Identifying codes for infinite triangular grids with a finite number of rows

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Extended Abstract

Let G be a graph G. The *neighborhood* of a vertex v in G, denoted by $N(v)$, is the set of vertices adjacent to v i G. It *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$.

A set $C \subseteq V(G)$ is an *identifying code* in G if

(i) for all $v \in V(G)$, $N[v] \cap C \neq \emptyset$, and

(ii) for all $u, v \in V(G)$, $N[u] \cap C \neq N[v] \cap C$.

The *identifier* of v by C, denoted by $C[v]$, is the set $N[v] \cap C$. Hence a identifying code is a set such that the vertices have non-empty distinct identifiers.

Let G be a (finite or infinite) graph with bounded maximum degree. For any non-negative integer r and vertex v, we denote by $B_r(v)$ the ball of radius r in G, that is $B_r(v) = \{x \mid \mathrm{dist}(v,x) \leq r\}$. For any set of vertices $C \subseteq V(G)$, the *density* of C in G, denoted by $d(C,G)$, is defined by

$$d(C,G) = \limsup_{r \to +\infty} \frac{|C \cap B_r(v_0)|}{|B_r(v_0)|} \ ,$$

where v_0 is an arbitrary vertex in G. The infimum of the density of an identifying code in G is denoted by $d^*(G)$. Observe that if G is finite, then $d^*(G) = |C^*|/|V(G)|$, where C^* is a minimum-size identifying code in G.

The problem of finding low-density identifying codes was introduced in [9] in relation to fault diagnosis in arrays of processors. Here the vertices of an identifying code correspond to controlling processors able to check themselves and their neighbors. Thus the identifying property guarantees location of a faulty processor from the set of "complaining" controllers. Identifying codes are also used in [10] to model a location detection problem with sensor networks.

Particular interest was dedicated to grids as many processor networks have a grid topology. There are three regular infinite grids in the plane, namely the hexagonal grid, the square grid and the triangular grid.

Regarding the infinite hexagonal grid $\mathcal{G}_H$, the best upper bound on $d^*(\mathcal{G}_H)$ is $3/7$ and comes from two identifying codes constructed by Cohen et al. [4]; these authors also proved a lower bound of $16/39$. This lower bound was improved to $12/29$ by Cranston and Yu [6]. Cukierman and Yu [7] further improved it to $5/12$.

The *infinite square grid* $\mathcal{G}_S$ is the infinite graph with vertices in $\mathbb{Z}\times\mathbb{Z}$ such that $N((x,y)) = \{(x, y\pm 1), (x\pm 1, y)\}$. Given an integer $k \geq 2$, let $[k] = \{1,\dots,k\}$ and let S_k be the subgraph of $\mathcal{G}_S$ induced by the vertex set $\{(x,y) \in \mathbb{Z} \times [k]\}$. In [3], Cohen et al. gave a periodic identifying code of $\mathcal{G}_S$ with density $7/20$. This density was later proved to be optimal by Ben-Haim and Litsyn [1]. Daniel, Gravier, and Moncel [8] showed that $d^*(S_1) = \frac{1}{2}$ and $d^*(S_2) = \frac{3}{7}$. They also showed that for every $k \geq 3$, $\dfrac{7}{20} - \dfrac{1}{2k} \leq d^*(S_k) \leq \min\left\{\dfrac{2}{5}, \dfrac{7}{20} + \dfrac{2}{k}\right\}$. These bounds were recently improved by Bouznif et al. [2] who established

$$\frac{7}{20} + \frac{1}{20k} \leq d^*(S_k) \leq \min\left\{\frac{2}{5}, \frac{7}{20} + \frac{3}{10k}\right\}.$$

They also proved $d^*(S_3) = \frac{3}{7}$.

The *infinite triangular grid* $\mathcal{G}_T$ is the infinite graph with vertices in $\mathbb{Z} \times \mathbb{Z}$ such that $N((x,y)) = \{(x, y \pm 1), (x \pm 1, y), (x-1, y+1), (x+1, y-1)\}$. Given an integer $k \geq 2$, let $[k] = \{1, \ldots, k\}$ and let T_k be the subgraph of $\mathcal{G}_T$ induced by the vertex set $\{(x,y) \in \mathbb{Z} \times [k]\}$. Karpovsky et al. [9] showed that $d^*(\mathcal{G}_T) = 1/4$. Trivially, $T_1 = S_1$. Hence $d^*(T_1) = \frac{1}{2}$ In this paper, we prove the following results regarding the density of an identifying code of T_k, $k > 1$.

Theorem 1

- *$d^*(T_k) = \frac{1}{4} + \frac{1}{4k}$ for every odd k.*
- *$d^*(T_2) = \frac{1}{2}$ and $d^*(T_4) = d^*(T_6) = \frac{1}{3}$.*
- *$\frac{1}{4} + \frac{1}{4k} \leq d^*(T_k) \leq \frac{1}{4} + \frac{1}{2k}$ for every even k.*

The upper bounds are obtained by showing periodic identifying codes with the desired density. Consider the sets (see Figure 1)

$$\begin{aligned}
C_2 &= \{(x,1) \mid x \equiv 1,3 \mod 5\} \cup \{(x,2) \mid x \equiv 1,2,4 \mod 5\};\\
C_{2k-1} &= \{(x,y) \mid x \mid x,y \text{ odd and } 1 \leq x,y \leq 2k-1\};\\
C_4 &= \{(x,2) \mid x \equiv 0,3 \mod 3\} \cup \{(x,3) \mid x \equiv 0,1 \mod 3\};\\
C_{2k} &= \{(x,y),(x,y) \mid x,y \text{ odd }, 1 \leq x \leq 2k \text{ and } 1 \leq y \leq 2k-3\} \cup \{(x, 2k-1) \mid x \in \mathbb{Z}\}.
\end{aligned}$$

It is easy to check that the above defined sets C_2 is an identifying codes of T_2 with density $1/2$, C_3 is identifying codes of T_3 with density $1/3$, C_4 is an identifying code of T_4 with density $3/10$. and C_{2k-1} is an identifying code of T_{2k-1} with density $\frac{1}{4} + \frac{1}{4k}$ and C_{2k} is an identifying code of T_{2k} with density $\frac{1}{4} + \frac{1}{2k}$.

Our lower bounds are obtained via the Discharging Method. The general idea is the following. We consider any identifying code C of T_k. The vertices in C receive a certain value $q_k > 0$ of charge and the vertices not in C receive charge 0. Then we apply some local discharging rules. Here local means that there is no charge transfer from a vertex to a vertex at distance more than d_k for some fixed constant d_k, and that the total charge sent by a vertex is bounded by some fixed value m_k. Finally, we prove that after the discharging, every vertex v has final charge $\text{chrg}^*(v)$ at least p_k for some fixed $p_k > 0$. We claim that it implies $d(C,G) \geq \frac{p_k}{q_k}$. Since a vertex sends charge at most m_k to vertices at distance at most d_k, a charge of at most $m_k \cdot |B_{r+s}(v_0) \setminus B_r(v_0)| \leq 2d_k \cdot k \cdot m_k$ enters $B_r(v_0)$ during the discharging phase. Thus

$$\begin{aligned}
|C \cap B_r(v_0)| = \frac{1}{q_k} \sum_{v \in B_r(v_0)} \text{chrg}_0(v) &\geq \frac{1}{q_k} \left(\sum_{v \in B_r(v_0)} \text{chrg}^*(v) - m_k \cdot |B_{r+s}(v_0) \setminus B_r(v_0)| \right)\\
&\geq \frac{p_k |B_r(v_0)| - 2d_k \cdot k \cdot m_k}{q_k}.
\end{aligned}$$

But $|B_r(v_0)| \geq 2(k+1)r - k^2$, thus $d(C, \mathcal{S}_k) \geq \limsup_{r \to +\infty} \left(\frac{p_k}{q_k} - \frac{1}{q_k} \cdot \frac{2d_k \cdot k \cdot m_k}{2(k+1)r - k^2} \right) = \frac{p_k}{q_k}$. This proves our claim. As the claim holds for any identifying code, we have $d^*(T_k) \geq p_k/q_k$.

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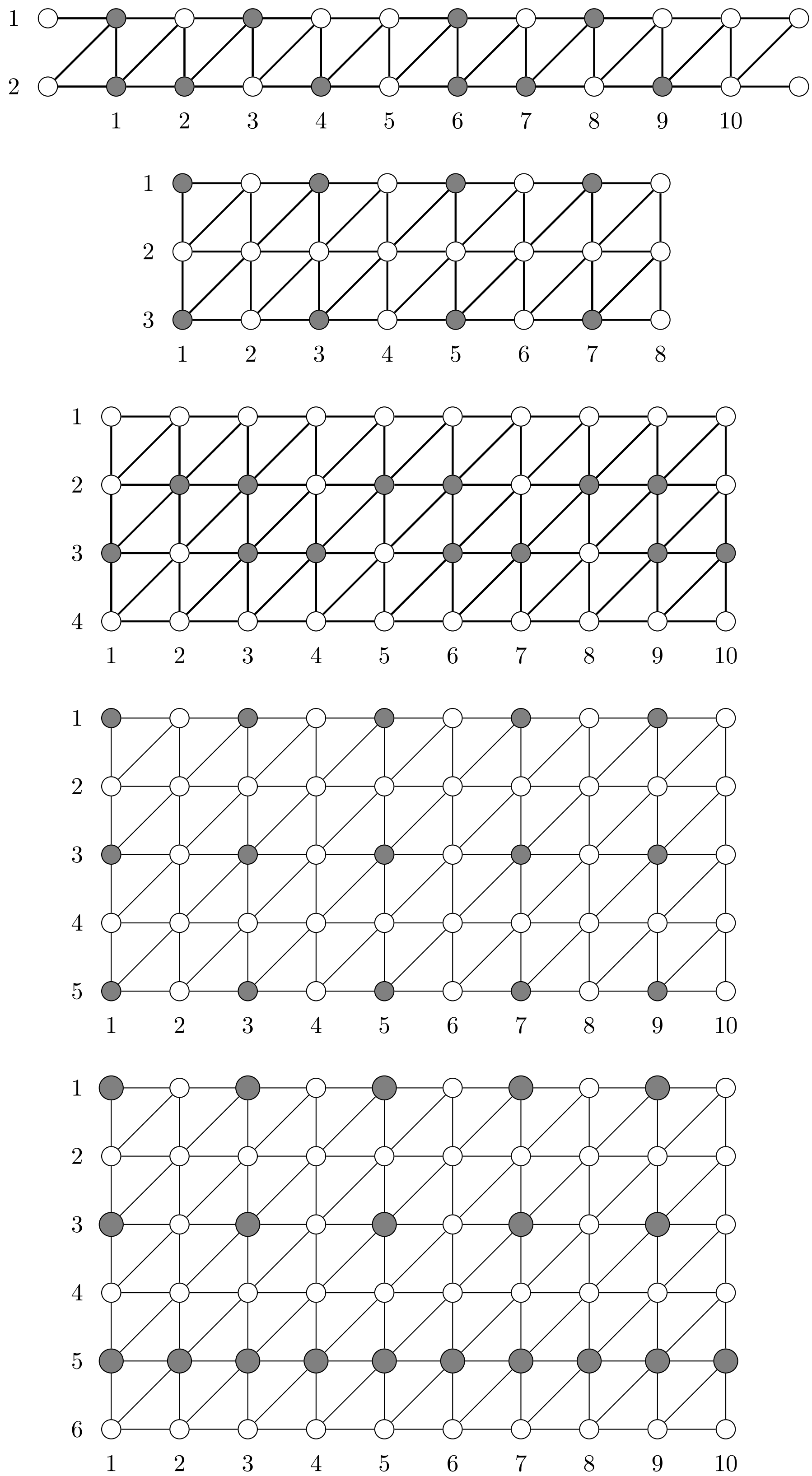

Figure 1: Optimal identifying codes of T_2, T_3, T_4, T_5 and T_6

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