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To cite this version:
Rennan Dantas, Frédéric Havet, Rudini Sampaio. Identifying codes for infinite triangular grids with a finite number of rows. Bordeaux Graph Workshop 2016, Nov 2016, Bordeaux, France. hal-01411109

HAL Id: hal-01411109
https://hal.archives-ouvertes.fr/hal-01411109
Submitted on 7 Dec 2016

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Identifying codes for infinite triangular grids with a finite number of rows

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Extended Abstract

Let G be a graph G. The neighborhood of a vertex v in G, denoted by N(v), is the set of vertices adjacent to v in G. It closed neighborhood is the set N[v] = N(v) ∪ {v}.

A set C ⊆ V(G) is an identifying code in G if

(i) for all v ∈ V(G), N[v] ∩ C ≠ ∅, and
(ii) for all u, v ∈ V(G), N[u] ∩ C ≠ N[v] ∩ C.

The identifier of v by C, denoted by C[v], is the set N[v] ∩ C. Hence a identifying code is a set such that the vertices have non-empty distinct identifiers.

Let G be a (finite or infinite) graph with bounded maximum degree. For any non-negative integer r and vertex v, we denote by B_r(v) the ball of radius r in G, that is B_r(v) = {x | dist(v, x) ≤ r}. For any set of vertices C ⊆ V(G), the density of C in G, denoted by d(C, G), is defined by

\[ d(C, G) = \limsup_{r \to +\infty} \frac{|C \cap B_r(v_0)|}{|B_r(v_0)|} \]

where v_0 is an arbitrary vertex in G. The infimum of the density of an identifying code in G is denoted by d^*(G). Observe that if G is finite, then d^*(G) = |C^*|/|V(G)|, where C^* is a minimum-size identifying code in G.

The problem of finding low-density identifying codes was introduced in [9] in relation to fault diagnosis in arrays of processors. Here the vertices of an identifying code correspond to controlling processors able to check themselves and their neighbors. Thus the identifying property guarantees location of a faulty processor from the set of “complaining” controllers. Identifying codes are also used in [10] to model a location detection problem with sensor networks.

Particular interest was dedicated to grids as many processor networks have a grid topology. There are three regular infinite grids in the plane, namely the hexagonal grid, the square grid and the triangular grid.

Regarding the infinite hexagonal grid G_H, the best upper bound on d^*(G_H) is 3/7 and comes from two identifying codes constructed by Cohen et al. [4]; these authors also proved a lower bound of 16/30. This lower bound was improved to 12/29 by Cranston and Yu [6]. Cukiernian and Yu [7] further improved it to 5/12.

The infinite square grid G_S is the infinite graph with vertices in Z × Z such that N((x, y)) = {(x, y ± 1), (x ± 1, y)}. Given an integer k ≥ 2, let [k] = {1, ..., k} and let S_k be the subgraph of G_S induced by the vertex set \{(x, y) ∈ Z × [k]\}. In [3], Cohen et al. gave a periodic identifying code of G_S with density 7/20. This density was later proved to be optimal by Ben-Haim and Litvin [1]. Daniel, Gravier, and Moncel [8] showed that d^*(S_1) = 1/2 and d^*(S_2) = 4/7. They also showed that for every k ≥ 3, \[ \frac{7}{20} - \frac{1}{2k} \leq d^*(S_k) \leq \min \left\{ \frac{2}{5}, \frac{7}{20} + \frac{2}{k} \right\} \]

These bounds were recently improved by Bouznif et al. [2] who established

\[ \frac{7}{20} + \frac{1}{20k} \leq d^*(S_k) \leq \min \left\{ \frac{2}{5}, \frac{7}{20} + \frac{3}{10k} \right\} \]

They also proved d^*(S_3) = 4/7.
The infinite triangular grid \( G_T \) is the infinite graph with vertices in \( \mathbb{Z} \times \mathbb{Z} \) such that \( N((x, y)) = \{(x, y \pm 1), (x \pm 1, y), (x - 1, y + 1), (x + 1, y - 1)\} \). Given an integer \( k \geq 2 \), let \( [k] = \{1, \ldots, k\} \) and let \( T_k \) be the subgraph of \( G_T \) induced by the vertex set \( \{(x, y) \in \mathbb{Z} \times [k]\} \). Karpovsky et al. \cite{9} showed that \( d^*(G_T) = 1/4 \). Trivially, \( T_1 = S_1 \). Hence \( d^*(T_1) = 1/4 \) in this paper, we prove the following results regarding the density of an identifying code of \( T_k \), \( k > 1 \).

**Theorem 1**  
- \( d^*(T_k) = \frac{1}{4} + \frac{1}{4k} \) for every odd \( k \).  
- \( d^*(T_2) = \frac{1}{2} \) and \( d^*(T_4) = d^*(T_6) = \frac{1}{3} \).  
- \( \frac{1}{4} + \frac{1}{4k} \leq d^*(T_k) \leq \frac{1}{4} + \frac{1}{4k} \) for every even \( k \).

The upper bounds are obtained by showing periodic identifying codes with the desired density. Consider the sets (see Figure 1)

| \( C_2 \) | \( \{(x, l) \mid x \equiv 1, 3 \mod 5\} \cup \{(x, 2) \mid x \equiv 1, 2, 4 \mod 5\}; \) |
| \( C_{2k-1} \) | \( \{(x, y) \mid x, y \text{ odd and } 1 \leq x, y \leq 2k - 1\}; \) |
| \( C_4 \) | \( \{(x, 2) \mid x \equiv 0, 3 \mod 3\} \cup \{(x, 3) \mid x \equiv 0, 1 \mod 3\}; \) |
| \( C_{2k} \) | \( \{(x, y), (x, y) \mid x, y \text{ odd}, 1 \leq x \leq 2k \text{ and } 1 \leq y \leq 2k - 3\} \cup \{(x, 2k - 1) \mid x \in \mathbb{Z}\}. \) |

It is easy to check that the above defined sets \( C_2 \) is an identifying codes of \( T_2 \) with density 1/2, \( C_3 \) is identifying codes of \( T_3 \) with density 1/3, \( C_4 \) is an identifying code of \( T_4 \) with density 3/10, and \( C_{2k-1} \) is an identifying code of \( T_{2k-1} \) with density \( \frac{1}{4} + \frac{1}{4k} \) and \( C_{2k} \) is an identifying code of \( T_{2k} \) with density \( \frac{1}{4} + \frac{1}{4k} \).

Our lower bounds are obtained via the Discharging Method. The general idea is the following. We consider any identifying code \( C \) of \( T_k \). The vertices in \( C \) receive a certain value \( q_k > 0 \) of charge and the vertices not in \( C \) receive charge 0. Then we apply some local discharging rules. Here local means that there is no charge transfer from a vertex to a vertex at distance more than \( d_k \) for some fixed constant \( d_k \), and that the total charge sent by a vertex is bounded by some fixed value \( m_k \). Finally, we prove that after the discharging, every vertex \( v \) has final charge \( \text{chrg}^*(v) \) at least \( p_k \) for some fixed \( p_k > 0 \). We claim that it implies \( d(C, G) \geq \frac{p_k}{q_k} \). Since a vertex sends charge at most \( m_k \) to vertices at distance at most \( d_k \), a charge of at most \( m_k \cdot |B_{r+\delta}(v_0) \cap B_r(v_0)| \leq 2d_k \cdot k \cdot m_k \) enters \( B_r(v_0) \) during the discharging phase. Thus

\[
|C \cap B_r(v_0)| = \frac{1}{q_k} \sum_{v \in B_r(v_0)} \text{chrg}_0(v) \geq \frac{1}{q_k} \left( \sum_{v \in B_r(v_0)} \text{chrg}^*(v) - m_k \cdot |B_{r+\delta}(v_0) \cap B_r(v_0)| \right) \\
\geq \frac{p_k|B_r(v_0)| - 2d_k \cdot k \cdot m_k}{q_k}.
\]

But \( |B_r(v_0)| \geq 2(k+1)r - k^2 \), thus \( d(C, S_k) \geq \lim_{r \to +\infty} \left( \frac{p_k}{q_k} - \frac{1}{q_k} \cdot \frac{2d_k \cdot k \cdot m_k}{2(k+1)r - k^2} \right) = \frac{p_k}{q_k} \). This proves our claim. As the claim holds for any identifying code, we have \( d^*(T_k) \geq \frac{p_k}{q_k} \).

**Acknowledgements**

This research was supported by FAPERJ, Capes (Project STIC-AmSud 2012), CNPq (Project Universal/Bolsa de Produtividade), and the FUNCAP/CNRS project GAIATO INC-0083-00047.01.00/13.
Figure 1: Optimal identifying codes of $T_2$, $T_3$, $T_4$, $T_5$ and $T_6$
References


