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Subdivisions of oriented cycles in digraphs with large chromatic number

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Extended Abstract

The chromatic number $\chi(D)$ of a digraph $D$ is the chromatic number of its underlying graph. The chromatic number of a class of digraphs $\mathcal{D}$, denoted by $\chi(\mathcal{D})$, is the smallest $k$ such that $\chi(D) \leq k$ for all $D \in \mathcal{D}$, or $+\infty$ if no such $k$ exists. If $\chi(D) \neq +\infty$, we say that $\mathcal{D}$ has bounded chromatic number.

We are interested in the following question : which are the digraph classes $\mathcal{D}$ such that every digraph with sufficiently large chromatic number contains an element of $\mathcal{D}$ ? Let us denote by $\text{Forb}(\mathcal{H})$ (resp. $\text{Forb}(\mathcal{H})$) the class of digraphs that do not contain $\mathcal{H}$ (resp. any element of $\mathcal{H}$) as a subdigraph. We are particularly interested in the chromatic number of $\text{Forb}(\mathcal{C})$, where $\mathcal{C}$ is a family of oriented cycles.

**Problem 1.** Which are the classes of digraphs $\mathcal{D}$ such that $\chi(\text{Forb}(\mathcal{D})) < +\infty$ ?

An oriented graph is an orientation of a (simple) graph. An oriented path (resp., an oriented cycle) is said directed if all vertices have in-degree and out-degree at most 1.

Observe that if $D$ is an orientation of a graph $G$ and $\text{Forb}(D)$ has bounded chromatic number, then $\text{Forb}(G)$ has also bounded chromatic number. A classical result by Erdős implies that $G$ must be a tree. Burr proved that every $(k - 1)^2$-chromatic digraph contains every oriented tree of order $k$ and conjectured Burr [3] that it could be further improved to $(2k - 2)$-chromatic digraphs.

For special oriented trees $T$, better bounds on the chromatic number of $\text{Forb}(T)$ are known. The most famous one, known as Gallai-Hasse-Roy-Vitaver Theorem [6] states that $\chi(\text{Forb}(P^+(k))) = k$, where $P^+(k)$ is the directed path of length $k$ (a directed path is an oriented path in which all arcs are in the same direction).

The chromatic number of the class of digraphs not containing a prescribed oriented path $P$ on $n$ vertices with two blocks (blocks are maximal directed subpaths) has been determined by Addario-Berry et al. [1] :

**Theorem 2** (Addario-Berry et al. [1]). Let $P$ be an oriented path with two blocks on $n \geq 4$ vertices, then $\chi(\text{Forb}(P)) = n - 1$.

In this paper, we are interested in the chromatic number of $\text{Forb}(\mathcal{H})$ when $\mathcal{H}$ is an infinite family of oriented cycles. Let us denote by $\text{S-Forb}(D)$ (resp. $\text{S-Forb}(\mathcal{D})$) the class of digraphs that contain no subdivision of $D$ (resp. any element of $\mathcal{D}$) as a subdigraph. We are particularly interested in the chromatic number of $\text{S-Forb}(\mathcal{C})$, where $\mathcal{C}$ is a family of oriented cycles.

Let us denote by $C_k$ the directed cycle of length $k$. For all $k$, $\chi(\text{S-Forb}(C_k)) = +\infty$ because transitive tournaments have no directed cycle. Let us denote by $C(k, \ell)$ the oriented cycle with two blocks, one of length $k$ and the other of length $\ell$. Observe that the oriented cycles with two blocks are the subdivisions of $C(1, 1)$. As pointed by Gyárfás and Thomassen (see [1]), there are acyclic oriented graphs with arbitrarily large chromatic number and no oriented cycles with two blocks. Therefore $\chi(\text{S-Forb}(C(k, \ell))) = +\infty$. We first generalise this result to every oriented cycle.

**Theorem 3.** For any oriented cycle $C$, $\chi(\text{S-Forb}(C)) = +\infty$.

In fact, we show the following stronger theorem.

**Theorem 4.** For any positive integers $b, c$, there exists an acyclic digraph $D_c$ with $\chi(D_c) \geq c$ in which all oriented cycles have more than $b$ blocks.
We need a construction due to Erdős and Lovász [5] of hypergraphs with high girth and large chromatic number. 

**Theorem 5.** [5, Theorem 1'] For $k,g,c \in \mathbb{N}$, there exists a $k$-uniform hypergraph with girth larger than $g$ and weak chromatic number larger than $c$.

We assume $g$ is being fixed, the following construction allow us to find $D_{c+1}$ from $D_c$.

Let $p$ be the number of proper $c$-colourings of $D_c$ and let those colourings be denoted by $col^1_c, \ldots, col^p_c$. By Theorem 5 there exists a $c \times p$-uniform hypergraph $\mathcal{H}$ with weak chromatic number $> p$ and girth $> g/2$. Let $X = \{x_1, \ldots, x_n\}$ be the ground set of $\mathcal{H}$.

We construct $D_{c+1}$ from $n$ disjoint copies $D^1_c, \ldots, D^n_c$ of $D_c$ as follows. For each hyperedge $S \in \mathcal{H}$, we do the following:

- We partition $S$ into $p$ sets $S_1, \ldots, S_p$ of cardinality $c$.
- For each set $S_i = \{x_{k_1}, \ldots, x_{k_c}\}$, we choose vertices $v_{k_1} \in D^1_c, \ldots, v_{k_c} \in D^p_c$ such that $col^1_c(v_{k_1}) = 1, \ldots, col^p_c(v_{k_c}) = c$, and add a new vertex $w_{S,i}$ with $v_{k_1}, \ldots, v_{k_c}$ as in-neighbours.

On the other hand, considering strongly connected (strong for short) digraphs may lead to dramatically different result. An example is provided by the following celebrated result due to Bondy [2], which can be rephrased as follows when denoting the class of strong digraphs by $S$.

**Theorem 6** (Bondy [2]). $\chi(S\text{-Forb}(\mathcal{C}_k) \cap S) = k - 1$.

Inspired by this theorem, Addario-Berry et al. [1] posed the following problem.

**Problem 7.** Let $k$ and $\ell$ be two positive integers then $\chi(S\text{-Forb}(C(k,\ell) \cap S)) < k + \ell$.

We give evidence for this problem by showing the following weaker statement.

**Theorem 8.** Let $k$ and $\ell$ be two positive integers such that $k \geq \max\{\ell, 3\}$, and let $D$ be a digraph in $S\text{-Forb}(C(k,\ell)) \cap S$. Then, $\chi(D) \leq (k + \ell - 2)(k + \ell - 3)(2\ell + 2)(k + \ell + 1)$.

We need the following lemma.

The union of two digraphs $D_1$ and $D_2$ is the digraph $D_1 \cup D_2$ with vertex set $V(D_1) \cup V(D_2)$ and arc set $A(D_1) \cup A(D_2)$.

**Lemma 9.** Let $D_1$ and $D_2$ be two digraphs. $\chi(D_1 \cup D_2) \leq \chi(D_1) \times \chi(D_2)$.

A consequence of the previous lemma is that, if we partition the arc set of $D$ into set $A_1 \cdots A_k$, then bounding the chromatic number of all digraphs induced by the $A_i$ implies that $D$ has bounded chromatic number.

**Proof.** Let $D$ be a strong digraph without any copy of $C(k,\ell)$, we exhibit a colouring of $D$ using a bounded number of colours. The proof heavily relies on the technique of levelling. Let $u$ be a vertex of $D$. The level of a vertex $x$, noted $\text{lvl}(x)$ is the length of the shortest dipath from $u$ to $x$. $L(i)$ is the set of vertices at level $i$.

Since $D$ is strongly connected, it has an out-generator $u$. Let $T$ be a BFS-tree with root $u$. We define the following sets of arcs.

$$
A_0 = \{xy \in A(D) \mid \text{lvl}(x) = \text{lvl}(y)\};
A_1 = \{xy \in A(D) \mid 0 < |\text{lvl}(x) - \text{lvl}(y)| < k + \ell - 3\};
A' = \{xy \in A(D) \mid \text{lvl}(x) - \text{lvl}(y) \geq k + \ell - 3\}.
$$

Since $k + \ell - 3 > 0$ and there is no arc $xy$ with $\text{lvl}(y) > \text{lvl}(x) + 1$, $(A_0, A_1, A')$ is a partition of $A(D)$. Observe moreover that $A(T) \subseteq A_1$. We further partition $A'$ into two sets $A_2$ and $A_3$, where $A_2 = \{xy \in A' \mid y$ is an ancestor of $x$ in $T\}$ and $A_3 = A' \setminus A_2$. Then $(A_0, A_1, A_2, A_3)$ is a partition of $A(D)$. Let $D_j = (V(D), A_j)$ for all $j \in \{0, 1, 2, 3\}$. 
Claim 10. \( \chi(D_0) \leq k + \ell - 2. \)

Proof. Observe that \( D_0 \) is the disjoint union of the \( D[L_i] \) where \( L_i = \{ v \mid \text{dist}_D(u, v) = i \} \). Therefore it suffices to prove that \( \chi(D[L_i]) \leq k + \ell - 2 \) for all non-negative integer \( i \).

\( L_0 = \{ u \} \) so the result holds trivially for \( i = 0 \).

Assume now \( i \geq 1 \). Suppose for a contradiction \( \chi(D[L_i]) \geq k + \ell - 1 \). Since \( k \geq 3 \), by Theorem 2, \( D[L_i] \) contains a copy \( Q \) of \( P^+(k-1, \ell-1) \), the path on two blocks of length \( k-1 \) and \( \ell-1 \) with one vertex of indegree 2. Let \( v_1 \) and \( v_2 \) be the initial and terminal vertices of \( Q \), let \( x \) be the least common ancestor of \( v_1 \) and \( v_2 \). By definition, for \( j \in \{1, 2\} \), there exists a dipath \( P_j \) from \( x \) to \( v_j \) in \( T \). By definition of least common ancestor, \( V(P_1) \cap V(P_2) = \{ x \} \), \( V(P_j) \cap L_i = \{ v_j \} \), \( j = 1, 2 \), and both \( P_1 \) and \( P_2 \) have length at least 1. Consequently, \( P_1 \cup P_2 \cup Q \) is a subdivision of \( C(k, \ell) \), a contradiction. \( \square \)

Claim 11. \( \chi(D_1) \leq k + \ell - 3. \)

Proof. Let \( \phi_1 \) be the colouring of \( D_1 \) defined by \( \phi_1(x) = \text{lvl}(x) \pmod{k+\ell-3} \). By definition of \( D_1 \), this is clearly a proper colouring of \( D_1 \). \( \square \)

The following two claims are more complicated, we refer the reader to [4] for the complete proofs.

Claim 12. \( \chi(D_2) \leq 2\ell + 2. \)

Claim 13. \( \chi(D_3) \leq k + \ell + 1. \)

Claims 10, 11, 12, and 13, together with Lemma 9 yield the result. \( \square \)

More generally, one may wonder what happens for other oriented cycles. Our next result generalises Theorem 8 for \( C_4 \) the cycle with 4 blocks.

Theorem 14. Let \( D \) be a digraph in \( S-\text{Forb}(\hat{C}_4) \). If \( D \) admits an out-generator, then \( \chi(D) \leq 24. \)

Proof. The general idea is the same as in the proof of Theorem 8.

Suppose that \( D \) admits an out-generator \( u \) and let \( T \) be an BFS-tree with root \( u \). We partition \( A(D) \) into three sets according to the levels of \( u \).

\begin{align*}
A_0 &= \{(x, y) \in A(D) \mid \text{lvl}(x) = \text{lvl}(y)\}; \\
A_1 &= \{(x, y) \in A(D) \mid |\text{lvl}(x) - \text{lvl}(y)| = 1\}; \\
A_2 &= \{(x, y) \in A(D) \mid \text{lvl}(y) \leq \text{lvl}(x) - 2\}.
\end{align*}

For \( i = 0, 1, 2 \), let \( D_i = (V(D), A_i) \).

Claim 15. \( \chi(D_0) \leq 3. \)

Proof. Suppose for a contradiction that \( \chi(D) \geq 4. \) By Theorem 2, it contains a \( P^-(1, 1) \) \( (y_1, y_2, y_2) \), that is \( (y, y_1) \) and \( (y, y_2) \) are in \( A(D_0) \). Let \( x \) be the least common ancestor of \( y_1 \) and \( y_2 \) in \( T \). The union of \( T[x, y_1] \), \( (y, y_1) \), \( (y, y_2) \), and \( T[x, y_2] \) is a subdivision of \( \hat{C}_4 \), a contradiction. \( \square \)

Claim 16. \( \chi(D_1) \leq 2. \)

Proof. Since the arc are between consecutive levels, then the colouring \( \phi_1 \) defined by \( \phi_1(x) = \text{lvl}(x) \pmod{2} \) is a proper 2-colouring of \( D_1 \). \( \square \)

Let \( y \in V_i \) we denote by \( N'(y) \) the out-degree of \( y \) in \( \bigcup_{0 \leq j \leq i-1} V_j \). Let \( D' = (V, A') \) with \( A' = \bigcup_{x \in V} \{ (x, y) \mid y \in N'(x) \} \) and \( D_x = (V, A_x) \) where \( A_x \) is the set of arc inside the level and from \( V_i \) to \( V_{i+1} \) for all \( i \). Note that \( A = A' \cup A_x \) and
Claim 17. \( \chi(D_2) \leq 4. \)

Proof. We refer to [4] for the proof of this statement. \( \square \)

Claims 15, 16, 17, and Lemma 9 implies \( \chi(D) \leq 24. \) \( \square \)

References


