira dimension

$$\kappa(X_0) = \limsup \frac{\log \dim H^0(X_0, K^{\otimes \ell})}{\log \ell}$$

This can be shown to coincide with the maximal complex dimension of the image of X_0 under pluri-canonical maps to complex projective space, so that $\kappa(X_0) \in \{-\infty, 0, 1, ..., m\}$.

Lelong number: Let $W \subset \mathbb{C}^n$ be a domain, and Θ a positive current of degree (q, q) on W. For a point $p \in W$ one defines

$$\mathfrak{v}(\Theta, p, r) = \frac{1}{r^{2(n-q)}} \int_{|z-p|<r} \Theta(z) \wedge (dd^c|z|^2)^{n-q}$$

The Lelong number of Θ at p is defined as

$$\mathfrak{v}(\Theta, p) = \lim_{r \to 0} \mathfrak{v}(\Theta, p, r)$$

Let Θ be the curvature of singular hermitian metric $h = e^{-u}$, one has

$$\mathfrak{v}(\Theta, p) = \sup\{\lambda \geq 0 : u \leq \lambda \log(|z-p|^2) + O(1)\}$$

Christophe Mourougane and Shigeharu Takayama, introduced the notion of relative Kähler metric as follows **[20]**.

Definition 1.1. — Let $\pi : X \to Y$ be a holomorphic map of complex manifolds. A real d-closed $(1,1)$-form ω on X is said to be a relative Kähler form for π, if for every point $y \in Y$, there exists an open neighbourhood W of y and a smooth plurisubharmonic function Ψ on W such that $\omega + \pi^*(\sqrt{-1}\partial\bar{\partial}\Psi)$ is a Kähler form on $\pi^{-1}(W)$. A morphism π is said to be Kähler, if there exists a relative Kähler form for π, and $\pi : X \to Y$ is said to be a Kähler fiber space, if π is proper, Kähler, and surjective with connected fibers.

We consider an effective holomorphic family of complex manifolds. This means we have a holomorphic map $\pi : X \to Y$ between complex manifolds such that

1.The rank of the Jacobian of π is equal to the dimension of Y everywhere.

2.The fiber $X_t = \pi^{-1}(t)$ is connected for each $t \in Y$

3.X_t is not biholomorphic to $X_{t'}$ for distinct points $t; t' \in B$.

It is worth to mention that Kodaira showed that all fibers are dieomorphic to each other.

The relative Kähler form is denoted by

$$\omega_{X/Y} = \sqrt{-1} g_{\alpha,\bar{\beta}}(z,s) dz^\alpha \wedge d\bar{z}^\beta$$

Moreover take $\omega_X = \sqrt{-1}\partial\bar{\partial}\log\det g_{\alpha,\bar{\beta}}(z,y)$ on the total space X. The fact is ω_X in general is not Kähler on total space and $\omega_X|_{X_y} = \omega_{X_y}$. More precisely $\omega_X = \omega_F + \omega_H$ where ω_F is a form along fiber direction and ω_H is a form along horizontal direction. ω_H may not be Kähler metric in general, but ω_F is Kähler metric. Now let ω be a relative Kähler form on X and $m := \dim X - \dim Y$, We define the relative Ricci form $Ric_{X/Y,\omega}$ of ω by

$$Ric_{X/Y,\omega} = -\sqrt{-1}\partial\bar{\partial}\log(\omega^m \wedge \pi^*|dy_1 \wedge dy_2 \wedge ... \wedge dy_k|^2)$$

where $(y_1, ..., y_k)$ is a local coordinate of Y, where Y is a curve. See [**35**]

Let for family $\pi : \mathcal{X} \to Y$

$$\rho_{y_0} : T_{y_0}Y \to H^1(X, TX) = \mathcal{H}^{0,1}_{\bar{\sigma}}(TX)$$

be the KodairaâĂŞSpencer map for the corresponding deformation of X over Y at the point $y_0 \in Y$ where $\mathcal{X}_{y_0} = X$

If $v \in T_{y_0}Y$ is a tangent vector, say $v = \frac{\partial}{\partial y}|_{y_0}$ and $\frac{\partial}{\partial s} + b^\alpha \frac{\partial}{\partial z^\alpha}$ is any lift to $\mathcal{X}$ along X, then

$$\bar{\partial}\left(\frac{\partial}{\partial s} + b^\alpha \frac{\partial}{\partial z^\alpha}\right) = \frac{\partial b^\alpha(z)}{\partial z^{\bar{\beta}}}\frac{\partial}{\partial z^\alpha} dz^{\bar{\beta}}$$

is a $\bar{\partial}$-closed form on X, which represents $\rho_{y_0}(\partial/\partial y)$.

The Kodaira-Spencer map is induced as edge homomorphism by the short exact sequence

$$0 \to T_{X/Y} \to TX \to \pi^* T_Y \to 0$$

Weil-Petersson metric when fibers are Calabi-Yau manifolds can be defined as follows[**6**].

Definition 1.2. — Calabi-Yau manifold is a compact Kähler manifold with trivial canonical bundle. The local Kuranishi family of polarized Calabi-Yau manifolds $\mathcal{X} \to Y$ is smooth (unobstructed) by the Bogomolov-Tian-Todorov theorem. Let each fibers is a Calabi-Yau manifold. One can assign the unique (Ricci-flat) Yau metric $g(y)$ on X_y. The metric $g(y)$ induces a metric on $\wedge^{0,1}(TX)$. For $v, w \in T_y(Y)$, one then defines the Weil-Petersson metric on the base Y by

$$g_{WP}(v,w)=\int_X <\rho(v),\rho(w)>_{g(y)}$$

2. Fiberwise Calabi-Yau metric

The volume of fibers $\pi^{-1}(y)=X_y$ is a homological constant independent of y, and we assume that it is equal to 1. Since fibers are Calabi-Yau manifolds so $c_1(X_y)=0$, hence there is a smooth function F_y such that $Ric(\omega_y)=\sqrt{-1}\partial\bar{\partial}F_y$ and $\int_{X_y}(e^{F_y}-1)\omega_y^{n-m}=0$. The function F_y vary smoothly in y. By Yau's theorem there is a unique Ricci-flat Kähler metric $\omega_{SRF,y}$ on X_y cohomologous to ω_0. So there is a smooth function ρ_y on $\pi^{-1}(y)=X_y$ such that $\omega_0\mid_{X_y}+\sqrt{-1}\partial\bar{\partial}\rho_y=\omega_{SRF,y}$ is the unique Ricci-flat Kähler metric on X_y. If we normalize by $\int_{X_y}\rho_y\omega_0^n\mid_{X_y}=0$ then ρ_y varies smoothly in y and defines a smooth function ρ on X and we let

$$\omega_{SRF}|_{X_y}=\omega_0+\sqrt{-1}\partial\bar{\partial}\rho$$

which is called as Semi-Ricci Flat metric. Such Semi-Flat Calabi-Yau metrics were first constructed by Greene-Shapere-Vafa-Yau on surfaces [**4**]. More precisely, a closed real $(1,1)$-form ω_{SRF} on open set $U\subset X\setminus S$, (where S is proper analytic subvariety contains singular points of X) will be called semi-Ricci flat if its restriction to each fiber $X_y\cap U$ with $y\in f(U)$ be Ricci-flat. Notice that ω_{SRF} is positive in fiber direction, but it is still open problem that such current to be semi-positive in horizontal direction. Moreover $[\omega_{SRF}]\neq[\omega_0]$.

For the log-Calabi-Yau fibration $f:(X,D)\to Y$, such that (X_t,D_t) are log Calabi-Yau varieties and central pair (X_0,D_0) has simple normal crossing singularities, if (X,ω) be a Kähler variety with PoincarÃl' singularities then the semi-Ricci flat metric has $\omega_{SRF}|_{X_t}$ is quasi-isometric with the following model which we call it **fibrewise Poincare singularities.**

$$\frac{\sqrt{-1}}{\pi}\sum_{k=1}^n\frac{dz_k\wedge d\bar{z}_k}{|z_k|^2(\log|z_k|^2)^2}+\frac{\sqrt{-1}}{\pi}\frac{1}{(\log|t|^2-\sum_{k=1}^n\log|z_k|^2)^2}\left(\sum_{k=1}^n\frac{dz_k}{z_k}\wedge\sum_{k=1}^n\frac{d\bar{z}_k}{\bar{z}_k}\right)$$

We can define the same **fibrewise conical singularities.** and the semi-Ricci flat metric has $\omega_{SRF}|_{X_t}$ is quasi-isometric with the following model

$$\frac{\sqrt{-1}}{\pi}\sum_{k=1}^n\frac{dz_k\wedge d\bar{z}_k}{|z_k|^2}+\frac{\sqrt{-1}}{\pi}\frac{1}{(\log|t|^2-\sum_{k=1}^n\log|z_k|^2)^2}\left(\sum_{k=1}^n\frac{dz_k}{z_k}\wedge\sum_{k=1}^n\frac{d\bar{z}_k}{\bar{z}_k}\right)$$

In fact the previous remark will tell us that the semi Ricci flat metric ω_{SRF} has pole singularities with Poincare growth.

Remark: Note that we can always assume the central fiber has simple normal crossing singularities(when dimension of base is one) up to birational modification and base change due to semi-stable reduction of Grothendieck, Kempf, Knudsen, Mumford and Saint-Donat as follows.

Theorem (Grothendieck, Kempf, Knudsen, Mumford and Saint-Donat[**24**])Let k be an algebraically closed field of characteristic 0 (e.g. $k = \mathbb{C}$). Let $f : X \to C$ be a surjective morphism from a k-variety X to a non-singular curve C and assume there exists a closed point $z \in C$ such that $f_{|X\setminus f^{-1}(z)} : X \setminus f^{-1}(z) \to C \setminus \{z\}$ is smooth. Then we find a commutative diagram

$$\begin{tikzcd}
X \arrow[d, "f"'] & X \times_C C' \arrow[l] \arrow[d] & X' \arrow[l, "p"'] \arrow[dl, "f'"] \\
C & C' \arrow[l, "\pi"]
\end{tikzcd}$$

with the following properties

1. $\pi : C' \to C$ is a finite map, C' is a non-singular curve and $\pi^{-1}(z) = \{z'\}$.

2. p is projective and is an isomorphism over $C'\setminus\{z'\}$. X' is non-singular and $f'^{-1}(z')$ is a reduced divisor with simple normal crossings, i.e., we can write $f'^{-1}(z') = \sum_i E_i$ where the E_i are 1-codimensional subvarieties (i.e., locally they are defined by the vanishing of a single equation), which are smooth and, for all r, all the intersections $E_{i_1} \cap \ldots \cap E_{i_r}$ are smooth and have codimension r.

Now if the dimension of smooth base be bigger than one, then we don't know the semi-stable reduction and instead we can use weak Abramovich-Karu reduction or Kawamata's unipotent reduction theorem. In fact when the dimension of base is one we know from Fujino's recent result that if we allow semi-stable reduction and MMP on the family of Calabi-Yau varieties then the central fiber will be Calabi-Yau variety. But If the dimension of smooth base be bigger than one on the family of Calabi-Yau fibers , then if we apply MMP and weak Abramovich-Karu semi-stable reduction [**31**] then the special fiber can have simple nature. But if the dimension of base be singular then we don't know about semi-stable reduction which seems is very important for finding canonical metric along Iitaka fibration.

3. Relative Kähler-Einstein metric

Definition 3.1. — Let X be a smooth projective variety with $\kappa(X) \geq 0$. Then for a sufficiently large $m > 0$, the complete linear system $|m!K_X|$ gives

a rational fibration with connected fibers $f : X \dashrightarrow Y$. We call $f : X \dashrightarrow Y$ the Iitaka fibration of$X$. Iitaka fibration is unique in the sense of birational equivalence. We may assume that f is a morphism and Y is smooth. For Iitaka fibration f we have
1. For a general fiber F, $\kappa(F) = 0$ holds.
2. $\dim Y = \kappa(Y)$.

Let X be a Kähler variety with an intermediate Kodaira dimension $\kappa(X) > 0$ then we have an Iitaka fibration $\pi : X \to Y = \mathrm{Proj}R(X, K_X) = X_{can}$ such that fibers are Calabi-Yau varieties. We set $K_{X/Y} = K_X \otimes \pi^* K_Y^{-1}$ and call it the relative canonical bundle of $\pi : X \to Y$

Definition 3.2. — Let X be a Kähler variety with $\kappa(X) > 0$ then the **relative Kähler-Einstein metric** is defined as follows

$$Ric_{X/Y}^{h^{\omega}_{X/Y}}(\omega) = -\omega$$

where

$$Ric_{X/Y}^{h^{\omega}_{X/Y}}(\omega) = \sqrt{-1}\partial\bar{\partial}\log(\frac{\omega^n \wedge \pi^*\omega_{can}^m}{\pi^*\omega_{can}^m})$$

and ω_{can} is a canonical metric on $Y = X_{can}$.

$$Ric_{X/Y}^{h^{\omega_{SRF}}_{X/Y}}(\omega) = \sqrt{-1}\partial\bar{\partial}\log(\frac{\omega_{SRF}^n \wedge \pi^*\omega_{can}^m}{\pi^*\omega_{can}^m}) = \omega_{WP}$$

here ω_{WP} is a Weil-Petersson metric[**6**].

Note that if $\kappa(X) = -\infty$ then along Mori fibre space $f : X \to Y$ we can define Relative Kähler-Einstein metric as

$$Ric_{X/Y}^{h^{\omega}_{X/Y}}(\omega) = \omega$$

when fibers and base are K-poly-stable, see[**5**]

Note that, if X be a Calabi-Yau variety and we have a holomorphic fibre space $\pi : X \to Y$, which fibres are Calabi-Yau varieties, then we have the relative Ricci flat metric $Ric_{X/Y}(\omega) = 0$, which turns out to be $Ric(\omega) = 0 + \pi^*(\omega'_Y)$ where $\omega'_Y = \omega_{WP}$, see[**3**]

Then by the definition of Relative Kähler metric $Ric(\omega_X) = -\omega_Y + \pi^*(\omega'_Y)$ which $\omega'_Y = \omega_{WP}$ is Weil-Petersson metric(we can define Weil-Petersson metric completely on the base of Iitaka fibration) by using higher canonical bundle formula of Fujino-Mori. In fact ω_X in the volume sense can be written as a volume on the total space with wedge product with pull pack of canonical volume form(which satisfies in $Ric(\omega_Y) = -\omega_Y$) of the base. In fact $Ric(\omega) = -\omega + \pi^*(\omega_{WP})$ or $Ric(\omega) = -\omega + \alpha$, is not a right caninical metric for varieties with an intermediate Kodaira dimension. In fact even Fujino-Mori's canonical bundle formula affirm that the canonical metric introduced by Song-Tian[**1**][**2**][**46**], and also Hajime Tsuji[**42**], Berman[**48**], Eyssidieux-Guedj-Zeriahi [**47**] fail to be right canonical metric metric for Kähler varieties of positive Kodaira dimension.

For the existence of Kähler-Einstein metric when our variety is of general type, we need to the nice deformation of Kähler-Ricci flow and for intermidiate Kodaira dimension we need to work on relative version of Kähler Ricci flow which is as the complex Monge-Ampere equation introduced by-Song-Tian and highlighted by R.Berman [**7**] . i.e

$$\frac{\partial \omega}{\partial t} = -Ric_{X/Y}(\omega) - \omega$$

take the reference metric as $\tilde{\omega_t} = e^{-t}\omega_0 + (1 - e^{-t})Ric(\frac{\omega_{SRF}^n \wedge \pi^* \omega_{can}^m}{\pi^* \omega_{can}^m})$ then the version of Kähler Ricci flow is equivalent to the following relative Monge-Ampere equation

$$\frac{\partial \phi_t}{\partial t} = \frac{(\tilde{\omega_t} + \sqrt{-1}\partial\bar{\partial}\phi_t)^n \wedge \pi^* \omega_{can}^m}{\omega_{SRF}^n \wedge \pi^* \omega_{can}^m} - \phi_t$$

Take the relative canonical volume form $\Omega_{X/Y} = \frac{\omega_{SRF}^n \wedge \pi^* \omega_{can}^m}{\pi^* \omega_{can}^m}$ and $\omega_t = \tilde{\omega_t} + \sqrt{-1}\partial\bar{\partial}\phi_t$, then

$$\frac{\partial \omega_t}{\partial t} = \frac{\partial \tilde{\omega_t}}{\partial t} + \sqrt{-1}\partial\bar{\partial}\frac{\partial \phi_t}{\partial t}$$

By taking $\omega_\infty = -Ric(\Omega_{X/Y}) + \sqrt{-1}\partial\bar{\partial}\phi_\infty$ we obtain after using estimates

$$\log \frac{\omega_\infty^n}{\Omega_{X/Y}} - \phi_\infty = 0$$

By taking $-\sqrt{-1}\partial\bar{\partial}$ of both sides we get

$$Ric_{X/Y}(\omega_\infty) = -\omega_\infty$$

hence by the definition of relative Kähler-metric and higher canonical bundle formula we can not have the Song-Tian metric[**1, 2**]

$$Ric(\omega_\infty) = -\omega_\infty + \pi^*(\omega_{WP})$$

More explicitly on pair (X, D) where D is a snc divisor, we can write

$$Ric(\omega_{(X,D)}) = -\omega_Y + \omega_{WP}^D + \sum_P (b(1 - t_P^D))[\pi^*(P)] + [B^D]$$

where B^D is $\mathbb{Q}$-divisor on X such that $\pi_* \mathcal{O}_X([iB_+^D]) = \mathcal{O}_B$ $(\forall i > 0)$. Here $s_P^D := b(1 - t_P^D)$ where t_P^D is the log-canonical threshold of $\pi^* P$ with respect to $(X, D - B^D/b)$ over the generic point η_P of P. i.e.,

$$t_P^D := \max\{t \in \mathbb{R} \mid \big(X, D - B^D/b + t\pi^*(P)\big) \text{ is sub log canonical over } \eta_P\}$$

and ω_{can} has zero Lelong number[**45**]

Remark:Note that the log semi-Ricci flat metric ω_{SRF}^D is not continuous in general. But if the central fiber has at worst canonical singularities and the central fiber (X_0, D_0) be itself as Calabi-Yau pair, then by open condition property of Kahler-Einstein metrics, semi-Ricci flat metric is smooth in an open Zariski subset.

Remark:So by applying the previous remark, the relative volume form

$$\Omega_{(X,D)/Y} = \frac{(\omega_{SRF}^D)^n \wedge \pi^*\omega_{can}^m}{\pi^*\omega_{can}^m \mid S \mid^2}$$

is not smooth in general, where $S \in H^0(X, L_N)$ and N is a divisor which come from canonical bundle formula of Fujino-Mori. Note that Song-Tian measure is invariant under birational change

Now we try to extend the Relative Ricci flow to the fiberwise conical relative Ricci flow. We define the conical Relative Ricci flow on pair $\pi : (X, D) \to Y$ where D is a simple normal crossing divisor as follows

$$\frac{\partial \omega}{\partial t} = -Ric_{(X,D)/Y}(\omega) - \omega + [N]$$

where N is a divisor which come from canonical bundle formula of Fujino-Mori.

Take the reference metric as $\tilde{\omega_t} = e^{-t}\omega_0 + (1 - e^{-t})Ric(\frac{\omega_{SRF}^n \wedge \pi^*\omega_{can}^m}{\pi^*\omega_{can}^m})$ then the conical relative Kähler Ricci flow is equivalent to the following relative Monge-Ampere equation

$$\frac{\partial \phi_t}{\partial t} = \log \frac{(\tilde{\omega_t} + Ric(h_N) + \sqrt{-1}\partial\bar{\partial}\phi_t)^n \wedge \pi^*\omega_{can}^m \mid S_N \mid^2}{(\omega_{SRF}^D)^n \wedge \pi^*\omega_{can}^m} - \phi_t$$

Now we prove the C^0-estimate for this relative Monge-Ampere equation due to Tian's C^0-estimate

By approximation our Monge-Ampere equation, we can write

$$\frac{\partial \varphi_\epsilon}{\partial t} = \log \frac{(\omega_{t,\epsilon} + \sqrt{-1}\partial\bar{\partial}\varphi_t)^m \wedge \pi^*\omega_{can}^n \left(||S||^2 + \epsilon^2\right)^{(1-\beta)}}{(\omega_{SRF}^D)^m \wedge \pi^*\omega_{can}^n} - \delta \left(||S||^2 + \epsilon^2\right)^\beta - \varphi_{t,\epsilon}$$

So by applying maximal principle we get an upper bound for $\varphi_{t,\epsilon}$ as follows

$$\frac{\partial}{\partial t} \sup \varphi_\epsilon \leq \sup \log \frac{\omega_{t,\epsilon}^m \wedge \pi^*(\omega_{can})^n \left(||S||^2 + \epsilon^2\right)^{(1-\beta)}}{(\omega_{SRF}^D)^m \wedge \pi^*\omega_{can}^n} - \delta \left(||S||^2 + \epsilon^2\right)^\beta$$

and by expanding $\omega_{t,\epsilon}^n$ we have a constant C independent of ϵ such that the following expression is bounded if and only if the Song-Tian-Tsuji measure be bonded, so to get C^0 estimate we need special fiber has mild singularities in the sense of MMP

$$\frac{\omega_{t,\epsilon}^m \wedge \pi^*(\omega_{can})^n \left(||S||^2+\epsilon^2\right)^{(1-\beta)}}{(\omega_{SRF}^D)^m \wedge \pi^*\omega_{can}^n} \approx C$$

and also $\delta \to 0$ so $\delta\left(||S||^2+\epsilon^2\right)^{\beta}$ is too small. So we can get a uniform upper bound for φ_ϵ. By applying the same argument for the lower bound, and using maximal principle again, we get a C^0 estimate for φ_ϵ. Moreover if central fiber X_0 has canonical singularities then Song-Tian-Tsuji measure is continuous.

So this means that we have C^0-estimate for relative Kähler-Ricci flow if and only if the central fiber has at worst canonical singularities. Note that to get C^∞-estimate we need just check that our reference metric is bounded and Song-Tian-Tsuji measure is $\mathbb{C}^\infty$-smooth . So it just remain to see that ω_{WP} is bounded. But when fibers are not smooth in general, Weil-Petersson metric is not bounded and Yoshikawa in Proposition 5.1 in **[27]** showed that under the some additional condition when central fiber X_0 is reduced and irreducible and has only canonical singularities we have

$$0 \le \omega_{WP} \le C\frac{\sqrt{-1} \mid s \mid^{2r} ds \wedge d\bar{s}}{\mid s \mid^2 (-\log \mid s \mid)^2}$$

It is worth to mention that for varieties with an intermediate Kodaira dimension Birkar-Cascini-Hacon-McKernan program **[50]** does not work and we need to use Relative Kähler Ricci flow instead Kähler Ricci flow. Since for such varieties Kähler Ricci flow is not a right surgery and never can be deformed to canonical metric as we introduced.

4. Fiberwise Calabi-Yau foliation

Note that the main difficulty of the solution of C^∞ for the solution of relative Kähler-Einstein metric is that the null direction of fiberwise Calabi-Yau metric ω_{SRF} gives a foliation along Iitaka fibration $\pi : X \to Y$ and we call it fiberwise Calabi-Yau foliation(due to H.Tsuji) and can be defined as follows

$$\mathcal{F} = \{\theta \in T_{X/Y} | \omega_{SRF}(\theta, \bar{\theta}) = 0\}$$

and along log Iitaka fibration $\pi : (X, D) \to Y$, we can define the following foliation

$$\mathcal{F}' = \{\theta \in T_{X'/Y} | \omega_{SRF}^D(\theta, \bar{\theta}) = 0\}$$

where $X' = X \setminus D$. In fact the method of Song-Tian (only in fiber direction and they couldn't prove the estimates in horizontal direction which is the main part of computation) works when $\omega_{SRF} > 0$. More precisely, in null direction, the function φ satisfies in the complex Monge-Ampere foliation

$$(\omega_{SRF})^{\kappa} = 0$$

gives rise to a foliation by X by complex sub-manifolds.

For the null direction we need to an extension of Monge-Ampere foliation method of Gang Tian in [**18**]. It will appear in my new paper [**19**]

A complex analytic space is a topological space such that each point has an open neighborhood homeomorphic to some zero set $V(f_1, \dots, f_k)$ of finitely many holomorphic functions in $\mathbb{C}^n$, in a way such that the transition maps (restricted to their appropriate domains) are biholomorphic functions.

Definition: Let X be normal variety. A foliation on X is a nonzero coherent subsheaf $\mathcal{F} \subset T_X$ satisfying

(1) $\mathcal{F}$ is closed under the Lie bracket, and

(2) $\mathcal{F}$ is saturated in T_X (i.e., $T_X/\mathcal{F}$ is torsion free). The Condition (2) above implies that $\mathcal{F}$ is reflexive, i.e. $\mathcal{F} = \mathcal{F}^{**}$.

The canonical class $K_{\mathcal{F}}$ of $\mathcal{F}$ is any Weil divisor on X such that $\mathcal{O}_X(-K_{\mathcal{F}}) \cong \det(\mathcal{F})$.

Definition 4.1. — Let $\pi : X \to Y$ be a dominant morphism of normal varieties. Suppose that π is equidimensional. relative canonical bundle can be defined as follows

$$K_{X/Y} := K_X - \pi^* K_Y$$

Let $\mathcal{F}$ be the foliation on X induced by π, then

$$K_{\mathcal{F}} = K_{X/Y} - R(\pi)$$

where $R(\pi) = \cup_D ((\pi)^* D - ((\pi)^* D)_{red})$ is the ramification divisor of π. Here D runs through all prime divisors on Y. The canonical class $K_{\mathcal{F}}$ of $\mathcal{F}$ is any Weil divisor on X such that $\mathcal{O}_X(-K_{\mathcal{F}}) \cong \det(\mathcal{F}) := (\wedge^r \mathcal{F})^{**}$ See [**41**]

Now take a C^∞ $(1,1)$-form ω on a complex manifold X of complex dimension n and let

$$\mathrm{ann}(\omega) = \{W \in TX | \omega(W, \bar{V}) = 0, \forall V \in TX\}$$

Now we have the following lemma due to Schwarz inequality [**16**]

Lemma 4.2. — *If ω is non-negative then we can write,*

$$ann(\omega) = \{W \in TX | \omega(W, \bar{W}) = 0, \forall W \in TX\}$$

Moreover, if we assume $\omega^{n-1} \neq 0$ and $\omega^n = 0$ then $ann(\omega)$ is subbundle of TX.

Furthermore, we have the following straightforward lemma which make $\mathrm{ann}(\omega)$ to be as foliation

Lemma 4.3. — *If ω is non-negative, $\omega^{n-1} \neq 0$, $\omega^n = 0$, and $d\omega = 0$, then*

$$\mathcal{F} = ann(\omega) = \{W \in TX | \omega(W, \bar{W}) = 0, \forall W \in TX\}$$

define a foliation $\mathcal{F}$ on X and each leaf of $\mathcal{F}$ being a Riemann surface

Now Tsuji [**10**][**42**] took relative form $\omega_{X/Y}$ instead ω in previous lemma and wrote it as a foliation. In my opinion Tsuji's foliation is fail to be right foliation and we need to revise it. First of all we don't know such metric ω_{SRF} is non-negative and second we must take $W \in T_{X/Y}$ in relative tangent bundle and we don't have in general $d\omega_{SRF} = 0$, In fact we know just that $d_{X/Y}\omega_{SRF} = 0$. Moreover ω_{SRF} is not smooth in general and it is a $(1,1)$-current with log pole singularities.

Hence on Calabi-Yau fibration, we can introduce the following bundle

$$\mathcal{F} = \mathrm{ann}(\omega_{SRF}) = \{W \in T_{X/Y} | \omega_{SRF}(W, \bar{W}) = 0, \forall W \in T_{X/Y}\}$$

in general is the right bundle to be considered and not something Tsuji wrote in [**14**]. It is not a foliation in general. In fact it is a foliation is fiber direction and may not be a foliation in horizontal direction, but it generalize the notion of foliation. The correct solution of it as Monge-Ampere foliation still remained as open problem.

In the fibre direction, $\mathcal{F}$ is a foliation and we have the following straightforward theorem due to Bedford-Kalka.[**40**][**15**][**4**]

Theorem 4.4. — *Let $\mathcal{L}$ be a leaf of $f_*\mathcal{F}$, then $\mathcal{L}$ is a closed complex submanifold and the leaf $\mathcal{L}$ can be seen as fiber on the moduli map*

$$\eta : \mathcal{Y} \to \mathcal{M}_{CY}^D$$

where $\mathcal{M}_{CY}$ is the moduli space of calabi-Yau fibers with at worst canonical singularites and

$$\mathcal{Y} = \{y \in Y_{reg} | X_y \ \textit{has Kawamata log terminal singularities}\}$$

5. Smoothness of fiberwise integral of Calabi-Yau volume

Let X be a closed normal analytic subspace in some open subset U of $\mathbb{C}^N$ with an isolated singularity. Take $f : X \to \Delta$ be a degeneration of smooth Calabi-Yau manifolds, then

$$s \to \int_{X_s} \Omega_s \wedge \bar{\Omega}_s \in C^\infty$$

if and only if the monodromy M acting on the cohomology of the Milnor fibre of f is the identity and the restriction map $j : H^n(X^*) \to H^n(F)^M$ is surjective, where $X^* = X \setminus \{0\}$ and M denotes monodromy acting on $H^n(F)$ and $H^n(F)^M$ is the M-invariant subgroup and F is the Milnor fiber at zero(see Corollary 6.2. [**21**]).

Note that to get C^∞-estimate for the solution of CMA along fibration $f : X \to Y$ we need to have C^∞-smooth relative volume form $\Omega_{X/Y}$. So such volume forms are not unique and in fact Song-Tian-Tsuji measure has minimal singularites. If we consider a CMA equation with the relative volume form constructed by $\int_{X_s} \Omega_s \wedge \overline{\Omega_s}$, then such fiberwise integral volume forms must be smooth and in an special case when X is an analytical subspace of $\mathbb{C}^N$ with an isolated singularities we get the C^∞-smoothness of such fiberwise integral.

Note that, If X_0 only has canonical singularities, or if X is smooth and X_0 only has isolated ordinary quadratic singularities, then if $\pi : X \to \mathbb{C}^*$ be a family of degeneration of of Calabi-Yau fibers. Then the L^2-metric

$$\int_{X_s} \Omega_s \wedge \bar{\Omega}_s$$

is continuous. See Remark 2.10. of [**26**].

In fact Tsuji introduced a canonial volume form along Iitaka fibration as pull back of a canonical volume form of canonical model X_{can} over the fiberwise integral of relative volume forms. See Definition 1.6 of Tsuji's paper [**42**]. So this fact tells us that the C^0-estimate of Tsuji [**51**] and Eyssidieux-Guedj-Zeriahi [**47**] is not correct.

Moreover, Song-Tian-Tsuji measure is not smooth and nor continuous and implicit function theorem does not work here. In fact we are facing with two different singularities , one singularity arise from fiber direction near central fiber and also we have another type of singularity in horizontal direction near central fiber. So this comment tells us that Kolodziej's C^0-estimate does not work for finding canonical metric along Calabi-Yau fibration. Hence C^0 estimate of Song-Tian is not correct. In fact Song-Tian-Tsuji measure has log log singularieties and with additional mild singular condition in the sense of MMP it can be continuous . Moreover Boucksom-Tsuji [**49**] assumed relative volume form is smooth which in general such canonical relative volume form is not smooth. So we need to study the algebraic equivalency of smoothness of Song-Tian-Tsuji measure.

6. Fiberwise Kähler-Einstein stability

Now we use the Wang[**32**], Takayama[**29**], and Tosatti [**30**] result for the following definition.

Definition 6.1. — Let $\pi : X \to B$ be a family of Kähler-Einstein varieties, then we introduce the new notion of stability and call it fiberwise KE-stability, if the Weil-Petersson distance $d_{WP}(B,0) < \infty$(which is equivalent to say Song-Tian-Tsuji measure is bounded near central fiber). Note when fibers are Calabi-Yau varities, Takayama, by using Tian's Kähler-potential for Weil-Petersson metric for moduli space of Calabi-Yau varieties showed that Fiberwise KE-Stability is as same as when the central fiber is Calabi-Yau variety with at worst canonical singularities. So this definition work when the dimension of base is one. But if the dimension of base be bigger than one, then it is better to replace boundedness of Weil-Petersson distance with boundedness of Song-Tian-Tsuji measure which seems to be more natural to me. We mention that the Song-Tian-Tsuji measure is bounded near origin if and only if after a finite base change the Calabi-Yau family is birational to one with central fiber a Calabi-Yau variety with at worst canonical singularities.

So along canonical model $\pi : X \to X_{can}$ for mildly singular variety X, we have $Ric(\omega_X) = -\omega_Y + \omega_{WP} + [N]$ if and only if our family of fibers be fiberwise KE-stable

Let $\pi : (X, D) \to B$ is a holomorphic submersion onto a compact Kähler manifold B with $c_1(K_B) < 0$ where the fibers are log Calabi-Yau manifolds and D is a simple normal crossing divisor in X. Let our family of fibers is fiberwise KE-stable. Then (X, D) admits a unique twisted Kähler-Einstein metric ω_B solving

$$Ric(\omega_{(X,D)}) = -\omega_B + \omega_{WP}^D + (1-\beta)[N]$$

where ω_{WP} is the logarithmic Weil-Petersson form on the moduli space of log Calabi-Yau fibers and $[D]$ is the current of integration over D.

More precisely, we have

$$Ric(\omega_{(X,D)}) = -\omega_B + \omega_{WP}^D + \sum_P (b(1-t_P^D))[\pi^*(P)] + [B^D]$$

where B^D is $\mathbb{Q}$-divisor on X such that $\pi_* \mathcal{O}_X([iB_+^D]) = \mathcal{O}_B$ $(\forall i > 0)$. Here $s_P^D := b(1-t_P^D)$ where t_P^D is the log-canonical threshold of $\pi^* P$ with respect to $(X, D - B^D/b)$ over the generic point η_P of P. i.e.,

$$t_P^D := \max\{t \in \mathbb{R} \mid \big(X, D - B^D/b + t\pi^*(P)\big) \text{ is sub log canonical over } \eta_P\}$$

and ω_{can} has zero Lelong number.

With cone angle $2\pi\beta$, $(0 < \beta < 1)$ along the divisor D, where h is an Hermitian metric on line bundle corresponding to divisor N, i.e., L_N. This equation can be solved. Take, $\omega = \omega(t) = \omega_B + (1-\beta)Ric(h) + \sqrt{-1}\partial\bar{\partial}v$ where $\omega_B = e^{-t}\omega_0 + (1-e^{-t})Ric(\frac{(\omega^D_{SRF})^n \wedge \pi^*\omega^m_{can}}{\pi^*\omega^m_{can}})$, by using Poincare-Lelong equation,

$$\sqrt{-1}\partial\bar{\partial}\log|s_N|^2_h = -c_1(L_N, h) + [N]$$

we have

$$\begin{aligned} Ric(\omega) = \\ &= -\sqrt{-1}\partial\bar{\partial}\log\omega^m \\ &= -\sqrt{-1}\partial\bar{\partial}\log\pi_*\Omega_{(X,D)/B} - \sqrt{-1}\partial\bar{\partial}v - (1-\beta)c_1([N], h) + (1-\beta)\{N\} \end{aligned}$$

and

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial}\log\pi_*\Omega_{(X,D)/B} + \sqrt{-1}\partial\bar{\partial}v = \\ &= \sqrt{-1}\partial\bar{\partial}\log\pi_*\Omega_{(X,D)/B} + \omega - \omega_B - Ric(h) \end{aligned}$$

Hence, by using

$$\omega^D_{WP} = \sqrt{-1}\partial\bar{\partial}\log(\frac{(\omega^D_{SRF})^n \wedge \pi^*\omega^m_{can}}{\pi^*\omega^m_{can} \mid S \mid^2})$$

we get

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial}\log\pi_*\Omega_{(X,D)/B} + \sqrt{-1}\partial\bar{\partial}v = \\ &= \omega_B - \omega^D_{WP} - (1-\beta)c_1(N) \end{aligned}$$

So,

$$Ric(\omega_{(X,D)}) = -\omega_B + \omega^D_{WP} + (1-\beta)[N]$$

which is equivalent with

$$Ric_{(X,D)/Y}(\omega) = -\omega + [N]$$

7. Existence of Initial Kähler metric along relative Kähler Ricci flow

Uniqueness of the solutions of relative Kahler Ricci flow along Iitaka fibration or $\pi : X \to X_{can}$ or along log canonical model $\pi : (X, D) \to X_{can}^D$. Let ϕ_0 and ψ_0 be be ω-plurisubharmonic functions such that $\mathfrak{v}(\phi_0, x) = 0$ for all $x \in X$, let ϕ_t, and ψ_t be the solutions of relative Kähler Ricci flow starting from ϕ_0 and ψ_0, respectively. Then in [**16**] it has been proven that if $\phi_0 < \psi_0$ then $\phi_t < \psi_t$ for all t. In particular, the flow is unique. So from the deep result of Tsuji-Schumacher[**15**], it has been showen that Weil-Petersson metric has zero Lelong number on moduli space of Calabi-Yau varieties, and by the same method we can show that logarithmic Weil-Petersson metric has zero Lelong number on moduli space of log Calabi-Yau varieties, hence by taking the initial metric to be Weil-Petersson metric or logarithmic Weil-Petersson metric and since Weil-Petersson metric or logarithmic Weil-Petersson metric are Kahler and semi-positive hence we get the uniqueness of the solutions of relative Kähler Ricci flow. Now we show how finite generation of canonical ring can be solved by positivity theory and Analytical Minimal Model Program via Kähler- Ricci flow.

Now we give a relation between the existence of Zariski Decomposition and the existence of initial Kähler metric along relative Kähler Ricci flow:

Finding an initial Kähler metric ω_0 to run the Kähler Ricci flow is important. Along holomorphic fibration with Calabi-Yau fibres, finding such initial metric is a little bit mysterious. In fact, we show that how the existence of initial Kähler metric is related to finite generation of canonical ring along singularities.

Let $\pi : X \to Y$ be an Iitaka fibration of projective varieties X, Y,(possibily singular) then is there always the following decomposition

$$K_Y + \frac{1}{m!}\pi_*\mathcal{O}_X(m!K_{X/Y}) = P + N$$

where P is semiample and N is effective. The reason is that, If X is smooth projective variety, then as we mentioned before, the canonical ring $R(X, K_X)$ is finitely generated. We may thus assume that $R(X, kK_X)$ is generated in degree 1 for some $k > 0$. Passing to a log resolution of $|kK_X|$ we may assume that $|kK_X| = M + F$ where F is the fixed divisor and M is base point free and so M defines a morphism $f : X \to Y$ which is the Iitaka fibration. Thus $M = f^*O_Y(1)$ is semiample and F is effective.

In singular case, if X is log terminal. By using Fujino-Mori's higher canonical bundle formula, after resolving X', we get a morphism $X' \to Y'$ and a klt pair $K'_Y + B'_Y$. The Y described above is the log canonical model of $K'_Y + B'_Y$ and so in fact (assuming as above that $Y' \to Y$ is a morphism), then $K'_Y + B'_Y \sim_{\mathbb{Q}} P+N$ where P is the pull-back of a rational multiple of $O_Y(1)$ and N is effective (the stable fixed divisor). If $Y'- \to Y$ is not a morphism, then P will have a base locus corresponding to the indeterminacy locus of this map.(Thanks of Hacon answer to my Mathoverflow question **[22]** which is due to E.Viehweg **[23]**)

So the existence of Zariski decomposition is related to the finite generation of canonical ring (when X is smooth or log terminal). Now if such Zariski decomposition exists then, there exists a singular hermitian metric h, with semi-positive Ricci curvature$\sqrt{-1}\Theta_h$ on P, and it is enough to take the initial metric $\omega_0 = \sqrt{-1}\Theta_h + [N]$ or $\omega_0 = \sqrt{-1}\Theta_h + \sqrt{-1}\delta\partial\bar{\partial}\|S_N\|^{2\beta}$ along relative Kähler Ricci flow

$$\frac{\partial\omega(t)}{\partial t} = -Ric_{X/Y}(\omega(t)) - \omega(t)$$

with log terminal singularities.

So when X, Y have at worst log terminal singularities(hence canonical ring is f.g and we have initial Kähler metric to run Kähler Ricci flow with starting metric ω_0) and central fibre is Calabi-Yau variety, and $-K_Y < 0$, then all the fibres are Calabi-Yau varieties and the relative Kähler-Ricci flow converges to ω which satisfies in

$$Ric(\omega_X) = -\omega_Y + f^*\omega_{WP}$$

Remark: The fact is that the solutions of relative Kähler-Einstein metric or Song-Tian metric $Ric(\omega_X) = -\omega_Y + f^*\omega_{WP}$ may not be C^∞. In fact we have C^∞ of solutions if and only if the Song-Tian measure or Tian's Kähler potential be C^∞. Now we explain that under some following algebraic condition we have C^∞-solutions for

$$Ric(\omega_X) = -\omega_Y + f^*\omega_{WP} + [N]$$

along Iitaka fibration. We recall the following Kawamata's theorem **[17]**.

Theorem 7.1. — *Let $f : X \to B$ be a surjective morphism of smooth projective varieties with connected fibers. Let $P = \sum_j P_j$, $Q = \sum_l Q_l$, be normal crossing divisors on X and B, respectively, such that $f^{-1}(Q) \subset P$ and f is smooth over $B \setminus Q$. Let $D = \sum_j d_j P_j$ be a $\mathbb{Q}$-divisor on X, where d_j may be positive, zero or negative, which satisfies the following conditions A,B,C:*

***A**) $D = D^h + D^v$ such that any irreducible component of D^h is mapped surjectively onto B by f , $f : Supp(D^h) \to B$ is relatively normal crossing over $B \setminus Q$, and $f(Supp(Dv)) \subset Q$. An irreducible component of D^h (resp. D^v) is called horizontal (resp. vertical)*

***B**)$d_j < 1$ for all j*

***C**) The natural homomorphism $\mathcal{O}_B \to f_*\mathcal{O}_X(\lceil -D \rceil)$ is surjective at the generic point of B.*

***D**) $K_X + D \sim_{\mathbb{Q}} f^*(K_B + L)$ for some $\mathbb{Q}$-divisor L on B.*

Let

$$
\begin{aligned}
f^*Q_l &= \sum_j w_{lj}P_j \\
\bar{d_j} &= \frac{d_j + w_{lj} - 1}{w_{lj}}, \ \textit{if}\, f(P_j) = Q_l \\
\delta_l &= max\{\bar{d_j}; f(P_j) = Q_l\}. \\
\Delta &= \sum_l \delta_l Q_l. \\
M &= L - \Delta.
\end{aligned}
$$

Then M is nef.

The following theorem is straightforward from Kawamata's theorem

Theorem 7.2. — *Let $d_j < 1$ for all j be as above in Theorem 0.11, and fibers be log Calabi-Yau pairs, then*

$$\int_{X_s \setminus D_s} (-1)^{n^2/2} \frac{\Omega_s \wedge \overline{\Omega_s}}{\mid S_s \mid^2}$$

is continuous on a nonempty Zariski open subset of B.

Since the inverse of volume gives a singular hermitian line bundle, we have the following theorem from Theorem 0.11

Theorem 7.3. — *Let $K_X + D \sim_{\mathbb{Q}} f^*(K_B + L)$ for some $\mathbb{Q}$-divisor L on B and*

$$
\begin{aligned}
f^*Q_l &= \sum_j w_{lj}P_j \\
\bar{d_j} &= \frac{d_j + w_{lj} - 1}{w_{lj}}, \ \textit{if}\, f(P_j) = Q_l \\
\delta_l &= max\{\bar{d_j}; f(P_j) = Q_l\}. \\
\Delta &= \sum_l \delta_l Q_l. \\
M &= L - \Delta.
\end{aligned}
$$

Then

$$\left(\int_{X_s \setminus D_s} (-1)^{n^2/2} \frac{\Omega_s \wedge \overline{\Omega_s}}{\mid S_s \mid^2}\right)^{-1}$$

is a continuous hermitian metric on the $\mathbb{Q}$*-line bundle* $K_B + \Delta$ *when fibers are log Calabi-Yau pairs.*

8. Stable family and Relative Kähler-Einstein metric

For compactification of the moduli spaces of polarized varieties Alexeev,and Kollar-Shepherd-Barron,**[25]** started a program by using new notion of moduli space of "stable family". They needed to use the new class of singularities, called semi-log canonical singularities.

Let X be an equidimensional algebraic variety that satisfies Serre's S_2 condition and is normal crossing in codimension one. Let Δ be an effective $\mathbb{R}$-divisor whose support does not contain any irreducible components of the conductor of X. The pair (X, Δ) is called a semi log canonical pair (an slc pair, for short) if

(1) $K_X + \Delta$ is $\mathbb{R}$-Cartier;

(2) (X^v, Θ) is log canonical, where $v : X^v \to X$ is the normalization and $K_{X^v} + \Theta = v^*(KX + \Delta)$

Note that, the conductor $\mathcal{C}_X$ of X is the subscheme defined by, $\mathfrak{cond}_X := Hom_{\mathcal{O}_X}(v_*\mathcal{O}_{X^v}, \mathcal{O}_X)$.

A morphism $f : X \to B$ is called a weakly stable family if it satisfies the following conditions:

1. f is flat and projective
2. $\omega_{X/B}$ is a relatively ample $\mathbb{Q}$-line bundle
3. X_b has semi log canonical singularities for all $b \in B$

A weakly stable family $f : X \to B$ is called a stable family if it satisfies KollarâĂŹs condition, that is, for any $m \in \mathbb{N}$

$$\omega_{X/B}^{[m]}|_{X_b} \cong \omega_{X_b}^{[m]}.$$

Note that, if the central fiber be Gorenstein and stable variety, then all general fibers are stable varieties, i.e, stability is an open condition

Conjecture: Weil-Petersson metric (or logarithmic Weil-Petersson metric)on stable family is semi-positive as current and such family has finite distance from zero i.e $d_{WP}(B, 0) < \infty$ when central fiber is stable variety also.

Moreover we predict the following conjecture holds true.

Conjecture: Let $f : X \to B$ is a stable family of polarized Calabi-Yau varieties, and let B is a smooth disc. then if the central fiber be stable variety as polarized Calabi-Yau variety, then we have following canonical metric on total space.

$$Ric(\omega_X) = -\omega_B + f^*(\omega_{WP}) + [N]$$

Moreover, if we have such canonical metric then our family of fibers is stable.

We predict that if the base be singular with mild singularites of general type(for example $B = X_{can}$) then we have such canonical metric on the stable family

Conjecture: The twisted Kähler-Einstein metric $Ric(\omega_X) = -\omega_Y + \alpha$ where α is a semi-positive current has unique solution if and only if α has zero Lelong number

Now the following formula is cohomological characterization of Relative Kähler-Ricci flow due to Tian

Theorem 8.1. — *The maximal time existence T for the solutions of relative Kähler Ricci flow is*

$$T = \sup\{t \mid e^{-t}[\omega_0] + (1 - e^{-t})c_1(K_{X/Y} + D) \in \mathcal{K}((X, D)/Y)\}$$

where $\mathcal{K}((X, D)/Y)$ denote the relative Kähler cone of $f : (X, D) \to Y$

Now take we have holomorphic fibre space $f : X \to Y$ such that fibers and base are Fano K-poly stable, then we have the relative Kähler-Einstein metric

$$Ric_{X/Y}(\omega) = \omega$$

we need to work on relative version of Kähler Ricci flow. i.e

$$\frac{\partial \omega}{\partial t} = -Ric_{X/Y}(\omega) + \omega$$

take the reference metric as $\tilde{\omega_t} = e^t\omega_0 + (1 + e^t)Ric(\frac{(\omega^n_{SKE})\wedge\pi^*\omega^m_Y}{\pi^*\omega^m_Y})$ then the version of Kähler Ricci flow is equivalent to the following relative Monge-Ampere equation

$$\frac{\partial \phi_t}{\partial t} = \frac{(\tilde{\omega_t} + \sqrt{-1}\partial\bar{\partial}\phi_t)^n \wedge \pi^*\omega^m_Y}{(\omega^n_{SKE}) \wedge \pi^*\omega^m_Y} + \phi_t$$

where ω_Y is the Kähler-Einstein metric corresponding to $Ric(\omega_Y) = \omega_Y$ and ω_{SKE} is the fiberwise Fano Kähler-Einstein metric.

In fact the relative volume form is $\Omega_{X/Y} = \frac{(\omega^n_{SKE})\wedge\pi^*\omega^m_Y}{\pi^*\omega^m_Y}$ and we have the following relative Monge-Ampere equation

$$\frac{\partial \phi_t}{\partial t} = \frac{(\tilde{\omega_t} + \sqrt{-1}\partial\bar{\partial}\phi_t)^n}{\Omega_{X/Y}} + \phi_t$$

Hence from $Ric_{X/Y}(\omega) = \omega$ by using the definition of relative Kähler metric we obtain $Ric(\omega_X) = \omega_Y + f^*(\omega'_Y)$, for some ω'_Y on the base which this metric is correspond to canonical metric on moduli part of family of fibers, which is Weil-Petersson metric $\omega_{WP} = \int_{X/Y} c_1(K_{X/Y}, h)^{n+1}$. Moreover I think the equation of $Ric(\omega) = \omega + f^*(\omega_{WP})$ work when we have $0 < \int_{X/Y} \omega_{SKE}^{n+1} < C$ which in general ω_{SKE} may not be semi-positive and semi-positivity of fiberwise Fano Kähler-Einstein metric is correspond to K-poly stability of total space X(this is my conjecture). The same assumption must holds when fibers are of general types. See [**5**]

Remark: Note that we still don't know canonical bundle type formula along Mori-fiber space. So finding explicit Song-Tian type metric on pair (X, D) along Mori fiber space when base and fibers are K-poly stable is not known yet. In fact when fibers and base is K-stable, then the relative Kähler-Einstein metric can not brake up to twisted Kähler-Einstein metric, due to luck of canonical bundle formula for such type of fibration.

Conjecture:Let $\pi : X \to B$ is smooth, and every X_t is K-poly stable. Then the plurigenera $P_m(X_t) = \dim H^0(X_t, -mK_{X_t})$ is independent of $t \in B$ for any m.

Idea of proof. We can apply the relative Kähler Ricci flow method for it. In fact if we prove that

$$\frac{\partial \omega(t)}{\partial t} = -Ric_{X/Y}(\omega(t)) + \omega(t)$$

has long time solution along Fano fibration such that the fibers are K-poly stable then we can get the invariance of plurigenera in the case of K-poly stability

9. Bogomolov-Miyaoka-Yau inequality for minimal varieties with intermediate Kodaira dimension

From the differential geometric proof of Yau [**44**] and the algebraic proof of Miyaoka [**43**] for minimal varieties of general type $\kappa(X) = \dim X$, we know that by using Kähler Ricci flow method we can get the following inequality

$$(-1)^n c_1^n(X) \leq (-1)^n \frac{2(n+1)}{n} c_1^{n-2}(X) c_2(X)$$

So we can extend this idea for the Bogomolov-Miyaoka-Yau inequality for minimal varieties with an intermediate Kodaira dimension $0 < \kappa(X) < \dim X$

So, we have the following inequality as soon as relative Kähler Ricci flow has C^∞-solution:

$$\left(\frac{2(n+1)}{n}c_2(\mathcal{T}_{X/X_{can}}) - c_1^2(\mathcal{T}_{X/X_{can}})\right).[\omega]^{n-2} \geq 0$$

where ω is a relative Kähler form on the minimal projective variety $X = X_{min}$ and $X_{can} = \mathrm{Proj} \bigoplus_{m\geq 0} H^0(X, K_X^m)$ is the canonical model of X (here $\mathcal{T}_{X/X_{can}} = Hom(\Omega^1_{X/X_{can}}, \mathcal{O}_X)$ mean relative tangent sheaf) via Iitaka fibration $\pi : X \to X_{can}$.

Certainly we must require stability in order that this inequality holds true. The stability must be equivalent with the fact that the following flow converges in C^∞

$$\frac{\partial\omega(t)}{\partial t} = -Ric_{X/X_{can}}(\omega(t)) - \omega(t)$$

Here $Ric_{X/X_{can}} = dd^c \log \Omega_{X/X_{can}}$(where $\Omega_{X/X_{can}}$ is the relative volume form) means relative Ricci form. Note that if such relative Kähler Ricci flow has solution then $K_{X/X_{can}}$ is psudo-effective I think that the analytical minimal model program can prove this.

In fact, if we have relative Kähler-Einstein metric $Ric_{X/X_{can}}\omega = -\omega$, then Bogomolov-Miyaoka-Yau inequality for minimal varieties with intermediate Kodaira dimension $0 < \kappa(X) < \dim X$ holds true.

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