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Subdivisions of oriented cycles in digraphs with large chromatic number

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Extended Abstract

The chromatic number \(\chi(D)\) of a digraph \(D\) is the chromatic number of its underlying graph. The bounded chromatic number of a class of digraphs \(\mathcal{D}\), denoted by \(\chi(\mathcal{D})\), is the smallest \(k\) such that \(\chi(D) \leq k\) for all \(D \in \mathcal{D}\), or \(+\infty\) if no such \(k\) exists. If \(\chi(D) \neq +\infty\), we say that \(\mathcal{D}\) has bounded chromatic number.

We are interested in the following question: which are the digraph classes \(\mathcal{D}\) such that every digraph with sufficiently large chromatic number contains an element of \(\mathcal{D}\)? Let us denote by \(\text{Forb}(H)\) (resp. \(\text{Forb}(\mathcal{H})\)) the class of digraphs that do not contain \(H\) (resp. any element of \(\mathcal{H}\)) as a subdigraph. The above question can be restated as follows:

**Problem 1.** Which are the classes of digraphs \(\mathcal{D}\) such that \(\chi(\text{Forb}(\mathcal{D})) < +\infty\)?

An oriented graph is an orientation of a (simple) graph. An oriented path (resp., an oriented cycle) is said directed if all vertices have in-degree and out-degree at most 1.

Observe that if \(\mathcal{D}\) is an orientation of a graph \(G\) and \(\text{Forb}(\mathcal{D})\) has bounded chromatic number, then \(\text{Forb}(G)\) has also bounded chromatic number. A classical result by Erdős implies that \(G\) must be a tree. Burr proved that every \((k-1)^2\)-chromatic digraph contains every oriented tree of order \(k\) and conjectured Burr [3] that it could be further improved to \((2k-2)\)-chromatic digraphs.

For special oriented trees \(T\), better bounds on the chromatic number of \(\text{Forb}(T)\) are known. The most famous one, known as Gallai-Hasse-Roy-Vitaver Theorem [6] states that \(\chi(\text{Forb}(P^+(k))) = k\), where \(P^+(k)\) is the directed path of length \(k\) (a directed path is an oriented path in which all arcs are in the same direction).

The chromatic number of the class of digraphs not containing a prescribed oriented path \(P\) on \(n\) vertices with two blocks (blocks are maximal directed subpaths) has been determined by Addario-Berry et al. [1]:

**Theorem 2** (Addario-Berry et al. [1]). Let \(P\) be an oriented path with two blocks on \(n \geq 4\) vertices, then \(\chi(\text{Forb}(P)) = n - 1\).

In this paper, we are interested in the chromatic number of \(\text{Forb}(\mathcal{H})\) when \(\mathcal{H}\) is an infinite family of oriented cycles. Let us denote by \(\text{S-Forb}(\mathcal{D})\) (resp. \(\text{S-Forb}(\mathcal{D})\)) the class of digraphs that contain no subdivision of \(D\) (resp. any element of \(\mathcal{D}\)) as a subdigraph. We are particularly interested in the chromatic number of \(\text{S-Forb}(C)\), where \(C\) is a family of oriented cycles.

Let us denote by \(C_k\) the directed cycle of length \(k\). For all \(k\), \(\chi(\text{S-Forb}(C_k)) = +\infty\) because transitive tournaments have no directed cycle. Let us denote by \(C(k, \ell)\) the oriented cycle with two blocks, one of length \(k\) and the other of length \(\ell\). Observe that the oriented cycles with two blocks are the subdivisions of \(C(1,1)\). As pointed by Gyárfás and Thomassen (see [1]), there are acyclic oriented graphs with arbitrarily large chromatic number and no oriented cycles with two blocks. Therefore \(\chi(\text{S-Forb}(C(k, \ell))) = +\infty\). We first generalise this result to every oriented cycle.

**Theorem 3.** For any oriented cycle \(C\), \(\chi(\text{S-Forb}(C)) = +\infty\).

In fact, we show the following stronger theorem.

**Theorem 4.** For any positive integers \(b, c\), there exists an acyclic digraph \(D_c\) with \(\chi(D_c) \geq c\) in which all oriented cycles have more than \(b\) blocks.
We need a construction due to Erdős and Lovász [5] of hypergraphs with high girth and large chromatic number.

**Theorem 5.** [5, Theorem 1'] For $k, g, c \in \mathbb{N}$, there exists a $k$-uniform hypergraph with girth larger than $g$ and weak chromatic number larger than $c$.

We assume $g$ is being fixed, the following construction allow us to find $D_{c+1}$ from $D_c$.

Let $p$ be the number of proper $c$-colourings of $D_c$, and let those colourings be denoted by $\text{col}_1^c, \ldots, \text{col}_p^c$. By Theorem 5 there exists a $c \times p$-uniform hypergraph $\mathcal{H}$ with weak chromatic number $> p$ and girth $> g/2$. Let $X = \{x_1, \ldots, x_n\}$ be the ground set of $\mathcal{H}$.

We construct $D_{c+1}$ from $n$ disjoint copies $D_{c+1}^1, \ldots, D_{c+1}^n$ of $D_c$ as follows. For each hyperedge $S \in \mathcal{H}$, we do the following:

- We partition $S$ into $p$ sets $S_1, \ldots, S_p$ of cardinality $c$.
- For each set $S_i = \{x_{k_1}, \ldots, x_{k_c}\}$, we choose vertices $v_{k_1} \in D_{c+1}^1, \ldots, v_{k_c} \in D_{c+1}^p$ such that $\text{col}_v^c(v_{k_1}) = 1, \ldots, \text{col}_v^c(v_{k_c}) = c$, and add a new vertex $w_{S,i}$ with $v_{k_1}, \ldots, v_{k_c}$ as in-neighbours.

On the other hand, considering strongly connected (strong for short) digraphs may lead to dramatically different result. An example is provided by the following celebrated result due to Bondy [2], which can be rephrased as follows when denoting the class of strong digraphs by $\mathcal{S}$.

**Theorem 6** (Bondy [2]). $\chi(\text{S-Forb}(\mathcal{C}_k) \cap \mathcal{S}) = k - 1$.

Inspired by this theorem, Addario-Berry et al. [1] posed the following problem.

**Problem 7.** Let $k$ and $\ell$ be two positive integers then $\chi(\text{S-Forb}(C(k, \ell) \cap \mathcal{S})) < k + \ell$.

We give evidence for this problem by showing the following weaker statement.

**Theorem 8.** Let $k$ and $\ell$ be two positive integers such that $k \geq \max\{\ell, 3\}$, and let $D$ be a digraph in $\text{S-Forb}(C(k, \ell)) \cap \mathcal{S}$. Then, $\chi(D) \leq (k + \ell - 2)(k + \ell - 3)(2\ell + 2)(k + \ell + 1)$.

We need the following lemma.

The union of two digraphs $D_1$ and $D_2$ is the digraph $D_1 \cup D_2$ with vertex set $V(D_1) \cup V(D_2)$ and arc set $A(D_1) \cup A(D_2)$.

**Lemma 9.** Let $D_1$ and $D_2$ be two digraphs. $\chi(D_1 \cup D_2) \leq \chi(D_1) \times \chi(D_2)$.

A consequence of the previous lemma is that, if we partition the arc set of $D$ into set $A_1, \ldots, A_k$, then bounding the chromatic number of all digraphs induced by the $A_i$ implies that $D$ has bounded chromatic number.

**Proof.** Let $D$ be a strong digraph without any copy of $C(k, \ell)$, we exhibit a colouring of $D$ using a bounded number of colours. The proof heavily relies on the technique of levelling. Let $u$ be a vertex of $D$. The level of a vertex $x$, noted $\text{lvl}(x)$ is the length of the shortest dipath from $u$ to $x$. $L(i)$ is the set of vertices at level $i$.

Since $D$ is strongly connected, it has an out-generator $u$. Let $T$ be a BFS-tree with root $u$. We define the following sets of arcs.

$$
A_0 = \{xy \in A(D) \mid \text{lvl}(x) = \text{lvl}(y)\}; \\
A_1 = \{xy \in A(D) \mid 0 < |\text{lvl}(x) - \text{lvl}(y)| < k + \ell - 3; \}
A' = \{xy \in A(D) \mid \text{lvl}(x) - \text{lvl}(y) \geq k + \ell - 3\}.
$$

Since $k + \ell - 3 > 0$ and there is no arc $xy$ with $\text{lvl}(y) > \text{lvl}(x) + 1$, $(A_0, A_1, A')$ is a partition of $A(D)$. Observe moreover that $A(T) \subseteq A_1$. We further partition $A'$ into two sets $A_2$ and $A_3$, where $A_2 = \{xy \in A' \mid y$ is an ancestor of $x$ in $T\}$ and $A_3 = A' \setminus A_2$. Then $(A_0, A_1, A_2, A_3)$ is a partition of $A(D)$. Let $D_j = (V(D), A_j)$ for all $j \in \{0, 1, 2, 3\}$.
Claim 10. $\chi(D_0) \leq k + \ell - 2$.

Proof. Observe that $D_0$ is the disjoint union of the $D[L_i]$ where $L_i = \{ v \mid \text{dist}_D(u, v) = i \}$. Therefore it suffices to prove that $\chi(D[L_i]) \leq k + \ell - 2$ for all non-negative integer $i$.

Let $L_0 = \{ u \}$ so the result holds trivially for $i = 0$.

Assume now $i \geq 1$. Suppose for a contradiction $\chi(D[L_i]) \geq k + \ell - 1$. Since $k \geq 3$, by Theorem 2, $D[L_i]$ contains a copy $Q$ of $P^+(k-1, \ell-1)$, the path on two blocks of length $k-1$ and $\ell-1$ with one vertex of indegree 2. Let $v_1$ and $v_2$ be the initial and terminal vertices of $Q$, and let $x$ be the least common ancestor of $v_1$ and $v_2$. By definition, for $j \in \{1, 2\}$, there exists a dipath $P_j$ from $x$ to $v_j$ in $T$. By definition of least common ancestor, $V(P_1) \cap V(P_2) = \{ x \}$, $V(P_j) \cap L_i = \{ v_j \}, j = 1, 2$, and both $P_1$ and $P_2$ have length at least 1. Consequently, $P_1 \cup P_2 \cup Q$ is a subdivision of $C(k, \ell)$, a contradiction.

Claim 11. $\chi(D_1) \leq k + \ell - 3$.

Proof. Let $\phi_1$ be the colouring of $D_1$ defined by $\phi_1(x) = \text{lvl}(x) \pmod{k + \ell - 3}$. By definition of $D_1$, this is clearly a proper colouring of $D_1$.

The following two claims are more complicated, we refer the reader to [4] for the complete proofs.

Claim 12. $\chi(D_2) \leq 2\ell + 2$.

Claim 13. $\chi(D_3) \leq k + \ell + 1$.

Claims 10, 11, 12, and 13, together with Lemma 9 yield the result.

More generally, one may wonder what happens for other oriented cycles. Our next result generalises Theorem 8 for $\hat{C}_4$ the cycle with 4 blocks.

Theorem 14. Let $D$ be a digraph in $\text{S-Forb}(\hat{C}_4)$. If $D$ admits an out-generator, then $\chi(D) \leq 24$.

Proof. The general idea is the same as in the proof of Theorem 8.

Suppose that $D$ admits an out-generator $u$ and let $T$ be an BFS-tree with root $u$. We partition $A(D)$ into three sets according to the levels of $u$.

$$A_0 = \{(x, y) \in A(D) \mid \text{lvl}(x) = \text{lvl}(y)\};$$
$$A_1 = \{(x, y) \in A(D) \mid |\text{lvl}(x) - \text{lvl}(y)| = 1\};$$
$$A_2 = \{(x, y) \in A(D) \mid \text{lvl}(y) \leq \text{lvl}(x) - 2\}.$$

For $i = 0, 1, 2$, let $D_i = (V(D), A_i)$.

Claim 15. $\chi(D_0) \leq 3$.

Proof. Suppose for a contradiction that $\chi(D) \geq 4$. By Theorem 2, it contains a $P^-(1, 1)$ $(y_1, y, y_2)$, that is $(y, y_1)$ and $(y, y_2)$ are in $A(D_0)$. Let $x$ be the least common ancestor of $y_1$ and $y_2$ in $T$. The union of $T[x, y_1]$, $(y, y_1)$, $(y, y_2)$, and $T[x, y_2]$ is a subdivision of $\hat{C}_4$, a contradiction.

Claim 16. $\chi(D_1) \leq 2$.

Proof. Since the arc are between consecutive levels, then the colouring $\phi_1$ defined by $\phi_1(x) = \text{lvl}(x) \pmod{2}$ is a proper 2-colouring of $D_1$.

Let $y \in V_i$ we denote by $N'(y)$ the out-degree of $y$ in $\bigcup_{0 \leq j \leq i - 1} V_j$. Let $D' = (V, A')$ with $A' = \cup_{x \in V} \{ (x, y) \in N'(x) \}$ and $D_x = (V, A_x)$ where $A_x$ is the set of arc inside the level and from $V_i$ to $V_{i+1}$ for all $i$. Note that $A = A' \cup A_x$ and
Claim 17. $\chi(D_2) \leq 4$.

Proof. We refer to [4] for the proof of this statement.

Claims 15, 16, 17, and Lemma 9 implies $\chi(D) \leq 24$.

References


