Nonsmooth modal analysis for a one-dimensional finite bar subject to unilateral contact using the Wave Finite Element Method

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This contribution suggests a numerical procedure, involving the Wave Finite Element Method (WFEM), able to perform nonsmooth modal analysis [1] of a frictionless unilateral contact problem defined on a finite one-dimensional domain, as depicted in Fig 1.

The WFEM is a shock-fitting method which consists in locating and tracking shock waves propagating in a mechanical system [2]. In this approach, the space domain is discretized into cells of length $\Delta x = x_{i+1} - x_i$ and time is discretized into intervals of length $\Delta t = t_{n+1} - t_n$, where i is the index of the cell interface and n is the index of the time step. Then, a system of equations is derived for the calculation of the state (defined here by the stress $\boldsymbol{\sigma}$ and velocity $\mathbf{v}$) of the cells at each time step t_n. The simulation of an elastic wave propagating with a given finite speed c is performed by iterating these relations in discrete time with a prescribed time interval $\Delta t = \Delta x / c$. This method accurately captures potentially discontinuous wave fronts propagating at a finite speed. For an unforced system without unilateral constraints, the WFEM yields the typical system of equations

$$\mathbf{Q}^{(n+1)} = \mathbf{A}\mathbf{Q}^{(n)} \tag{1}$$

where $\mathbf{Q}^{(n)} = [\boldsymbol{\sigma}^{(n)} \ \mathbf{v}^{(n)}]^\top$ is the state of the system at time step t_n. The matrix $\mathbf{A} \in \mathbb{R}^{2N \times 2N}$, where N is the number of cells, embeds the dynamics of the system. Using Eq. (1), $\mathbf{Q}^{(n)}$ can be directly expressed in terms of the initial conditions $\mathbf{Q}^{(0)}$

$$\mathbf{Q}^{(n)} = \mathbf{A}^n \mathbf{Q}^{(0)} \tag{2}$$

where $\mathbf{A}^n$ is the matrix $\mathbf{A}$ to the power of n.

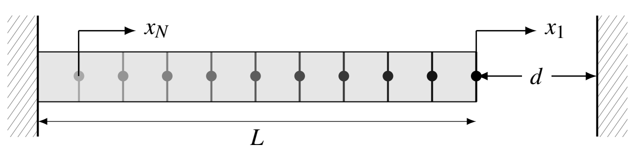


Fig. 1: Elastodynamic problem with unilateral contact constraints and free–fixed boundary conditions

Consider the free–fixed bar in Fig. 1: it is subject to a unilateral contact constraint. Once the WFEM discretization is performed, the contact constraints are enforced using the concept of *floating boundary conditions* [2], which loosely speaking, can be regarded as a conditional switch between free and fixed boundary conditions when a penetration is detected during the iterative process. From Eq. (2), two types of matrices $\mathbf{A}$ shall then be distinguished: $\mathbf{A}_\mathrm{f}$ for the free–fixed condition (no contact) and $\mathbf{A}_\mathrm{c}$ for fixed–fixed condition (contact). T-periodic solutions are sought using the condition $\mathbf{Q}^{(n_T)} = \mathbf{Q}^{(0)}$ where $n_T \Delta t = T$. The solution is assumed to be composed by r consecutive steps in the free phase and p consecutive steps in the contact phase, leading to the following periodicity condition

$$\mathbf{Q}^{(n_T)} = \mathbf{A}_\mathrm{c}^p \mathbf{A}_\mathrm{f}^r \mathbf{Q}^{(0)} \tag{3}$$

where $n_T = r + p$. The durations of free and contact phases corresponding to periodic motions are not known a priori; therefore, the values of r and p leading to solutions are unknowns. Also, note that periodic solutions with more than one contact phase per period are not targeted with the proposed formulation. Periodicity condition in Eq. (3) simplifies to

$$\left(\mathbf{A}_{\mathrm{c}}^{p}\mathbf{A}_{\mathrm{f}}^{r} - \mathbf{I}\right)\mathbf{Q}^{(0)} = \mathbf{0}. \tag{4}$$

The initial condition $\mathbf{Q}^{(0)}$ satisfying Eq. (4) is called a "potential solution". Potential solutions are *admissible* solutions of the formulation if they satisfy the following additional conditions:

COND1: Point of contact must not penetrate the wall during periodic motion: $x_1^{(n)} \leqslant d, \quad n = 0, 1, \ldots, r$
COND2: Point of contact cannot separate from the wall during contact: $x_1^{(n)} = d, \quad n = r+1, r+2, \ldots, r+p$
COND3: Contact interaction must be compressive: $\lambda^{(n)} \leqslant 0, \quad n = r+1, r+2, \ldots, r+p$

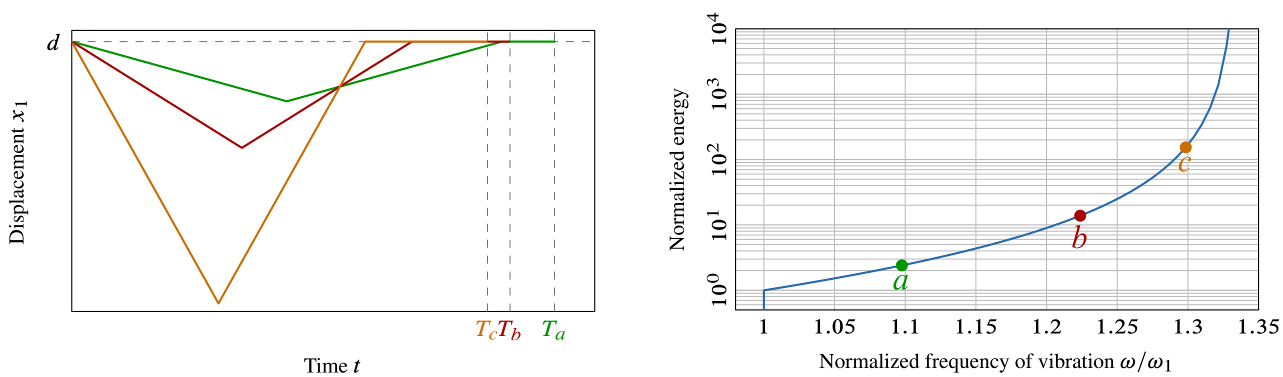
In the above conditions, $x_1^{(n)}$ is the displacement of the contact node at time step t_n, d is the initial gap between the bar tip and the rigid wall, and λ is the contact force.

As seen from Eq. (4), a periodic solution is necessarily an element of the kernel of the matrix $\mathbf{S}_T = \mathbf{A}_{\mathrm{c}}^{p}\mathbf{A}_{\mathrm{f}}^{r} - \mathbf{I}$, *i.e.* $\mathbf{Q}^{(0)} \in \ker \mathbf{S}_T$. The dimension m of $\ker \mathbf{S}_T$ depends on the combinations of r and p. A non-trivial solution may exist only if $\det(\mathbf{S}_T) = 0$, hence only the combinations of r and p that correspond to $m \geqslant 1$ are of interest. Given a basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$ of $\ker \mathbf{S}_T$, $\mathbf{Q}^{(0)}$ reads

$$\mathbf{Q}^{(0)} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \ldots + \alpha_m\mathbf{e}_m \tag{5}$$

Initial state $\mathbf{Q}^{(0)}$ is completely determined by the unknown coefficients $\alpha_1, \alpha_2, \ldots, \alpha_m$. The state $\mathbf{Q}^{(n)}$ is calculated at each time step through Eq. (2); hence this state can be expressed in terms of the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_m$. A solution is obtained by calculating the coefficients that satisfy the above necessary conditions.

It has been observed that when $\dim(\ker \mathbf{S}_T) = 1$, only one solution satisfies all the conditions. Additionally, by changing r and p, it is possible to find other admissible solutions in the vicinity of an existing solution. A continuum of periodic orbits then emerges: it defines a nonsmooth mode of vibration (NSM) [3]. The displacement x_1 is represented in Fig. 2(a) for three solutions belonging to the first NSM (continuation of the first linear modeshape of the bar) whose Frequency–Energy branch is shown in Fig. 2(b). The periodic solutions are traveling waves with discontinuous wave fronts interacting with each other and the boundaries.



(a) Periodic displacements of the contacting node

(b) Frequency–energy plot

Fig. 2: First nonsmooth mode of the free–fixed bar subjected to unilateral contact constraints. Energy is normalized with respect to the energy of the grazing solution

References

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