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To cite this version:
Nathann Cohen, Frédéric Havet, William Lochet, Nicolas Nisse. Subdivisions of oriented cycles in digraphs with large chromatic number. Bordeaux Graph Workshop 2016, Nov 2016, Bordeaux, France. pp.85-88. hal-01411115

HAL Id: hal-01411115
https://hal.archives-ouvertes.fr/hal-01411115
Submitted on 7 Dec 2016

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Subdivisions of oriented cycles in digraphs with large chromatic number

Nathann Cohen 1, Frédéric Havet 2, William Lochet 2 and Nicolas Nisse 3

1 CNRS, LRI, Univ. Paris Sud, Orsay, France 2 Univ. Côte d’Azur, CNRS, Inria, I3S, France 3 Univ. Côte d’Azur, Inria, CNRS, I3S, France

Extended Abstract

The chromatic number $\chi(D)$ of a digraph $D$ is the chromatic number of its underlying graph. The chromatic number of a class of digraphs $\mathcal{D}$, denoted by $\chi(\mathcal{D})$, is the smallest $k$ such that $\chi(D) \leq k$ for all $D \in \mathcal{D}$, or $+\infty$ if no such $k$ exists. If $\chi(D) \neq +\infty$, we say that $\mathcal{D}$ has bounded chromatic number.

We are interested in the following question: which are the digraph classes $\mathcal{D}$ such that every digraph with sufficiently large chromatic number contains an element of $\mathcal{D}$? Let us denote by $\text{Forb}(H)$ (resp. $\text{Forb}(\mathcal{H})$) the class of digraphs that do not contain $H$ (resp. any element of $\mathcal{H}$) as a subdigraph. The above question can be restated as follows:

**Problem 1.** Which are the classes of digraphs $\mathcal{D}$ such that $\chi(\text{Forb}(\mathcal{D})) < +\infty$?

An oriented graph is an orientation of a (simple) graph. An oriented path (resp., an oriented cycle) is said directed if all vertices have in-degree and out-degree at most 1.

Observe that if $D$ is an orientation of a graph $G$ and $\text{Forb}(D)$ has bounded chromatic number, then $\text{Forb}(G)$ has also bounded chromatic number. A classical result by Erdős implies that $G$ must be a tree. Burr proved that every $(k-1)^2$-chromatic digraph contains every oriented tree of order $k$ and conjectured Burr [3] that it could be further improved to $(2k-2)$-chromatic digraphs.

For special oriented trees $T$, better bounds on the chromatic number of $\text{Forb}(T)$ are known. The most famous one, known as Gallai-Hasse-Roy-Vitaver Theorem [6] states that $\chi(\text{Forb}(P^+(k))) = k$, where $P^+(k)$ is the directed path of length $k$ (a directed path is an oriented path in which all arcs are in the same direction).

The chromatic number of the class of digraphs not containing a prescribed oriented path $P$ on $n$ vertices with two blocks (blocks are maximal directed subpaths) has been determined by Addario-Berry et al. [1]:

**Theorem 2** (Addario-Berry et al. [1]). Let $P$ be an oriented path with two blocks on $n \geq 4$ vertices, then $\chi(\text{Forb}(P)) = n - 1$.

In this paper, we are interested in the chromatic number of $\text{Forb}(\mathcal{H})$ when $\mathcal{H}$ is an infinite family of oriented cycles. Let us denote by $\text{S-Forb}(D)$ (resp. S-Forb($\mathcal{D}$)) the class of digraphs that contain no subdivision of $D$ (resp. any element of $\mathcal{D}$) as a subdigraph. We are particularly interested in the chromatic number of S-Forb($\mathcal{C}$), where $\mathcal{C}$ is a family of oriented cycles.

Let us denote by $C^k_\ell$ the directed cycle of length $k$. For all $k$, $\chi(\text{S-Forb}(C^k_\ell)) = +\infty$ because transitive tournaments have no directed cycle. Let us denote by $C(k, \ell)$ the oriented cycle with two blocks, one of length $k$ and the other of length $\ell$. Observe that the oriented cycles with two blocks are the subdivisions of $C(1,1)$. As pointed by Gyárfás and Thomassen (see [1]), there are acyclic oriented graphs with arbitrarily large chromatic number and no oriented cycles with two blocks. Therefore $\chi(\text{S-Forb}(C(k, \ell))) = +\infty$. We first generalise this result to every oriented cycle.

**Theorem 3.** For any oriented cycle $C$, $\chi(\text{S-Forb}(C)) = +\infty$.

In fact, we show the following stronger theorem.

**Theorem 4.** For any positive integers $b, c$, there exists an acyclic digraph $D_c$ with $\chi(D_c) \geq c$ in which all oriented cycles have more than $b$ blocks.
We need a construction due to Erdős and Lovász [5] of hypergraphs with high girth and large chromatic number.

**Theorem 5.** [5, Theorem 1'] For \( k, g, c \in \mathbb{N} \), there exists a \( k \)-uniform hypergraph with girth larger than \( g \) and weak chromatic number larger than \( c \).

We assume \( g \) is being fixed, the following construction allow us to find \( D_{c+1} \) from \( D_c \).

Let \( p \) be the number of proper \( c \)-colourings of \( D_c \), and let those colourings be denoted by \( \text{col}_1^c, \ldots, \text{col}_p^c \). By Theorem 5 there exists a \( c \times p \)-uniform hypergraph \( \mathcal{H} \) with weak chromatic number \( p \) and girth \( g/2 \). Let \( X = \{ x_1, \ldots, x_n \} \) be the ground set of \( \mathcal{H} \).

We construct \( D_{c+1} \) from \( n \) disjoint copies \( D_1^c, \ldots, D_n^c \) of \( D_c \) as follows. For each hyperedge \( S \in \mathcal{H} \), we do the following:

- We partition \( S \) into \( p \) sets \( S_1, \ldots, S_p \) of cardinality \( c \).
- For each set \( S_i = \{ x_{i_1}, \ldots, x_{i_k} \} \), we choose vertices \( v_{k_1} \in D_1^{x_{i_1}}, \ldots, v_{k_c} \in D_c^{x_{i_c}} \) such that \( \text{col}_i^c(v_{k_1}) = 1, \ldots, \text{col}_i^c(v_{k_c}) = c \), and add a new vertex \( w_{S,i} \) with \( v_{k_1}, \ldots, v_{k_c} \) as in-neighbours.

On the other hand, considering strongly connected (strong for short) digraphs may lead to dramatically different result. An example is provided by the following celebrated result due to Bondy [2], which can be rephrased as follows when denoting the class of strong digraphs by \( S \).

**Theorem 6** (Bondy [2]). \( \chi(S-\text{Forb}(\mathcal{C}_k) \cap S) = k - 1 \).

Inspired by this theorem, Addario-Berry et al. [1] posed the following problem.

**Problem 7.** Let \( k \) and \( \ell \) be two positive integers then \( \chi(S-\text{Forb}(C(k, \ell) \cap S)) < k + 1 \).

We give evidence for this problem by showing the following weaker statement.

**Theorem 8.** Let \( k \) and \( \ell \) be two positive integers such that \( k \geq \max\{ \ell, 3 \} \), and let \( D \) be a digraph in \( S-\text{Forb}(C(k, \ell)) \cap S \). Then, \( \chi(D) \leq (k + \ell - 2)(k + \ell - 3)(2\ell + 2)(k + \ell + 1) \).

We need the following lemma.

The union of two digraphs \( D_1 \) and \( D_2 \) is the digraph \( D_1 \cup D_2 \) with vertex set \( V(D_1) \cup V(D_2) \) and arc set \( A(D_1) \cup A(D_2) \).

**Lemma 9.** Let \( D_1 \) and \( D_2 \) be two digraphs. \( \chi(D_1 \cup D_2) \leq \chi(D_1) \times \chi(D_2) \).

A consequence of the previous lemma is that, if we partition the arc set of \( D \) into set \( A_1 \cdots A_k \), then bounding the chromatic number of all digraphs induced by the \( A_i \) implies that \( D \) has bounded chromatic number.

**Proof.** Let \( D \) be a strong digraph without any copy of \( C(k, \ell) \), we exhibit a colouring of \( D \) using a bounded number of colours. The proof heavily relies on the technique of levelling.

Let \( u \) be a vertex of \( D \). The level of a vertex \( x \), noted \( \text{lvl}(x) \) is the length of the shortest dipath from \( u \) to \( x \). \( L(i) \) is the set of vertices at level \( i \).

Since \( D \) is strongly connected, it has an out-generator \( u \). Let \( T \) be a BFS-tree with root \( u \). We define the following sets of arcs.

\[
\begin{align*}
A_0 &= \{ xy \in A(D) \mid \text{lvl}(x) = \text{lvl}(y) \}; \\
A_1 &= \{ xy \in A(D) \mid 0 < \text{lvl}(x) - \text{lvl}(y) < k + \ell - 3 \}; \\
A' &= \{ xy \in A(D) \mid \text{lvl}(x) - \text{lvl}(y) \geq k + \ell - 3 \}.
\end{align*}
\]

Since \( k + \ell - 3 > 0 \) and there is no arc \( xy \) with \( \text{lvl}(y) > \text{lvl}(x) + 1 \), \( (A_0, A_1, A') \) is a partition of \( A(D) \). Observe moreover that \( A(T) \subseteq A_1 \). We further partition \( A' \) into two sets \( A_2 \) and \( A_3 \), where \( A_2 = \{ xy \in A' \mid y \text{ is an ancestor of } x \text{ in } T \} \) and \( A_3 = A' \setminus A_2 \). Then \( (A_0, A_1, A_2, A_3) \) is a partition of \( A(D) \). Let \( D_j = (V(D), A_j) \) for all \( j \in \{ 0, 1, 2, 3 \} \).
Claim 10. $\chi(D_0) \leq k + \ell - 2$.

Proof. Observe that $D_0$ is the disjoint union of the $D[L_i]$ where $L_i = \{v \mid \text{dist}_D(u, v) = i\}$. Therefore it suffices to prove that $\chi(D[L_i]) \leq k + \ell - 2$ for all non-negative integer $i$.

$L_0 = \{u\}$ so the result holds trivially for $i = 0$.

Assume now $i \geq 1$. Suppose for a contradiction $\chi(D[L_i]) \geq k + \ell - 1$. Since $k \geq 3$, by Theorem 2, $D[L_i]$ contains a copy $Q$ of $P^*(k-1, \ell - 1)$, the path on two blocks of length $k-1$ and $\ell-1$ with one vertex of indegree 2. Let $v_1$ and $v_2$ be the initial and terminal vertices of $Q$, and let $x$ be the least common ancestor of $v_1$ and $v_2$. By definition, for $j \in \{1, 2\}$, there exists a dipath $P_j$ from $x$ to $v_j$ in $T$. By definition of least common ancestor, $V(P_j) \cap V(P_2) = \{x\}$, $V(P_j) \cap L_i = \{v_j\}$, $j = 1, 2$, and both $P_1$ and $P_2$ have length at least 1. Consequently, $P_1 \cup P_2 \cup Q$ is a subdivision of $C(k, \ell)$, a contradiction.

Claim 11. $\chi(D_1) \leq k + \ell - 3$.

Proof. Let $\phi_1$ be the colouring of $D_1$ defined by $\phi_1(x) = \text{lvl}(x) \pmod{k + \ell - 3}$. By definition of $D_1$, this is clearly a proper colouring of $D_1$.

The following two claims are more complicated, we refer the reader to [4] for the complete proofs.

Claim 12. $\chi(D_2) \leq 2\ell + 2$.

Claim 13. $\chi(D_3) \leq k + \ell + 1$.

Claims 10, 11, 12, and 13, together with Lemma 9 yield the result.

More generally, one may wonder what happens for other oriented cycles. Our next result generalises Theorem 8 for $C_4$ the cycle with 4 blocks.

Theorem 14. Let $D$ be a digraph in $\text{S-Forb}(C_4)$. If $D$ admits an out-generator, then $\chi(D) \leq 24$.

Proof. The general idea is the same as in the proof of Theorem 8.

Suppose that $D$ admits an out-generator $u$ and let $T$ be an BFS-tree with root $u$. We partition $A(D)$ into three sets according to the levels of $u$.

$$
A_0 = \{(x, y) \in A(D) \mid \text{lvl}(x) = \text{lvl}(y)\};
$$

$$
A_1 = \{(x, y) \in A(D) \mid |\text{lvl}(x) - \text{lvl}(y)| = 1\};
$$

$$
A_2 = \{(x, y) \in A(D) \mid \text{lvl}(y) \leq \text{lvl}(x) - 2\}.
$$

For $i = 0, 1, 2$, let $D_i = (V(D), A_i)$.

Claim 15. $\chi(D_0) \leq 3$.

Proof. Suppose for a contradiction that $\chi(D) \geq 4$. By Theorem 2, it contains a $P^*(1, 1)$ $(y_1, y, y_2)$, that is $(y, y_1)$ and $(y, y_2)$ are in $A(D_0)$. Let $x$ be the least common ancestor of $y_1$ and $y_2$ in $T$. The union of $T[x, y_1]$, $(y, y_1)$, $(y, y_2)$, and $T[x, y_2]$ is a subdivision of $C_4$, a contradiction.

Claim 16. $\chi(D_1) \leq 2$.

Proof. Since the arc are between consecutive levels, then the colouring $\phi_1$ defined by $\phi_1(x) = \text{lvl}(x) \pmod{2}$ is a proper 2-colouring of $D_1$.

Let $y \in V_i$ we denote by $N^i(y)$ the out-degree of $y$ in $\bigcup_{0 \leq j \leq i-1} V_j$. Let $D' = (V, A')$ with $A' = \cup_{x \in V} \{x, y \in N^i(x)\}$ and $D_x = (V, A_x)$ where $A_x$ is the set of arc inside the level and from $V_i$ to $V_{i+1}$ for all $i$. Note that $A = A' \cup A_x$ and
Claim 17. $\chi(D_2) \leq 4$.

Proof. We refer to [4] for the proof of this statement.

Claims 15, 16, 17, and Lemma 9 imply $\chi(D) \leq 24$.

References


