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Abstract

This work focuses on the asymptotic behavior of the density in small time of a stochastic differential equation driven by an α-stable process with index $\alpha \in (0, 2)$. We assume that the process depends on a parameter $\beta = (\theta, \sigma)^T$ and we study the sensitivity of the density with respect to this parameter. This extends the results of [5] which was restricted to the index $\alpha \in (1, 2)$ and considered only the sensitivity with respect to the drift coefficient. By using Malliavin calculus, we obtain the representation of the density and its derivative as an expectation and a conditional expectation. This permits to analyze the asymptotic behavior in small time of the density, using the time rescaling property of the stable process.

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Keywords. Lévy process, Density in small time, Stable process, Malliavin calculus for jump processes.

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1 Introduction

We consider the following stochastic differential equation (SDE)

$$X_t^\beta = x_0 + \int_0^t b(X_s^\beta, \theta) ds + \sigma L_t \tag{1.1}$$

for $t \in [0,1]$, where $(L_t)_{t\in[0,1]}$ is a Lévy process whose Lévy measure is similar near zero to the one of an α-stable process with exponent $0 < \alpha < 2$.

In the last decades, a large literature has been devoted to the existence and regularity of the density to the solution $(X_t)_t$, for $t > 0$, of a general stochastic equation driven by pure jump Lévy processes. We can mention the works of Bichteler, Gravereaux and Jacod [2], Picard [13], Denis [8], Ishikawa-Kunita [10], Fournier-Printems [9] and more recently the works of Debussche-Fournier [7] and Kulik [12], under Hölder continuity assumptions on the coefficients of the equation and assuming that the equation is driven by an α-stable process.

In this paper, our aim is to study the asymptotic behavior, in small time, of the density of (X_t^β), the solution of (1.1), as well as its derivative with respect to the parameter $\beta = (\theta, \sigma)^T$. This problem plays an important role in asymptotic Statistics based on high frequency observations. Indeed, considering the estimation of β from the discrete time observations $(X_{i/n}^\beta)_{0\le i\le n}$, and denoting by $p_{1/n}^\beta(x,y)$ the transition density of the discrete time process, the estimation rate of the parameter β strongly relies on the asymptotic behavior of the derivative $\nabla_\beta p_{1/n}^\beta(x,y)$, as n goes to infinity. Based on the results established in the present paper, we derive, in [6], an asymptotic expansion of the log-likelihood ratio and we prove the LAMN property for the parameter β.

The main contributions of this paper are obtained by using the Malliavin calculus for jump processes developed by Bichteler, Gravereaux and Jacod [2] and adapted to the particular case of equation (1.1) by Clément-Gloter [5]. Although it requires some strong derivability assumptions on the coefficients of the equation, it leads to some explicit representation formulas for the density and its derivative (see also Ivanenko - Kulik [11]). Let us mention that alternative representations for the density can be obtained by other methods, for example the method proposed by Bouleau-Denis [3] based on Dirichlet forms or the parametrix method used by Kulik [12].

This paper is made up of two parts. In the first part we establish some representation formulas for the density and its derivative. This extends the results of Clément-Gloter [5] where only the derivative with respect to the drift parameter θ was considered. These representation formulas involve some Malliavin weights whose expressions are given explicitly. This permits to identify in the Malliavin weights a main part and a negligible part in small time asymptotics.

In the second part of the paper, we study the asymptotic behavior of the transition density of X_t^β and its derivative, in small time. This was done in [5] with the restriction $\alpha > 1$ and for a derivative with respect to parameter in the drift part of the SDE only. In Theorem 3.2 and Theorem 3.3, we obtain asymptotic results in small time for $0 < \alpha < 2$ and for the derivatives with respect to parameters in the drift and the Lévy part of the SDE. In contrast to [5], the exposition now involves the solution of the ordinary differential equation defined by the deterministic part of (1.1). Our results are established through a careful study of each terms appearing in the Malliavin weights, which is complicated by the non integrability of the α-stable process as $\alpha \leq 1$.

The present paper is organized as follows : in Section 2, we recall the Malliavin integration by parts setting developed by [2] and used in [5], and give some representations of the transition density, its derivative, as well as its logarithm derivative. The main contribution of this section is to explicit the iterated Malliavin weights appearing in the expression of the derivative of the density. Section 3 studies their asymptotic behavior in small time by decomposing the Malliavin weights into a main part and a negligible part, and contains the main results of the paper (Theorem 3.2 and Theorem 3.3). It is worth to note that the rate of convergence for the derivative of the density with respect to θ and σ are different and that the derivative with respect to σ involves a more careful study. Finally, Section 4 contains some more technical proof.

2 Representation of the transition density via Malliavin calculus

The main aim of this section is to represent the density of a pure jump Lévy process as well as its derivative and its logarithm derivative as an expectation, using the Malliavin calculus for jump processes developed by Bichteler, Gravereaux and Jacod [2] and used by Clément-Gloter [5]. Due to the singularity of the Lévy measure of (L_t) at zero, we are not exactly in the context of [2], and we first recall the appropriate integration by parts setting developed in [5] for the reader convenience.
We first introduce some notations which are used throughout this article. For a vector $h \in \mathbb{R}^k$, h^T denotes the transpose of h, and $|h|$ denotes the euclidean norm. For a function f defined on $\mathbb{R} \times \mathbb{R}^2$ depending on both variables (x, β), here $\beta = (\theta, \sigma)^T \in \mathbb{R} \times (0, +\infty)$, we denote by f' the derivative of f with respect to the variable x, by $\partial_\theta f$ the derivative of f with respect to the parameter θ, by $\partial_\sigma f$ the derivative of f with respect to the parameter σ, and $\nabla_\beta f = \begin{pmatrix} \partial_\theta f \\ \partial_\sigma f \end{pmatrix}$.

2.1 Integration by parts setting

We consider a filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,1]}, \mathbb{P})$ endowed with a Poisson random measure μ on $[0,1]\times E$, where E is an open subset of $\mathbb{R}$, with compensator v on $[0,1]\times E$ and with compensated measure $\tilde{\mu} = \mu - v$. We now consider the process $(Y_t^\beta)_{t\in[0,1]}$, the solution of

$$Y_t^\beta = y_0 + \int_0^t a(Y_s^\beta, \theta)ds + c\sigma \int_0^t \int_E z\tilde{\mu}(ds, dz), \tag{2.1}$$

where the parameter $\beta = (\theta, \sigma)^T$ belongs to $\mathbb{R}\times(0,\infty)$, a is a function and c is a constant.

This is the framework of Clément-Gloter [5] and our aim is to give some explicit representation formulas for the density of Y_1^β and its derivative with respect to β.

We assume that the following assumptions are fulfilled.

H: (a) The function a has bounded derivatives up to order five with respect to both variables.

(b) The compensator of the Poisson random measure μ is given by $v(dt, dz) = dt \times g(z)dz$ with $g \geq 0$ on E, $\mathcal{C}^1$ on E and such that

$$\forall p \geq 2, \int_E |z|^p g(z)dz < \infty.$$

Note that comparing to the assumptions of [5], we relax the boundedness assumption on a.

We now recall the Malliavin operators L and Γ and their basic properties (see Bichteler, Gravereaux, Jacod [2], Chapter IV, Section 8-9-10). For a test function $f : [0,1]\times E \mapsto \mathbb{R}$ (f is measurable, $\mathcal{C}^2$ with respect to the second variable, with bounded derivative, and $f \in \cap_{p\geq 1}\mathbf{L}^p(\nu)$) we set $\mu(f) = \int_0^1 \int_E f(t,z)\mu(dt, dz)$. We introduce an auxiliary function $\rho : E \mapsto (0,\infty)$ such that ρ admits a derivative and ρ, ρ' and $\rho\frac{g'}{g}$ belong to $\cap_{p\geq 1}\mathbf{L}^p(g(z)dz)$. With these notations, we define the Malliavin operator L, on a simple functional $\mu(f)$, in the same way as in [5] by the following equations :

$$L(\mu(f)) = \frac{1}{2}\mu\left(\rho' f' + \rho\frac{g'}{g}f' + \rho f''\right),$$

where f' and f'' are the derivatives with respect to the second variable. For $\Phi = F(\mu(f_1), .., \mu(f_k))$, with F of class $\mathcal{C}^2$, we set

$$L\Phi = \sum_{i=1}^k \frac{\partial F}{\partial x_i}(\mu(f_1), ..., \mu(f_k))L(\mu(f_i)) + \frac{1}{2}\sum_{i,j=1}^k \frac{\partial^2 F}{\partial x_i \partial x_j}(\mu(f_1), ..., \mu(f_k))\mu(\rho f_i' f_j').$$

These definitions permit to construct a linear operator L on a space $D \subset \cap_{p\geq 1}\mathbf{L}^p$ with the same basic properties as in [5, equations (i)-(iii), p.2322].

We associate to L, the symmetric bilinear operator Γ:

$$\Gamma(\Phi, \Psi) = L(\Phi\Psi) - \Phi L\Psi - \Psi L\Phi.$$

Moreover, if f and h are two test functions, we have:

$$\Gamma(\mu(f), \mu(h)) = \mu(\rho f' h').$$

These operators satisfy the following properties (see [2, equation (8-3)])

$$\begin{aligned} LF(\Phi) &= F'(\Phi)L\Phi + \frac{1}{2}F''(\Phi)\Gamma(\Phi, \Phi), \\ \Gamma(F(\Phi), \Psi) &= F'(\Phi)\Gamma(\Phi, \Psi), \\ \Gamma(F(\Phi_1, \Phi_2), \Psi) &= \partial_{\Phi_1} F(\Phi_1, \Phi_2)\Gamma(\Phi_1, \Psi) + \partial_{\Phi_2} F(\Phi_1, \Phi_2)\Gamma(\Phi_2, \Psi). \end{aligned} \tag{2.2}$$

The operator L and the operator Γ permit to establish the following integration by parts formula (see [2, Propositions 8-10, p.103]).

Proposition 2.1. *For Φ and Ψ in D, and f bounded with bounded derivatives up to order two, we have*

$$\mathbb{E} f'(\Phi)\Psi\Gamma(\Phi, \Phi) = \mathbb{E} f(\Phi)(-2\Psi L\Phi - \Gamma(\Phi, \Psi)).$$

Morover, if $\Gamma(\Phi, \Phi)$ is invertible and $\Gamma^{-1}(\Phi, \Phi) \in \cap_{p\geq 1} \boldsymbol{L}^p$, we have

$$\mathbb{E} f'(\Phi)\Psi = \mathbb{E} f(\Phi)\mathcal{H}_\Phi(\Psi), \tag{2.3}$$

with

$$\begin{aligned} \mathcal{H}_\Phi(\Psi) &= -2\Psi\Gamma^{-1}(\Phi, \Phi)L\Phi - \Gamma(\Phi, \Psi\Gamma^{-1}(\Phi, \Phi)) && (2.4) \\ &= -2\Psi\Gamma^{-1}(\Phi, \Phi)L\Phi - \frac{1}{\Gamma(\Phi, \Phi)}\Gamma(\Phi, \Psi) + \frac{\Psi}{\Gamma(\Phi, \Phi)^2}\Gamma(\Phi, \Psi). && (2.5) \end{aligned}$$

2.2 Representation of the density of Y_1^β and its derivative

The integration by parts setting of the preceding section permits to derive the existence of the density of Y_1^β given by (2.1), and gives a representation of this density as an expectation. From Bichteler, Gravereaux, Jacod [2, Section 10, p.130], we know that $\forall t > 0$, the variable Y_t^β, the solution of (2.1), belongs to the domain of the operator L, and we can compute LY_t^β and $\Gamma(Y_t^\beta, Y_t^\beta)$ as in [5]. We recall the representation formula for the density of Y_1^β (see [5]).

Theorem 2.1. *[Clément-Gloter [5]]: Let us denote by q^β the density of Y_1^β. We assume that **H** holds and that the auxiliary function ρ satisfies:*

$$\liminf_{u\to\infty} \frac{1}{\ln u} \int_E 1_{\{\rho(z)\geq 1/u\}} g(z)dz = +\infty. \tag{2.6}$$

Then,

$$q^\beta(u) = \mathbb{E}(1_{\{Y_1^\beta \geq u\}} \mathcal{H}_{Y_1^\beta}(1)),$$

with,

$$\mathcal{H}_{Y_1^\beta}(1) = \frac{\Gamma(Y_1^\beta, \Gamma(Y_1^\beta, Y_1^\beta))}{\Gamma(Y_1^\beta, Y_1^\beta)^2} - 2\frac{LY_1^\beta}{\Gamma(Y_1^\beta, Y_1^\beta)} = \frac{W_1^\beta}{(U_1^\beta)^2} - 2\frac{LY_1^\beta}{U_1^\beta}, \tag{2.7}$$

where the processes (LY_t^β) *and* $(U_t^\beta) = \Gamma(Y_t^\beta, Y_t^\beta)$ *are solutions of the linear equations:*

$$LY_t^\beta = \int_0^t a'(Y_s^\beta, \theta) LY_s^\beta ds + \frac{1}{2}\int_0^t a''(Y_s^\beta, \theta) U_s^\beta ds + \frac{c\sigma}{2}\int_0^t \int_E \left(\rho'(z) + \rho(z)\frac{g'(z)}{g(z)}\right)\mu(ds, dz), \tag{2.8}$$

$$U_t^\beta = 2\int_0^t a'(Y_s^\beta, \theta) U_s^\beta ds + c^2\sigma^2 \int_0^1 \int_E \rho(z)\mu(ds, dz). \tag{2.9}$$

The process $(W_t^\beta) = \Gamma(Y_t^\beta, U_t^\beta)$ *is the solution of the linear equation:*

$$W_t^\beta = 3\int_0^t a'(Y_s^\beta, \theta) W_s^\beta ds + 2\int_0^t a''(Y_s^\beta, \theta)(U_s^\beta)^2 ds + c^3\sigma^3 \int_0^t \int_E \rho(z)\rho'(z)\mu(ds, dz). \tag{2.10}$$

In [5], the authors studied the derivative of q^β with respect to the drift parameter θ only. Here, we intend to study the derivative of q^β with respect to both parameters θ and σ. We first remark that $(Y_t^\beta)_t$ admits derivatives with respect to θ and σ (see [2, Theorem 5.24 p.51]), denoted by $(\partial_\theta Y_t^\beta)_t$ and $(\partial_\sigma Y_t^\beta)_t$ respectively. Moreover, $(\partial_\theta Y_t^\beta)_t$, $(\partial_\sigma Y_t^\beta)_t$ are respectively the unique solutions of

$$\partial_\theta Y_t^\beta = \int_0^t a'(Y_s^\beta, \theta)\partial_\theta Y_s^\beta ds + \int_0^t \partial_\theta a(Y_s^\beta, \theta) ds, \tag{2.11}$$

$$\partial_\sigma Y_t^\beta = \int_0^t a'(Y_s^\beta, \theta)\partial_\sigma Y_s^\beta ds + c\int_0^t \int_E z\tilde{\mu}(ds, dz). \tag{2.12}$$

By iterating the integration by parts formula, since Y_1^β admits derivatives with respect to θ and σ, one can prove, under the assumption **H**, the existence and the continuity in β of $\nabla_\beta q^\beta$ (see Theorem 4-21 in [2]), moreover, we will represent it as an expectation in Theorem 2.3. The next result extends the result of Theorem 5 in [5], by giving an expression for the logarithm derivatives of the density w.r.t. (θ, σ) in terms of a conditional expectation.

Theorem 2.2. *Under the assumptions of Theorem 2.1,*

$$\frac{\nabla_\beta q^\beta}{q^\beta}(u) = \begin{pmatrix} \frac{\partial_\theta q^\beta}{q^\beta}(u) \\ \frac{\partial_\sigma q^\beta}{q^\beta}(u) \end{pmatrix} = \mathbb{E}(\mathcal{H}_{Y_1^\beta}(\nabla_\beta Y_1^\beta) | Y_1^\beta = u), \tag{2.13}$$

where

$$\mathcal{H}_{Y_1^\beta}(\nabla_\beta Y_1^\beta) := \begin{pmatrix} \mathcal{H}_{Y_1^\beta}(\partial_\theta Y_1^\beta) \\ \mathcal{H}_{Y_1^\beta}(\partial_\sigma Y_1^\beta) \end{pmatrix} = -2\begin{pmatrix} \partial_\theta Y_1^\beta \\ \partial_\sigma Y_1^\beta \end{pmatrix}\frac{LY_1^\beta}{U_1^\beta} + \begin{pmatrix} \partial_\theta Y_1^\beta \\ \partial_\sigma Y_1^\beta \end{pmatrix}\frac{W_1^\beta}{(U_1^\beta)^2} - \frac{1}{U_1^\beta}\begin{pmatrix} \Gamma(Y_1^\beta, \partial_\theta Y_1^\beta) \\ \Gamma(Y_1^\beta, \partial_\sigma Y_1^\beta) \end{pmatrix}, \tag{2.14}$$

LY_1^β, U_1^β and W_1^β are given in Theorem 2.1, the process $(V_t^\theta) = \Gamma(Y_t^\beta, \partial_\theta Y_t^\beta)$ is the solution of

$$V_t^\theta = 2\int_0^t a'(Y_s^\beta, \theta) V_s^\theta ds + \int_0^t U_s^\beta \left[(\partial_\theta a)'(Y_s^\beta, \theta) + a''(Y_s^\beta, \theta)\partial_\theta Y_s^\beta \right] ds, \tag{2.15}$$

and the process $(V_t^\sigma) = \Gamma(Y_t^\beta, \partial_\sigma Y_t^\beta)$ is the solution of

$$V_t^\sigma = 2\int_0^t a'(Y_s^\beta, \theta) V_s^\sigma ds + \int_0^t a''(Y_s^\beta, \theta)\partial_\sigma Y_s^\beta U_s^\beta ds + c^2\sigma \int_0^t \int_E \rho(z)\mu(ds, dz). \tag{2.16}$$

Proof. Theorem 2.2 is an extension of Theorem 5 in [5] where the main novelty is the expression for $\frac{\partial_\sigma q^\beta}{q^\beta}$. For the computation of the new term $\mathcal{H}_{Y_1^\beta}(\partial_\sigma Y_1^\beta)$, we apply Theorem 10-3 in [2] to the stochastic differential equation satisfied by the vector $(Y_t^\beta, U_t^\beta, \partial_\sigma Y_t^\beta)^T$, this gives the above expression for (V_t^σ). □

We end this subsection with an explicit representation of $\nabla_\beta q^\beta(u)$ which gives a computation of the iterated Malliavin weight $\mathcal{H}_{Y_1^\beta}(\mathcal{H}_{Y_1^\beta}(\nabla_\beta Y_1^\beta))$.

Theorem 2.3. *Under the assumptions of Theorem 2.1,*

$$\nabla_\beta q^\beta(u) = \begin{pmatrix} \partial_\theta q^\beta(u) \\ \partial_\sigma q^\beta(u) \end{pmatrix} = \mathbb{E}\left[1_{\{Y_1^\beta \geq u\}} \mathcal{H}_{Y_1^\beta}(\mathcal{H}_{Y_1^\beta}(\nabla_\beta Y_1^\beta)) \right], \tag{2.17}$$

where

$$\mathcal{H}_{Y_1^\beta}(\mathcal{H}_{Y_1^\beta}(\nabla_\beta Y_1^\beta)) = -2\mathcal{H}_{Y_1^\beta}(\nabla_\beta Y_1^\beta)\frac{LY_1^\beta}{U_1^\beta} + \mathcal{H}_{Y_1^\beta}(\nabla_\beta Y_1^\beta)\frac{W_1^\beta}{(U_1^\beta)^2} - \begin{pmatrix} \Gamma(Y_1^\beta, \mathcal{H}_{Y_1^\beta}(\partial_\theta Y_1^\beta)) \\ \Gamma(Y_1^\beta, \mathcal{H}_{Y_1^\beta}(\partial_\sigma Y_1^\beta)) \end{pmatrix} \frac{1}{U_1^\beta}, \tag{2.18}$$

where $\partial_\theta Y_1^\beta, \partial_\sigma Y_1^\beta$ are respectively given by equations (2.11), (2.12) and U_1^β, W_1^β are computed in Theorem 2.1, $\mathcal{H}_{Y_1^\beta}(\nabla_\beta Y_1^\beta)$ is given in Theorem 2.2.

Proof. Let f be a smooth functions with compact support. Then,

$$\nabla_\beta \mathbb{E}\left[f(Y_1^\beta) \right] = \int_{\mathbb{R}} du \nabla_\beta q^\beta(u) f(u).$$

On the other hand, using the integration by parts formula of the Malliavin calculus, we have

$$\begin{aligned} \nabla_\beta \mathbb{E}\left[f(Y_1^\beta) \right] &= \mathbb{E}\left[f'(Y_1^\beta)\nabla_\beta Y_1^\beta \right] \\ &= \mathbb{E}\left[f(Y_1^\beta)\mathcal{H}_{Y_1^\beta}(\nabla_\beta Y_1^\beta) \right] \\ &= \mathbb{E}\left[F(Y_1^\beta)\mathcal{H}_{Y_1^\beta}\left(\mathcal{H}_{Y_1^\beta}(\nabla_\beta Y_1^\beta) \right) \right] \end{aligned}$$

where F denotes a primitive function of f. If f converges to Dirac mass at some point u, from the estimates above, we can deduce (2.17). Moreover, from (2.5) we also get (2.18). □

To complete the result of Theorem 2.3, we give the expressions for $\Gamma(Y_1^\beta, \mathcal{H}_{Y_1^\beta}(\partial_\theta Y_1^\beta))$ and $\Gamma(Y_1^\beta, \mathcal{H}_{Y_1^\beta}(\partial_\sigma Y_1^\beta))$.

Lemma 2.1. *Under the assumptions of Theorem 2.1,*

$$\begin{pmatrix}\Gamma(Y_1^\beta, \mathcal{H}_{Y_1^\beta}(\partial_\theta Y_1^\beta)) \\ \Gamma(Y_1^\beta, \mathcal{H}_{Y_1^\beta}(\partial_\sigma Y_1^\beta))\end{pmatrix} = \begin{pmatrix}V_1^\theta \\ V_1^\sigma\end{pmatrix} \mathcal{H}_{Y_1^\beta}(1) - \begin{pmatrix}\partial_\theta Y_1^\beta \\ \partial_\sigma Y_1^\beta\end{pmatrix} \frac{2D_1^\beta}{U_1^\beta} - \begin{pmatrix}\partial_\theta Y_1^\beta \\ \partial_\sigma Y_1^\beta\end{pmatrix} \frac{\mathcal{H}_{Y_1^\beta}(1) W_1^\beta}{U_1^\beta} + \begin{pmatrix}\partial_\theta Y_1^\beta \\ \partial_\sigma Y_1^\beta\end{pmatrix} \frac{Q_1^\beta}{(U_1^\beta)^2} -$$
$$- \begin{pmatrix}T_1^\theta \\ T_1^\sigma\end{pmatrix} \frac{1}{U_1^\beta} + \begin{pmatrix}V_1^\theta \\ V_1^\sigma\end{pmatrix} \frac{W_1^\beta}{(U_1^\beta)^2}, \quad (2.19)$$

where $\partial_\theta Y_1^\beta, \partial_\sigma Y_1^\beta$ are respectively given in (2.11), (2.12), *U_1^β, W_1^β are computed in Theorem 2.1, V_1^θ, V_1^σ are computed in Theorem 2.2, $\mathcal{H}_{Y_1^\beta}(1)$ is given in* (2.7) *and $D_1^\beta = \Gamma(Y_1^\beta, LY_1^\beta)$, $Q_1^\beta = \Gamma(Y_1^\beta, W_1^\beta)$, $T_1^\theta = \Gamma(Y_1^\beta, V_1^\theta)$ and $T_1^\sigma = \Gamma(Y_1^\beta, V_1^\sigma)$.*

Proof. From the basic properties of the operators L and Γ (linearity and the chain rule property) stated in Section 2.1, we get that

$$\Gamma(Y_1^\beta, \mathcal{H}_{Y_1^\beta}(\partial_\theta Y_1^\beta)) = \Gamma\left[Y_1^\beta, -2\partial_\theta Y_1^\beta \frac{LY_1^\beta}{U_1^\beta}\right] + \Gamma\left[Y_1^\beta, \partial_\theta Y_1^\beta \frac{W_1^\beta}{(U_1^\beta)^2}\right] + \Gamma\left[Y_1^\beta, -\frac{\Gamma(Y_1^\beta, \partial_\theta Y_1^\beta)}{U_1^\beta}\right],$$

$$\Gamma(Y_1^\beta, \mathcal{H}_{Y_1^\beta}(\partial_\sigma Y_1^\beta)) = \Gamma\left[Y_1^\beta, -2\partial_\sigma Y_1^\beta \frac{LY_1^\beta}{U_1^\beta}\right] + \Gamma\left[Y_1^\beta, \partial_\sigma Y_1^\beta \frac{W_1^\beta}{(U_1^\beta)^2}\right] + \Gamma\left[Y_1^\beta, -\frac{\Gamma(Y_1^\beta, \partial_\sigma Y_1^\beta)}{U_1^\beta}\right],$$

where

$$\begin{aligned}\Gamma\left[Y_1^\beta, -2\partial_\theta Y_1^\beta \frac{LY_1^\beta}{U_1^\beta}\right] &= -2\frac{LY_1^\beta}{U_1^\beta}\Gamma(Y_1^\beta, \partial_\theta Y_1^\beta) - 2\frac{\partial_\theta Y_1^\beta}{U_1^\beta}\Gamma(Y_1^\beta, LY_1^\beta) + 2\partial_\theta Y_1^\beta \frac{LY_1^\beta}{(U_1^\beta)^2}\Gamma(Y_1^\beta, U_1^\beta) \\ &= -2\frac{LY_1^\beta}{U_1^\beta}V_1^\theta - 2\frac{\partial_\theta Y_1^\beta}{U_1^\beta}D_1^\beta + 2\partial_\theta Y_1^\beta \frac{LY_1^\beta}{(U_1^\beta)^2}W_1^\beta.\end{aligned}$$

$$\begin{aligned}\Gamma\left[Y_1^\beta, \partial_\theta Y_1^\beta \frac{W_1^\beta}{(U_1^\beta)^2}\right] &= \frac{W_1^\beta}{(U_1^\beta)^2}\Gamma(Y_1^\beta, \partial_\theta Y_1^\beta) + \frac{\partial_\theta Y_1^\beta}{(U_1^\beta)^2}\Gamma(Y_1^\beta, W_1^\beta) - \frac{2\partial_\theta Y_1^\beta W_1^\beta}{(U_1^\beta)^3}\Gamma(Y_1^\beta, U_1^\beta) \\ &= \frac{W_1^\beta}{(U_1^\beta)^2}V_1^\theta + \frac{\partial_\theta Y_1^\beta}{(U_1^\beta)^2}Q_1^\beta - \frac{2\partial_\theta Y_1^\beta W_1^\beta}{(U_1^\beta)^3}W_1^\beta.\end{aligned}$$

$$\Gamma\left[Y_1^\beta, -\frac{\Gamma(Y_1^\beta, \partial_\theta Y_1^\beta)}{U_1^\beta}\right] = -\frac{\Gamma(Y_1^\beta, \Gamma(Y_1^\beta, \partial_\theta Y_1^\beta))}{U_1^\beta} + \frac{\Gamma(Y_1^\beta, \partial_\theta Y_1^\beta)}{(U_1^\beta)^2}\Gamma(Y_1^\beta, U_1^\beta) = -\frac{T_1^\theta}{U_1^\beta} + \frac{V_1^\theta}{(U_1^\beta)^2}W_1^\beta.$$

Similarly, we have

$$\Gamma\left[Y_1^\beta, -2\partial_\sigma Y_1^\beta \frac{LY_1^\beta}{U_1^\beta}\right] = -2\frac{LY_1^\beta}{U_1^\beta}V_1^\sigma - 2\frac{\partial_\sigma Y_1^\beta}{U_1^\beta}D_1^\beta + 2\partial_\sigma Y_1^\beta \frac{LY_1^\beta}{(U_1^\beta)^2}W_1^\beta.$$

$$\Gamma\left[Y_1^\beta, \partial_\sigma Y_1^\beta \frac{W_1^\beta}{(U_1^\beta)^2}\right] = \frac{W_1^\beta}{(U_1^\beta)^2} V_1^\sigma + \frac{\partial_\sigma Y_1^\beta}{(U_1^\beta)^2} Q_1^\beta - \frac{2\partial_\sigma Y_1^\beta W_1^\beta}{(U_1^\beta)^3} W_1^\beta.$$

$$\Gamma\left[Y_1^\beta, -\frac{\Gamma(Y_1^\beta, \partial_\sigma Y_1^\beta)}{U_1^\beta}\right] = -\frac{T_1^\sigma}{U_1^\beta} + \frac{V_1^\sigma}{(U_1^\beta)^2} W_1^\beta.$$

Then, from (2.7) and the above estimates, we get the formula (2.19), after some calculus and the proof is complete. □

Lemma 2.2. *Under the assumptions of Theorem 2.1, there are versions of the processes* $(D_t^\beta) = (\Gamma(Y_t^\beta, LY_t^\beta))_t$, $(Q_t^\beta) = \Gamma(Y_t^\beta, W_t^\beta)_t$, $(T_t^\theta)_t = (\Gamma(Y_t^\beta, V_t^\theta))_t$ *and* $(T_t^\sigma)_t = (\Gamma(Y_t^\beta, V_t^\sigma))_t$ *that are solutions of the linear equations:*

$$D_t^\beta = 2\int_0^t a'(Y_s^\beta,\theta) D_s^\beta ds + \int_0^t a''(Y_s^\beta,\theta) LY_s^\beta U_s^\beta ds + \frac{1}{2}\int_0^t a''(Y_s^\beta,\theta) W_s^\beta ds + \frac{1}{2}\int_0^t a'''(Y_s^\beta,\theta)(U_s^\beta)^2 ds$$
$$+ \frac{c^2\sigma^2}{2}\int_0^t \int_E \rho(z)\left(\rho'(z) + \rho(z)\frac{g'(z)}{g(z)}\right)' \mu(ds,dz), \tag{2.20}$$

$$Q_t^\beta = 4\int_0^t a'(Y_s^\beta,\theta) Q_s^\beta ds + 7\int_0^t a''(Y_s^\beta,\theta) W_s^\beta U_s^\beta ds + 2\int_0^t a'''(Y_s^\beta,\theta)(U_s^\beta)^3 ds$$
$$+ c^4\sigma^4 \int_0^t \int_E \rho(z)\left[(\rho(z)')^2 + \rho(z)\rho(z)''\right]\mu(ds,dz), \tag{2.21}$$

$$T_t^\theta = 3\int_0^t a'(Y_s^\beta,\theta) T_s^\theta ds + 3\int_0^t a''(Y_s^\beta,\theta) V_s^\theta U_s^\beta ds + \int_0^t (\partial_\theta a)'(Y_s^\beta,\theta) W_s^\beta ds + \int_0^t a''(Y_s^\beta,\theta)\partial_\theta Y_s^\beta W_s^\beta ds$$
$$+ \int_0^t (\partial_\theta a)''(Y_s^\beta,\theta)(U_s^\beta)^2 + \int_0^t a'''(Y_s^\beta,\theta)\partial_\theta Y_s^\beta (U_s^\beta)^2 ds, \tag{2.22}$$

$$T_t^\sigma = 3\int_0^t a'(Y_s^\beta,\theta) T_s^\beta ds + 3\int_0^t a''(Y_s^\beta,\theta) V_s^\sigma U_s^\beta ds + \int_0^t a''(Y_s^\beta,\theta)\partial_\sigma Y_s^\beta W_s^\beta ds$$
$$+ \int_0^t a'''(Y_s^\beta,\theta)\partial_\sigma Y_s^\beta (U_s^\beta)^2 ds + c^3\sigma^2 \int_0^t \int_E \rho(z)\rho(z)'\mu(ds,dz). \tag{2.23}$$

Proof. The proof of Lemma 2.2 is a direct consequence of Theorem 10-3 in [2]. Indeed, considering the stochastic differential equation satisfied by the vector $\left(Y_t^\beta, LY_t^\beta, U_t^\beta, W_t^\beta, V_t^\theta, V_t^\sigma, \partial_\theta Y_t^\beta, \partial_\sigma Y_t^\beta\right)^T$ and using Theorem 10-3 in [2], we prove that the processes $(D_t^\beta) = (\Gamma(Y_t^\beta, LY_t^\beta))_t$, $(Q_t^\beta) = \Gamma(Y_t^\beta, W_t^\beta)_t$, $(T_t^\theta)_t = (\Gamma(Y_t^\beta, V_t^\theta))_t$ and $(T_t^\sigma)_t = (\Gamma(Y_t^\beta, V_t^\sigma))_t$ are solutions of linear equations, respectively, given by (2.20)-(2.23). □

3 Application to the asymptotic behavior of the transition density and its derivative in small time

We will study the density in small time of the process

$$X_t^\beta = x_0 + \int_0^t b(X_s^\beta, \theta)ds + \sigma L_t \quad t \in [0,1],$$

where $(L_t)_{t\in[0,1]}$ is a pure jump Lévy process and we assume that the following assumptions are fulfilled.

$\mathbf{H}_1$: (a) The function b has bounded derivatives up to order five with respect to both variables.

(b_i) The Lévy process $(L_t)_{t\in[0,1]}$ is given by $L_t = \int_0^t \int_{[-1,1]} z\{\overline{\mu}(ds,dz) - \overline{v}(ds,dz)\} + \int_0^t \int_{[-1,1]^c} z\overline{\mu}(ds,dz)$ where $\overline{\mu}$ is a Poisson random measure, with compensator $\overline{v}(dt,dz) = dt \times F(z)dz$ where $F(z)$ is given on $\mathbb{R}^*$ by $F(z) = \frac{1}{|z|^{\alpha+1}}\tau(z), \quad \alpha \in (0,2)$. Moreover, we assume that τ is a non negative smooth function equal to 1 on [-1,1], vanishing on $[-2,2]^c$ such that $0 \le \tau \le 1$.

(b_{ii}) We assume that $\forall p \ge 1$, $\int_{\mathbb{R}} \left|\frac{\tau'(u)}{\tau(u)}\right|^p \tau(u)du < \infty$, $\int_{\mathbb{R}} \left|\frac{\tau''(u)}{\tau(u)}\right|^p \tau(u)du < \infty$.

Remark 3.1. *The introduction of the truncation function τ in the density of the Lévy measure is a technical tool to ensure the integrability of $|L_t|^p, \forall p \ge 1$. These assumptions will guarantee that* (1.1) *has an unique solution belonging to $\boldsymbol{L}^p, \forall p \ge 1$ and ensure that our variables are in the domain of the Malliavin operators which are introduced in the previous section. Moreover, under these assumptions, X_t^β admits a smooth density, for $t > 0$.*

3.1 Rescaled process

We can observe that the process $(n^{1/\alpha}L_{t/n})$ equals in law to a centered Lévy process with Lévy measure

$$F_n(z) = \frac{1}{|z|^{1+\alpha}}\tau(\frac{z}{n^{1/\alpha}}). \tag{3.1}$$

This clearly suggests that when n grows, the process $(n^{1/\alpha}L_{t/n})$ converges to an α-stable process. In the sequel, it will be convenient to construct a family of Lévy processes $(L_t^n)_{n\ge1}$ with the same law as $(n^{1/\alpha}L_{t/n})$, on a common probability space where the limiting α-stable process exists as well, and where the convergence holds true in a path-wise sense, as done in [5].

Let us consider $\mu^e(dt,dz,du)$ a Poisson measure on $[0,\infty) \times \mathbb{R}^* \times [0,1]$ with compensating measure $v^e(dt,dz,du) = dt\frac{dz}{|z|^{1+\alpha}}du$ and we denote by $\tilde{\mu}^e(dt,dz,du) = \mu^e(dt,dz,du) - v^e(dt,dz,du)$ the compensated Poisson random measure. This measure corresponds to the jump measure of an α-stable process, where each jump is marked with a uniform variable on [0,1] .

We define the Poisson measures $\mu^{(n)}$, for all $n \geq 1$, and μ by setting :

$$\forall A \subset [0,\infty) \times \mathbb{R}, \quad \mu^{(n)}(A) = \int_{[0,\infty)} \int_{\mathbb{R}} \int_{[0,1]} 1_A(t,z) 1_{\{u \leq \tau(\frac{z}{n^{1/\alpha}})\}} \mu^e(dt, dz, du),$$

$$\forall A \subset [0,\infty) \times \mathbb{R}, \quad \mu(A) = \int_{[0,\infty)} \int_{\mathbb{R}} \int_{[0,1]} 1_A(t,z) \mu^e(dt, dz, du).$$

By simple computation, one can check that the compensator of the measure $\mu^{(n)}(dt, dz)$ is $\upsilon^{(n)}(dt, dz) = dt \times \tau(\frac{z}{n^{1/\alpha}})\frac{dz}{|z|^{1+\alpha}} = dt \times F_n(z)dz$ and the compensator of $\mu(dt, dz)$ is $\upsilon(dt, dz) = dt \times \frac{dz}{|z|^{1+\alpha}}$. Moreover, we note $\tilde{\mu}^{(n)}(dt, dz) = \mu^{(n)}(dt, dz) - \upsilon^{(n)}(dt, dz)$ and $\tilde{\mu}(dt, dz) = \mu(dt, dz) - \upsilon(dt, dz)$ the compensated Poisson random measures. Remark that since $\tau(z) = 1$ for $|z| \leq 1$, the measures $\mu^{(n)}(dt, dz)$ and $\mu(dt, dz)$ coincide on the set $\{(t, z) | t \in [0, 1], |z| \leq n^{1/\alpha}\}$.

Now we define the stochastic processes associated to these random measures,

$$L_t^{\alpha} = \int_0^t \int_{[-1,1]} z\tilde{\mu}(ds, dz) + \int_0^t \int_{[-1,1]^c} z\mu(ds, dz). \tag{3.2}$$

$$L_t^n = \int_0^t \int_{[-n^{1/\alpha}, n^{1/\alpha}]} z\tilde{\mu}^{(n)}(ds, dz) + \int_0^t \int_{[-n^{1/\alpha}, n^{1/\alpha}]^c} z\mu^{(n)}(ds, dz). \tag{3.3}$$

By construction, the process (L_t^{α}) is a centered α-stable process, and the process (L_t^n) is equal in law to the process $(n^{1/\alpha} L_{t/n})_{t\in[0,1]}$, since they are based on random measures with the same compensator. Remark that the jumps of L_t^n with size smaller than $n^{1/\alpha}$ exactly coincide with the jumps of L^{α} with size smaller than $n^{1/\alpha}$. On the other hand, the process L^n has no jump with a size greater than $2n^{1/\alpha}$. Using that the measures μ and $\mu^{(n)}$ coincide on the subsets of $\{(t, z); |z| \leq n^{1/\alpha}\}$, and the function $\tau(\frac{z}{n^{1/\alpha}})\frac{1}{|z|^{1+\alpha}} = \frac{1}{|z|^{1+\alpha}}$ is symmetric on $|z| \leq n^{1/\alpha}$, we can rewrite:

$$L_t^n = \int_0^t \int_{[-1,1]} z\{\mu(ds, dz) - \upsilon(ds, dz)\} + \int_0^t \int_{1<|z|<n^{1/\alpha}} z\mu(ds, dz) + \int_0^t \int_{n^{1/\alpha} \leq |z| \leq 2n^{1/\alpha}} z\mu^{(n)}(ds, dz). \tag{3.4}$$

The following simple lemma gives a connection between L^n and the stable process L^{α}.

Lemma 3.1. *On the event* $A_n = \left\{\mu(\{(t, z) | 0 \leq t \leq 1, |z| \geq n^{1/\alpha}\}) = 0\right\}$*, we have*

$$\mu^{(n)} = \mu, \quad L_t^n = L_t^{\alpha}, \tag{3.5}$$

and,

$$\mathbb{P}(A_n) = 1 + O(1/n). \tag{3.6}$$

Furthermore, let $(f_n)_{n\in\mathbb{N}}$ and f be measurable functions from $\Omega\times[0,1]\times\mathbb{R}$ to $\mathbb{R}$ such that there exists C with $\mathbb{P}(C)=1$ and $\forall\omega\in C$, $\forall s\in[0,1],\forall|z|>1$ $f_n(\omega,s,z)\xrightarrow{n\to\infty} f(\omega,s,z)$. Then

$$\int_0^1\int_{|z|>1} f_n(\omega,s,z)\mu^{(n)}(ds,dz)\xrightarrow[a.s.]{n\to\infty}\int_0^1\int_{|z|>1} f(\omega,s,z)\mu(ds,dz). \tag{3.7}$$

Moreover, we have $L_1^n\xrightarrow[a.s]{n\to\infty} L_1^\alpha$.

Proof. We know that the measure $\mu^{(n)}(ds,dz)$ and $\mu(ds,dz)$ coincide on the set $\{(s,z)|s\in[0,1],|z|\le n^{1/\alpha}\}$, and by comparison of the representations (3.2) and (3.4), it is clear that equation (3.5) holds true on the event that the supports of the random measure μ and $\mu^{(n)}$ do not intersect $\{(t,z)|0\le t\le 1,|z|\ge n^{1/\alpha}\}$. On the other hand, the support of $\mu^{(n)}$ is included in the support of μ, and thus (3.5) is true on the event $\mu(\{(t,z)|0\le t\le 1,|z|\ge n^{1/\alpha}\})=0$. The probability of the latter event is $e^{-2/\alpha n}$ which converges to 1 at rate $1/n$ as stated. Then we also get (3.6).
Let $A=\cup_{n=1}^\infty A_n$, we get that $P(A)=1$ since $A_n\subseteq A_{n+1}$ for each $n\in\mathbb{N}$ and (3.6) holds. Thus, for all $\omega\in A\cap C,\exists n_0(\omega)\ge 1,\forall n\ge n_0(\omega),\mu^{(n)}=\mu$ and $f_n(\omega,s,z)\to f(\omega,s,z)\forall s\in[0,1],\forall|z|>1$. And then we deduce that

$$\int_0^1\int_{|z|>1} f_n(\omega,s,z)\mu^{(n)}(ds,dz)\xrightarrow[a.s.]{n\to\infty}\int_0^1\int_{|z|>1} f(\omega,s,z)\mu(ds,dz).$$

As a consequence, it is easy to see that L_1^n converges almost surely to L_1^α. □

3.2 Representation of the density in small time and first approximation

Our main aim is to study the asymptotic behavior of the transition density of the random variable $X^\beta_{\frac{1}{n}}$. In that follows, we denote by $p^\beta_{\frac{1}{n}}(x,y)$ the transition density of the homogeneous Markov chain $(X^\beta_{\frac{i}{n}})_{i=0,\dots,n}$. We observe that $(X^\beta_{\frac{t}{n}})_{t\in[0,1]}$ equals in law $(\overline{Y}_t^{n,\beta,x_0})_{t\in[0,1]}$ where the process $(\overline{Y}_t^{n,\beta,x_0})_{t\in[0,1]}$ is given by

$$\overline{Y}_t^{n,\beta,x_0}=x_0+\frac{1}{n}\int_0^t b(\overline{Y}_s^{n,\beta,x_0},\theta)ds+\frac{\sigma}{n^{1/\alpha}}L_t^n \quad t\in[0,1], \tag{3.8}$$

where (L_t^n) is defined by (3.4) and is such that $\frac{1}{n^{1/\alpha}}(L_t^n)$ equals in law to $(L_{t/n})$.
Let q^{n,β,x_0} be the density of $\overline{Y}_1^{n,\beta,x_0}$ then the connection between the densities of $X^\beta_{\frac{1}{n}}$ and $\overline{Y}_1^{n,\beta,x_0}$ is given by

$$p^\beta_{1/n}(x_0,x)=q^{n,\beta,x_0}(x). \tag{3.9}$$

We are in the framework of Sections 2.1 and 2.2, with $g(z) := F_n(z) = \frac{1}{|z|^{1+\alpha}}\tau(\frac{z}{n^{1/\alpha}})$.

We choose the auxiliary function ρ^n as

$$\rho^n(z) = \begin{cases} z^4 & \text{if} \quad |z| < 1 \\ \zeta(z) & \text{if} \quad 1 \leq |z| \leq 2 \\ z^2\tau(\frac{z}{2n^{1/\alpha}}) & \text{if} \quad |z| > 2, \end{cases} \tag{3.10}$$

where τ is defined in the assumption $\mathbf{H}_1(b_i)$, and ζ is a non negative function belonging to $\mathbf{C}^\infty$ such that the function ρ^n belongs to $\mathbf{C}^\infty$. Note that ζ is defined such that $\rho^n(z)$ satisfies all conditions of Section 2.1. From the assumptions of τ, we can easily deduce that $z^2\tau(\frac{z}{2n^{1/\alpha}}) = \begin{cases} z^2 & \text{if} \quad 2 \leq |z| \leq 2n^{1/\alpha} \\ 0 & \text{if} \quad |z| > 4n^{1/\alpha}. \end{cases}$

Moreover, we can see that $\rho^n(z) \xrightarrow{n\to\infty} \rho(z)$ where

$$\rho(z) = \begin{cases} z^4 & \text{if} \quad |z| < 1 \\ \zeta(z) & \text{if} \quad 1 \leq |z| \leq 2, \\ z^2 & \text{if} \quad |z| > 2. \end{cases} \tag{3.11}$$

Note that from the definition of ρ^n and ρ, we can easily see that $\rho^n(z) = \rho(z)$ if $|z| \leq 2n^{1/\alpha}$.

Remark 3.2. *The choice of the auxiliary function ρ^n for $|z| < 1$ ensures that the non-degeneracy condition* (2.6) *is satisfied. It will appear later that the choice of the auxiliary function ρ^n for $|z| > 2$ permits to obtain Malliavin weights sufficiently integrable to compensate the lack of integrability of L_1^α [see remark 3.5 below].*

From now on, the function a and the constant c appearing in Section 2.1 will be given explicitly as

$$a(x,\theta) = \frac{1}{n}b(x,\theta), \quad c = \frac{1}{n^{1/\alpha}}.$$

Using the results in Section 2.2, we get a representation of the density of $X^\beta_{\frac{1}{n}}$. Moreover, we obtain a first approximation for the weight $\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(1)$. This leads to the decomposition of the density into a main part and a remainder part.

Theorem 3.1. *Under the assumption $\boldsymbol{H}_1$, we have*

$$p^\beta_{\frac{1}{n}}(x_0,u) = q^{n,\beta,x_0}(u) = \mathbb{E}(1_{\{\overline{Y}_1^{n,\beta,x_0} \geq u\}}\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(1)), \tag{3.12}$$

with

$$\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(1) = \frac{1}{\sigma} n^{1/\alpha} \left[\widehat{\mathcal{H}}_{1,\beta}^n(1) + \widehat{\mathcal{H}}_{2,\beta}^n(1)\right] + \mathcal{R}_{1,\beta}^n(1) + \mathcal{R}_{2,\beta}^n(1) + \mathcal{R}_{3,\beta}^n(1). \tag{3.13}$$

The main terms $\widehat{\mathcal{H}}_{1,\beta}^n(1), \widehat{\mathcal{H}}_{2,\beta}^n(1)$ *are given by*

$$\widehat{\mathcal{H}}_{1,\beta}^n(1) = \left[\frac{\int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-3} \rho^n(z)(\rho^n)'(z)\mu^{(n)}(ds,dz)}{\epsilon_1^n \left[\int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z)\mu^{(n)}(ds,dz)\right]^2}\right], \tag{3.14}$$

$$\widehat{\mathcal{H}}_{2,\beta}^n(1) = \left[-\frac{\int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-1} \left[(\rho^n)'(z) - \frac{1+\alpha}{z}\rho^n(z)\right] \mu^{(n)}(ds,dz)}{\epsilon_1^n \int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z)\mu^{(n)}(ds,dz)}\right], \tag{3.15}$$

where

$$\epsilon_s^n = \exp\left(\frac{1}{n}\int_0^s b'(\overline{Y}_u^{n,\beta,x_0}, \theta)du\right). \tag{3.16}$$

The remainder terms satisfy

$$\forall p \geq 2, \quad \mathbb{E}\left|\mathcal{R}_{1,\beta}^n(1)\right|^p \leq \frac{C}{n}, \qquad |\mathcal{R}_{2,\beta}^n(1)| \leq \frac{C}{n}, \qquad |\mathcal{R}_{3,\beta}^n(1)| \leq \frac{C}{n}, \tag{3.17}$$

where C is some deterministic constant.

Proof. Under the assumptions $\mathbf{H}_1$, we can apply the results of Theorem 2.1 to $\overline{Y}_1^{n,\beta,x_0}$. The non degeneracy assumption is verified by the choice of $\rho^n(z)$ near zero [see (3.10)]. Let us denote by $U_t^{n,\beta} = \Gamma[\overline{Y}_t^{n,\beta,x_0}, \overline{Y}_t^{n,\beta,x_0}]$, and $W_t^{n,\beta} = \Gamma[\overline{Y}_t^{n,\beta,x_0}, U_t^{n,\beta}]$, then we obtain:

$$p_{\frac{1}{n}}^{\beta}(x_0,u) = q^{n,\beta,x_0}(u) = \mathbb{E}(1_{\{\overline{Y}_1^{n,\beta,x_0} \geq u\}} \mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(1));$$

with

$$\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(1) = \frac{W_1^{n,\beta}}{(U_1^{n,\beta})^2} - 2\frac{L\overline{Y}_1^{n,\beta,x_0}}{U_1^{n,\beta}}. \tag{3.18}$$

Applying the results of Theorem 2.1 and solving the linear equations (2.8)-(2.10) we get,

$$U_1^{n,\beta} = \frac{(\epsilon_1^n)^2\sigma^2}{n^{2/\alpha}} \int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z)\mu^{(n)}(ds,dz), \tag{3.19}$$

$$L(\overline{Y}_1^{n,\beta,x_0}) = \frac{\epsilon_1^n}{2n}\int_0^1 b''(\overline{Y}_s^{n,\beta,x_0},\theta)(U_s^{n,\beta})(\epsilon_s^n)^{-1}ds + \frac{\sigma\epsilon_1^n}{2n^{1/\alpha}}\int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-1}[(\rho^n)'(z) + \frac{F_n'(z)}{F_n(z)}\rho^n(z)]\mu^{(n)}(ds,dz), \tag{3.20}$$

$$W_1^{n,\beta} = \frac{\sigma^3(\epsilon_1^n)^3}{n^{3/\alpha}}\int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-3}(\rho^n)'(z)\rho^n(z)\mu^{(n)}(ds,dz) + \frac{2(\epsilon_1^n)^3}{n}\int_0^1 b''(\overline{Y}_s^{n,\beta,x_0},\theta)(U_s^{n,\beta})^2(\epsilon_s^n)^{-3}ds. \tag{3.21}$$

Recalling that $F_n(z) = \frac{1}{|z|^{1+\alpha}}\tau(\frac{z}{n^{1/\alpha}})$ [see Eq.(3.1) in Section 3.2], then $\frac{F_n'(z)}{F_n(z)} = -\frac{1+\alpha}{z} + \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\frac{1}{n^{1/\alpha}}$ if $|z| \leq 2n^{1/\alpha}$. Based on these expressions and (3.18) we deduce, after some calculus, the decomposition (3.13), where the remainder terms are given by,

$$\mathcal{R}^n_{1,\beta}(1) = -\frac{\int_0^1 \int_{\mathbb{R}} (\epsilon^n_s)^{-1}\rho^n(z)\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\mu^{(n)}(ds,dz)}{\sigma\epsilon^n_1 \int_0^1 \int_{\mathbb{R}} (\epsilon^n_s)^{-2}\rho^n(z)\mu^{(n)}(ds,dz)}, \tag{3.22}$$

$$\mathcal{R}^n_{2,\beta}(1) = \frac{2(\epsilon^n_1)^3 \int_0^1 b''(\overline{Y}^{n,\beta,x_0}_s,\theta)\left(U^{n,\beta}_s\right)^2(\epsilon^n_s)^{-3}ds}{n(U^{n,\beta}_1)^2}, \tag{3.23}$$

$$\mathcal{R}^n_{3,\beta}(1) = -\frac{(\epsilon^n_1)\int_0^1 b''(\overline{Y}^{n,\beta,x_0}_s,\theta)(U^{n,\beta}_s)(\epsilon^n_s)^{-1}ds}{2n(U^{n,\beta}_1)}. \tag{3.24}$$

We now consider the properties of the remainder terms.

For $\mathcal{R}^n_{1,\beta}(1)$, since (ϵ^n_s) is bounded by above and below (recall (3.16)), and since $\tau'(z) = 0$ on $[-1,1]$ then for M a positive constant we have

$$\left|\mathcal{R}^n_{1,\beta}(1)\right| \leq M\left(\frac{\int_0^1 \int_{|z|>2} z^2\left|\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right|\mu^{(n)}(ds,dz)}{\sigma\int_0^1 \int_{|z|>2} z^2\mu^{(n)}(ds,dz)}\right). \tag{3.25}$$

Assume that there exists a jump of the Lévy process L^n_1 in $[-2n^{1/\alpha}, -n^{1/\alpha}) \cup (n^{1/\alpha}, 2n^{1/\alpha}]$, then we get $\int_0^1 \int_{|z|>2} z^2\mu^{(n)}(ds,dz) > n^{2/\alpha}$. Thus,

$$\frac{\int_0^1 \int_{|z|>2} z^2\left|\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right|\mu^{(n)}(ds,dz)}{\sigma\int_0^1 \int_{|z|>2} z^2\mu^{(n)}(ds,dz)} \leq \frac{1}{\sigma}\int_0^1 \int_{|z|>2}\left(\frac{z}{n^{1/\alpha}}\right)^2\left|\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right|\mu^{(n)}(ds,dz). \tag{3.26}$$

Assume that there are no jumps in $[-2n^{1/\alpha}, -n^{1/\alpha}) \cup (n^{1/\alpha}, 2n^{1/\alpha}]$, since $\tau(z/n^{1/\alpha}) = 1$ if $|z| \leq n^{1/\alpha}$, then $\tau'(z/n^{1/\alpha}) = 0$ and as a consequence, the right-hand side of (3.25) equals zero in this case.

In both cases, for any $p \geq 1$

$$\mathbb{E}\left(\frac{\int_0^1 \int_{|z|>2} z^2\left|\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right|\mu^{(n)}(ds,dz)}{\sigma\int_0^1 \int_{|z|>2} z^2\mu^{(n)}(ds,dz)}\right)^{2p} \leq \mathbb{E}\left(\frac{1}{\sigma}\int_0^1 \int_{|z|>2}\left(\frac{z}{n^{1/\alpha}}\right)^2\left|\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right|\mu^{(n)}(ds,dz)\right)^{2p}. \tag{3.27}$$

Now from $\mu^{(n)}(ds,dz) = \tilde{\mu}^{(n)}(ds,dz) + \upsilon^{(n)}(ds,dz)$, by convexity inequality, we have for $C(p)$ a positive constant

$$\mathbb{E}\left[\frac{1}{\sigma}\int_0^1\int_{|z|>2}\left(\frac{z}{n^{1/\alpha}}\right)^2\left|\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right|\mu^{(n)}(ds,dz)\right]^{2p}$$
$$\leq \frac{C(p)}{\sigma^{2p}}\mathbb{E}\left[\int_0^1\int_{|z|>2}\left(\frac{z}{n^{1/\alpha}}\right)^2\left|\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right|\tilde{\mu}^{(n)}(ds,dz)\right]^{2p} \tag{3.28}$$
$$+\frac{C(p)}{\sigma^{2p}}\left[\int_0^1\int_{|z|>2}\left(\frac{z}{n^{1/\alpha}}\right)^2\left|\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right|\upsilon^{(n)}(ds,dz)\right]^{2p}.$$

Using Kunita's first inequality (see Theorem 4.4.23 in [1]), there exists a constant $D(2p)>0$ such that

$$\mathbb{E}\left[\int_0^1\int_{|z|>2}\left(\frac{z}{n^{1/\alpha}}\right)^2\left|\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right|\tilde{\mu}^{(n)}(ds,dz)\right]^{2p}$$
$$\leq D(2p)\left[\int_0^1\int_{|z|>2}\left(\frac{z}{n^{1/\alpha}}\right)^4\left|\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right|^2\upsilon^{(n)}(ds,dz)\right]^{p}$$
$$+D(2p)\left[\int_0^1\int_{|z|>2}\left(\frac{z}{n^{1/\alpha}}\right)^{4p}\left|\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right|^{2p}\upsilon^{(n)}(ds,dz)\right]$$
$$=D(2p)\left[\int_0^1\int_{|z|>2}\left(\frac{z}{n^{1/\alpha}}\right)^4\left|\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right|^2\frac{1}{|z|^{1+\alpha}}\tau(z/n^{1/\alpha})dzds\right]^{p}$$
$$+D(2p)\left[\int_0^1\int_{|z|>2}\left(\frac{z}{n^{1/\alpha}}\right)^{4p}\left|\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right|^{2p}\frac{1}{|z|^{1+\alpha}}\tau(z/n^{1/\alpha})dzds\right]$$
$$=\frac{D(2p)}{n^p}\left[\int_0^1\int_1^2\left(\frac{1}{u^{\alpha-3}}\left|\frac{\tau'(u)}{\tau(u)}\right|^2\tau(u)\right)duds\right]^p+\left[\frac{D(2p)}{n}\int_0^1\int_1^2\left(\frac{1}{u^{\alpha+1-4p}}\left|\frac{\tau'(u)}{\tau(u)}\right|^{2p}\tau(u)\right)duds\right].$$

where at the last line we have used the change of the variable $u=\frac{z}{n^{1/\alpha}}$.

Moreover, we have

$$\left[\int_0^1\int_{|z|>2}\left(\frac{z}{n^{1/\alpha}}\right)^2\left|\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right|\upsilon^{(n)}(ds,dz)\right]^{2p}=\left[\int_0^1\int_{|z|>2}\left(\frac{z}{n^{1/\alpha}}\right)^2\left|\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right|\frac{1}{|z|^{1+\alpha}}\tau(z/n^{1/\alpha})dzds\right]^{2p}$$
$$=\left[\frac{1}{n}\int_0^1\int_1^2\frac{1}{u^{\alpha-1}}\left|\frac{\tau'(u)}{\tau(u)}\right|\tau(u)duds\right]^{2p}. \tag{3.29}$$

Under the assumption $\mathbf{H}_1(b_{ii})$, we can deduce the bound for $\mathbb{E}\left|\mathcal{R}^n_{1,\beta}(1)\right|^p, \forall p\geq 2$.

Finally, using that b has bounded derivatives and that $\sup_{0\leq s\leq 1}\frac{U^{n,\beta}_s}{U^{n,\beta}_1}$ is bounded, the remainder terms $R^n_{2,\beta}(1), R^n_{3,\beta}(1)$ satisfy the upper bound

$$|R^n_{2,\beta}(1)|\leq\frac{C}{n},\quad |R^n_{3,\beta}(1)|\leq\frac{C}{n},$$

where C is some deterministic constant. □

3.3 Asymptotic behavior of the transition density and its derivative

In this section, we study the asymptotic behavior of $p^{\beta}_{\frac{1}{n}}(x_0, u)$ (the density of $X^{\beta}_{\frac{1}{n}}$) and its derivative with respect to the parameter β. From the connection (3.9), instead of working directly with the density $p^{\beta}_{\frac{1}{n}}$, we consider the asymptotic behavior of q^{n,β,x_0}, the density of $\overline{Y}_1^{n,\beta,x_0}$ given by (3.8). Considering the equation (3.8), one can prove that if $\alpha > 1$ then $n^{1/\alpha}\left(\overline{Y}_1^{n,\beta,x_0} - x_0\right)$ is close to a stable Lévy process [see [5]]. If $\alpha \leq 1$, this is no longer the case and we have to introduce the solution of the ordinary differential equation

$$\varsigma_t^{n,\beta,x_0} = x_0 + \frac{1}{n}\int_0^t b(\varsigma_s^{n,\beta,x_0}, \theta)ds \quad t \in [0,1]. \tag{3.30}$$

We prove that $n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\beta,x_0})$ is close to a stable Lévy process in Lemma 3.2 below.

Lemma 3.2. *Let* $(\varsigma_t^{n,\beta,x_0})$ *be the solution of the ordinary differential equation* (3.30)*, then*

$$n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\beta,x_0}) \xrightarrow[a.s.]{n\to\infty} \sigma L_1^{\alpha}, \tag{3.31}$$

and this convergence is uniform with respect to x_0.

Proof. We have

$$\begin{aligned}
&\left|n^{1/\alpha}\left(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\beta,x_0}\right) - \sigma L_1^{\alpha}\right| = \left|\frac{1}{n}\int_0^1 n^{1/\alpha}\left[b(\overline{Y}_s^{n,\beta,x_0}, \theta) - b(\varsigma_s^{n,\beta,x_0}, \theta)\right] ds + \sigma\left[L_1^n - L_1^{\alpha}\right]\right| \\
&\leq \frac{1}{n}\int_0^1 ||b'||_\infty \left[\left|n^{1/\alpha}\left(\overline{Y}_s^{n,\beta,x_0} - \varsigma_s^{n,\beta,x_0}\right) - \sigma L_s^{\alpha}\right|\right] ds + \frac{\sigma||b'||_\infty}{n}\int_0^1 |L_s^{\alpha}|\, ds + \sigma\left|L_1^n - L_1^{\alpha}\right|,
\end{aligned}$$

where $||b'||_\infty = \sup_{x\in\mathbb{R}} |b'(x,\theta)|$. Applying the Gronwall's lemma and using the boundedness of b', we get

$$\sup_{x_0}\left|n^{1/\alpha}\left(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\beta,x_0}\right) - \sigma L_1^{\alpha}\right| \leq C\left[\frac{\sigma}{n}\int_0^1 |L_s^{\alpha}|ds + \sigma|L_1^n - L_1^{\alpha}|\right], \tag{3.32}$$

where C is a positive constant. From Lemma 3.1, we have $L_1^n \xrightarrow{a.s.} L_1^{\alpha}$, and from the construction of the α-stable process L_t^{α} (recall (3.2)), we get $\int_0^1 |L_s^{\alpha}|ds < \infty$ a.s. Then $\frac{\sigma}{n}\int_0^1 |L_s^{\alpha}|ds \xrightarrow{a.s.} 0$ and we get the result of Lemma 3.2. $\square$

Remark 3.3. *If we assume that the function b is of class $\mathcal{C}^{1+k}$ with respect to x ($k > 0$) and setting*

$A(f) = bf'$ such that $f(\varsigma_t^{n,\beta,x_0}, \theta) = f(\varsigma_0^{n,\beta,x_0}, \theta) + \int_0^t (Af)(\varsigma_s^{n,\beta,x_0}, \theta) ds$. Then we obtain

$$\begin{aligned}
\varsigma_t^{n,\beta,x_0} &= x_0 + \frac{tb(x_0,\theta)}{n} + \frac{1}{n}\int_0^t \int_0^{t_1} \frac{(Ab)}{n}(\varsigma_{t_2}^{n,\beta,x_0}, \theta) dt_2 dt_1 \\
&= x_0 + \frac{tb(x_0,\theta)}{n} + \frac{t^2 (Ab)(x_0,\theta)}{2n^2} + \frac{1}{n^3}\int_0^t \int_0^{t_1} \int_0^{t_2} (A(Ab))\,(\varsigma_{t_3}^{n,\beta,x_0}, \theta) dt_3 dt_2 dt_1 \\
&= x_0 + \frac{tb(x_0,\theta)}{n} + \frac{t^2 (Ab)(x_0,\theta)}{2n^2} + ... + \frac{t^k (A^k b)(x_0,\theta)}{k! n^k} + \frac{1}{n^{k+1}}\int_0^t \int_0^{t_1} ... \int_0^{t_k} (A^{k+1} b)(\varsigma_{t_k}^{n,\beta,x_0}, \theta) dt_k dt_{k-1} ... dt_{t_1} \\
&= \varsigma_t^{(k),n,\beta,x_0} + \frac{1}{n^{k+1}}\int_0^t \int_0^{t_1} ... \int_0^{t_k} (A^{k+1} b)(\varsigma_{t_k}^{n,\beta,x_0}, \theta) dt_k dt_{k-1} ... dt_{t_1}
\end{aligned}$$

with $\varsigma_t^{(k),n,\beta,x_0} = x_0 + \frac{tb(x_0,\theta)}{n} + \frac{t^2(Ab)(x_0,\theta)}{2n^2} + ... + \frac{t^k (A^k b)(x_0,\theta)}{k! n^k}$.
Under the assumption that the function b is of class $\mathcal{C}^{1+k}$ with respect to x ($k > 0$), we deduce that $\left|\varsigma_t^{n,\beta,x_0} - \varsigma_t^{(k),n,\beta,x_0}\right| \leq \frac{C}{n^{k+1}}$. Combining this with Lemma 3.2, we get

$$n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{(k),n,\beta,x_0}) \xrightarrow[a.s.]{n\to\infty} \sigma L_1^\alpha, \quad \textit{as soon as } \frac{1}{\alpha} < k+1. \tag{3.33}$$

Hence we can replace in Lemma 3.2 the solution of the ordinary differential equation by its explicit short time approximation $\varsigma_1^{(k),n,\beta,x_0}$ as soon as k is large enough.

We will now state the main result of this section about the asymptotic behavior of the transition density and its derivative with respect to the parameter β. In order to apply these results in statistics, we need some uniformity with respect to the parameter β and consequently we study the asymptotic behavior of $p_{\frac{1}{n}}^{\beta_n}$ where $(\beta_n)_{n\geq 1}$ is a sequence such that $\beta_n \xrightarrow{n\to\infty} \beta$.

Theorem 3.2. *Let $(\varsigma_t^{n,\beta,x_0})$ be the solution of the ordinary differential equation* (3.30) *and let $(\beta_n)_{n\geq 1}$ be a sequence such that $\beta_n \xrightarrow{n\to\infty} \beta$. For all $(x_0, u) \in \mathbb{R}^2$,*

1. *$\frac{\sigma_n}{n^{1/\alpha}} p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\beta_n,x_0}) \xrightarrow{n\to\infty} \varphi_\alpha(u)$,*

2. *$\sup_{u\in\mathbb{R}} \sup_n \frac{\sigma_n}{n^{1/\alpha}} p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\beta_n,x_0}) < \infty$,*

where φ_α is the density of L_1^α.

Proof. From (3.9) and Theorem 3.1, we have

$$\frac{\sigma_n}{n^{1/\alpha}} p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\beta_n,x_0}) = \frac{\sigma_n}{n^{1/\alpha}} q^{n,\beta,x_0}(\frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\beta_n,x_0}) = \mathbb{E}\left(1_{\{\overline{Y}_1^{n,\beta_n,x_0} \geq \frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\beta_n,x_0}\}} \frac{\sigma_n}{n^{1/\alpha}} \mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(1)\right), \tag{3.34}$$

where $\frac{\sigma_n}{n^{1/\alpha}} \mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(1) = \widehat{\mathcal{H}}_{1,\beta_n}^n(1) + \widehat{\mathcal{H}}_{2,\beta_n}^n(1) + \frac{\sigma_n}{n^{1/\alpha}} \mathcal{R}_{1,\beta_n}^n(1) + \frac{\sigma_n}{n^{1/\alpha}} \mathcal{R}_{2,\beta_n}^n(1) + \frac{\sigma_n}{n^{1/\alpha}} \mathcal{R}_{3,\beta_n}^n(1)$, with $\widehat{\mathcal{H}}_{1,\beta_n}^n(1), \widehat{\mathcal{H}}_{2,\beta_n}^n(1)$ given by (3.14), (3.15) and $\mathcal{R}_{1,\beta_n}^n(1)$, $\mathcal{R}_{2,\beta_n}^n(1)$, $\mathcal{R}_{3,\beta_n}^n(1)$ satisfy the bounds (3.17).

Then from (3.16), the boundedness of b', and Lemma 3.1, it is immediate that

$$\widehat{\mathcal{H}}^n_{1,\beta_n}(1) \xrightarrow[a.s.]{n\to\infty} \mathcal{H}_{1,L^\alpha}(1), \tag{3.35}$$

$$\widehat{\mathcal{H}}^n_{2,\beta_n}(1) \xrightarrow[a.s.]{n\to\infty} \mathcal{H}_{2,L^\alpha}(1). \tag{3.36}$$

where $\mathcal{H}_{1,L^\alpha}(1), \mathcal{H}_{2,L^\alpha}(1)$ are given by

$$\mathcal{H}_{1,L^\alpha}(1) = \frac{\int_0^1 \int_{\mathbb{R}} \rho(z)\rho'(z)\mu(ds,dz)}{\left[\int_0^1 \int_{\mathbb{R}} \rho(z)\mu(ds,dz)\right]^2}, \tag{3.37}$$

$$\mathcal{H}_{2,L^\alpha}(1) = -\frac{\int_0^1 \int_{\mathbb{R}} \left[\rho'(z) - \frac{1+\alpha}{z}\rho(z)\right]\mu(ds,dz)}{\int_0^1 \int_{\mathbb{R}} \rho(z)\mu(ds,dz)}. \tag{3.38}$$

Moreover, using again the boundedness of b' and the fact that $\rho^n(z)$ is a non negative function, we deduce the upper bounds

$$\left|\widehat{\mathcal{H}}^n_{1,\beta_n}(1)\right| \leq C^* \left[\frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z)|(\rho^n)'(z)|\mu^{(n)}(ds,dz)}{\left[\int_0^1 \int_{\mathbb{R}} \rho^n(z)\mu^{(n)}(ds,dz)\right]^2}\right], \tag{3.39}$$

$$\left|\widehat{\mathcal{H}}^n_{2,\beta_n}(1)\right| \leq C^* \left[\frac{\int_0^1 \int_{\mathbb{R}} \left[|\rho^{n\prime}(z)| + \frac{1+\alpha}{|z|}\rho^n(z)\right]\mu^{(n)}(ds,dz)}{\int_0^1 \int_{\mathbb{R}} \rho^n(z)\mu^{(n)}(ds,dz)}\right], \tag{3.40}$$

for some constant $C^* > 0$.

We now show that $\sup_n \left|\widehat{\mathcal{H}}^n_{1,\beta_n}(1)\right|^p$ and $\sup_n \left|\widehat{\mathcal{H}}^n_{2,\beta_n}(1)\right|^p$ are integrable $\forall p \geq 1$. The proof will be divided into the two following steps:

Step 1.1: We show that the right-hand side of (3.39) is bounded by a random variable independent of n and belonging to $\cap_{p\geq 1}\mathbf{L}^p$. In fact, since the measures $\mu^{(n)}(ds,dz)$ and $\mu(ds,dz)$ coincide on the set $\{(s,z)|s \in [0,1], |z| \leq n^{1/\alpha}\}$, and $\rho^n(z) = \rho(z)$ on the support of the Poisson measure $\mu^{(n)}$, we have

$$\frac{\int_0^1 \int_{\mathbb{R}} \rho^n(z)|(\rho^n)'(z)|\mu^{(n)}(ds,dz)}{\left[\int_0^1 \int_{\mathbb{R}} \rho^n(z)\mu^{(n)}(ds,dz)\right]^2} \leq \left(\frac{\int_0^1 \int_{|z|\leq 2} |\rho'(z)|\,\rho(z)\mu(ds,dz)}{\left(\int_0^1 \int_{|z|\leq 2} \rho(z)\mu(ds,dz)\right)^2} + \frac{\int_0^1 \int_{|z|>2} 2|z|^3\mu^{(n)}(ds,dz)}{\left(\int_0^1 \int_{|z|>2} z^2\mu^{(n)}(ds,dz)\right)^2}\right). \tag{3.41}$$

We now consider in the right-hand side of (3.41). Using that ρ, ρ' belongs to $\cap_{p\geq 1}\mathbf{L}^p(1_{|z|\leq 2}|z|^{-1-\alpha}dz)$, we get

$$\mathbb{E}\left[\left(\int_0^1 \int_{|z|\leq 2} |\rho'(z)|\rho(z)\mu(ds,dz)\right)^p\right] < \infty, \quad \forall p \geq 1. \tag{3.42}$$

On the other hand, since ρ satisfies the non degeneracy assumption (2.6), $[\int_0^1 \int_{|z|\leq 2} \rho(z)\mu(ds,dz)]^{-1}$ belongs to $\cap_{p\geq 1}\mathbf{L}^p$ [see [5, Theorem 4 p.2323]], we deduce that the first term of (3.41) belongs to

$\cap_{p\geq 1}\mathbf{L}^p$, moreover, it does not depend on n .

We now consider the second term of the right-hand side of (3.41). From the fact that $v^{(n)}(\{(t,z)|0\leq t\leq 1, |z|>2\})<\infty$, we can construct the integral with respect to the random measure $\mu^{(n)}$ as follows [see Chapter VI in [4]]

$$\begin{aligned} \int_0^1\int_{|z|>2} 2|z|^3\mu^{(n)}(ds,dz) &= \sum_{i=1}^{N_1} 2|Z_i|^3 \quad \text{a.s.},\\ \int_0^1\int_{|z|>2} z^2\mu^{(n)}(ds,dz) &= \sum_{i=1}^{N_1} Z_i^2 \quad \text{a.s.}, \end{aligned} \tag{3.43}$$

where $N=(N_t)_{1\geq t\geq 0}$ is a Poisson process with intensity $\lambda_n=\int_{|z|>2}F_n(z)dz<\infty$, and $(Z_i)_{i\geq 0}$ are i.i.d. random variable independent of N with probability measure $\frac{F_n(z)1_{|z|>2}dz}{\lambda_n}$. Thus,

$$\frac{\int_0^1\int_{|z|>2}2|z|^3\mu^{(n)}(ds,dz)}{\left(\int_0^1\int_{|z|>2}z^2\mu^{(n)}(ds,dz)\right)^2}=\frac{\sum_{i=1}^{N_1}2|Z_i|^3}{\left(\sum_{i=1}^{N_1}Z_i^2\right)^2}\leq\frac{\sum_{i=1}^{N_1}2|Z_i|^3}{\sum_{i=1}^{N_1}Z_i^4}\leq 1.$$

where we used the fact that $Z_i^2\geq 0$, and $|Z_i|>2$. We deduce $\sup_n\left|\widehat{\mathcal{H}}^n_{1,\beta_n}(1)\right|^p$ is integrable $\forall p\geq 1$.

Step 1.2: We show that $\sup_n\left|\widehat{\mathcal{H}}^n_{2,\beta_n}(1)\right|^p$ is integrable.

Using the definitions of ρ^n (recall (3.10)), ρ (recall (3.11)) and $\rho^n=\rho$ on the support of the Poisson measure $\mu^{(n)}$ [see Section 3.2], we have

$$\begin{aligned} &\frac{\int_0^1\int_{\mathbb{R}}\left[|\rho^{n\prime}(z)|+\frac{1+\alpha}{|z|}\rho^n(z)\right]\mu^{(n)}(ds,dz)}{\int_0^1\int_{\mathbb{R}}\rho^n(z)\mu^{(n)}(ds,dz)}\\ &\qquad\leq\frac{\int_0^1\int_{|z|\leq 2}\left(|\rho(z)'|+\rho(z)\frac{1+\alpha}{|z|}\right)\mu(ds,dz)}{\int_0^1\int_{|z|\leq 2}\rho(z)\mu(ds,dz)}+\frac{\int_0^1\int_{|z|>2}(3+\alpha)|z|\mu^{(n)}(ds,dz)}{\int_0^1\int_{|z|>2}z^2\mu^{(n)}(ds,dz)} \end{aligned} \tag{3.44}$$

where we used the fact that $\int_0^1\int_{|z|\leq 2}\rho(z)\mu(ds,dz)\geq 0$, $\int_0^1\int_{|z|>2}\rho^n(z)\mu^{(n)}(ds,dz)\geq 0$, and the measures $\mu^{(n)}(ds,dz)$ and $\mu(ds,dz)$ coincide on the set $\{(s,z)|s\in[0,1],|z|\leq n^{1/\alpha}\}$.

Proceeding as for the first term in the right-hand side of (3.41), we also get that the first term of (3.44) belongs to $\cap_{p\geq 1}\mathbf{L}^p$.

On the other hand, for the second term of (3.44) we have:

$$\frac{\int_0^1\int_{|z|>2}(3+\alpha)|z|\mu^{(n)}(ds,dz)}{\int_0^1\int_{|z|>2}z^2\mu^{(n)}(ds,dz)}\leq\frac{\int_0^1\int_{|z|>2}(3+\alpha)z^2\mu^{(n)}(ds,dz)}{\int_0^1\int_{|z|>2}z^2\mu^{(n)}(ds,dz)}=3+\alpha.$$

And this completes the proof of **Step 1.2.**

We finally deduce that $\sup_n\left|\widehat{\mathcal{H}}^n_{1,\beta_n}(1)\right|^p$ and $\sup_n\left|\widehat{\mathcal{H}}^n_{2,\beta_n}(1)\right|^p$ are integrable for all $p\geq 1$. Applying the dominated convergence theorem, we get

$$\widehat{\mathcal{H}}^n_{1,\beta_n}(1)\xrightarrow[\mathbf{L}^p]{n\to\infty}\mathcal{H}_{1,L^\alpha}(1),\quad\forall p\geq 1. \tag{3.45}$$

$$\widehat{\mathcal{H}}^{n}_{2,\beta_n}(1) \xrightarrow[\mathbf{L}^p]{n\to\infty} \mathcal{H}_{2,L^\alpha}(1), \quad \forall p \geq 1. \tag{3.46}$$

On the other hand, Lemma 3.2 implies that $n^{1/\alpha}(\overline{Y}_1^{n,\beta_n,x_0} - \varsigma_1^{n,\beta,x_0})$ converges almost surely to σL_1^α. Then, an easy computation, using that $P(L_1^\alpha = u) = 0$, shows the almost sure convergence

$$1_{\{\overline{Y}_1^{n,\beta_n,x_0} \geq \frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\beta_n,x_0}\}} = 1_{[u,\infty)}\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta_n,x_0} - \varsigma_1^{n,\beta_n,x_0})}{\sigma_n}\right) \xrightarrow[n\to\infty]{\textbf{a.s.}} 1_{[u,\infty)}(L_1^\alpha). \tag{3.47}$$

Moreover from the boundedness property of the variables, applying the dominated convergence theorem, we get the latter convergence in $\mathbf{L}^p$, $\forall p \geq 1$. We finally get that :

$$\frac{\sigma_n}{n^{1/\alpha}} q^{n,\beta_n,x_0}\left(\frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\beta_n,x_0}\right) \xrightarrow{n\to\infty} \mathbb{E}[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}_{L^\alpha}(1)]. \tag{3.48}$$

where $\mathcal{H}_{L^\alpha}(1) = \mathcal{H}_{1,L^\alpha}(1) + \mathcal{H}_{2,L^\alpha}(1)$ and $\mathcal{H}_{1,L^\alpha}(1), \mathcal{H}_{2,L^\alpha}(1)$ are given by (3.37), (3.38), respectively. Remark that, we easily get from (3.12), (3.13), (3.17) and (3.45), (3.46) that

$$\sup_{u\in\mathbb{R}} \sup_{n} \frac{\sigma_n}{n^{1/\alpha}} q^{n,\beta_n,x_0}\left(\frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\beta_n,x_0}\right) < \infty. \tag{3.49}$$

To finish the proof of the convergence, it remains to show that the right-hand side of (3.48) is a representation for $\varphi_\alpha(u)$, the density of L_1^α. Let us denote by $\varphi^n(u)$ the density of the variable L_1^n. We consider the situation where the drift function $b \equiv 0$ and $x_0 = 0$ for which $n^{1/\alpha}\overline{Y}_1^{n,\beta,x_0} = \sigma L_1^n$. Then (3.48), (3.49) yield

$$\varphi^n(u) \xrightarrow{n\to\infty} \mathbb{E}[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}_{L^\alpha}(1)] := \psi(u), \tag{3.50}$$

$$\sup_{u\in\mathbb{R}} \sup_{n} \varphi^n(u) < \infty. \tag{3.51}$$

Assume by contradiction that, for some u, we have $\psi(u) \neq \varphi_\alpha(u)$. From the fact that $P(L_1^\alpha = u) = 0$, it can be seen that ψ is continuous at the point u. Hence, one can find a continuous, compactly supported, function f such that

$$\int f(x)\psi(x)dx \neq \int f(x)\varphi_\alpha(x)dx. \tag{3.52}$$

On the one hand we have, $\mathbb{E}[f(L_1^n)] = \int f(x)\varphi^n(x)dx \xrightarrow{n\to\infty} \int f(x)\psi(x)dx$ where we have used the dominated convergence theorem with (3.50)-(3.51). On the other hand, we can write

$$\mathbb{E}[f(L_1^n)] = \mathbb{E}[f(L_1^n)1_{\{L_1^n = L_1^\alpha\}}] + \mathbb{E}[f(L_1^n)1_{\{L_1^n \neq L_1^\alpha\}}]. \tag{3.53}$$

By Lemma 3.1, we have $\mathbb{P}(L_1^n = L_1^\alpha) \xrightarrow{n\to\infty} 1$. We deduce that,

$$\mathbb{E}[f(L_1^n)] \xrightarrow{n\to\infty} \mathbb{E}[f(L_1^\alpha)] = \int f(x)\varphi_\alpha(x)dx. \tag{3.54}$$

This last convergence result clearly contradicts (3.52). And consequently we get that $\mathbb{E}[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}_{L^\alpha}(1)] = \varphi_\alpha(u)$. Combining the preceding results with (3.48), we can easily get the results of Theorem 3.2. $\square$

Remark 3.4. *i) From the convergence* (3.33)*, proceeding as in the proof of Theorem 3.2, we can state that*

$$\frac{\sigma_n}{n^{1/\alpha}} p^{\beta_n}_{\frac{1}{n}}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{(k),n,\beta_n,x_0}) \xrightarrow{n\to\infty} \varphi_\alpha(u), \quad \text{if} \quad k > \frac{1}{\alpha} - 1.$$

ii) The results of Theorem 3.2 have been obtained by Kulik [12], using the parametrix method.

In the next theorem, we study the asymptotic behavior of the derivatives of the density with respect to the parameters θ and σ. Such results are crucial in asymptotic statistics.

Theorem 3.3. *Let* $(\beta_n)_{n\geq 1}$ *be a sequence such that* $\beta_n \xrightarrow{n\to\infty} \beta$*. For all* $(x_0, u) \in \mathbb{R}^2$*,*

i) $\frac{\sigma_n^2}{n^{\frac{2}{\alpha}-1}} \partial_\theta p^{\beta_n}_{\frac{1}{n}}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\beta_n,x_0}) \xrightarrow{n\to\infty} -\partial_\theta b(x_0,\theta) \times \varphi'_\alpha(u)$,
$\frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma p^{\beta_n}_{\frac{1}{n}}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\beta_n,x_0}) \xrightarrow{n\to\infty} -\varphi_\alpha(u) - u\varphi'_\alpha(u)$,

ii) $\sup_{u\in\mathbb{R}} \sup_n \left| \frac{\sigma_n^2}{n^{\frac{2}{\alpha}-1}} \partial_\theta p^{\beta_n}_{\frac{1}{n}}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\beta_n,x_0}) \right| < \infty$,
$\sup_{u\in\mathbb{R}} \sup_n \left| \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma p^{\beta_n}_{\frac{1}{n}}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\beta_n,x_0}) \right| < \infty$.

The proof of this theorem is postponed to Section 3.4.2 below. Let us first remark that from (3.9) and Theorem 2.3, we have

$$\begin{aligned}\nabla_\beta p^{\beta_n}_{\frac{1}{n}}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\beta_n,x_0}) &= \nabla_\beta q^{n,\beta_n,x_0}(\frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\beta_n,x_0}) \\ &= \mathbb{E}\left[\mathbf{1}_{\{\overline{Y}_1^{n,\beta_n,x_0} \geq \frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\beta_n,x_0}\}} \mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\nabla_\beta \overline{Y}_1^{n,\beta_n,x_0})) \right]. \quad (3.55)\end{aligned}$$

Moreover, from (2.7), (2.14) and (2.18), (2.19), by some simple calculus, we get the explicit formula for the iterated Malliavin weight

$$\begin{aligned}&\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\nabla_\beta \overline{Y}_1^{n,\beta,x_0})) \\ &= \begin{pmatrix} \partial_\theta \overline{Y}_1^{n,\beta,x_0} \\ \partial_\sigma \overline{Y}_1^{n,\beta,x_0} \end{pmatrix} \mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(1)^2 - \begin{pmatrix} V_1^{n,\theta} \\ V_1^{n,\sigma} \end{pmatrix} \frac{2\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(1)}{U_1^{n,\beta}} + \begin{pmatrix} \partial_\theta \overline{Y}_1^{n,\beta,x_0} \\ \partial_\sigma \overline{Y}_1^{n,\beta,x_0} \end{pmatrix} \frac{\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(1) W_1^{n,\beta}}{(U_1^{n,\beta})^2} + \begin{pmatrix} \partial_\theta \overline{Y}_1^{n,\beta,x_0} \\ \partial_\sigma \overline{Y}_1^{n,\beta,x_0} \end{pmatrix} \frac{(W_1^{n,\beta})^2}{(U_1^{n,\beta})^4} \\ &- \begin{pmatrix} V_1^{n,\theta} \\ V_1^{n,\sigma} \end{pmatrix} \frac{W_1^{n,\beta}}{(U_1^{n,\beta})^3} + \begin{pmatrix} \partial_\theta \overline{Y}_1^{n,\beta,x_0} \\ \partial_\sigma \overline{Y}_1^{n,\beta,x_0} \end{pmatrix} \frac{2D_1^{n,\beta}}{(U_1^{n,\beta})^2} - \begin{pmatrix} \partial_\theta \overline{Y}_1^{n,\beta,x_0} \\ \partial_\sigma \overline{Y}_1^{n,\beta,x_0} \end{pmatrix} \frac{Q_1^{n,\beta}}{(U_1^{n,\beta})^3} + \begin{pmatrix} T_1^{n,\theta} \\ T_1^{n,\sigma} \end{pmatrix} \frac{1}{(U_1^{n,\beta})^2}.\end{aligned} \quad (3.56)$$

where $\mathcal{H}^n_{\overline{Y}_1^{n,\beta,x_0}}(1)$, $U_1^{n,\beta}, W_1^{n,\beta}$ are given by (3.13), (3.19), (3.21), respectively.
The expression of $\partial_\theta \overline{Y}_t^{n,\beta,x_0}$ is given by solving

$$\partial_\theta \overline{Y}_t^{n,\beta,x_0} = \frac{1}{n}\int_0^t b'(\overline{Y}_s^{n,\beta,x_0},\theta)\partial_\theta \overline{Y}_s^{n,\beta,x_0} ds + \frac{1}{n}\int_0^t \partial_\theta b(\overline{Y}_s^{n,\beta,x_0},\theta) ds, \quad (3.57)$$

we get

$$\partial_\theta \overline{Y}_1^{n,\beta,x_0} = \frac{1}{n}\epsilon_1^n \int_0^1 (\epsilon_s^n)^{-1}\partial_\theta b(\overline{Y}_s^{n,\beta,x_0},\theta)ds. \tag{3.58}$$

The expression of $\partial_\sigma \overline{Y}_t^{n,\beta,x_0}$ is given by solving $\partial_\sigma \overline{Y}_t^{n,\beta,x_0} = \frac{1}{n}\int_0^t b'(\overline{Y}_s^{n,\beta,x_0},\theta)\partial_\sigma \overline{Y}_s^{n,\beta,x_0}ds + \frac{L_t^n}{n^{1/\alpha}}$, and we get that

$$\partial_\sigma \overline{Y}_1^{n,\beta,x_0} = \frac{1}{n^{1/\alpha}}\epsilon_1^n \int_0^1 (\epsilon_s^n)^{-1}dL_s^n. \tag{3.59}$$

For the computations of $V_1^{n,\theta} = \Gamma(\overline{Y}_1^{n,\beta,x_0}, \partial_\theta \overline{Y}_1^{n,\beta,x_0})$ and $V_1^{n,\sigma} = \Gamma(\overline{Y}_1^{n,\beta,x_0}, \partial_\sigma \overline{Y}_1^{n,\beta,x_0})$, using (2.15), (2.16) we have

$$V_1^{n,\theta} = \frac{1}{n}(\epsilon_1^n)^2 \int_0^1 (\epsilon_s^n)^{-2}\left(U_s^{n,\beta}\left[(\partial_\theta b)'(\overline{Y}_s^{n,\beta,x_0},\theta) + b''(\overline{Y}_s^{n,\beta,x_0},\theta)\partial_\theta \overline{Y}_s^{n,\beta,x_0}\right]\right)ds, \tag{3.60}$$

$$V_1^{n,\sigma} = \frac{1}{n}(\epsilon_1^n)^2 \int_0^1 (\epsilon_s^n)^{-2}\left(b''(\overline{Y}_s^{n,\beta,x_0},\theta)\partial_\sigma \overline{Y}_s^{n,\beta,x_0}U_s^{n,\beta}\right)ds + \frac{\sigma}{n^{2/\alpha}}(\epsilon_1^n)^2 \int_0^t \int_{\mathbb{R}} (\epsilon_s^n)^{-2}\rho^n(z)\mu^{(n)}(ds,dz). \tag{3.61}$$

Finally from Eq.((2.20) - (2.23)) we compute explicitly $D_1^{n,\beta} = \Gamma(\overline{Y}_1^{n,\beta,x_0}, L\overline{Y}_1^{n,\beta,x_0})$, $Q_1^{n,\beta} = \Gamma(\overline{Y}_1^{n,\beta,x_0}, W_1^{n,\beta})$, $T_1^{n,\theta} = \Gamma(\overline{Y}_1^{n,\beta,x_0}, V_1^{n,\theta})$ $T_1^{n,\sigma} = \Gamma(\overline{Y}_1^{n,\beta,x_0}, V_1^{n,\sigma})$ we get:

$$\begin{aligned} D_1^{n,\beta} &= \frac{(\epsilon_1^n)^2}{n}\int_0^1 (\epsilon_s^n)^{-2}b''(\overline{Y}_s^{n,\beta,x_0},\theta)L\overline{Y}_s^{n,\beta,x_0}U_s^{n,\beta}ds + \frac{(\epsilon_1^n)^2}{2n}\int_0^1 (\epsilon_s^n)^{-2}b''(\overline{Y}_s^{n,\beta,x_0},\theta)W_s^{n,\beta}ds \\ &+ \frac{(\epsilon_1^n)^2}{2n}\int_0^1 (\epsilon_s^n)^{-2}b'''(\overline{Y}_s^{n,\beta,x_0},\theta)(U_s^{n,\beta})^2ds + \frac{\sigma^2(\epsilon_1^n)^2}{2n^{2/\alpha}}\int_0^1\int_{\mathbb{R}}(\epsilon_s^n)^{-2}\rho^n(z)\left((\rho^n)'(z) + \rho^n(z)\frac{F_n'(z)}{F_n(z)}\right)'\mu^{(n)}(ds,dz), \end{aligned} \tag{3.62}$$

$$\begin{aligned} Q_1^{n,\beta} &= \frac{7(\epsilon_1^n)^4}{n}\int_0^1 (\epsilon_1^n)^{-4}b''(\overline{Y}_s^{n,\beta,x_0},\theta)W_s^{n,\beta}U_s^{n,\beta}ds + \frac{2(\epsilon_1^n)^4}{n}\int_0^1 (\epsilon_1^n)^{-4}b'''(\overline{Y}_s^{n,\beta,x_0},\theta)(U_s^{n,\beta})^3ds \\ &+ \frac{\sigma^4(\epsilon_1^n)^4}{n^{4/\alpha}}\int_0^1\int_{\mathbb{R}}(\epsilon_1^n)^{-4}\rho^n(z)\left[\left((\rho^n)'(z)\right)^2 + \rho^n(z)(\rho^n)''(z)\right]\mu^{(n)}(ds,dz), \end{aligned} \tag{3.63}$$

$$\begin{aligned} T_1^{n,\theta} &= \frac{3(\epsilon_1^n)^3}{n}\int_0^1 (\epsilon_1^n)^{-3}b''(\overline{Y}_s^{n,\beta,x_0},\theta)V_s^{n,\theta}U_s^{n,\beta}ds + \frac{(\epsilon_1^n)^3}{n}\int_0^1 (\epsilon_1^n)^{-3}(\partial_\theta b)'(\overline{Y}_s^{n,\beta,x_0},\theta)W_s^{n,\beta}ds \\ &+ \frac{(\epsilon_1^n)^3}{n}\int_0^1 (\epsilon_1^n)^{-3}b''(\overline{Y}_s^{n,\beta,x_0},\theta)\partial_\theta \overline{Y}_s^{n,\beta,x_0}W_s^{n,\beta}ds + \frac{(\epsilon_1^n)^3}{n}\int_0^1 (\epsilon_1^n)^{-3}(\partial_\theta b)''(\overline{Y}_s^{n,\beta,x_0},\theta)(U_s^{n,\beta})^2ds \\ &+ \frac{(\epsilon_1^n)^3}{n}\int_0^1 (\epsilon_1^n)^{-3}b'''(\overline{Y}_s^{n,\beta,x_0},\theta)\partial_\theta \overline{Y}_s^{n,\beta,x_0}(U_s^{n,\beta})^2ds, \end{aligned} \tag{3.64}$$

$$\begin{aligned} T_1^{n,\sigma} &= \frac{3(\epsilon_1^n)^3}{n}\int_0^1 (\epsilon_1^n)^{-3}b''(\overline{Y}_s^{n,\beta,x_0},\theta)V_s^{n,\sigma}U_s^{n,\beta}ds + \frac{(\epsilon_1^n)^3}{n}\int_0^1 (\epsilon_1^n)^{-3}b''(\overline{Y}_s^{n,\beta,x_0},\theta)\partial_\sigma \overline{Y}_s^{n,\beta,x_0}W_s^{n,\beta}ds \\ &+ \frac{(\epsilon_1^n)^3}{n}\int_0^1 (\epsilon_1^n)^{-3}b'''(\overline{Y}_s^{n,\beta,x_0},\theta)\partial_\sigma \overline{Y}_s^{n,\beta,x_0}(U_s^{n,\beta})^2ds + \frac{\sigma^2(\epsilon_1^n)^3}{n^{3/\alpha}}\int_0^1\int_{\mathbb{R}}(\epsilon_1^n)^{-3}\rho^n(z)(\rho^n)'(z)\mu^{(n)}(ds,dz). \end{aligned} \tag{3.65}$$

From the above calculus and combining with (3.55) and (3.56) we have a representation for the derivative of the density with respect to parameter β as an expectation and this permits to analyze its asymptotic behavior in small time. To obtain the results of Theorem 3.3, we have to consider the convergence of the iterated Malliavin weights and this is based on the preceding explicit expressions. In the sequel, we prove that all the terms involving the derivatives of b with respect to x are remainder terms.

3.4 Asymptotic behavior of the iterated Malliavin weight and the proof of Theorem 3.3

3.4.1 Preliminary lemmas

In this section, we study the convergence of the iterated Malliavin weight $\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\nabla_\beta \overline{Y}_1^{n,\beta_n,x_0}))$ which is the cornerstone of the proof for the convergence of $\nabla_\beta p_{\frac{1}{n}}^{\beta_n}$ later. Firstly, we state some technical lemmas useful for our aim. The proofs of these lemmas are postponed at the end of the paper.

Lemma 3.3. *We have*

i) $|\partial_\theta \overline{Y}_1^{n,\beta,x_0}| \leq \frac{C}{n}$, *where C is some deterministic constant.*

ii) $\sup_{s\in[0,1]} \left|\partial_\sigma \overline{Y}_s^{n,\beta,x_0}\right| \xrightarrow[L^{2p}]{n\to\infty} 0, \quad \forall p \geq 1.$

Lemma 3.4. *The following decompositions and estimates hold*

i) $\frac{1}{n^{1/\alpha}} \frac{D_1^{n,\beta}}{\left(U_1^{n,\beta}\right)^2} = \frac{n^{1/\alpha}}{2\sigma^2} \widehat{\mathcal{H}}_{3,\beta}^n(1) + \mathcal{R}_{4,\beta}^n(1) + \mathcal{R}_{5,\beta}^n(1) + \mathcal{R}_{6,\beta}^n(1).$

ii) $\frac{1}{n^{1/\alpha}} \frac{Q_1^{n,\beta}}{\left(U_1^{n,\beta}\right)^3} = \frac{n^{1/\alpha}}{\sigma^2} \widehat{\mathcal{H}}_{4,\beta}^n(1) + \mathcal{R}_{7,\beta}^n(1) + \mathcal{R}_{8,\beta}^n(1).$

iii) $\frac{1}{n^{1/\alpha+1}} \frac{\sup_{s\in[0,1]} |\partial_\sigma \overline{Y}_s^{n,\beta,x_0} W_s^{n,\beta}|}{\left(U_1^{n,\beta}\right)^2} \xrightarrow[L^{2p}]{n\to\infty} 0, \quad \forall p \geq 1.$

The main terms $\widehat{\mathcal{H}}_{3,\beta}^n(1), \widehat{\mathcal{H}}_{4,\beta}^n(1)$ are given by

$$\widehat{\mathcal{H}}_{3,\beta}^n(1) = \frac{\int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z) \left[(\rho^n)''(z) - (\rho^n)'(z)\frac{(1+\alpha)}{z} + \rho^n(z)\frac{(1+\alpha)}{z^2}\right] \mu^{(n)}(ds,dz)}{(\epsilon_1^n)^2 \left(\int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z) \mu^{(n)}(ds,dz)\right)^2}, \tag{3.66}$$

$$\widehat{\mathcal{H}}_{4,\beta}^n(1) = \frac{\int_0^1 \int_{\mathbb{R}} (\epsilon_1^n)^{-4} \rho^n(z) \left[((\rho^n)'(z))^2 + \rho^n(z)(\rho^n)''(z)\right] \mu^{(n)}(ds,dz)}{(\epsilon_1^n)^2 \left(\int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z) \mu^{(n)}(ds,dz)\right)^3}. \tag{3.67}$$

where $(\epsilon_s^n)_{s\in[0,1]}$ *is given by* (3.16)*, and the main and remainder terms satisfy for* $p \geq 1$*,* C *is some deterministic constant*

$$\widehat{\mathcal{H}}^n_{3,\beta}(1) \xrightarrow[\boldsymbol{L}^p]{n\to\infty} \mathcal{H}_{3,L^\alpha}(1), \quad \widehat{\mathcal{H}}^n_{4,\beta}(1) \xrightarrow[\boldsymbol{L}^p]{n\to\infty} \mathcal{H}_{4,L^\alpha}(1), \tag{3.68}$$

$$\mathcal{R}^n_{4,\beta}(1) \xrightarrow[\boldsymbol{L}^{2p}]{n\to\infty} 0, \quad \mathcal{R}^n_{5,\beta}(1) \xrightarrow[\boldsymbol{L}^p]{n\to\infty} 0, \quad |\mathcal{R}^n_{6,\beta}(1)| \leq \frac{C}{2n^{1+1/\alpha}}, \tag{3.69}$$

$$\mathcal{R}^n_{7,\beta}(1) \xrightarrow[\boldsymbol{L}^p]{n\to\infty} 0, \quad |\mathcal{R}^n_{8,\beta}(1)| \leq \frac{C}{n^{1+1/\alpha}}, \tag{3.70}$$

with

$$\mathcal{H}_{3,L^\alpha}(1) = \frac{\int_0^1 \int_{\mathbb{R}} \left(\rho(z)\rho''(z) - \rho(z)\rho'(z)\frac{(1+\alpha)}{z} + (\rho(z))^2\,\frac{(1+\alpha)}{z^2}\right)\mu(ds,dz)}{\left(\int_0^1 \int_{\mathbb{R}} \rho(z)\mu(ds,dz)\right)^2}, \tag{3.71}$$

$$\mathcal{H}_{4,L^\alpha}(1) = \frac{\int_0^1 \int_{\mathbb{R}} \rho(z)\left[(\rho'(z))^2 + \rho(z)\rho''(z)\right]\mu(ds,dz)}{\left(\int_0^1 \int_{\mathbb{R}} \rho(z)\mu(ds,dz)\right)^3}. \tag{3.72}$$

Lemma 3.5. *The following estimates hold:*

i) $\left|\frac{V_1^{n,\theta}}{U_1^{n,\beta}}\right| \leq \frac{C}{n}$,

ii) $\frac{1}{n^{2/\alpha-1}}\frac{T_1^{n,\theta}}{\left(U_1^{n,\beta}\right)^2} \xrightarrow[\boldsymbol{L}^p]{n\to\infty} 0, \quad \forall p \geq 1$,

iii) $\frac{V_1^{n,\sigma}}{U_1^{n,\beta}} = \frac{1}{\sigma} + \mathcal{R}^n_{9,\beta}(1)$,

iv) $\frac{1}{n^{1/\alpha}}\frac{T_1^{n,\sigma}}{\left(U_1^{n,\beta}\right)^2} = \frac{1}{\sigma^2}\widehat{\mathcal{H}}^n_{5,\beta}(1) + \mathcal{R}^n_{10,\beta}(1) + \mathcal{R}^n_{11,\beta}(1) + \mathcal{R}^n_{12,\beta}(1)$,

where C *is some deterministic constant. The main term* $\widehat{\mathcal{H}}^n_{5,\beta}(1)$ *is given by*

$$\widehat{\mathcal{H}}^n_{5,\beta}(1) = \frac{\int_0^1 \int_{\mathbb{R}} (\epsilon_1^n)^{-3}\rho^n(z)(\rho^n)'(z)\mu^{(n)}(ds,dz)}{\epsilon_1^n\left(\int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2}\rho^n(z)\mu^{(n)}(ds,dz)\right)^2} \tag{3.73}$$

with (ϵ_s^n) *is given by* (3.16)*. Moreover, the remainder terms* $(\mathcal{R}^n_{i,\beta}(1))_{9\leq i\leq 12}$ *converge to zero as* $n \to \infty$ *in* $\boldsymbol{L}^p, \forall p \geq 2$ *and* $\widehat{\mathcal{H}}^n_{5,\beta}(1) \xrightarrow[\boldsymbol{L}^p,\forall p\geq 1]{n\to\infty} \mathcal{H}_{5,L^\alpha}(1)$*, with*

$$\mathcal{H}_{5,L^\alpha}(1) = \frac{\int_0^1 \int_{\mathbb{R}} \rho(z)\rho'(z)\mu(ds,dz)}{\left(\int_0^1 \int_{\mathbb{R}} \rho(z)\mu(ds,dz)\right)^2}. \tag{3.74}$$

Lemma 3.6. *For all* $p \geq 1$*, the following convergences hold uniformly with respect to* x_0*:*

$$n\partial_\theta \overline{Y}_1^{n,\beta,x_0}\left(\widehat{\mathcal{H}}^n_\beta(1)\right)^2 \xrightarrow[\boldsymbol{L}^p]{n\to\infty} \partial_\theta b(x_0,\theta)\left(\mathcal{H}_{L^\alpha}(1)\right)^2, \tag{3.75}$$

$$n\partial_\theta \overline{Y}_1^{n,\beta,x_0} \widehat{\mathcal{H}}_\beta^n(1)\widehat{\mathcal{H}}_{1,\beta}^n(1) \xrightarrow[\boldsymbol{L^p}]{n\to\infty} \partial_\theta b(x_0,\theta)\mathcal{H}_{L^\alpha}(1)\mathcal{H}_{1,L^\alpha}(1), \tag{3.76}$$

$$n^{1/\alpha}\partial_\sigma \overline{Y}_1^{n,\beta,x_0} \left(\widehat{\mathcal{H}}_\beta^n(1)\right)^2 \xrightarrow[\boldsymbol{L^p}]{n\to\infty} L_1^\alpha \left(\mathcal{H}_{L^\alpha}(1)\right)^2, \tag{3.77}$$

$$n^{1/\alpha}\partial_\sigma \overline{Y}_1^{n,\beta,x_0} \widehat{\mathcal{H}}_\beta^n(1)\widehat{\mathcal{H}}_{1,\beta}^n(1) \xrightarrow[\boldsymbol{L^p}]{n\to\infty} L_1^\alpha \mathcal{H}_{L^\alpha}(1)\mathcal{H}_{1,L^\alpha}(1), \tag{3.78}$$

where $\widehat{\mathcal{H}}_\beta^n(1) = \widehat{\mathcal{H}}_{1,\beta}^n(1) + \widehat{\mathcal{H}}_{2,\beta}^n(1)$ *with* $\widehat{\mathcal{H}}_{1,\beta}^n(1), \widehat{\mathcal{H}}_{2,\beta}^n(1)$ *given by* (3.14), (3.15)*;* $\mathcal{H}_{L^\alpha}(1) = \mathcal{H}_{1,L^\alpha}(1) + \mathcal{H}_{2,L^\alpha}(1)$ *where* $\mathcal{H}_{1,L^\alpha}(1), \mathcal{H}_{2,L^\alpha}(1)$ *are defined by* (3.37), (3.38)*,* $\partial_\theta \overline{Y}_1^{n,\beta,x_0}$ *is given by* (3.58) *and* $\partial_\sigma \overline{Y}_1^{n,\beta,x_0}$ *is given by* (3.59).

Lemma 3.7. *For all* $p \geq 1$ *then the following convergences hold uniformly with respect to* x_0*:*

i) $n\partial_\theta \overline{Y}_1^{n,\beta,x_0} \widehat{\mathcal{H}}_{3,\beta}^n(1) \xrightarrow[\boldsymbol{L^p}]{n\to\infty} \partial_\theta b(x_0,\theta)\mathcal{H}_{3,L^\alpha}(1),$

ii) $n\partial_\theta \overline{Y}_1^{n,\beta,x_0} \widehat{\mathcal{H}}_{4,\beta}^n(1) \xrightarrow[\boldsymbol{L^p}]{n\to\infty} \partial_\theta b(x_0,\theta)\mathcal{H}_{4,L^\alpha}(1),$

iii) $n^{1/\alpha}\partial_\sigma \overline{Y}_1^{n,\beta,x_0} \widehat{\mathcal{H}}_{3,\beta}^n(1) \xrightarrow[\boldsymbol{L^p}]{n\to\infty} L_1^\alpha \mathcal{H}_{3,L^\alpha}(1),$

iv) $n^{1/\alpha}\partial_\sigma \overline{Y}_1^{n,\beta,x_0} \widehat{\mathcal{H}}_{4,\beta}^n(1) \xrightarrow[\boldsymbol{L^p}]{n\to\infty} L_1^\alpha \mathcal{H}_{4,L^\alpha}(1),$

where $\widehat{\mathcal{H}}_{3,\beta}^n(1), \widehat{\mathcal{H}}_{4,\beta}^n(1)$ *are given by* (3.66), (3.67)*, and* $\mathcal{H}_{3,L^\alpha}(1), \mathcal{H}_{4,L^\alpha}(1)$*, are defined by* (3.71), (3.72).

Remark 3.5. *We observe that although* L_1^α *does not belong to* $\boldsymbol{L^p}$*, the choice of the auxiliary function* ρ *permits to prove that* $L_1^\alpha(\mathcal{H}_{L^\alpha}(1))^2$*,* $L_1^\alpha \mathcal{H}_{3,L^\alpha}(1)$ *and* $L_1^\alpha \mathcal{H}_{4,L^\alpha}(1)$ *belong to* $\boldsymbol{L^p}, \forall p \geq 1$.

Based on the preceding lemmas, we can prove the following convergence result.

Proposition 3.1. *Let* $(\beta_n)_{n\geq 1}$ *be a sequence such that* $\beta_n \xrightarrow{n\to\infty} \beta$ *then for all* $p \geq 2$

$$\frac{\sigma_n^2}{n^{1/\alpha}} \mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0})) \xrightarrow[\boldsymbol{L^p}]{n\to\infty} \mathcal{H}^{(2)}, \tag{3.79}$$

$$\frac{\sigma_n^2}{n^{2/\alpha-1}} \mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta_n,x_0})) \xrightarrow[\boldsymbol{L^p}]{n\to\infty} \partial_\theta b(x_0,\theta)\mathcal{H}_1^{(2)}, \tag{3.80}$$

where $\mathcal{H}^{(2)}$ *and* $\mathcal{H}_1^{(2)}$ *are some random variables whose expressions do not depend on* β *and* b.

Proof. From the equation (3.56), we have

$$\begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}}\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta_n,x_0})) \\ \frac{\sigma_n^2}{n^{1/\alpha}}\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0})) \end{pmatrix} = \begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(1)^2 + \begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}} V_1^{n,\theta_n} \\ \frac{\sigma_n^2}{n^{1/\alpha}} V_1^{n,\sigma_n} \end{pmatrix} \frac{-2\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(1)}{(U_1^{n,\beta_n})}$$
$$+ \begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \frac{\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(1) W_1^{n,\beta_n}}{(U_1^{n,\beta_n})^2} + \begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \frac{(W_1^{n,\beta_n})^2}{(U_1^{n,\beta_n})^4} - \begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}} V_1^{n,\theta_n} \\ \frac{\sigma_n^2}{n^{1/\alpha}} V_1^{n,\sigma_n} \end{pmatrix} \frac{W_1^{n,\beta_n}}{(U_1^{n,\beta_n})^3}$$
$$+ \begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \frac{2D_1^{n,\beta_n}}{(U_1^{n,\beta_n})^2} - \begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \frac{Q_1^{n,\beta_n}}{(U_1^{n,\beta_n})^3} + \begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}} T_1^{n,\theta_n} \\ \frac{\sigma_n^2}{n^{1/\alpha}} T_1^{n,\sigma_n} \end{pmatrix} \frac{1}{(U_1^{n,\beta_n})^2}. \tag{3.81}$$

We will prove the convergence of each term in the right-hand side of (3.81)

Term 1: Recall (3.13) and set $\widehat{\mathcal{H}}_{\beta_n}^n(1) = \widehat{\mathcal{H}}_{1,\beta_n}^n(1) + \widehat{\mathcal{H}}_{2,\beta_n}^n(1)$, $\mathcal{R}_{\beta_n}^n(1) = \mathcal{R}_{2,\beta_n}^n(1) + \mathcal{R}_{3,\beta_n}^n(1)$. Remark that by (3.17), we have $\left|\mathcal{R}_{\beta_n}^n(1)\right| \leq \frac{C}{n}$ where C is some deterministic constant. Moreover, we can rewrite the first term as

$$\begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \left[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(1)\right]^2$$
$$= \begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \left[\frac{n^{1/\alpha}}{\sigma_n}\left[\widehat{\mathcal{H}}_{1,\beta_n}^n(1) + \widehat{\mathcal{H}}_{2,\beta_n}^n(1)\right] + \mathcal{R}_{1,\beta_n}^n(1) + \mathcal{R}_{2,\beta_n}^n(1) + \mathcal{R}_{3,\beta_n}^n(1)\right]^2$$
$$= \begin{pmatrix} n\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ n^{1/\alpha}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \widehat{\mathcal{H}}_{\beta_n}^n(1)^2 + \begin{pmatrix} \frac{2\sigma_n}{n^{1/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ 2\sigma_n\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \widehat{\mathcal{H}}_{\beta_n}^n(1)\mathcal{R}_{\beta_n}^n(1) + \begin{pmatrix} \frac{2\sigma_n}{n^{1/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ 2\sigma_n\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \widehat{\mathcal{H}}_{\beta_n}^n(1)\mathcal{R}_{1,\beta_n}^n(1)$$
$$+ \begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \mathcal{R}_{1,\beta_n}^n(1)^2 + \begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \left(\mathcal{R}_{\beta_n}^n(1)\right)^2 + \begin{pmatrix} \frac{2\sigma_n^2}{n^{2/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{2\sigma_n^2}{n^{1/\alpha}}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \mathcal{R}_{1,\beta_n}^n(1)\mathcal{R}_{\beta_n}^n(1). \tag{3.82}$$

where $\mathcal{R}_{1,\beta_n}^n(1)$ is given by (3.22). We can deduce from (3.17), (3.45), (3.46), Lemma 3.6 and Lemma 3.3 that

$$\begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \left(\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(1)\right)^2 \xrightarrow[\mathbf{L}^p, \forall p\geq 2]{n\to\infty} \begin{pmatrix} \partial_\theta b(x_0,\theta) \\ L_1^\alpha \end{pmatrix} \left(\mathcal{H}_{L^\alpha}(1)\right)^2.$$

Term 2: From (3.13) and Lemma 3.5 part i) and part iii), we can estimate the second term as

$$\begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}} V_1^{n,\theta_n} \\ \frac{\sigma_n^2}{n^{1/\alpha}} V_1^{n,\sigma_n} \end{pmatrix} \frac{-2\mathcal{H}^n_{\overline{Y}_1^{n,\beta_n,x_0}}(1)}{(U_1^{n,\beta_n})} = \begin{pmatrix} O(\frac{1}{n^{2/\alpha}}) \\ \frac{-2\sigma_n^2}{n^{1/\alpha}} \left[\mathcal{R}^n_{9,\beta_n}(1) + \frac{1}{\sigma_n}\right] \end{pmatrix} \left[\frac{1}{\sigma_n} n^{1/\alpha} \widehat{\mathcal{H}}^n_{\beta_n}(1) + \mathcal{R}^n_{1,\beta_n}(1) + \mathcal{R}^n_{\beta_n}(1)\right]$$

$$= \begin{pmatrix} O(\frac{1}{n^{1/\alpha}}) \widehat{\mathcal{H}}^n_{\beta_n}(1) + O(\frac{1}{n^{2/\alpha}}) \mathcal{R}^n_{1,\beta_n}(1) + O(\frac{1}{n^{2/\alpha}}) \mathcal{R}^n_{\beta_n}(1) \\ -2\widehat{\mathcal{H}}^n_{\beta_n}(1) - \frac{2\sigma_n \mathcal{R}^n_{1,\beta_n}(1)}{n^{1/\alpha}} - \frac{2\sigma_n \mathcal{R}^n_{\beta_n}(1)}{n^{1/\alpha}} - 2\sigma_n \mathcal{R}^n_{9,\beta_n}(1) \widehat{\mathcal{H}}^n_{\beta_n}(1) - \frac{2\sigma_n^2 \mathcal{R}^n_{1,\beta_n}(1) \mathcal{R}^n_{9,\beta_n}(1)}{n^{1/\alpha}} - \frac{2\sigma_n^2 \mathcal{R}^n_{\beta_n}(1) \mathcal{R}^n_{9,\beta_n}(1)}{n^{1/\alpha}} \end{pmatrix}$$

where C is some deterministic constant and $O(\frac{1}{n^{2/\alpha}})$ is a random variable bounded by $\frac{C}{n^{2/\alpha}}$. From (3.17), (3.45), (3.46) and Lemma 3.5, we also conclude that

$$\begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}} V_1^{n,\theta_n} \\ \frac{\sigma_n^2}{n^{1/\alpha}} V_1^{n,\sigma_n} \end{pmatrix} \frac{-2\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(1)}{(U_1^{n,\beta_n})} \xrightarrow[\mathbf{L}^p, \forall p \geq 2]{n\to\infty} \begin{pmatrix} 0 \\ -2\mathcal{H}_{L^\alpha}(1) \end{pmatrix}$$

Term 3: From (3.13) and $\frac{W_1^{n,\beta_n}}{(U_1^{n,\beta_n})^2} = \frac{n^{1/\alpha}}{\sigma_n} \widehat{\mathcal{H}}^n_{1,\beta_n}(1) + \mathcal{R}^n_{2,\beta_n}(1)$ where $\widehat{\mathcal{H}}^n_{1,\beta_n}(1)$ and $\mathcal{R}^n_{2,\beta_n}(1)$ are given by (3.14) and (3.23), we have

$$\begin{aligned}
&\begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \frac{\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(1) W_1^{n,\beta_n}}{(U_1^{n,\beta_n})^2} \\
&= \begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \left[\frac{n^{1/\alpha} \widehat{\mathcal{H}}^n_{\beta_n}(1)}{\sigma_n} + \mathcal{R}^n_{1,\beta_n}(1) + \mathcal{R}^n_{\beta_n}(1)\right] \left[\frac{n^{1/\alpha}}{\sigma_n} \widehat{\mathcal{H}}^n_{1,\beta_n}(1) + \mathcal{R}^n_{2,\beta_n}(1)\right] \\
&= \begin{pmatrix} n \partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ n^{1/\alpha} \partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \widehat{\mathcal{H}}^n_{\beta_n}(1) \widehat{\mathcal{H}}^n_{1,\beta_n}(1) + \begin{pmatrix} \frac{\sigma_n}{n^{1/\alpha-1}} \partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \sigma_n \partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \widehat{\mathcal{H}}^n_{\beta_n}(1) \mathcal{R}^n_{2,\beta_n}(1) \\
&+ \begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \mathcal{R}^n_{1,\beta_n}(1) \mathcal{R}^n_{2,\beta_n}(1) + \begin{pmatrix} \frac{\sigma_n}{n^{1/\alpha-1}} \partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \sigma_n \partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \mathcal{R}^n_{1,\beta_n}(1) \widehat{\mathcal{H}}^n_{1,\beta_n}(1) \\
&+ \begin{pmatrix} \frac{\sigma_n}{n^{1/\alpha-1}} \partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \sigma_n \partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \mathcal{R}^n_{\beta_n}(1) \widehat{\mathcal{H}}^n_{1,\beta_n}(1) + \begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \mathcal{R}^n_{\beta_n}(1) \mathcal{R}^n_{2,\beta_n}(1).
\end{aligned}$$

From (3.17), (3.45), (3.46), Lemma 3.3 and Lemma 3.6, we also conclude that

$$\begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}} \partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}} \partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \frac{\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(1) W_1^{n,\beta_n}}{(U_1^{n,\beta_n})^2} \xrightarrow[\mathbf{L}^p, \forall p \geq 2]{n\to\infty} \begin{pmatrix} \partial_\theta b(x_0,\theta) \mathcal{H}_{1,L_\alpha}(1) \mathcal{H}_{L_\alpha}(1) \\ L_1^\alpha \mathcal{H}_{1,L_\alpha}(1) \mathcal{H}_{L_\alpha}(1). \end{pmatrix}$$

Term 4: Using $\frac{W_1^{n,\beta_n}}{(U_1^{n,\beta_n})^2} = \frac{n^{1/\alpha}}{\sigma_n}\widehat{\mathcal{H}}^n_{1,\beta_n}(1) + \mathcal{R}^n_{2,\beta_n}(1)$ again, we can rewrite

$$\begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \frac{(W_1^{n,\beta_n})^2}{(U_1^{n,\beta_n})^4} = \begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \left[\frac{n^{1/\alpha}}{\sigma_n}\widehat{\mathcal{H}}^n_{1,\beta_n}(1) + \mathcal{R}^n_{2,\beta_n}(1)\right]^2$$
$$= \begin{pmatrix} n\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ n^{1/\alpha}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} (\widehat{\mathcal{H}}^n_{1,\beta_n}(1))^2 + \begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} (\mathcal{R}^n_{2,\beta_n}(1))^2 + \begin{pmatrix} \frac{2\sigma_n}{n^{1/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ 2\sigma_n\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \mathcal{R}^n_{2,\beta_n}(1)\widehat{\mathcal{H}}^n_{1,\beta_n}(1).$$

From (3.17), (3.45), (3.46), Lemma 3.6 and Lemma 3.3, we also conclude that

$$\begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \frac{(W_1^{n,\beta_n})^2}{(U_1^{n,\beta_n})^4} \xrightarrow[\mathbf{L}^p,\forall p\geq 2]{n\to\infty} \begin{pmatrix} \partial_\theta b(x_0,\theta)\,(\mathcal{H}_{1,L^\alpha}(1))^2 \\ L_1^\alpha\,(\mathcal{H}_{1,L^\alpha}(1))^2 \end{pmatrix}$$

Term 5: From Lemma 3.5 we can estimate the fifth term as

$$\begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}}V_1^{n,\theta_n} \\ \frac{\sigma_n^2}{n^{1/\alpha}}V_1^{n,\sigma_n} \end{pmatrix} \frac{W_1^{n,\beta_n}}{(U_1^{n,\beta_n})^3} = \begin{pmatrix} O(\frac{1}{n^{2/\alpha}}) \\ \frac{\sigma_n^2}{n^{1/\alpha}}\left(\mathcal{R}^n_{9,\beta_n}(1) + \frac{1}{\sigma_n}\right) \end{pmatrix} \left(\frac{n^{1/\alpha}}{\sigma_n}\widehat{\mathcal{H}}^n_{1,\beta_n}(1) + \mathcal{R}^n_{2,\beta_n}(1)\right)$$
$$= \begin{pmatrix} O(\frac{1}{n^{1/\alpha}})\widehat{\mathcal{H}}^n_{1,\beta_n}(1) + O(\frac{1}{n^{2/\alpha}})\mathcal{R}^n_{2,\beta_n}(1) \\ \sigma_n\mathcal{R}^n_{9,\beta_n}(1)\widehat{\mathcal{H}}^n_{1,\beta_n}(1) + \frac{\sigma_n^2}{n^{1/\alpha}}\mathcal{R}^n_{9,\beta_n}(1)\mathcal{R}^n_{2,\beta_n}(1) + \widehat{\mathcal{H}}^n_{1,\beta_n}(1) + \frac{\sigma_n}{n^{1/\alpha}}\mathcal{R}^n_{2,\beta_n}(1) \end{pmatrix}$$

where C is some deterministic constant. From (3.17), (3.45), Lemma 3.5, we also conclude that

$$\begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}}V_1^{n,\theta_n} \\ \frac{\sigma_n^2}{n^{1/\alpha}}V_1^{n,\sigma_n} \end{pmatrix} \frac{W_1^{n,\beta_n}}{(U_1^{n,\beta_n})^3} \xrightarrow[\mathbf{L}^p,\forall p\geq 2]{n\to\infty} \begin{pmatrix} 0 \\ \mathcal{H}_{1,L^\alpha}(1) \end{pmatrix}$$

Term 6: Using Lemma 3.4 we write the sixth term as

$$\begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \frac{D_1^{n,\beta_n}}{(U_1^{n,\beta_n})^2}$$
$$= \begin{pmatrix} \frac{n}{2}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{n^{1/\alpha}}{2}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \widehat{\mathcal{H}}^n_{3,\beta_n}(1) + \begin{pmatrix} \frac{\sigma_n^2}{n^{1/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \sigma_n^2\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \left(\mathcal{R}^n_{4,\beta_n}(1) + \mathcal{R}^n_{5,\beta_n}(1) + \mathcal{R}^n_{6,\beta_n}(1)\right).$$

Applying Lemma 3.4, Lemma 3.7 and Lemma 3.3 we obtain that

$$\begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \frac{2D_1^{n,\beta_n}}{(U_1^{n,\beta_n})^2} \xrightarrow[\mathbf{L}^p,\forall p\geq 2]{n\to\infty} \begin{pmatrix} \partial_\theta b(x_0,\theta)\mathcal{H}_{3,L^\alpha}(1) \\ L_1^\alpha\mathcal{H}_{3,L^\alpha}(1) \end{pmatrix}$$

where $\mathcal{H}_{3,L^\alpha}(1)$ is defined in Lemma 3.4.

Term 7: From Lemma 3.4, we can rewrite the seventh term as

$$\begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \frac{Q_1^{n,\beta_n}}{(U_1^{n,\beta_n})^3}$$

$$= \begin{pmatrix} n\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ n^{1/\alpha}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \widehat{\mathcal{H}}_{4,\beta_n}^n(1) + \begin{pmatrix} \frac{\sigma_n^2}{n^{1/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \sigma_n^2\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \left(\mathcal{R}_{7,\beta_n}^n(1) + \mathcal{R}_{8,\beta_n}^n(1)\right).$$

Applying Lemma 3.4, Lemma 3.7 and Lemma 3.3 we obtain that

$$\begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \frac{Q_1^{n,\beta_n}}{(U_1^{n,\beta_n})^3} \xrightarrow[\mathbf{L}^p, \forall p\geq 2]{n\to\infty} \begin{pmatrix} \partial_\theta b(x_0,\theta)\mathcal{H}_{4,L^\alpha}(1) \\ L_1^\alpha \mathcal{H}_{4,L^\alpha}(1) \end{pmatrix}$$

where $\mathcal{H}_{4,L^\alpha}(1)$ is defined in Lemma 3.4.

Term 8: From Lemma 3.5, we have

$$\begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}} T_1^{n,\theta_n} \\ \frac{\sigma_n^2}{n^{1/\alpha}} T_1^{n,\sigma_n} \end{pmatrix} \frac{1}{(U_1^{n,\beta_n})^2} = \begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}} \frac{T_1^{n,\theta_n}}{(U_1^{n,\beta_n})^2} \\ \widehat{\mathcal{H}}_{5,\beta_n}^n(1) + \sigma_n^2\mathcal{R}_{10,\beta_n}^n(1) + \sigma_n^2\mathcal{R}_{11,\beta_n}^n(1) + \sigma_n^2\mathcal{R}_{12,\beta_n}^n(1) \end{pmatrix}$$

Using the results of Lemma 3.5, we easily deduce that

$$\begin{pmatrix} \frac{\sigma_n^2}{n^{2/\alpha-1}} T_1^{n,\theta_n} \\ \frac{\sigma_n^2}{n^{1/\alpha}} T_1^{n,\sigma_n} \end{pmatrix} \frac{1}{(U_1^{n,\beta_n})^2} \xrightarrow[\mathbf{L}^p, \forall p\geq 2]{n\to\infty} \begin{pmatrix} 0 \\ \mathcal{H}_{5,L^\alpha}(1) \end{pmatrix}$$

where $\mathcal{H}_{5,L^\alpha}(1)$ is defined in Lemma 3.5.

Finally from the above convergences, we can deduce the result of Proposition 3.1. $\square$

3.4.2 Proof of Theorem 3.3

We will first prove part *ii*) and then give a proof for part *i*).

ii) Remark that from (3.79), (3.80)

$$\sup_{u\in\mathbb{R}} \sup_n \mathbb{E}\left| 1_{\{\overline{Y}_1^{n,\beta_n,x_0} \geq \frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\beta_n,x_0}\}} \frac{\sigma_n^2}{n^{2/\alpha-1}} \mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta_n,x_0}))\right| < \infty,$$

and,

$$\sup_{u\in\mathbb{R}} \sup_n \mathbb{E}\left| 1_{\{\overline{Y}_1^{n,\beta_n,x_0} \geq \frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\beta_n,x_0}\}} \frac{\sigma_n^2}{n^{1/\alpha}} \mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0}))\right| < \infty,$$

by representation (3.55) this leads to

$$\sup_{u\in\mathbb{R}}\sup_n\left|\frac{\sigma_n^2}{n^{\frac{2}{\alpha}-1}}\partial_\theta p_{\frac{1}{n}}^{\beta_n}(x_0,\frac{u\sigma_n}{n^{1/\alpha}}+\varsigma_1^{n,\beta_n,x_0})\right|<\infty,$$

and

$$\sup_{u\in\mathbb{R}}\sup_n\left|\frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma p_{\frac{1}{n}}^{\beta_n}(x_0,\frac{u\sigma_n}{n^{1/\alpha}}+\varsigma_1^{n,\beta_n,x_0})\right|<\infty.$$

i) From (3.55), (3.47) and Proposition 3.1, we easily deduce that

$$\frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma p_{\frac{1}{n}}^{\beta_n}(x_0,\frac{u\sigma_n}{n^{1/\alpha}}+\varsigma_1^{n,\beta_n,x_0})=\frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma q^{n,\beta_n,x_0}(\frac{u\sigma_n}{n^{1/\alpha}}+\varsigma_1^{n,\beta_n,x_0})\xrightarrow{n\to\infty}\mathbb{E}[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}^{(2)}],\quad(3.83)$$

$$\frac{\sigma_n^2}{n^{\frac{2}{\alpha}-1}}\partial_\theta p_{\frac{1}{n}}^{\beta_n}(x_0,\frac{u\sigma_n}{n^{1/\alpha}}+\varsigma_1^{n,\beta_n,x_0})=\frac{\sigma_n^2}{n^{\frac{2}{\alpha}-1}}\partial_\theta q^{n,\beta_n,x_0}(\frac{u\sigma_n}{n^{1/\alpha}}+\varsigma_1^{n,\beta_n,x_0})\xrightarrow{n\to\infty}\partial_\theta b(x_0,\theta)\times\mathbb{E}[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}_1^{(2)}],\tag{3.84}$$

where $\mathcal{H}^{(2)}$ and $\mathcal{H}_1^{(2)}$ are defined in Proposition 3.1.
To finish the proof of Theorem 3.3, it remains to show that $\mathbb{E}[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}_1^{(2)}]=-\varphi_\alpha'(u)$ and $\mathbb{E}[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}^{(2)}]=-\left[\varphi_\alpha(u)+u{\varphi_\alpha}'(u)\right]$. This is done in Lemma 3.8 below.

Lemma 3.8. *We have*

$$\varphi_\alpha'(u)=-\mathbb{E}[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}_1^{(2)}],$$
$$-\left[\varphi_\alpha(u)+u{\varphi_\alpha}'(u)\right]=\mathbb{E}[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}^{(2)}],$$

where φ_α is the density of L_1^α and $\mathcal{H}^{(2)}$ and $\mathcal{H}_1^{(2)}$ are defined in Proposition 3.1.

Proof. Let us consider the situation where $b(x,\theta)=\theta$ and $x_0=0$. In that case, we have $\overline{Y_1}^{n,\beta,x_0}=\frac{\theta}{n}+\frac{\sigma}{n^{1/\alpha}}L_1^n$ and thus the density of $\overline{Y_1}^{n,\beta,x_0}$ is related to the density of L_1^n by the relation,

$$q^{n,\beta,x_0}(u)=\frac{n^{1/\alpha}}{\sigma}\varphi^n\left(\frac{n^{1/\alpha}}{\sigma}\left(u-\frac{\theta}{n}\right)\right).$$

Then,

$$\partial_\theta q^{n,\beta,x_0}(u)=-\frac{n^{2/\alpha-1}}{\sigma^2}(\varphi^n)'\left(\frac{n^{1/\alpha}}{\sigma}\left(u-\frac{\theta}{n}\right)\right),$$
$$\partial_\sigma q^{n,\beta,x_0}(u)=-\frac{n^{1/\alpha}}{\sigma^2}\varphi^n\left(\frac{n^{1/\alpha}}{\sigma}\left(u-\frac{\theta}{n}\right)\right)-\frac{(n^{1/\alpha})^2}{\sigma^3}\left(u-\frac{\theta}{n}\right)(\varphi^n)'\left(\frac{n^{1/\alpha}}{\sigma}\left(u-\frac{\theta}{n}\right)\right),$$

By a change of variables, we get

$$\partial_\theta q^{n,\beta,x_0}\left(\frac{u\sigma}{n^{1/\alpha}}+\frac{\theta}{n}\right)=-\frac{n^{2/\alpha-1}}{\sigma^2}\left(\varphi^n\right)'(u)$$
$$\partial_\sigma q^{n,\beta,x_0}(\frac{u\sigma}{n^{1/\alpha}}+\frac{\theta}{n})=-\frac{n^{1/\alpha}}{\sigma^2}\left[\varphi^n(u)+u(\varphi^n)'(u)\right].$$

Hence, we can apply the results of part *ii)* of Theorem 3.3 and (3.83), (3.84) in this specific setting. This yields

$$\forall u, \quad (\varphi^n)'(u) \xrightarrow{n\to\infty} -\mathbb{E}[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}_1^{(2)}], \tag{3.85}$$

$$\forall u, \quad \left[\varphi^n(u) + u(\varphi^n)'(u)\right] \xrightarrow{n\to\infty} -\mathbb{E}[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}^{(2)}], \tag{3.86}$$

$$\sup_{u,n} |(\varphi^n)'(u)| < \infty, \tag{3.87}$$

$$\sup_{u,n} \left|\varphi^n(u) + u(\varphi^n)'(u)\right| < \infty. \tag{3.88}$$

Let us denote $\mathcal{X}(u) = -\mathbb{E}[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}_1^{(2)}]$ and assume by contradiction that $\mathcal{X} \neq \varphi_\alpha'$. Using the continuity of $u \mapsto \mathcal{X}(u)$, there exists a smooth, compactly supported function f, such that $\int \mathcal{X}(u)f(u)du \neq \int \varphi_\alpha'(u)f(u)du$. Now, on the one hand we have

$$\int (\varphi^n)'(u)f(u)du \xrightarrow{n\to\infty} \int \mathcal{X}(u)f(u)du, \tag{3.89}$$

where we have used the dominated convergence theorem, together with (3.85), (3.87).

On the other hand, we can write,

$$\begin{aligned} \int (\varphi^n)'(u)f(u)du &= -\int \varphi^n(u)f'(u)du \\ &= -\mathbb{E}[f'(L_1^n)] \xrightarrow{n\to\infty} -\mathbb{E}[f'(L_1^\alpha)] && (3.90)\\ &= -\int \varphi_\alpha(u)f'(u)du = \int \varphi_\alpha'(u)f(u)du && (3.91) \end{aligned}$$

where the convergence (3.90) is obtained in the same way as (3.54). Clearly (3.91) contradicts (3.89), and we get $\mathbb{E}[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}_1^{(2)}] = -\varphi_\alpha'(u)$.

By the same method, let us denote $\mathcal{X}_1(u) = -\mathbb{E}[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}^{(2)}]$ and assume by contradiction that $u \mapsto \mathcal{X}_1(u)$ is different from $u \mapsto \left[\varphi_\alpha(u) + u(\varphi_\alpha)'(u)\right]$. Using the continuity of $u \mapsto \mathcal{X}_1(u)$, there exists a smooth, compactly supported function f, such that $\int \mathcal{X}_1(u)f(u)du \neq \int \left[\varphi_\alpha(u) + u(\varphi_\alpha)'(u)\right] f(u)du$. Now, we have

$$\int \left[\varphi^n(u) + u(\varphi^n)'(u)\right] f(u)du \xrightarrow{n\to\infty} \int \mathcal{X}_1(u)f(u)du, \tag{3.92}$$

where we have used the dominated convergence theorem, together with (3.86), (3.88).

On the other hand, letting $g(u) = uf(u)$ and using the integration by parts formula, we can write,

$$\int \left[\varphi^n(u) + u(\varphi^n)'(u)\right] f(u)du = \int \varphi^n(u)f(u)du + \int (\varphi^n)'(u)g(u)du$$

$$= \mathbb{E}[f(L_1^n)] - \int \varphi^n(u)g'(u)du = \mathbb{E}[f(L_1^n)] - \mathbb{E}[g'(L_1^n)] \xrightarrow{n\to\infty} \mathbb{E}[f(L_1^\alpha)] - \mathbb{E}[g'(L_1^\alpha)] \tag{3.93}$$

$$= \int \varphi_\alpha(u) f(u) du - \int \varphi_\alpha(u) g'(u) du = \int \varphi_\alpha(u) f(u) du + \int \varphi'_\alpha(u) g(u) du \tag{3.94}$$

where the convergence (3.93) is obtained in the same way as (3.54). Clearly (3.94) contradicts (3.92), and the lemma is proved. □

4 Appendix

In this appendix, we give the proofs of Lemmas 3.3 - 3.7 of Section 3.4.1.

Proof of Lemma 3.3: *i*): Using the fact that b has bounded derivatives, the boundedness of $(\epsilon_t^n)_{t\in[0,1]}$, $((\epsilon_t^n))^{-1})_{t\in[0,1]}$ and from (3.58) we obtain that $|\partial_\theta \overline{Y}_1^{n,\beta,x_0}| \leq \frac{C}{n}$.
ii): From (3.59) and the boundedness of $((\epsilon_t^n)^r)_{t\in[0,1]} \forall r \in \mathbb{Z}$, we have

$$\begin{aligned} \sup_{s\in[0,1]} \left|\partial_\sigma \overline{Y}_s^{n,\beta,x_0}\right| &= \frac{1}{n^{1/\alpha}} \sup_{s\in[0,1]} \left|\epsilon_s^n \int_0^s (\epsilon_u^n)^{-1} dL_u^n\right| \\ &\leq \frac{C}{n^{1/\alpha}} \sup_{s\in[0,1]} \left|\int_0^s \int_{|z|\leq 1} (\epsilon_u^n)^{-1} z\tilde{\mu}(du,dz)\right| + \frac{C}{n^{1/\alpha}} \sup_{s\in[0,1]} \left|\int_0^s \int_{|z|>1} (\epsilon_u^n)^{-1} z\mu^{(n)}(du,dz)\right| \end{aligned} \tag{4.1}$$

We now consider the first term of (4.1).
Using Doob's martingale inequality, we have

$$\frac{1}{n^{2p/\alpha}} \mathbb{E}\left[\sup_{s\in[0,1]} \left(\left|\int_0^s \int_{|z|\leq 1} (\epsilon_u^n)^{-1} z\tilde{\mu}(du,dz)\right|\right)^{2p}\right] \leq \frac{D(p)}{n^{2p/\alpha}} \mathbb{E}\left(\left|\int_0^1 \int_{|z|\leq 1} (\epsilon_u^n)^{-1} z\tilde{\mu}(du,dz)\right|^{2p}\right)$$

where $D(p) = \left(\frac{2p}{2p-1}\right)^{2p}$. And then using Kunita's first inequality (see Theorem 4.4.23 in [1]), there exists a constant $M(2p) > 0$ such that

$$\begin{aligned} &\frac{D(p)}{n^{2p/\alpha}} \mathbb{E}\left(\left|\int_0^1 \int_{|z|\leq 1} (\epsilon_u^n)^{-1} z\tilde{\mu}(du,dz)\right|^{2p}\right) \\ &\leq \frac{D(p)M(2p)}{n^{2p/\alpha}} \left[\int_0^1 \int_{|z|\leq 1} (\epsilon_u^n)^{-2} z^2 \frac{1}{|z|^{1+\alpha}} \tau(z/n^{1/\alpha}) dz du\right]^p \\ &+ \frac{D(p)M(2p)}{n^{2p/\alpha}} \left[\int_0^1 \int_{|z|\leq 1} (\epsilon_u^n)^{-2p} z^{2p} \frac{1}{|z|^{1+\alpha}} \tau(z/n^{1/\alpha}) dz du\right] \\ &\leq \frac{C_1 D(p) M(2p)}{n^{2p/\alpha}} \left[\left(\int_0^1 \int_{|z|\leq 1} \frac{1}{|z|^{\alpha-1}} dz du\right)^p + \int_0^1 \int_{|z|\leq 1} \frac{1}{|z|^{1+\alpha-2p}} dz du\right] \xrightarrow{n\to\infty} 0. \end{aligned}$$

where C_1 is some deterministic constant. Thus, we can deduce that the first term of (4.1) converges to zero in $\mathbf{L}^{2p}, \forall p \geq 1$.

We now consider the second term of (4.1).

From the fact that $\mu^{(n)}(du, dz)$ is a positive measure, and the boundedness of $(\epsilon_t^n)_{t\in[0,1]}$, we get that for C^* a positive constant

$$\frac{1}{n^{1/\alpha}} \sup_{s\in[0,1]} \left| \epsilon_s^n \int_0^s \int_{|z|>1} (\epsilon_u^n)^{-1} z \mu^{(n)}(du, dz) \right| \leq \frac{C^*}{n^{1/\alpha}} \int_0^1 \int_{|z|>1} |z| \mu^{(n)}(du, dz). \tag{4.2}$$

Then,

$$\frac{1}{n^{2p/\alpha}} \mathbb{E} \left(\sup_{s\in[0,1]} \left| \epsilon_s^n \int_0^s \int_{|z|>1} (\epsilon_u^n)^{-1} z \mu^{(n)}(du, dz) \right| \right)^{2p} \leq \frac{C^*}{n^{2p/\alpha}} \mathbb{E} \left(\int_0^1 \int_{|z|>1} |z| \mu^{(n)}(du, dz) \right)^{2p}.$$

Moreover, from $\mu^{(n)}(ds, dz) = \tilde{\mu}^{(n)}(ds, dz) + \upsilon^{(n)}(ds, dz)$ then for $C_2(p)$ a positive constant, we have

$$\begin{aligned} &\frac{1}{n^{2p/\alpha}} \mathbb{E} \left(\int_0^1 \int_{|z|>1} |z| \mu^{(n)}(du, dz) \right)^{2p} \\ &\leq \frac{C_2(p)}{n^{2p/\alpha}} \left[\mathbb{E} \left(\int_0^1 \int_{|z|>1} |z| \tilde{\mu}^{(n)}(du, dz) \right)^{2p} + \mathbb{E} \left(\int_0^1 \int_{|z|>1} |z| \upsilon^{(n)}(du, dz) \right)^{2p} \right]. \end{aligned}$$

Using Kunita's first inequality (see Theorem 4.4.23 in [1]), there exists a positive constant $C_3(2p)$ such that

$$\begin{aligned} &\frac{1}{n^{2p/\alpha}} \mathbb{E} \left[\left| \int_0^1 \int_{|z|>1} |z| \tilde{\mu}^{(n)}(ds, dz) \right| \right]^{2p} \\ &\leq \frac{C_3(2p)}{n^{2p/\alpha}} \left[\left(\int_0^1 \int_{|z|>1} z^2 \upsilon^{(n)}(ds, dz) \right)^p + \left(\int_0^1 \int_{|z|>1} z^{2p} \upsilon^{(n)}(ds, dz) \right) \right] \\ &= \frac{C_3(2p)}{n^{2p/\alpha}} \left(\int_0^1 \int_{|z|>1} z^2 \frac{1}{|z|^{1+\alpha}} \tau(z/n^{1/\alpha}) dz ds \right)^p + \frac{C_3(2p)}{n^{2p/\alpha}} \left[\int_0^1 \int_{|z|>1} z^{2p} \frac{1}{|z|^{1+\alpha}} \tau(z/n^{1/\alpha}) dz ds \right] \\ &\leq \frac{2C_3(2p)}{n^{2p/\alpha}} \left[\int_0^1 \int_1^{2n^{1/\alpha}} \frac{1}{z^{\alpha-1}} dz ds \right]^p + \left[\frac{2C_3(2p)}{n^{2p/\alpha}} \int_0^1 \int_1^{2n^{1/\alpha}} \frac{1}{z^{\alpha+1-2p}} dz ds \right] \xrightarrow{n\to\infty} 0. \end{aligned} \tag{4.3}$$

where we used the fact that τ is a non negative function equal to 1 on [-1,1], vanishing on $[-2, 2]^c$ and satisfying $0 \leq \tau \leq 1$ and M is some deterministic constant. Hence, we get that the second term of (4.1) also converges to zero in $\mathbf{L}^{2p}, \forall p \geq 1$. And this finishes the proof of the part ii). □

Proof of Lemma 3.4: Recall that $D_1^{n,\beta}$ and $U_1^{n,\beta}$ are given by (3.62) and (3.19). The part i) of this lemma is proved by decomposing $\frac{D_1^{n,\beta}}{(U_1^{n,\beta})^2}$, then we obtain that the main term is (3.66) and the remainder terms are

$$\mathcal{R}^n_{4,\beta}(1) = \frac{\int_0^1 \int_{|z|>2} (\epsilon^n_s)^{-2} \rho^n(z) \rho^{n\prime}(z) \frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \mu^{(n)}(ds, dz)}{2\sigma^2 n^{1/\alpha} (\epsilon^n_1)^2 \left(\int_0^1 \int_{\mathbb{R}} (\epsilon^n_s)^{-2} \rho^n(z) \mu^{(n)}(ds, dz) \right)^2} + \\ + \frac{\int_0^1 \int_{|z|>2} (\epsilon^n_s)^{-2} (\rho^n(z))^2 \left[\frac{\tau''(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} - \left(\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})} \right)^2 \right] \mu^{(n)}(ds, dz)}{2\sigma^2 n^{1/\alpha} (\epsilon^n_1)^2 \left(\int_0^1 \int_{\mathbb{R}} (\epsilon^n_s)^{-2} \rho^n(z) \mu^{(n)}(ds, dz) \right)^2},$$

$$\mathcal{R}^n_{5,\beta}(1) = \frac{(\epsilon^n_1)^2 \int_0^1 (\epsilon^n_s)^{-2} b''(\overline{Y}^{n,\beta,x_0}_s, \theta) \left(2L\overline{Y}^{n,\beta,x_0}_s U^{n,\beta}_s + W^{n,\beta}_s \right) ds}{2n^{1+1/\alpha} \left(U^{n,\beta}_1 \right)^2},$$

$$\mathcal{R}^n_{6,\beta}(1) = \frac{(\epsilon^n_1)^2 \int_0^1 (\epsilon^n_s)^{-2} b'''(\overline{Y}^{n,\beta,x_0}_s, \theta) (U^{n,\beta}_s)^2 ds}{2n^{1+1/\alpha} \left(U^{n,\beta}_1 \right)^2}.$$

The part $ii)$ of this lemma is proved by decomposing $\frac{Q^{n,\beta}_1}{(U^{n,\beta}_1)^3}$, then we obtain that the main term is (3.67) and the remainder terms are

$$\mathcal{R}^n_{7,\beta}(1) = \frac{7(\epsilon^n_1)^4 \int_0^1 (\epsilon^n_1)^{-4} b''(\overline{Y}^{n,\beta,x_0}_s, \theta) W^{n,\beta}_s U^{n,\beta}_s ds}{n^{1+1/\alpha} \left(U^{n,\beta}_1 \right)^3}$$

$$\mathcal{R}^n_{8,\beta}(1) = \frac{2(\epsilon^n_1)^4 \int_0^1 (\epsilon^n_1)^{-4} b'''((\overline{Y}^{n,\beta,x_0}_s, \theta) (U^{n,\beta}_s)^3 ds}{n^{1+1/\alpha} \left(U^{n,\beta}_1 \right)^3}.$$

We now study the convergence of the main terms.

From (3.16), the boundedness of b', and Lemma 3.1, it is clear that $\widehat{\mathcal{H}}^n_{3,\beta_n}(1)$ converges almost surely to $\mathcal{H}_{3,L^\alpha}(1)$. Moreover, using again the boundedness of b', the upper and lower bounds of $(\epsilon^n_s)_{s\in[0,1]}$ and the fact that $\rho^n(z)$ is a non negative function, we deduce the upper bound, for some constant $C > 0$,

$$\left| \widehat{\mathcal{H}}^n_{3,\beta}(1) \right| \leq C \left[\frac{\int_0^1 \int_{\mathbb{R}} \left(\rho^n(z) \left| (\rho^n)''(z) \right| + \rho^n(z) \left| \rho^{n\prime}(z) \right| \frac{(1+\alpha)}{|z|} + (\rho^n(z))^2 \frac{(1+\alpha)}{z^2} \right) \mu^{(n)}(ds, dz)}{\left(\int_0^1 \int_{\mathbb{R}} \rho^n(z) \mu^{(n)}(ds, dz) \right)^2} \right]. \tag{4.4}$$

Now we show that $\sup_n \left| \widehat{\mathcal{H}}^n_{3,\beta_n}(s) \right|^p$ is integrable. This is done by the same method as in **Step 1.1** in the proof of Theorem 3.2. And then applying the dominated convergence theorem, we get $\widehat{\mathcal{H}}^n_{3,\beta}(1) \xrightarrow[\mathbf{L}^p, \forall p\geq 1]{n\to\infty} \mathcal{H}_{3,L^\alpha}(1)$. In the same way we prove that $\widehat{\mathcal{H}}^n_{4,\beta}(1) \xrightarrow[\mathbf{L}^p, \forall p\geq 1]{n\to\infty} \mathcal{H}_{4,L^\alpha}(1)$.

Moreover, using that b has bounded derivatives and $\frac{U^{n,\beta}_s}{U^{n,\beta}_1}$ is bounded for $0 \leq s \leq 1$, the remainder terms satisfy the upper bounds

$$|\mathcal{R}^n_{5,\beta}(1)| \le C\left[\frac{\sup_{s\in[0,1]}|L\overline{Y}^{n,\beta,x_0}_s|}{n^{1+1/\alpha}U^{n,\beta}_1} + \frac{\sup_{s\in[0,1]}|W^{n,\beta}_s|}{n^{1+1/\alpha}(U^{n,\beta}_1)^2}\right], \quad |\mathcal{R}^n_{6,\beta}(1)| \le \frac{C}{2n^{1+1/\alpha}}, \tag{4.5}$$

$$|\mathcal{R}^n_{7,\beta}(1)| \le C\left[\frac{\sup_{s\in[0,1]}|W^{n,\beta}_s|}{n^{1+1/\alpha}(U^{n,\beta}_1)^2}\right], \quad |\mathcal{R}^n_{8,\beta}(1)| \le \frac{C}{n^{1+1/\alpha}}. \tag{4.6}$$

Now from (3.19), (3.20) and (3.21), using that b has bounded derivatives and $\sup_{0\le s\le 1}\frac{U^{n,\beta}_s}{U^{n,\beta}_1}$ is bounded again , we get that

$$\frac{\sup_{s\in[0,1]}|L\overline{Y}^{n,\beta,x_0}_s|}{n^{1+1/\alpha}U^{n,\beta}_1} \le C\left[\frac{1}{n^{2+1/\alpha}} + \frac{\int_0^1\int_{\mathbb{R}}\left[|\rho^{n\prime}(z)| + \frac{1+\alpha}{|z|}\rho^n(z)\right]\mu^{(n)}(ds,dz)}{n\sigma\int_0^1\int_{\mathbb{R}}\rho^n(z)\mu^{(n)}(ds,dz)}\right]$$
$$+ C\left[\frac{\int_0^1\int_{|z|>2} z^2\left|\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right|\mu^{(n)}(ds,dz)}{\sigma n^{1+1/\alpha}\int_0^1\int_{|z|>2} z^2\mu^{(n)}(ds,dz)}\right]$$

$$\frac{\sup_{s\in[0,1]}|W^{n,\beta}_s|}{n^{1+1/\alpha}(U^{n,\beta}_1)^2} \le C\left[\frac{1}{n^{2+1/\alpha}} + \frac{\int_0^1\int_{\mathbb{R}}\rho^n(z)|(\rho^n)'(z)|\mu^{(n)}(ds,dz)}{n\sigma\left[\int_0^1\int_{\mathbb{R}}\rho^n(z)\mu^{(n)}(ds,dz)\right]^2}\right]. \tag{4.7}$$

From the results of **Step 1.1**, **Step 1.2** in the proof of Theorem 3.2, and the control given in the proof of Theorem 3.1 for (3.27) we can easily deduce that $\frac{\sup_{s\in[0,1]}|L\overline{Y}^{n,\beta,x_0}_s|}{n^{1+1/\alpha}U^{n,\beta}_1}$ and $\frac{\sup_{s\in[0,1]}|W^{n,\beta}_s|}{n^{1+1/\alpha}(U^{n,\beta}_1)^2}$ converge to zero in $\mathbf{L}^p, \forall p\ge 1$. Clearly, $\mathcal{R}^n_{5,\beta}(1)$ and $\mathcal{R}^n_{7,\beta}(1)$ also converge to zero in $\mathbf{L}^p, \forall p \ge 1$.

We now consider the convergence to zero of $\mathcal{R}^n_{4,\beta}(1)$.

From the boundedness of $(\epsilon^n_t)_{t\in[0,1]}$, the definition of ρ^n [see (3.10)], and from the fact that $\mu^{(n)}(ds,dz)$ is a positive measure, we have

$$\left|\mathcal{R}^n_{4,\beta}(1)\right| \le \frac{C}{2\sigma^2}\left[\frac{\int_0^1\int_{|z|>2}\left[\frac{|z|^3}{n^{1/\alpha}}\left|\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right| + \frac{z^4}{n^{1/\alpha}}\left(\left|\frac{\tau''(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right| + \left(\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right)^2\right)\right]\mu^{(n)}(ds,dz)}{\left(\int_0^1\int_{|z|>2} z^2\mu^{(n)}(ds,dz)\right)^2}\right] \tag{4.8}$$

$$\le \frac{C}{2\sigma^2}\left[\frac{\int_0^1\int_{|z|>2}\left[\frac{|z|^3}{n^{1/\alpha}}\left|\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right| + \frac{z^4}{n^{1/\alpha}}\left(\left|\frac{\tau''(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right| + \left(\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right)^2\right)\right]\mu^{(n)}(ds,dz)}{n^{4/\alpha}}\right] \tag{4.9}$$

where we used the fact that $\int_0^1\int_{|z|>2} z^2\mu^{(n)}(ds,dz) > n^{2/\alpha}$, if there exists a jump of the Lévy process in $[-2n^{1/\alpha}, -n^{1/\alpha}) \cup (n^{1/\alpha}, 2n^{1/\alpha}]$. And if there are no jumps in $[-2n^{1/\alpha}, -n^{1/\alpha}) \cup (n^{1/\alpha}, 2n^{1/\alpha}]$, since

$\tau(z/n^{1/\alpha}) = 1$ if $|z| \le n^{1/\alpha}$, we have $\tau'(z/n^{1/\alpha}) = 0$ and $\tau''(z/n^{1/\alpha}) = 0$. Thus, for M(p) a positive constant, we have

$$\mathbb{E}\left(\mathcal{R}^{1,n}_{4,\beta}(1)\right)^{2p} \le \frac{C^{2p}}{2^{2p}\sigma^{2p}n^{2p/\alpha}}\mathbb{E}\left[\int_0^1\int_{|z|>2}\left[\frac{1}{n^{1/\alpha}}\left(\left|\frac{z}{n^{1/\alpha}}\right|\right)^3\left|\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right|\right.\right.$$

$$\left.\left.+\left(\frac{z}{n^{1/\alpha}}\right)^4\left(\left|\frac{\tau''(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right|+\left(\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right)^2\right)\right]\mu^{(n)}(ds,dz)\right]^{2p}$$

$$\le \frac{M(p)}{\sigma^{2p}n^{2p/\alpha}}\left[\mathbb{E}\left[\int_0^1\int_{|z|>2}\frac{1}{n^{1/\alpha}}\left(\left|\frac{z}{n^{1/\alpha}}\right|\right)^3\left|\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right|\mu^{(n)}(ds,dz)\right]^{2p}+\right.$$

$$\left.\mathbb{E}\left[\int_0^1\int_{|z|>2}\left(\frac{z}{n^{1/\alpha}}\right)^4\left|\frac{\tau''(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right|\mu^{(n)}(ds,dz)\right]^{2p}+\mathbb{E}\left[\int_0^1\int_{|z|>2}\left(\frac{z}{n^{1/\alpha}}\right)^4\left(\frac{\tau'(z/n^{1/\alpha})}{\tau(z/n^{1/\alpha})}\right)^2\mu^{(n)}(ds,dz)\right]^{2p}\right]$$

Similarly to the proof of Theorem 3.1, we show that under assumption $\mathbf{H}_1(b_{ii})$, $\mathcal{R}^{1,n}_{4,\beta}(1)$ converges to zeros as $n\to\infty$ in $\mathbf{L}^{2p}$ for all $p\ge 1$ and this completes the proof of this part.

iii) From Lemma 3.3-*ii*) and the estimation (4.7), we easily deduce the result of this part. □

Proof of Lemma 3.5: *i*) From (3.60), the fact that b has bounded derivatives, $\sup_{0\le s\le 1}\frac{U_s^{n,\beta}}{U_1^{n,\beta}}$ is bounded, the upper and lower bounds of $((\epsilon_t^n))_{t\in[0,1]}$, we easily deduce the result of the part *i*).

ii) From (3.19), (3.64) we have

$$\frac{T_1^{n,\theta}}{n^{\frac{2}{\alpha}-1}\left(U_1^{n,\beta}\right)^2} = \frac{3(\epsilon_1^n)^3\int_0^1(\epsilon_1^n)^{-3}b''(\overline{Y}_s^{n,\beta,x_0},\theta)V_s^{n,\theta}U_s^{n,\beta}ds}{n^{\frac{2}{\alpha}}\left(U_1^{n,\beta}\right)^2}+\frac{(\epsilon_1^n)^3\int_0^1(\epsilon_1^n)^{-3}(\partial_\theta b)'(\overline{Y}_s^{n,\beta,x_0},\theta)W_s^{n,\beta}ds}{n^{\frac{2}{\alpha}}\left(U_1^{n,\beta}\right)^2}$$

$$+\frac{(\epsilon_1^n)^3\int_0^1(\epsilon_1^n)^{-3}b''(\overline{Y}_s^{n,\beta,x_0},\theta)\partial_\theta\overline{Y}_s^{n,\beta,x_0}W_s^{n,\beta}ds}{n^{\frac{2}{\alpha}}\left(U_1^{n,\beta}\right)^2}+\frac{(\epsilon_1^n)^3\int_0^1(\epsilon_1^n)^{-3}(\partial_\theta b)''(\overline{Y}_s^{n,\beta,x_0},\theta)(U_s^{n,\beta})^2ds}{n^{\frac{2}{\alpha}}\left(U_1^{n,\beta}\right)^2}$$

$$+\frac{(\epsilon_1^n)^3\int_0^1(\epsilon_1^n)^{-3}b'''(\overline{Y}_s^{n,\beta,x_0},\theta)\partial_\theta\overline{Y}_s^{n,\beta,x_0}(U_s^{n,\beta})^2ds}{n^{\frac{2}{\alpha}}\left(U_1^{n,\beta}\right)^2}.$$

We deduce, using Lemma 3.3-*i*) and Lemma 3.5-*i*) that

$$\left|\frac{T_1^{n,\theta}}{n^{\frac{2}{\alpha}-1}\left(U_1^{n,\beta}\right)^2}\right| \le \frac{C_1}{n^{\frac{2}{\alpha}}}+C_2\frac{\sup_{s\in(0,1]}|W_s^{n,\beta}|}{n^{\frac{2}{\alpha}}(U_1^{n,\beta})^2}, \tag{4.10}$$

where C_1, C_2 are some deterministic constants. Now from the estimation (4.7), we easily deduce that $\frac{\sup_{s\in(0,1]}|W_s^{n,\beta}|}{n^{\frac{2}{\alpha}}(U_1^{n,\beta})^2}$ tends to zero as $n\to\infty$ and then we get the result of this part.

iii) and *iv*) From (3.19), (3.61), (3.65), an easy computation shows the decomposition of $\frac{V_1^{n,\sigma}}{U_1^{n,\beta}}$ and $\frac{1}{n^{1/\alpha}} \frac{T_1^{n,\sigma}}{\left(U_1^{n,\beta}\right)^2}$, where the leading term is (3.73) and the remainder terms are given by

$$\mathcal{R}_{9,\beta}^n(1) = \frac{(\epsilon_1^n)^2 \int_0^1 (\epsilon_s^n)^{-2} \left(b''(\overline{Y}_s^{n,\beta,x_0}, \theta) \partial_\sigma \overline{Y}_s^{n,\beta,x_0} U_s^{n,\beta}\right) ds}{n U_1^{n,\beta}}$$

$$\mathcal{R}_{10,\beta}^n(1) = \frac{3(\epsilon_1^n)^3 \int_0^1 (\epsilon_1^n)^{-3} b''(\overline{Y}_s^{n,\beta,x_0}, \theta) V_s^{n,\sigma} U_s^{n,\beta} ds}{n^{1+1/\alpha} \left(U_1^{n,\beta}\right)^2}$$

$$\mathcal{R}_{11,\beta}^n(1) = \frac{(\epsilon_1^n)^3 \int_0^1 (\epsilon_1^n)^{-3} b''(\overline{Y}_s^{n,\beta,x_0}, \theta) \partial_\sigma \overline{Y}_s^{n,\beta,x_0} W_s^{n,\beta} ds}{n^{1+1/\alpha} \left(U_1^{n,\beta}\right)^2}$$

$$\mathcal{R}_{12,\beta}^n(1) = \frac{(\epsilon_1^n)^3 \int_0^1 (\epsilon_1^n)^{-3} b'''(\overline{Y}_s^{n,\beta,x_0}, \theta) \partial_\sigma \overline{Y}_s^{n,\beta,x_0} (U_s^{n,\beta})^2 ds}{n^{1+1/\alpha} \left(U_1^{n,\beta}\right)^2}$$

Moreover, using that b has bounded derivatives and $\sup_{0 \le s \le 1} \frac{U_s^{n,\beta}}{U_1^{n,\beta}}$ is bounded, the remainder terms satisfy the upper bounds

$$\left|\mathcal{R}_{9,\beta}^n(1)\right| \le \frac{C}{n} \sup_{s \in [0,1]} \left|\partial_\sigma \overline{Y}_s^{n,\beta,x_0}\right|, \quad \left|\mathcal{R}_{10,\beta}^n(1)\right| \le \frac{C_1}{n^{1+1/\alpha}} \left[\frac{\sup_{s \in [0,1]} |V_s^{n,\sigma}|}{U_1^{n,\beta}}\right],$$

$$|\mathcal{R}_{11,\beta}^n(1)| \le \frac{C_2}{n^{1+1/\alpha}} \left[\frac{\sup_{s \in [0,1]} |\partial_\sigma \overline{Y}_s^{n,\beta,x_0} W_s^{n,\beta}|}{\left(U_1^{n,\beta}\right)^2}\right], \quad |\mathcal{R}_{12,\beta}^n(1)| \le \frac{C_3}{n^{1+1/\alpha}} \sup_{s \in [0,1]} \left|\partial_\sigma \overline{Y}_s^{n,\beta,x_0}\right|$$

where C, C_1, C_2, C_3 are deterministic constants.

We observe that from Lemma 3.3 and Lemma 3.4-*iii*), we can deduce immediately the convergences to zero in $\mathbf{L}^p, \forall p \ge 2$ of the remainder terms $\mathcal{R}_{9,\beta}^n(1), \mathcal{R}_{11,\beta}^n(1)$ and $\mathcal{R}_{12,\beta}^n(1)$.

For $\mathcal{R}_{10,\beta}^n(1)$, the proof strongly relies on the Lemma 3.3-*ii*), (3.61), (3.19), the boundedness of (ϵ_s^n), the fact that b has bounded derivatives and $\frac{U_s^{n,\beta}}{U_1^{n,\beta}}$ is bounded for $0 \le s \le 1$. Then we easily deduce the result of this step.

From the above estimations, it follows that the remainder terms converge to zero in $\mathbf{L}^p$ for all $p \ge 2$. The convergence of $\widehat{\mathcal{H}}_{5,\beta}^n(1)$ follows by the same method as in the proof of the convergence of $\widehat{\mathcal{H}}_{3,\beta}^n(1)$ in the proof of Lemma 3.3 and this completes the proof of this lemma. □

Proof of Lemma 3.6: We first prove (3.75). From the fact that $\sup_{x_0} \sup_{s \in [0,1]} |\epsilon_s^n - 1| + |(\epsilon_s^n)^{-1} - 1| \xrightarrow{n \to \infty} 0$, the explicit expression of $\partial_\theta \overline{Y}_1^{n,\beta,x_0}$ given in (3.58) we easily get

$$\sup_{x_0} |n \partial_\theta \overline{Y}_1^{n,\beta_n,x_0} - \partial_\theta b(x_0, \theta)| \xrightarrow[a.s]{n \to \infty} 0.$$

From the expressions (3.14), (3.15), using the fact that $s \mapsto \epsilon^n_s$ converges uniformly with respect to x_0 to the constant 1 and Lemma 3.1, it can be seen that

$$\sup_{x_0} |\widehat{\mathcal{H}}^n_\beta(1) - \mathcal{H}_{L^\alpha}(1)| \xrightarrow[a.s]{n\to\infty} 0.$$

We deduce that almost surely, one has the convergence

$$\sup_{x_0} \left| n\partial_\theta \overline{Y}^{n,\beta,x_0}_1 \left(\widehat{\mathcal{H}}^n_\beta(1)\right)^2 - \partial_\theta b(x_0,\theta)\left(\mathcal{H}_{L^\alpha}(1)\right)^2 \right| \xrightarrow{n\to\infty} 0. \tag{4.11}$$

Since $\sup_n \left|\widehat{\mathcal{H}}^n_\beta(1)\right|^p$ is integrable for all $p \geq 1$, and using $|\partial_\theta \overline{Y}^{n,\beta,x_0}_1| \leq \frac{C}{n}$, we can apply the dominated convergence theorem and see that the convergence (4.11) holds in L^p-norm for all $p \geq 1$.

For (3.76): the proof is similar to (3.75).

For (3.77): let us recall that $L^n_s = \int_0^s \int_{|z|\leq 1} z\tilde{\mu}^{(n)}(dt,dz) + \int_0^s \int_{|z|>1} z\mu^{(n)}(dt,dz)$. Then, from (3.59) we have

$$\begin{aligned} n^{1/\alpha}\partial_\sigma \overline{Y}^{n,\beta,x_0}_1 \left(\widehat{\mathcal{H}}^n_\beta(1)\right)^2 &= \left(\widehat{\mathcal{H}}^n_\beta(1)\right)^2 \epsilon^n_1 \int_0^1 (\epsilon^n_s)^{-1} dL^n_s \\ &= \left(\widehat{\mathcal{H}}^n_\beta(1)\right)^2 \epsilon^n_1 \int_0^1 \int_{|z|\leq 1} (\epsilon^n_s)^{-1} z\tilde{\mu}(ds,dz) + \left(\widehat{\mathcal{H}}^n_\beta(1)\right)^2 \epsilon^n_1 \int_0^1 \int_{|z|>1} (\epsilon^n_s)^{-1} z\mu^{(n)}(ds,dz) \end{aligned} \tag{4.12}$$

where we used the fact that the measures $\mu^{(n)}$ and μ coincide on the set $\{(t,z) | t \in [0,1], |z| \leq n^{1/\alpha}\}$.

We now consider the first term of (4.12). We will prove that $\left(\widehat{\mathcal{H}}^n_\beta(1)\right)^2 \epsilon^n_1 \int_0^1 \int_{|z|\leq 1} (\epsilon^n_s)^{-1} z\tilde{\mu}(ds,dz) \xrightarrow[\mathbf{L}^p,\forall p\geq 1]{n\to\infty}$ $(\mathcal{H}_{L^\alpha}(1))^2 \int_0^1 \int_{|z|\leq 1} z\tilde{\mu}(ds,dz)$ which reduces to prove that $\left(\widehat{\mathcal{H}}^n_\beta(1)\right)^2 \xrightarrow[\mathbf{L}^{2p},\forall p\geq 1]{n\to\infty} (\mathcal{H}_{L^\alpha}(1))^2$ and $\epsilon^n_1 \int_0^1 \int_{|z|\leq 1} (\epsilon^n_s)^{-1} z\tilde{\mu}(ds,dz) \xrightarrow[\mathbf{L}^{2p},\forall p\geq 1]{n\to\infty} \int_0^1 \int_{|z|\leq 1} z\tilde{\mu}(ds,dz)$.

From (3.45), (3.46) and the fact that $\sup_{x_0} \sup_{s\in[0,1]} |\epsilon^n_s - 1| + \left|(\epsilon^n_s)^{-1} - 1\right| \xrightarrow{n\to\infty} 0$ we easily get the result about the convergence of the first term in the right-hand side of (4.12).

For the second term in the right-hand side of (4.12), we show that $\left(\widehat{\mathcal{H}}^n_\beta(1)\right)^2 \epsilon^n_1 \int_0^1 \int_{|z|>1} (\epsilon^n_s)^{-1} z\mu^{(n)}(ds,dz) \xrightarrow[\mathbf{L}^p,\forall p\geq 1]{n\to\infty} (\mathcal{H}_{L^\alpha}(1))^2 \int_0^1 \int_{|z|>1} z\mu(ds,dz)$ which reduces to prove

$$\widehat{\mathcal{H}}^n_\beta(1) \xrightarrow[\mathbf{L}^{2p}]{n\to\infty} \mathcal{H}_{L^\alpha}(1), \quad \forall p \geq 1, \tag{4.13}$$

$$\widehat{\mathcal{H}}^n_\beta(1)\epsilon^n_1 \int_0^1 \int_{|z|>1} (\epsilon^n_s)^{-1} z\mu^{(n)}(ds,dz) \xrightarrow[\mathbf{L}^{2p}]{n\to\infty} \mathcal{H}_{L^\alpha}(1) \int_0^1 \int_{|z|>1} z\mu(ds,dz), \quad ,\forall p \geq 1. \tag{4.14}$$

For (4.13), this follows from (3.45), (3.46).

For (4.14), applying Lemma 3.1, the fact that $s \mapsto \epsilon^n_s$ converges uniformly to the constant 1 and (3.35), (3.36), it follows easily that

$$\widehat{\mathcal{H}}^n_\beta(1)\epsilon^n_1 \int_0^1 \int_{|z|>1} (\epsilon^n_s)^{-1} z\mu^{(n)}(ds,dz) \xrightarrow[a.s.]{n\to\infty} \mathcal{H}_{L^\alpha}(1) \int_0^1 \int_{|z|>1} z\mu(ds,dz). \tag{4.15}$$

Now using (3.39) we can deduce that for C a positive constant,

$$\left|\widehat{\mathcal{H}}^n_\beta(1)\epsilon^n_1\int_0^1\int_{|z|>1}(\epsilon^n_s)^{-1}z\mu^{(n)}(ds,dz)\right|$$
$$\leq C\left[\left(\frac{\int_0^1\int_{\mathbb{R}}\rho^n(z)|(\rho^n)'(z)|\mu^{(n)}(ds,dz)}{\left[\int_0^1\int_{\mathbb{R}}\rho^n(z)\mu^{(n)}(ds,dz)\right]^2}+\frac{\int_0^1\int_{\mathbb{R}}\left[|\rho^{n\prime}(z)|+\frac{1+\alpha}{|z|}\rho^n(z)\right]\mu^{(n)}(ds,dz)}{\int_0^1\int_{\mathbb{R}}\rho^n(z)\mu^{(n)}(ds,dz)}\right)\int_0^1\int_{2\geq|z|>1}|z|\mu(ds,dz)\right]$$
$$+C\left[\left(\frac{\int_0^1\int_{\mathbb{R}}\rho^n(z)|(\rho^n)'(z)|\mu^{(n)}(ds,dz)}{\left[\int_0^1\int_{\mathbb{R}}\rho^n(z)\mu^{(n)}(ds,dz)\right]^2}+\frac{\int_0^1\int_{\mathbb{R}}\left[|\rho^{n\prime}(z)|+\frac{1+\alpha}{|z|}\rho^n(z)\right]\mu^{(n)}(ds,dz)}{\int_0^1\int_{\mathbb{R}}\rho^n(z)\mu^{(n)}(ds,dz)}\right)\int_0^1\int_{|z|>2}|z|\mu^{(n)}(ds,dz)\right] \tag{4.16}$$

Considering the first term in the right-hand side of (4.16), from the proofs of **Step 1.1** and **Step 1.2** in Theorem 3.2, we deduce that it is bounded by a random variable independent of n and belonging to $\cap_{p\geq1}\mathbf{L}^{2p}$.

We now consider the second term in the right-hand side of (4.16). From (3.10), we have

$$\left|\left(\frac{\int_0^1\int_{\mathbb{R}}\rho^n(z)|(\rho^n)'(z)|\mu^{(n)}(ds,dz)}{\left[\int_0^1\int_{\mathbb{R}}\rho^n(z)\mu^{(n)}(ds,dz)\right]^2}+\frac{\int_0^1\int_{\mathbb{R}}\left[|\rho^{n\prime}(z)|+\frac{1+\alpha}{|z|}\rho^n(z)\right]\mu^{(n)}(ds,dz)}{\int_0^1\int_{\mathbb{R}}\rho^n(z)\mu^{(n)}(ds,dz)}\right)\int_0^1\int_{|z|>2}|z|\mu^{(n)}(ds,dz)\right|$$
$$\leq\left[\frac{\int_0^1\int_{|z|\leq2}\rho(z)\,|\rho'(z)|\,\mu(ds,dz)}{\left(\int_0^1\int_{|z|>2}z^2\mu^{(n)}(ds,dz)\right)^2}+\frac{\int_0^1\int_{|z|\leq2}\left(|\rho'(z)|+\rho(z)\frac{1+\alpha}{|z|}\right)\mu(ds,dz)}{\int_0^1\int_{|z|>2}z^2\mu^{(n)}(ds,dz)}\right]\int_0^1\int_{|z|>2}|z|\mu^{(n)}(dt,dz)$$
$$+\left[\frac{\int_0^1\int_{|z|>2}2|z|^3\mu^{(n)}(ds,dz)}{\left(\int_0^1\int_{|z|>2}z^2\mu^{(n)}(ds,dz)\right)^2}+\frac{\int_0^1\int_{|z|>2}(3+\alpha)\,|z|\mu^{(n)}(ds,dz)}{\int_0^1\int_{|z|>2}z^2\mu^{(n)}(ds,dz)}\right]\int_0^1\int_{|z|>2}|z|\mu^{(n)}(dt,dz) \tag{4.17}$$

Using the Cauchy - Schwarz inequality $\int_0^1\int_{|z|>2}\mu^n(dt,dz)\times\int_0^1\int_{|z|>2}z^2\mu^{(n)}(ds,dz)\geq\left(\int_0^1\int_{|z|>2}|z|\mu^n(dt,dz)\right)^2$ we get:

$$\frac{\int_0^1\int_{|z|>2}2|z|^3\mu^{(n)}(ds,dz)}{\left(\int_0^1\int_{|z|>2}z^2\mu^{(n)}(ds,dz)\right)^2}\int_0^1\int_{|z|>2}|z|\mu^{(n)}(dt,dz)\leq\frac{\int_0^1\int_{|z|>2}2|z|^3\mu^{(n)}(ds,dz)}{\left(\int_0^1\int_{|z|>2}|z|\mu^{(n)}(dt,dz)\right)^3}\left(\int_0^1\int_{|z|>2}\mu^{(n)}(dt,dz)\right)^2$$
$$=\frac{\sum_{i=1}^{N_1}2|Z_i|^3}{\left(\sum_{i=1}^{N_1}|Z_i|\right)^3}\left(\int_0^1\int_{|z|>2}\mu^{(n)}(dt,dz)\right)^2\leq2\left(\int_0^1\int_{|z|>2}\mu(dt,dz)\right)^2 \tag{4.18}$$

and

$$\frac{\int_0^1\int_{|z|>2}(3+\alpha)|z|\mu^{(n)}(ds,dz)}{\int_0^1\int_{|z|>2}z^2\mu^{(n)}(ds,dz)}\int_0^1\int_{|z|>2}|z|\mu^n(dt,dz)\leq(3+\alpha)\int_0^1\int_{|z|>2}\mu(dt,dz). \tag{4.19}$$

Combining (4.18), (4.19) with (4.17), it follows that the second term in the right-hand side of (4.16) is also bounded by a random variable independent of n and belonging to $\cap_{p\geq 1}\mathbf{L}^{2p}$. As a consequence, we get that

$$\sup_n \left| \widehat{\mathcal{H}}^n_\beta(1)\epsilon^n_1 \int_0^1 \int_{|z|>1} (\epsilon^n_s)^{-1} z\mu^{(n)}(ds,dz)\right|^{2p} \quad \text{is integrable} \quad \forall p \geq 1. \tag{4.20}$$

Under (4.15) and (4.20), we can apply the dominated convergence theorem and the result on the convergence of the second term in the right-hand side of (4.12) follows. This achieves the proof of (3.77).

For (3.78), the proof is similar to (3.77). □

Proof of Lemma 3.7: *i*) and *ii*): From (3.68) and proceeding as in the proof of (3.75), we deduce the results of *i*) and *ii*).

iii) and *iv*): From (3.59) and the fact that the measures $\mu^{(n)}$ and μ coincide on the set $\{(t,z)|t \in [0,1], |z| \leq n^{1/\alpha}\}$, we have

$$\begin{aligned} n^{1/\alpha}\partial_\sigma \overline{Y}^{n,\beta,x_0}_1 \widehat{\mathcal{H}}^n_{3,\beta}(1) &= \widehat{\mathcal{H}}^n_{3,\beta}(1)\epsilon^n_1 \int_0^1 (\epsilon^n_s)^{-1} dL^n_s \\ &= \widehat{\mathcal{H}}^n_{3,\beta}(1)\epsilon^n_1 \int_0^1 \int_{|z|\leq 1} (\epsilon^n_s)^{-1} z\tilde{\mu}(ds,dz) + \widehat{\mathcal{H}}^n_{3,\beta}(1)\epsilon^n_1 \int_0^1 \int_{|z|>1} (\epsilon^n_s)^{-1} z\mu^{(n)}(ds,dz). \end{aligned} \tag{4.21}$$

From (3.68) and proceeding as in the proof of (3.77), we can deduce the result of *iii*). Moreover, in the same way we can complete the proof of part *iv*). □

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