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Existence of guided waves due to a lineic perturbation of a 3D periodic medium

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Abstract

In this note, we exhibit a three dimensional structure that permits to guide waves. This structure is obtained by a geometrical perturbation of a 3D periodic domain that consists of a three dimensional grating of equi-spaced thin pipes oriented along three orthogonal directions. Homogeneous Neumann boundary conditions are imposed on the boundary of the domain. The diameter of the section of the pipes, of order $\varepsilon > 0$, is supposed to be small. We prove that, for $\varepsilon$ small enough, shrinking the section of one line of the grating by a factor of $\sqrt{\mu}$ ($0 < \mu < 1$) creates guided modes that propagate along the perturbed line. Our result relies on the asymptotic analysis (with respect to $\varepsilon$) of the spectrum of the Laplace-Neumann operator in this structure. Indeed, as $\varepsilon$ tends to 0, the domain tends to a periodic graph, and the spectrum of the associated limit operator can be computed explicitly.

Keywords : guided waves, periodic media, spectral theory.

AMS codes : 78M35, 35J05, 58C40

1 Statement of the problem

Let $\omega_1$, $\omega_2$ and $\omega_3$ be three Lipschitz bounded domains of $\mathbb{R}^2$ of same area ($|\omega_1| = |\omega_2| = |\omega_3|$) containing the origin $(0,0)$, let $\varepsilon > 0$ be a parameter (that is going to be small), and let $a_1$, $a_2$ and $a_3$ be three positive real numbers. We denote by $(e_i)_{i \in \{1,2,3\}}$, the standard basis of $\mathbb{R}^3$. For any $(k, \ell) \in \mathbb{Z}^2$, we consider the three dimensional domain $D_{k,\ell,3}^\varepsilon$ defined by

$$D_{k,\ell,3}^\varepsilon = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \text{ such that } ((x_1 - a_1 k) / \varepsilon, (x_2 - 2a_2 \ell) / \varepsilon) \in \omega_3)\},$$

which is an unbounded cylinder of constant cross section $\varepsilon \omega_3$. It is infinite along the $e_3$ direction (invariant with respect to $x_3$) and contains the point $(a_1 k, a_2 \ell, 0)$. Similarly, for any $(k, \ell) \in \mathbb{Z}^2$, we define the domains

$$D_{k,\ell,1}^\varepsilon = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \text{ such that } ((x_2 - a_2 k) / \varepsilon, (x_3 - 3a_3 \ell) / \varepsilon) \in \omega_1)\},$$

$$D_{k,\ell,2}^\varepsilon = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \text{ such that } ((x_1 - a_1 k) / \varepsilon, (x_3 - 3a_3 \ell) / \varepsilon) \in \omega_2)\},$$

and we consider the periodic domain $\Omega_\varepsilon$ given by

$$\Omega_\varepsilon = \bigcup_{i \in \{1,2,3\}} \bigcup_{(k, \ell) \in \mathbb{Z}^2} D_{k,\ell,i}^\varepsilon.$$  \hfill (1)

The domain $\Omega_\varepsilon$ is a three dimensional grating of equi-spaced parallel pipes (of constant cross section) oriented along the three orthogonal directions $e_1$, $e_2$ and $e_3$. It is $a_j$-periodic with respect to $x_j$, $j = 1, 2, 3$. Moreover, the points $(ka_1, \ell a_2, ma_3)$, $(k, \ell, m) \in \mathbb{Z}^3$, belong to $\Omega_\varepsilon$.

In order to create guided modes, we introduce a linear defect (see [1]-[5]-[2]) in the periodic structure by modifying the section size of one pipe of the grating (it is conjectured that guided modes cannot appear in the purely periodic structure, see [4] for the proof in the case of a symmetric medium). More precisely, we assume that the domain $D_{0,0,3}^\varepsilon$ is replaced with the domain

$$D_{0,0,3}^{\varepsilon,\mu} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \text{ such that } (x_1 / (\sqrt{\mu} \varepsilon), x_2 / (\sqrt{\mu} \varepsilon)) \in \omega_3)\},$$
(b) The graph of the restriction of the domain \( \Omega \). We look for guided modes, i.e. solutions to the wave equation that tends to 0. 

\[ D_{\text{pipe}} \] with respect to \( x \). 

\( \Omega \) we search solutions of the form \( u(x) = e^{i\omega t - \beta x} \), where \( \omega \) is a real parameter and \( u(x, x_2, x_3) \in \text{L}_2(B^\varepsilon) \) is an \( a_3 \)-periodic function in \( x_3 \). In fact, it is easily seen that the \( \beta \)-quasiperiodic function \( v(x, x_2, x_3) e^{-i\beta x_3} \) is an eigenfunction of the operator

\[
A^\mu(\beta) : D(A^\mu(\beta)) \subset L_2(B^\varepsilon) \rightarrow L_2(B^\varepsilon), \quad A^\mu(\beta) = -\Delta u \text{ in } B^\varepsilon,
\]

we search solutions of the form \( u(x_1, x_2, x_3, t) = v(x_1, x_2, x_3) e^{i\omega t - \beta x_3} \), where \( \beta \) is a real parameter and \( v(x_1, x_2, x_3) \in \text{L}_2(B^\varepsilon) \) is an \( a_3 \)-periodic function in \( x_3 \). In fact, it is easily seen that the \( \beta \)-quasiperiodic function \( v(x_1, x_2, x_3) e^{-i\beta x_3} \) is an eigenfunction of the operator

\[
A^\mu(\beta) : D(A^\mu(\beta)) \subset L_2(B^\varepsilon) \rightarrow L_2(B^\varepsilon), \quad A^\mu(\beta) = -\Delta u \text{ in } B^\varepsilon,
\]

with \( D(A^\mu(\beta)) = \{ u \in H^1(B^\varepsilon), u|_{\Sigma^+} = e^{-i\beta}\ , \partial_{x_3} u|_{\Sigma^+} = e^{-i\beta}\partial_{x_3} u|_{\Sigma^-}, \partial_n u|_{\partial B^\varepsilon \setminus \Sigma^\pm} = 0 \} \), where

\[
H^1(B^\varepsilon) = \{ u \in H^1(B^\varepsilon), \text{s.t. } \Delta u \in \text{L}_2(B^\varepsilon) \} \quad \text{and} \quad \Sigma^\pm = \{ (x_1, x_2, x_3) \in \partial B^\varepsilon, x_3 = \pm a_3/2 \}.
\]

To study the spectral properties of \( A^\mu(\beta) \), we investigate its (formal) limit \( A^\mu(\beta) \) as \( \varepsilon \) tends to 0. The operator \( A^\mu(\beta) \) is defined on the limit graph \( G \) (see Fig. 1b) and its spectrum can be explicitly computed. In particular, its spectrum has infinitely many gaps (Lemma 2.1), i.e. open intervals \( (a, b) \subset \mathbb{R} \) such that the intersection of \( [a, b] \) with the spectrum is reduced to \( (a, b) \). Moreover, for \( \mu < 1 \), there is at least one eigenvalue in each gap (Lemma 2.5). Since, in addition, for \( \varepsilon > 0 \) sufficiently small, the spectrum of \( A^\mu(\beta) \) is close to the spectrum of \( A^\mu(\beta) \), the existence of guided modes is guaranteed (Theorem 3.1).

Figure 1: Illustration of the perturbed periodic domain \( \Omega^\mu \) and the limit graph \( G \).

\[ (a) \text{ The perturbed domain } \Omega^\mu \]

\[ (b) \text{ The graph } G. \]
2 The spectrum of the limit operator $A^\mu(\beta)$

2.1 Definition of the limit operator $A^\mu(\beta)$

The limit operator $A^\mu(\beta)$ is defined on the infinite periodic graph $G = \bigcap_{\varepsilon > 0} B^\mu_\varepsilon$ obtained as the limit of $B^\mu_\varepsilon$ as $\varepsilon$ tends to 0: $G$ is made of the vertices $\{v_{k,\ell} = (ka_1, la_2, 0), e^\pm_{k,\ell} = (ka_1, la_2, \pm a_3/2), (k, \ell) \in \mathbb{Z}^2\}$ connected by the edges $\{e_{k+1/2,\ell} = (v_{k,\ell}, v_{k+1,\ell}), e_{k,\ell+1/2} = (v_{k,\ell}, v_{k,\ell+1}), e^\pm_{k,\ell} = (v_{k,\ell}, v^\pm_{k,\ell}), (k, \ell) \in \mathbb{Z}^2\}$. It is $a_1$-periodic with respect to $x_1$ and $a_2$-periodic with respect to $x_2$ (see Fig. 1b).

For any function $u$ defined on $G$, we denote by $u_{k,\ell}$ (resp. $u^\pm_{k,\ell}$) its value at the vertex $v_{k,\ell}$ (resp. $v^\pm_{k,\ell}$). The restriction of $u$ to the edge $e_{k+1/2,\ell}$ (resp. $e_{k,\ell+1/2}$ and $e^\pm_{k,\ell}$) is denoted by $u_{k+1/2,\ell}(x_1)$ (resp. $u_{k,\ell+1/2}(x_2)$ and $u^\pm_{k,\ell}(x_3)$).

The definition of $A^\mu(\beta)$ also requires the introduction of the function spaces $L^2_2(G)$ and $H^2(G)$ defined as

$$
L^2_2(G) = \left\{ u : \|u\|_{L^2_2(G)} < +\infty \right\}, \quad H^2(G) = \left\{ u \in C(G) : \|u\|_{H^2(G)} < +\infty \right\},
$$

where,

$$
\|u\|^2_{L^2_2(G)} = \sum_{(k,\ell) \in \mathbb{Z}^2} \left( w^\mu_{k,\ell} \sum_{\pm} \|u^\pm_{k,\ell}\|_{L^2_2(e^\pm_{k,\ell})}^2 + \|u_{k+1/2,\ell}\|_{L^2_2(e_{k+1/2,\ell})}^2 + \|u_{k,\ell+1/2}\|_{L^2_2(e_{k,\ell+1/2})}^2 \right),
$$

$$
\|u\|^2_{H^2(G)} = \sum_{(k,\ell) \in \mathbb{Z}^2} \left( \sum_{\pm} \|u^\pm_{k,\ell}\|_{H^2(e^\pm_{k,\ell})}^2 + \|u_{k+1/2,\ell}\|_{H^2(e_{k+1/2,\ell})}^2 + \|u_{k,\ell+1/2}\|_{H^2(e_{k,\ell+1/2})}^2 \right),
$$

and, for any $(k, \ell) \in \mathbb{Z}^2$, $w^\mu_{k,\ell}$ is the weight coefficient equal to $\mu$ for $k = \ell = 0$ and 1 otherwise.

The unbounded limit operator in $L^2_2(G)$ has domain

$$
D(A^\mu(\beta)) = \left\{ u \in H^2(G) : \forall (k, \ell) \in \mathbb{Z}^2, \quad u^\pm_{k,\ell} = e^{-i\beta}u^\pm_{k,\ell}, \quad (u^+_k)^' - (u^-_k)^' = -a_3/2, \quad u^+_{k+1/2,\ell}(ka_1) - u^-_{k-1/2,\ell}(ka_1) + u^+_{k,\ell+1/2}(la_2) - u^-_{k,\ell-1/2}(la_2) + u^\mu_{k,\ell} \left( (u^+_k)^'(0) - (u^-_k)^'(0) \right) = 0 \right\},
$$

and is defined by

$$
\forall u \in D(A^\mu(\beta)), \quad A^\mu(\beta)u = -u'' \text{ on any edge of the graph } G.
$$

The functions of $D(A^\mu(\beta))$ are continuous on $G$ and $\beta$-periodic. Moreover, they satisfy the Kirchhoff conditions (8) that enforce the weighted sum of the outward derivatives of $u$ to vanish at each vertex $v_{k,\ell}$, $(k, \ell) \in \mathbb{Z}^2$. We point out that the perturbation, which results from a geometrical modification of the domain for the problem (4), is taken into account at the limit by means of the Kirchhoff condition (8) at the vertex $v_{0,0}$ $(w^\mu_{0,0} = \mu)$. The formal derivation of the limit model can be found in [6]. It is easily verified that the operator $A^\mu(\beta)$ is self-adjoint (for the weighted scalar product associated with (6)), see also [3]. The objective of the following two sections is to study the spectrum of $A^\mu(\beta)$.

2.2 Characterization and properties of the essential spectrum of $A^\mu(\beta)$

By a compact perturbation argument, one can prove that $\sigma_{ess}(A^\mu(\beta)) = \sigma(A(\beta))$, where $A(\beta) = A^1(\beta)$ is the purely periodic operator corresponding to $A^\mu(\beta)$ for $\mu = 1$. The computation of its spectrum relies on the Floquet-Bloch theory (see [9]). More precisely, we can prove that $\lambda = \omega^2 \in \sigma(A(\beta))$ if and only if either $\omega = 0$ and $\beta = 0$ or $\omega \neq 0$ and there exists $(k_1, k_2) \in [0, \pi]^2$ such that

$$
\sin(\omega a_2) \sin(\omega a_3) (\cos(\omega a_1) - \cos k_1) + \sin(\omega a_2) \sin(\omega a_1) (\cos(\omega a_2) - \cos k_2)
$$

$$
+ \sin(\omega a_1) \sin(\omega a_2) (\cos(\omega a_3) - \cos \beta) = 0. \tag{10}
$$

Based on the previous characterization, we prove that the operator $A(\beta)$ has a countable infinity of gaps that can be separated into three categories (see [10] for the proof):
Lemma 2.1 The following properties hold:
1. \( \sigma_1 \cup \sigma_2 \cup \sigma_3 \subset \sigma(A(\beta)) \), where
   \[
   \sigma_i = \{(\pi n/a_i)^2, n \in \mathbb{Z}\} \text{ for } i \in \{1, 2\}, \text{ and } \sigma_3 = \{((\pm \beta + 2\pi n)/a_3)^2, n \in \mathbb{Z}\}.
   \]
2. For any \( \beta \in [0, \pi] \), the operator \( A(\beta) \) has infinitely many gaps whose ends tend to infinity.
3. Let \( \mathcal{W}(\beta) = \{\pi n/a_3, n \in \mathbb{N}^*\} \) if \( \beta \notin \{0, \pi\} \) and \( \mathcal{W}(\beta) = \{\beta/a_3 + (2n + 1)\pi/a_3, n \in \mathbb{N}^*\} \) if \( \beta \in \{0, \pi\} \). If an interval \( (\omega^2, \omega^2) \) is a spectral gap of \( A(\beta) \), then, one of the following possibilities holds:
   (i) \( \omega^2 \notin \sigma_1 \cup \sigma_2 \), \( \omega^2 \notin \sigma_1 \cup \sigma_2 \), and there is a unique \( \omega_0 \in (\omega^2, \omega^2) \) \( \mathcal{W}(\beta) \).
   (ii) \( \omega_0 \in \sigma_1 \cup \sigma_2 \), \( \omega^2 \notin \sigma_1 \cup \sigma_2 \) and \( \mathcal{W}(\beta) \cap (\omega^2, \omega^2) = \emptyset \).
   (iii) \( \omega_0 \notin \sigma_1 \cup \sigma_2 \), \( \omega^2 \in \sigma_1 \cup \sigma_2 \) and \( \mathcal{W}(\beta) \cap (\omega^2, \omega^2) = \emptyset \).

2.3 Computation of the discrete spectrum
Let us now determine the discrete spectrum of \( A^\mu(\beta) \). If \( \lambda = \omega^2 \) is an eigenvalue of \( A^\mu(\beta) \), then the corresponding eigenfunction \( u \in D(A^\mu(\beta)) \) satisfies the linear differential equation \( u'' + \omega^2 u = 0 \) on each edge of the graph \( G \). Solving explicitly this equation (on each edge), taking into account the quasi-periodicity of \( u \) and the Kirchhoff conditions (8), we can replace the eigenvalue problem \( A^\mu(\beta)u = \lambda u \) with a set of finite differences equations for \( (u_{k,\ell}, t, t) \in \mathbb{Z}^2 \):

**Lemma 2.2** Assume that \( \omega \notin \{\pi n/a_1 \cup \{\pi n/a_2 \cup \{\pi n/a_3 \} \} \in D(A^\mu(\beta)) \) is an eigenfunction of \( A^\mu(\beta) \) if and only if \( (u_{k,\ell}, t, t) \in \mathbb{Z}^2 \) and satisfies

\[
\begin{align*}
\frac{u_{k+1,\ell}}{\sin(\omega_1)} + \frac{u_{k-1,\ell}}{\sin(\omega_1)} + \frac{u_{k,\ell+1}}{\sin(\omega_2)} + \frac{u_{k,\ell-1}}{\sin(\omega_2)} - \frac{2g_\beta(\omega)}{\omega} u_{k,\ell} &= 0, \quad \forall (k, \ell) \in \mathbb{Z}^2 \setminus \{(0, 0)\}, \\
\frac{u_{1,0}}{\sin(\omega_1)} - \frac{u_{-1,0}}{\sin(\omega_1)} + \frac{u_{0,1}}{\sin(\omega_2)} - \frac{u_{0,-1}}{\sin(\omega_2)} - 2g_\beta(\omega) u_{0,0} &= (\mu - 1) \frac{\cos(\omega_3) - \cos(\beta)}{\sin(\omega_3)} u_{0,0},
\end{align*}
\]

where we have defined \( g_\beta(\omega) = \frac{1}{\sin(\omega_3)} + \frac{1}{\tan(\omega_1)} \) and \( \frac{1}{\cos(\omega_3)} - \frac{1}{\sin(\omega_3)} \).

As well-known, finite difference schemes may be investigated using the discrete Fourier transform

\[
\mathcal{F} : \mathcal{V} = (v_{k,\ell})_{(k,\ell) \in \mathbb{Z}^2} \mapsto \mathcal{F}(\mathcal{V}) = \tilde{\mathcal{V}}, \quad \tilde{\mathcal{V}}(\xi, \eta) = \sum_{k,\ell \in \mathbb{Z}} e^{i(k\xi + \ell\eta)} v_{k,\ell}, \quad (\xi, \eta) \in [0, 2\pi]^2.
\]

where \( \mathcal{F} \) is an isometry between \( L_2(\mathbb{Z}^2) \) and \( L_2([0, 2\pi)^2) \). This, together with Lemma 2.2, provides the following characterization for the discrete spectrum of \( A^\mu(\beta) \):

**Lemma 2.3** Assume that \( \omega \notin \{\pi n/a_1 \cup \{\pi n/a_2 \cup \{\pi n/a_3 \} \} \in D(A^\mu(\beta)) \) is an eigenfunction of \( A^\mu(\beta) \) if and only if the discrete Fourier transform \( \tilde{\mathcal{V}} \) of \( (u_{k,\ell}, t, t) \in \mathbb{Z}^2 \) belongs to \( L_2([0, 2\pi)^2) \) and satisfies

\[
(f(\xi, \eta, \omega) - \phi_\beta(\omega)) \tilde{\mathcal{V}}(\xi, \eta) = (\mu - 1) \phi_\beta(\omega) u_{0,0},
\]

where \( \phi_\beta(\omega) = \frac{\cos(\omega_3) - \cos(\beta)}{\sin(\omega_3)} \) and \( f(\xi, \eta, \omega) = \frac{\cos(\xi - \cos(\omega_1))}{\sin(\omega_1)} + \frac{\cos(\eta - \cos(\omega_2))}{\sin(\omega_2)} \).

Under the assumption of Lemma 2.3, (10) can be written as \( f(\xi, \eta, \omega) - \phi_\beta(\omega) = 0 \). It follows that \( \lambda = \omega^2 \) does not belong to \( \sigma_{ess}(A^\mu(\beta)) \) if and only if, for any \( (\xi, \eta) \in [0, 2\pi]^2 \), \( \phi_\beta(\omega) - f(\xi, \eta, \omega) \) does not vanish. As a consequence, as soon as \( \lambda = \omega^2 \notin \sigma_{ess}(A^\mu(\beta)) \), the function \( (\xi, \eta) \mapsto \phi_\beta(\omega)/(\phi_\beta(\omega) - f(\xi, \eta, \omega)) \) is continuous and bounded. Then, the inverse discrete Fourier transform can be applied to (13) to obtain

\[
u_{k,\ell} = \frac{(1 - \mu) u_{0,0}}{4\pi^2} \int_{(0,2\pi)^2} \frac{\phi_\beta(\omega)}{\phi_\beta(\omega) - f(\xi, \eta, \omega)} e^{-i(k\xi + \ell\eta)} d\xi d\eta, \quad \forall (k, \ell) \in \mathbb{Z}^2.
\]

Writing the previous relation for \( k = \ell = 0 \) yields the following criterion of existence of an eigenvalue:
Lemma 2.4 Assume that $\omega \notin \{\pi \mathbb{Z}/a_1\} \cup \{\pi \mathbb{Z}/a_2\} \cup \{\pi \mathbb{Z}/a_3\}$ and that $\lambda = \omega^2 \notin \sigma_{\text{ess}}(A^\mu(\beta))$. Then, $\lambda$ is an eigenvalue of $A^\mu(\beta)$ if and only if

$$
\mu = 1 - F_\beta(\omega) \quad \text{where} \quad F_\beta(\omega) = \left( \frac{1}{4\pi^2} \int_{(0,2\pi)^2} \frac{\phi_\beta(\omega)}{\phi_\beta(\omega) - f(\xi,\eta)} \, d\xi d\eta \right)^{-1}.
$$

(14)

The study of the behavior of the function $F_\beta$ leads to the existence of at least one eigenvalue in each gap of $A^\mu(\beta)$ as soon as $\mu < 1$, the minimal number of eigenvalues in each gap depending on the type of gaps (cf. Lemma. 2.1-3 for the classification):

Lemma 2.5 For $\mu > 1$, the operator $A^\mu(\beta)$ has no eigenvalue. For $0 < \mu < 1$, let $(\omega_0^2, \omega_1^2)$ be a spectral gap of the operator $A^\mu(\beta)$:

(a) If $(\omega_0^2, \omega_1^2)$ is a gap of type (i), then $A^\mu(\beta)$ has at least two eigenvalues $\lambda_1 = \omega_1^2$ and $\lambda_2 = \omega_2^2$ that satisfy $\omega_b < \omega_1 < \omega_0 < \omega_2 < \omega_l$ (see Lemma. 2.1-3 for the definition of $\omega_0$).

(b) If $(\omega_0^2, \omega_1^2)$ is a gap of type (ii) or (iii), then $A^\mu(\beta)$ has at least one eigenvalue $\lambda_1 = \omega_1^2$ such that $\omega_b < \omega_1 < \omega_l$.

The sketch of the proof of the previous lemma is the following, a complete proof being available in [10] (Theorem 5.2.1): First, one can verify that $F_\beta(\omega) \geq 0$ in any gap, which, together with (14) proves that $A^\mu(\beta)$ has no eigenvalue for $\mu > 1$. Then, if $(\omega_b^2, \omega_l^2)$ is a gap of type (i), one can show that

$$
\lim_{\omega \to \omega_b^2} (1 - F_\beta(\omega)) = \lim_{\omega \to \omega_l^2} (1 - F_\beta(\omega)) = 1 \quad \text{and} \quad \lim_{\omega \to \omega_0} (1 - F_\beta(\omega)) = 0.
$$

By continuity of $F_\beta$ inside the gap, (a) directly results from the intermediate value theorem and (14). If $(\omega_b^2, \omega_l^2)$ is a gap of type (ii), the intermediate value theorem also permits us to conclude since

$$
\lim_{\omega \to \omega_b} (1 - F_\beta(\omega)) \leq 0 \quad \text{and} \quad \lim_{\omega \to \omega_l} (1 - F_\beta(\omega)) = 1.
$$

A similar argument works for a gap of type (iii).

3 Guided modes for the operator $A^\mu(\beta)$: an asymptotic result

Finally, thanks to the general result [8] (Theorem 2.13 convergence of the spectrum of $A^\mu(\beta)$ toward the spectrum of $A^\mu(\beta)$), we can prove the following result of existence of eigenvalue for the operator $A^\mu(\beta)$:

Theorem 3.1 Let $\mu \in (0,1)$, $(\lambda_b, \lambda_l)$ be a spectral gap of the operator $A^\mu(\beta)$ and $\lambda_0 \in (\lambda_b, \lambda_l)$ be a (simple) eigenvalue of this operator. Then, there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ the operator $A^\mu(\beta)$ has an eigenvalue $\lambda_\varepsilon$ inside a spectral gap $(\lambda_b', \lambda_l')$. Moreover, $\lambda_\varepsilon = \lambda_0 + O(\sqrt{\varepsilon})$.

References


