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ON A PRIMAL-MIXED, VORTICITY-BASED FORMULATION FOR REACTION-DIFFUSION-BRINKMAN SYSTEMS

VERÓNICA ANAYA*, MOSTAFA BENDAHMANE†, DAVID MORA‡, AND RICARDO RUIZ-BAIER§

Abstract. We are interested in modelling the interaction of bacteria and certain nutrient concentration within a porous medium admitting viscous flow. The governing equations consist of a reaction-diffusion system representing the bacteria-chemical mass exchange, coupled to the Brinkman problem written in terms of fluid vorticity, velocity and pressure, and describing the flow patterns driven by an external source depending on the local distribution of the chemical species. *A priori* stability bounds are derived for the uncoupled problems, and the solvability of the full system is analysed using a fixed-point approach. We introduce a primal-mixed finite element method to numerically solve the model equations, employing a primal scheme with piecewise linear approximation of the reaction-diffusion unknowns, while the discrete flow problem uses a mixed approach based on Raviart-Thomas elements for velocity, Nédélec elements for vorticity, and piecewise constant pressure approximations. In particular, this choice produces exactly divergence-free velocity approximations. Moreover, we establish existence of discrete solutions and show convergence to a weak solution of the original problem. Finally, we report several numerical experiments illustrating the behaviour of the proposed scheme.

Key words. Reaction-diffusion; Brinkman flows; Vorticity formulation; Mixed finite elements; Chemical reactions.

AMS subject classifications. 65M60; 35K57; 35Q35; 76S05.

1. Introduction.

Reaction-diffusion systems can explain many phenomena taking place in diverse disciplines such as industrial and environmental processes, biomedical applications, or population dynamics. For instance, non-equilibrium effects associated to mass exchange and local configuration modifications in the concentration of species are usually represented in terms of classical reaction-diffusion equations. These models allow to reproduce chaos, spatiotemporal patterns, rhythmic and oscillatory scenarios, and many other mechanisms. Nevertheless, in most of these applications the reactions do not occur in complete isolation. The species are rather immersed in a fluid, or they move within (and interact with) a fluid-solid continuum, and therefore the motion of the fluid affects that of the species. Up to some extent, reciprocal effects might be substantially large, thus leading to local changes in the flow patterns.

Here it is assumed that the medium where the chemical reactions develop is a porous material saturated with an incompressible viscous fluid. Consequently, the fluid flow can be governed by Brinkman equations (representing linear momentum and mass conservation for the fluid). As indicated above, we also suppose that the local fluctuations of a species' concentration is important enough to affect the fluid flow. In turn, the reaction-diffusion equations include additional terms accounting for the advection of each species with the fluid velocity, therefore improving the mixing and interaction properties with respect to those observed under pure diffusion effects [9]. Diverse types of continuum-based models resulting in reaction-diffusion-momentum equations have been applied for e.g. enzyme reactions advected with a known Poiseuille velocity profile [12], population kinetics on moving domains [19, 24], or biochemical interactions on growing surfaces [8, 28].

We aim at developing numerical solutions to a class of similar systems, also addressing the solvability of the governing equations, the expected properties of the underlying solutions, and the convergence behaviour of suitable finite element schemes. More precisely, at both continuous and discrete levels, the Brinkman equations are set in a mixed form (that is, the associated formulation possesses a saddle-point structure involving the vorticity as additional unknown) whereas the formulation of the reaction-diffusion system is written exclusively in terms of the primal variables, in this case the species' concentration.

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Such a structure is motivated by the need of accurate vorticity rendering, obtained directly without postprocessing it from typically low-order discrete velocities (which usually leads to insufficiently reliable approximations).

The nonlinear reaction-diffusion system will be placed in the framework of general semilinear parabolic equations and its well-posedness, for a fixed velocity, follows from the classical theory of Ladyženskaja [15]. On the other hand, for a fixed species' concentration, a number of Brinkman formulations based on vorticity, pressure and velocity are available from the literature [2, 4, 5, 27] (see also [11, 21]). Here we adopt the one introduced in [3], where the well-posedness of the flow equations is established thanks to the classical Babuška-Brezzi framework. In turn, the fully coupled problem is analysed using a Schauder fixed-point approach. Perhaps the closest contributions in the spirit of the present study are the error estimation for finite element formulations to doubly-diffusive Boussinesq flows presented in [1], and the two-sidedly degenerate chemotaxis-fluid coupled system from [7], whose solvability relies on a regularisation step combined with compactness arguments.

Regarding numerical approximation, we propose a method based on piecewise linear and continuous polynomials for the species' concentrations, whereas the set of flow equations is discretised using a mixed approach based on Raviart-Thomas elements for the approximation of the velocity, Nédélec elements for vorticity, and piecewise constants for the pressure. The computational burden of this algorithm is comparable to classical low-cost approximations as the so-called MINI element for velocity-pressure formulations. On top of that, the proposed finite element method provides divergence-free velocities, thus preserving an essential constraint of the underlying physical system.

After this introductory section, the remainder of the paper is structured in the following manner. Section 2 states the model problem and introduces the concept of weak solutions, together with a regularised auxiliary coupled problem. The solvability of these auxiliary equations is established in Section 3, using a fixed-point approach. The passage to the limit that transforms the regularised system into the original problem is addressed in Section 4, and the proof of uniqueness of weak solution is postponed to Section 5. Section 6 deals with the numerical approximation of the model problem, using mixed finite elements and a first-order backward Euler time advancing scheme. In this section we establish existence of discrete solutions and show convergence to a weak solution. Finally, we provide in Section 7 a set of numerical tests to illustrate the properties of the numerical scheme and the features of the coupled model.

2. Governing equations.

2.1. Problem statement. Let $\Omega \subset \mathbb{R}^3$, be a simply connected, and bounded porous domain saturated with a Newtonian incompressible fluid, where also a bacterium and some chemical species (diffusible agents or nutrients) are present. Viscous flow in the porous medium is usually modelled with Brinkman equations stating momentum and mass conservation of the fluid. In addition, the Reynolds transport principle applied to mass conservation of the interacting species yields an advection-reaction-diffusion system. The physical scenario of interest can be therefore described by the following coupled system, written in terms of the fluid velocity $\boldsymbol{u}$, the rescaled fluid vorticity $\boldsymbol{\omega}$, the fluid pressure p, and the volumetric fraction (or concentration) of the bacterium c and of the chemical substance s. For a.e. $(\boldsymbol{x}, t) \in \Omega_T := \Omega \times [0, T]$,

$$\begin{aligned} \partial_t c + \boldsymbol{u} \cdot \nabla c - \operatorname{div}(D_c(c)\nabla c) = G_c(c,s), \quad \partial_t s + \boldsymbol{u} \cdot \nabla s - \operatorname{div}(D_s(s)\nabla s) = G_s(c,s), \\ \mathbb{K}^{-1}\boldsymbol{u} + \sqrt{\mu}\,\mathbf{curl}\,\boldsymbol{\omega} + \nabla p = s\boldsymbol{g} + \boldsymbol{f}, \quad \boldsymbol{\omega} - \sqrt{\mu}\,\mathbf{curl}\,\boldsymbol{u} = \mathbf{0}, \quad \operatorname{div}\boldsymbol{u} = 0, \end{aligned} \tag{2.1}$$

where μ is the fluid viscosity, $\mathbb{K}(\boldsymbol{x})$ is the permeability tensor rescaled with viscosity, $s\boldsymbol{g}$ represents the force exerted by the bacteria on the fluid motion (where $\boldsymbol{g}$ is the gravitational acceleration, or any other particular force to which the species comply), and $\boldsymbol{f}(\boldsymbol{x}, t)$ is an external force (e.g. centrifugal) applied to the porous medium. Moreover, D_c, D_s, G_c, G_s are concentration-dependent coefficients determining, respectively, the nonlinear diffusivities and reaction kinetics (representing production and degradation) of bacteria and chemical species.

Equations (2.1) are complemented with the following boundary and initial data:

$$\begin{aligned} (c\boldsymbol{u} - D_c(c)\nabla c) \cdot \boldsymbol{n} = (s\boldsymbol{u} - D_s(s)\nabla s) \cdot \boldsymbol{n} = 0, \ \boldsymbol{u} \cdot \boldsymbol{n} = u_\partial, \ \boldsymbol{\omega} \times \boldsymbol{n} = \boldsymbol{\omega}_\partial \quad (\boldsymbol{x}, t) \in \partial\Omega \times [0, T], \\ c = c_0, \ s = s_0, \quad (\boldsymbol{x}, t) \in \Omega \times \{0\}, \end{aligned} \tag{2.2}$$

representing that the species cannot leave the medium and that a slip velocity and a compatible vorticity trace are imposed along the boundary.

2.2. Main assumptions. We suppose that the permeability tensor $\mathbb{K} \in [C(\bar{\Omega})]^{3\times 3}$ is symmetric and uniformly positive definite. So it is its inverse, i.e., there exists $C > 0$ such that

$$\boldsymbol{v}^{\mathrm{t}}\mathbb{K}^{-1}(\boldsymbol{x})\boldsymbol{v} \geq C|\boldsymbol{v}|^2 \quad \forall\, \boldsymbol{v} \in \mathbb{R}^3,\ \forall\, \boldsymbol{x} \in \Omega. \tag{2.3}$$

In addition, the diffusivities are assumed always positive, coercive, and continuous: for $i \in \{s, c\}$,

$$D_i : [0,1] \mapsto \mathbb{R}^+ \text{ is continuous}, \quad 0 < D^{\min} \leq D_i(m) \leq D^{\max} < \infty, \quad m \in \mathbb{R}. \tag{2.4}$$

Regarding the reaction terms G_c, G_s, we assume there exist constants $C_c, C_s >$ such that

$$\begin{aligned}
&|G_c(c,s)| \leq C_c(1 + |c| + |s|), \quad |G_s(c,s)| \leq C_s(1 + |c| + |s|) \text{ for all } c, s \geq 0,\\
&G_c(c,s), G_s(c,s) \geq 0 \text{ if } c \geq 0 \text{ and } s \geq 0,\\
&G_c(c,s) = \nu_1 \text{ and } G_s(c,s) = \nu_2 \text{ if } c \leq 0 \text{ or } s \leq 0,
\end{aligned} \tag{2.5}$$

for some parameters $\nu_1 > 0$ and $\nu_2 > 0$. Initial data are assumed nonnegative and regular enough

$$s_0, c_0 \geq 0, \quad c_0, s_0 \in \mathrm{L}^\infty(\Omega). \tag{2.6}$$

Observe that for constant coefficients D_c, D_s, G_c, G_s, problem (2.1) might also serve as a model for the coupling of Newtonian flows with mass and heat transport [22].

2.3. Weak solutions. According to (2.2), let us introduce the functional spaces

$$\mathbf{H}_0(\mathrm{div};\Omega) = \big\{\boldsymbol{v} \in \mathbf{H}(\mathrm{div};\Omega) : \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\big\},\ \mathbf{H}_0(\mathbf{curl};\Omega) = \big\{\boldsymbol{z} \in \mathbf{H}(\mathbf{curl};\Omega) : \boldsymbol{z}\times\boldsymbol{n} = \mathbf{0} \text{ on } \partial\Omega\big\}.$$

We then proceed, for a given $t > 0$, to test the first two equations in (2.1) with functions in $\mathrm{H}^1_0(\Omega)$, the third equation in (2.1) against functions in $\mathbf{H}_0(\mathrm{div};\Omega)$, the fourth equation in (2.1) against functions in $\mathbf{H}_0(\mathbf{curl};\Omega)$ and to integrate by parts. The last equation is tested with functions in $\mathrm{L}^2_0(\Omega)$ and no integration by parts is applied. Consequently, and using (2.2) we can give the following definition.

Definition 2.1. *In view of the properties outlined in Section 2.2, we shall say that the function $(c, s, \boldsymbol{u}, \boldsymbol{\omega}, p)$ is a weak solution to (2.1) if*

$$\begin{aligned}
&s, c \in \mathrm{L}^\infty(\Omega_T) \cap \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)), \quad \partial_t s, \partial_t c \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)'),\\
&\boldsymbol{u} \in \mathbf{L}^2(0,T;\mathbf{H}(\mathrm{div};\Omega)), \quad \boldsymbol{\omega} \in \mathbf{L}^2(0,T;\mathbf{H}(\mathbf{curl};\Omega)), \quad p \in \mathrm{L}^2(0,T;\mathrm{L}^2_0(\Omega)),
\end{aligned}$$

and

$$\begin{aligned}
\int_0^T \langle \partial_t c, m^c\rangle\,\mathrm{d}t + \iint_{\Omega_T} (D_c(c)\nabla c - c\boldsymbol{u})\cdot\nabla m^c\,\mathrm{d}\boldsymbol{x}\mathrm{d}t &= \iint_{\Omega_T} G_c(c,s) m^c\,\mathrm{d}\boldsymbol{x}\mathrm{d}t,\\
\int_0^T \langle \partial_t s, m^s\rangle\,\mathrm{d}t + \iint_{\Omega_T} (D_s(s)\nabla s - s\boldsymbol{u})\cdot\nabla m^s\,\mathrm{d}\boldsymbol{x}\mathrm{d}t &= \iint_{\Omega_T} G_s(c,s) m^s\,\mathrm{d}\boldsymbol{x}\mathrm{d}t,\\
\iint_{\Omega_T} \mathbb{K}^{-1}\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\mathrm{d}t + \sqrt{\mu}\iint_{\Omega_T} \mathbf{curl}\,\boldsymbol{\omega}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\mathrm{d}t - \iint_{\Omega_T} p\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\mathrm{d}t &= \iint_{\Omega_T} (s\boldsymbol{g} + \boldsymbol{f})\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\mathrm{d}t,\\
\sqrt{\mu}\iint_{\Omega_T} \mathbf{curl}\,\boldsymbol{z}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\mathrm{d}t - \iint_{\Omega_T} \boldsymbol{\omega}\cdot\boldsymbol{z}\,\mathrm{d}\boldsymbol{x}\mathrm{d}t &= 0,\\
-\iint_{\Omega_T} q\,\mathrm{div}\,\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\mathrm{d}t &= 0,
\end{aligned}$$

for all $m^i \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))$, $\boldsymbol{v} \in \mathbf{L}^2(0,T;\mathbf{H}_0(\mathrm{div};\Omega))$, $\boldsymbol{z} \in \mathbf{L}^2(0,T;\mathbf{H}_0(\mathbf{curl};\Omega))$, $q \in \mathrm{L}^2(0,T;\mathrm{L}^2_0(\Omega))$.

Our first main result states existence of weak solutions to the reaction-diffusion-Brinkman equations.

Theorem 2.2. *Assume that conditions (2.3), (2.4) and (2.5) hold. If $c_0, v_0 \in \mathrm{L}^\infty(\Omega)$ with $c_0 \geq 0$ and $s_0 \geq 0$ a.e. in Ω, then there exists a weak solution of (2.1)-(2.2) in the sense of Definition 2.1.*

The proof of this result is based on an application of Schauder's fixed-point theorem (in an appropriate functional setting) to the following approximate system

$$\begin{aligned} \partial_t c + \boldsymbol{u}\cdot\nabla c - \operatorname{div}(D_c(c)\nabla c) = G_{c,\varepsilon}(c,s), \quad \partial_t s + \boldsymbol{u}\cdot\nabla s - \operatorname{div}(D_s(s)\nabla s) = G_{s,\varepsilon}(c,s),\\ \mathbb{K}^{-1}\boldsymbol{u} + \sqrt{\mu}\,\mathbf{curl}\,\boldsymbol{\omega} + \nabla p = \hat{s}\boldsymbol{g} + \boldsymbol{f}, \quad \boldsymbol{\omega} - \sqrt{\mu}\,\mathbf{curl}\,\boldsymbol{u} = \mathbf{0}, \quad \operatorname{div}\boldsymbol{u} = 0, \end{aligned} \tag{2.7}$$

defined for each fixed $\varepsilon > 0$, where $\hat{s}$ is a fixed function, and

$$G_{c,\varepsilon}(c,s) = \frac{G_c(c,s)}{1+\varepsilon\,|G_c(c,s)|}, \qquad G_{s,\varepsilon}(c,s) = \frac{G_s(c,s)}{1+\varepsilon\,|G_s(c,s)|} \qquad \text{for a.e. } c,s\in\mathbb{R}.$$

The proof of Theorem 2.2 continues with the derivation of *a priori* estimates, and then passing to the limit in the approximate solutions using monotonicity and compactness arguments. Having proved existence of solutions to the system (2.7), the goal is to send the regularisation parameter ε to zero in sequences of such solutions to compose weak solutions of the original system (2.1)-(2.2). Again, convergence is achieved by means of *a priori* estimates and compactness arguments.

3. Existence result to the regularised problem. In this section we prove, for each fixed $\varepsilon > 0$, the existence of solutions to (2.7), looking first at the solvability of the uncoupled systems.

3.1. Preliminaries. Let us recall the following abstract result (see e.g. [13, Theorem 1.3]):

THEOREM 3.1. *Let $(\mathcal{X}, \langle\cdot,\cdot\rangle_{\mathcal{X}})$ be a Hilbert space. Let $\mathcal{A}:\mathcal{X}\times\mathcal{X}\to\mathbb{R}$ be a bounded symmetric bilinear form, and let $\mathcal{G}:\mathcal{X}\to\mathbb{R}$ be a bounded functional. Assume that there exists $\bar\beta>0$ such that*

$$\sup_{\substack{y\in\mathcal{X}\\ y\neq 0}} \frac{\mathcal{A}(x,y)}{\|y\|_{\mathcal{X}}} \geq \bar\beta\,\|x\|_{\mathcal{X}} \quad \forall\, x\in\mathcal{X}.$$

Then, there exists a unique $x\in\mathcal{X}$, such that

$$\mathcal{A}(x,y) = \mathcal{G}(y) \quad \forall\, y\in\mathcal{X}.$$

Moreover, there exists $C>0$, independent of the solution, such that

$$\|x\|_{\mathcal{X}} \leq C\|\mathcal{G}\|_{\mathcal{X}'}.$$

Let us also consider the kernel of the bilinear form $\int_\Omega q\operatorname{div}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}$, that is

$$\mathrm{X} := \{\boldsymbol{v}\in\mathbf{H}_0(\mathrm{div};\Omega) : \int_\Omega q\operatorname{div}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = 0,\ \forall q\in\mathrm{L}^2_0(\Omega)\} = \{\boldsymbol{v}\in\mathbf{H}_0(\mathrm{div};\Omega) : \operatorname{div}\boldsymbol{v} = 0 \text{ a.e. in } \Omega\}.$$

Moreover, we endow the space $\mathbf{H}(\mathbf{curl};\Omega)$ with the following μ-dependent norm:

$$\|\boldsymbol{z}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} := \|\boldsymbol{z}\|^2_{0,\Omega} + \mu\|\,\mathbf{curl}\,\boldsymbol{z}\|^2_{0,\Omega},$$

and recall the following inf-sup condition (*cf.* [13]): there exists $\beta_2>0$ only depending on Ω, such that

$$\sup_{\substack{\boldsymbol{v}\in\mathbf{H}(\mathrm{div};\Omega)\\ \boldsymbol{v}\neq 0}} \frac{|\int_\Omega q\operatorname{div}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}|}{\|\boldsymbol{v}\|_{\mathbf{H}(\mathrm{div};\Omega)}} \geq \beta_2\|q\|_{0,\Omega} \quad \forall q\in\mathrm{L}^2_0(\Omega). \tag{3.1}$$

3.2. The fixed-point method. In view of invoking Schauder's fixed-point theorem to establish solvability of (2.7), we introduce the following closed subset of the Banach space $\mathrm{L}^2(\Omega_T)$:

$$\mathcal{K}_\phi = \{\phi\in\mathrm{L}^2(\Omega_T)\ :\ 0\leq\phi(\boldsymbol{x},t)\leq e^{-(\lambda-\beta)t}k_m, \text{ for a.e. } (\boldsymbol{x},t)\in\Omega_T\}, \tag{3.2}$$

for $\phi\in\{s,c\}$, where $k_m \geq \sup\{\|c_0\|_{\mathrm{L}^\infty(\Omega)}, \|s_0\|_{\mathrm{L}^\infty(\Omega)}\}$, and λ,β are defined in (3.6) and Lemma 3.4, respectively. Next, and thanks to Theorem 3.1, we can assert the solvability of the Brinkman equations for a fixed $\hat{s}\in\mathcal{K}_s$ and for any $t>0$.

Lemma 3.2. *Assume that $\hat{s} \in \mathcal{K}_s$. Then, the variational problem*

$$\begin{aligned}
\int_\Omega \mathbb{K}^{-1}\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \sqrt{\mu}\int_\Omega \mathbf{curl}\,\boldsymbol{\omega}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} - \int_\Omega p\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} &= \int_\Omega (\hat{s}\boldsymbol{g}+\boldsymbol{f})\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x},\\
\sqrt{\mu}\int_\Omega \mathbf{curl}\,\boldsymbol{z}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{\omega}\cdot\boldsymbol{z}\,\mathrm{d}\boldsymbol{x} &= 0,\\
-\int_\Omega q\,\mathrm{div}\,\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} &= 0,
\end{aligned} \tag{3.3}$$

admits a unique solution $(\boldsymbol{u},\boldsymbol{\omega},p) \in \mathbf{H}(\mathrm{div};\Omega)\times\mathbf{H}(\mathbf{curl};\Omega)\times \mathrm{L}^2_0(\Omega)$. Moreover, there exists $C>0$ independent of μ such that

$$\|\boldsymbol{u}\|_{\mathbf{H}(\mathrm{div};\Omega)} + \|\boldsymbol{\omega}\|_{\mathbf{H}(\mathbf{curl};\Omega)} + \|p\|_{0,\Omega} \le C\big(\|\hat{s}\|_{0,\Omega}\|\boldsymbol{g}\|_{\infty,\Omega} + \|\boldsymbol{f}\|_{0,\Omega} + \|u_\partial\|_{-1/2,\partial\Omega} + \|\boldsymbol{\omega}_\partial\|_{-1/2,\partial\Omega}\big). \tag{3.4}$$

Proof. First, we observe that, owing to the inf-sup condition (3.1), problem (3.3) is equivalent to: Find $(\boldsymbol{u},\boldsymbol{\omega}) \in \mathrm{X}\times\mathbf{H}(\mathbf{curl};\Omega)$ such that

$$\begin{aligned}
\int_\Omega \mathbb{K}^{-1}\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \sqrt{\mu}\int_\Omega \mathbf{curl}\,\boldsymbol{\omega}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} &= \int_\Omega (\hat{s}\boldsymbol{g}+\boldsymbol{f})\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}, \qquad \forall \boldsymbol{v}\in\mathrm{X},\\
\sqrt{\mu}\int_\Omega \mathbf{curl}\,\boldsymbol{z}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{\omega}\cdot\boldsymbol{z}\,\mathrm{d}\boldsymbol{x} &= 0, \qquad \forall \boldsymbol{z}\in\mathbf{H}_0(\mathbf{curl};\Omega).
\end{aligned}$$

Then, the desired result follows from Theorem 3.1, repeating the argument employed in the proofs of [3, Theorem 2.2 and Corollary 2.1]. □

On the other hand, given a fixed velocity $\boldsymbol{u} \in \mathbf{L}^2(0,T;\mathbf{H}(\mathrm{div};\Omega))$, the following result establishes the solvability of the regularised reaction-diffusion system:

Lemma 3.3. *For any $\boldsymbol{u} \in \mathbf{L}^2(0,T;\mathbf{H}(\mathrm{div};\Omega))$, the system*

$$\begin{aligned}
\int_0^T \langle \partial_t c, m^c\rangle\,\mathrm{d}t + \iint_{\Omega_T} (D_c(c)\nabla c - c\boldsymbol{u})\cdot\nabla m^c\,\mathrm{d}\boldsymbol{x}\mathrm{d}t &= \iint_{\Omega_T} G_{c,\varepsilon}(c,s)m^c\,\mathrm{d}\boldsymbol{x}\mathrm{d}t,\\
\int_0^T \langle \partial_t s, m^s\rangle\,\mathrm{d}t + \iint_{\Omega_T} (D_s(s)\nabla s - s\boldsymbol{u})\cdot\nabla m^s\,\mathrm{d}\boldsymbol{x}\mathrm{d}t &= \iint_{\Omega_T} G_{s,\varepsilon}(c,s)m^s\,\mathrm{d}\boldsymbol{x}\mathrm{d}t,
\end{aligned} \tag{3.5}$$

is uniquely solvable and there exists $C>0$, depending on $\|c_0\|_{0,\Omega}$ and $\|s_0\|_{0,\Omega}$, such that

$$\|c\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))} + \|s\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))} \le C.$$

Proof. It suffices to combine assumptions (2.4), (2.6) with the general result for quasilinear parabolic problems given in [15, Section 5]. □

As the next step, we consider a constant $\lambda>0$ and define the auxiliary variables (ϕ_c,ϕ_s) by setting

$$c = e^{\lambda t}\phi_c \quad\text{and}\quad s = e^{\lambda t}\phi_s. \tag{3.6}$$

Then (c,s) satisfies the strong form of (3.5) with diffusion and reaction terms replaced, respectively, by

$$\begin{cases}
D_c(c) := D_c(e^{\lambda t}c) \text{ and } D_s(s) := D_s(e^{\lambda t}s),\\
G_{c,\varepsilon}(c,s) := -\lambda c + e^{-\lambda t}G_{c,\varepsilon}(e^{\lambda t}c, e^{\lambda t}s),\\
G_{s,\varepsilon}(c,s) := -\lambda s + e^{-\lambda t}G_{s,\varepsilon}(e^{\lambda t}c, e^{\lambda t}s).
\end{cases}$$

We now introduce a map $\mathcal{R}:\mathcal{K}_s\to\mathcal{K}_s$ such that $\mathcal{R}(\hat{s}) = s$, where s solves (3.5). The goal is to prove that such map has a fixed-point. First, let us show that $\mathcal{R}$ is a continuous mapping. Let $(\hat{s}_n)_n$ be a sequence in $\mathcal{K}_s$ and $\hat{s}\in\mathcal{K}_s$ be such that $\hat{s}_n \to \hat{s}$ in $\mathrm{L}^2(\Omega_T)$ as $n\to\infty$. Let us then define $s_n = \mathcal{R}(\hat{s}_n)$, *i.e.*, s_n is the solution of (3.5) associated with $\hat{s}_n$ and the solution $\boldsymbol{u}$ of (3.3). The objective is to show that s_n converges to $\mathcal{R}(\hat{s})$ in $\mathrm{L}^2(\Omega_T)$. We start with the following lemma:

Lemma 3.4. *Let $(c_n,s_n)_n$ be the solution to problem (3.5) and recall that $\hat{s}\in\mathcal{K}_s$. Then* *(i) There exists a constant $\gamma \geq 0$ such that $0 \leq c_n(\boldsymbol{x},t), s_n(\boldsymbol{x},t) \leq e^{\gamma t} k_m$, for a.e. $(\boldsymbol{x},t) \in \Omega_T$, where k_m is defined as in (3.2).*

(ii) The sequence $(c_n, s_n)_n$ is bounded in $\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)) \cap \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))$.

(iii) The sequence $(c_n, s_n)_n$ is relatively compact in $\mathrm{L}^2(\Omega_T)$.

Proof. (i) We replace the strong form of (3.5) by

$$\partial_t c_n - \operatorname{div}(D_c(c_n)\nabla c_n) + \boldsymbol{u}\cdot\nabla c_n = -\lambda c_n + e^{-\lambda t} G_{c,\varepsilon}(e^{\lambda t} c_n, e^{\lambda t} s_n), \qquad \text{in } \Omega_T. \tag{3.7}$$

Multiplying (3.7) by $-c_n^- = \dfrac{c_n - |c_n|}{2}$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\int_\Omega \left|c_n^-\right|^2 \mathrm{d}\boldsymbol{x} + D^{\min}\int_\Omega \left|\nabla c_n^-\right|^2 \mathrm{d}\boldsymbol{x} - \int_\Omega c_n^- \boldsymbol{u}\cdot\nabla c_n \mathrm{d}\boldsymbol{x} - \lambda\int_\Omega c_n\, c_n^-\,\mathrm{d}\boldsymbol{x} \leq -\int_\Omega e^{-\lambda t} G_{c,\varepsilon}(e^{\lambda t} c_n, e^{\lambda t} s_n) c_n^-\,\mathrm{d}\boldsymbol{x}. \tag{3.8}$$

Now, we use that $\operatorname{div}\boldsymbol{u} = 0$ and (2.5) to deduce

$$-\int_\Omega c_n^- \boldsymbol{u}\cdot\nabla c_n\,\mathrm{d}\boldsymbol{x} = \frac{1}{2}\int_\Omega \boldsymbol{u}\cdot\nabla (c_n^-)^2\,\mathrm{d}\boldsymbol{x} = 0,$$

and

$$\begin{cases} -\displaystyle\int_\Omega e^{-\lambda t} G_{c,\varepsilon}(e^{\lambda t} c_n, e^{\lambda t} s_n) c_n^-\,\mathrm{d}\boldsymbol{x} = -\int_\Omega e^{-\lambda t}\frac{\nu_1}{1+\varepsilon\left|G_c(e^{\lambda t} c_n, e^{\lambda t} s_n)\right|} c_n^-\,\mathrm{d}\boldsymbol{x} \leq 0 & \text{if } c_n \leq 0, \\ -\displaystyle\int_\Omega e^{-\lambda t} G_{c,\varepsilon}(e^{\lambda t} c_n, e^{\lambda t} s_n) c_n^-\,\mathrm{d}\boldsymbol{x} = 0 & \text{if } c_n > 0. \end{cases}$$

According to the positivity of the second and the fourth terms in the left-hand side of (3.8), we get

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\int_\Omega \left|c_n^-\right|^2\,\mathrm{d}\boldsymbol{x} \leq 0.$$

Since c_0 is nonnegative, we deduce that $c_n^- = 0$, and reasoning similarly we have that $s_n^- = 0$.

Next, we let $k_c, k_s \in \mathbb{R}$ such that $k_c \geq \|c_0\|_{\mathrm{L}^\infty(\Omega)}$ and $k_s \geq \|s_0\|_{\mathrm{L}^\infty(\Omega)}$. Let us consider $t \in (0,T)$, and proceed to multiply (3.7) by

$$\xi_c := (c_n - e^{-(\lambda-\beta)t} k)^+, \quad \text{with } \beta \geq \lambda, \text{ and with } k = \sup\{k_c, k_s\},$$

and to integrate over Ω, which yields that there exists some constant $C_6 > 0$ such that

$$\begin{aligned} &\frac{1}{2}\frac{d}{\mathrm{d}t}\int_\Omega \xi_c^2\,\mathrm{d}\boldsymbol{x} - (\lambda-\beta)\int_\Omega e^{-(\lambda-\beta)t} k\,\xi_c\,\mathrm{d}\boldsymbol{x} + D^{\min}\int_\Omega |\nabla\xi_c|^2\,\mathrm{d}\boldsymbol{x} + \int_\Omega \xi_c\,\boldsymbol{u}\cdot\nabla c_n\,\mathrm{d}\boldsymbol{x} + \lambda\int_\Omega c_n\,\xi_c \mathrm{d}\boldsymbol{x} \\ &= \frac{1}{2}\frac{d}{\mathrm{d}t}\int_\Omega \xi_c^2\,\mathrm{d}\boldsymbol{x} + \beta\int_\Omega e^{-(\lambda-\beta)t} k\,\xi_c\,\mathrm{d}\boldsymbol{x} + D^{\min}\int_\Omega |\nabla\xi_c|^2\,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_\Omega \boldsymbol{u}\cdot(\nabla\xi_c^2)\,\mathrm{d}\boldsymbol{x} + \lambda\int_\Omega \xi_c^2\,\mathrm{d}\boldsymbol{x} \\ &\leq \int_\Omega e^{-\lambda t} G_{c,\varepsilon}(e^{\lambda t} c_n, e^{\lambda t} s_n)\xi_c\,\mathrm{d}\boldsymbol{x} \leq C_6\left(\int_\Omega e^{-(\lambda-\beta)t} k\xi_c\,\mathrm{d}\boldsymbol{x} + \int_\Omega \xi_c^2\,\mathrm{d}\boldsymbol{x} + \int_\Omega \xi_s^2\,\mathrm{d}\boldsymbol{x}\right), \end{aligned} \tag{3.9}$$

where $\xi_s := (s_n - e^{-(\lambda-\beta)t} k)^+$. Following an analogous treatment for s_n, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{\mathrm{d}t}\int_\Omega \xi_s^2\,\mathrm{d}\boldsymbol{x} - (\lambda-\beta)\int_\Omega e^{-(\lambda-\beta)t} k\xi_s\,\mathrm{d}\boldsymbol{x} + D^{\min}\int_\Omega |\nabla\xi_s|^2\,\mathrm{d}\boldsymbol{x} + \int_\Omega \xi_s\boldsymbol{u}\cdot\nabla s_n\,\mathrm{d}\boldsymbol{x} + \lambda\int_\Omega s_n\xi_s\,\mathrm{d}\boldsymbol{x} \\ &\quad \leq C_7\left(\int_\Omega e^{-(\lambda-\beta)t} k\xi_s\,\mathrm{d}\boldsymbol{x} + \int_\Omega \xi_c^2\,\mathrm{d}\boldsymbol{x} + \int_\Omega \xi_s^2\,\mathrm{d}\boldsymbol{x}\right), \end{aligned} \tag{3.10}$$

for some constant $C_7 > 0$. We next observe that

$$\int_\Omega \boldsymbol{u}\cdot\nabla(\xi_c)^2\,\mathrm{d}\boldsymbol{x} = 0, \quad \int_\Omega \boldsymbol{u}\cdot\nabla(\xi_s)^2\,\mathrm{d}\boldsymbol{x} = 0,$$

which inserted into (3.9) and (3.10), implies that

$$\begin{aligned}\frac{1}{2}\frac{d}{\mathrm{d}t}\int_\Omega \xi_c^2\,\mathrm{d}\boldsymbol{x}+\frac{1}{2}\frac{d}{\mathrm{d}t}\int_\Omega \xi_s^2\,\mathrm{d}\boldsymbol{x}+(\beta-C_6-C_7)e^{-(\lambda-\beta)t}k\left(\int_\Omega \xi_c\,\mathrm{d}\boldsymbol{x}+\int_\Omega \xi_s\,\mathrm{d}\boldsymbol{x}\right)\\ +(\lambda-C_6-C_7)\int_\Omega \xi_c^2\,\mathrm{d}\boldsymbol{x}+(\lambda-C_6-C_7)\int_\Omega \xi_s^2\,\mathrm{d}\boldsymbol{x}\le 0.\end{aligned}\tag{3.11}$$

Finally for $\beta\ge\lambda\ge C_6+C_7$, an application of (3.11) yields

$$\frac{d}{\mathrm{d}t}\int_\Omega |\xi_c|^2\,\mathrm{d}\boldsymbol{x}+\frac{d}{\mathrm{d}t}\int_\Omega |\xi_s|^2\,\mathrm{d}\boldsymbol{x}\le 0,$$

and exploiting that $c_0, s_0\le k$ in Ω, we conclude that $c_n(\cdot,t)\le e^{-(\lambda-\beta)t}k$ and $s_n(\cdot,t)\le e^{-(\lambda-\beta)t}k$ in $\Omega\times(0,T)$.

(ii) We multiply equation (3.7) by c_n and integrate over Ω to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{\mathrm{d}t}\int_\Omega |c_n|^2\,\mathrm{d}\boldsymbol{x}+D^{\min}\int_\Omega |\nabla c_n|^2\,\mathrm{d}\boldsymbol{x}+\int_\Omega c_n\boldsymbol{u}\cdot\nabla c_n\,\mathrm{d}\boldsymbol{x}+\lambda\int_\Omega |c_n|^2\,\mathrm{d}\boldsymbol{x}\\ \le\int_\Omega e^{-\lambda t}G_{c,\varepsilon}(e^{\lambda t}c_n,e^{\lambda t}s_n)c_n\,\mathrm{d}\boldsymbol{x}.\end{aligned}\tag{3.12}$$

Invoking the boundedness of c_n and s_n, we get that the integral on the right-hand side is bounded independently of n. Then using that

$$\int_\Omega c_n\boldsymbol{u}\cdot\nabla c_n\,\mathrm{d}\boldsymbol{x}=\frac{1}{2}\int_\Omega \boldsymbol{u}\cdot\nabla(c_n)^2\,\mathrm{d}\boldsymbol{x}=0,$$

we deduce from (3.12) the following bound

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\int_\Omega |c_n|^2\,\mathrm{d}\boldsymbol{x}+D^{\min}\int_\Omega |\nabla c_n|^2\,\mathrm{d}\boldsymbol{x}\le C_8,$$

valid for some constant $C_8>0$ independent of n. This completes the proof of (ii).

(iii) Next we multiply (3.7) by $\varphi\in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))$ and use the boundedness of c_n and s_n to get

$$\begin{aligned}\left|\int_0^T\langle\partial_t c_n,\varphi\rangle\,\mathrm{d}t\right| &=\Bigg|-\int_\Omega D_c(c_n)\nabla c_n\cdot\varphi\,\mathrm{d}\boldsymbol{x}+\int_\Omega c_n\boldsymbol{u}\cdot\nabla\varphi\,\mathrm{d}\boldsymbol{x}\\ &\qquad -\lambda\int_\Omega c_n\varphi\,\mathrm{d}\boldsymbol{x}+\int_\Omega e^{-\lambda t}G_{c,\varepsilon}(e^{\lambda t}c_n,e^{\lambda t}s_n)\varphi\,\mathrm{d}\boldsymbol{x}\Bigg|\\ &\le D^{\max}\|\nabla c_n\|_{\mathrm{L}^2(\Omega_T)}\|\varphi\|_{\mathrm{L}^2(\Omega_T)}+C_9\|\boldsymbol{u}\|_{0,\Omega}\|\nabla\varphi\|_{\mathrm{L}^2(\Omega_T)}\\ &\qquad +C_{10}\Big(\|c_n\|_{\mathrm{L}^2(\Omega_T)}+\|s_n\|_{\mathrm{L}^2(\Omega_T)}\Big)\|\varphi\|_{\mathrm{L}^2(\Omega_T)}\\ &\le C_{11}\|\varphi\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))},\end{aligned}$$

for some constants $C_9, C_{10}, C_{11}>0$ independent of n. Consequently, we end up with the bound

$$\|\partial_t c_n\|_{\mathrm{L}^2(0,T;(H^1(\Omega))')}\le C_{11}.\tag{3.13}$$

Working on the same lines as for $(c_n)_n$, the counterpart of (ii) and (3.13) for $(s_n)_n$ can be obtained.

Finally, (iii) is deduced from (ii) and the uniform boundedness of $(\partial_t c_n)_n$ and $(\partial_t s_n)_n$ in $\mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega)')$. $\square$

In summary, Lemmas 3.2, and 3.4 imply that there exists $(c,s,\boldsymbol{u},\boldsymbol{\omega},p)\in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))\times\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))\times\mathbf{H}(\mathrm{div};\Omega)\times\mathbf{H}(\mathbf{curl};\Omega)\times\mathrm{L}^2_0(\Omega)$ such that, up to extracting subsequences if necessary,

$$\begin{cases}(c_n,s_n)\to(c,s)\text{ in }\mathrm{L}^2(\Omega_T,\mathbb{R}^2)\text{ strongly and a.e., and in }\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))\text{ weakly},\\ (\boldsymbol{u}_n,\boldsymbol{\omega}_n,p_n)\to(\boldsymbol{u},\boldsymbol{\omega},p)\text{ in }\mathbf{H}(\mathrm{div};\Omega)\times\mathbf{H}(\mathbf{curl};\Omega)\times\mathrm{L}^2_0(\Omega)\text{ weakly},\end{cases}$$

and consequently, the continuity of $\mathcal{R}$ on $\mathcal{K}_s$ holds. Moreover, Lemma 3.4 indicates the boundedness of $\mathcal{R}(\mathcal{K}_s)$ within the set

$$\mathcal{S} = \left\{ s \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)) \ : \ \partial_t s \in \mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega))') \right\}. \tag{3.14}$$

Appealing to the theory of compact sets [23], the compactness of the map $\mathcal{S} \hookrightarrow \mathrm{L}^2(\Omega_T)$ implies that of $\mathcal{R}$. Therefore, and thanks to Schauder's fixed-point theorem, the operator $\mathcal{R}$ has a fixed point s. That is, there exists a solution to

$$\begin{aligned}
\int_0^T \langle \partial_t c_\varepsilon, m^c \rangle \,\mathrm{d}t + \iint_{\Omega_T} (D_c(c_\varepsilon)\nabla c_\varepsilon - c_\varepsilon \boldsymbol{u}_\varepsilon) \cdot \nabla m^c \,\mathrm{d}\boldsymbol{x}\mathrm{d}t &= \iint_{\Omega_T} G_{c,\varepsilon}(c_\varepsilon, s_\varepsilon) m^c \,\mathrm{d}\boldsymbol{x}\mathrm{d}t,\\
\int_0^T \langle \partial_t s_\varepsilon, m^s \rangle \,\mathrm{d}t + \iint_{\Omega_T} (D_s(s_\varepsilon)\nabla s_\varepsilon - s_\varepsilon \boldsymbol{u}_\varepsilon) \cdot \nabla m^s \,\mathrm{d}\boldsymbol{x}\mathrm{d}t &= \iint_{\Omega_T} G_{s,\varepsilon}(c_\varepsilon, s_\varepsilon) m^s \,\mathrm{d}\boldsymbol{x}\mathrm{d}t,\\
\iint_{\Omega_T} \mathbb{K}^{-1}\boldsymbol{u}_\varepsilon \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}\mathrm{d}t + \sqrt{\mu}\iint_{\Omega_T} \boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{\omega}_\varepsilon \,\mathrm{d}\boldsymbol{x}\mathrm{d}t - \iint_{\Omega_T} p_\varepsilon \,\mathrm{div}\,\boldsymbol{v} \,\mathrm{d}\boldsymbol{x}\mathrm{d}t &= \iint_{\Omega_T} (s_\varepsilon \boldsymbol{g} + \boldsymbol{f}) \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}\mathrm{d}t,\\
\sqrt{\mu}\iint_{\Omega_T} \boldsymbol{u}_\varepsilon \cdot \mathbf{curl}\,\boldsymbol{z} \,\mathrm{d}\boldsymbol{x}\mathrm{d}t - \iint_{\Omega_T} \boldsymbol{\omega}_\varepsilon \cdot \boldsymbol{z} \,\mathrm{d}\boldsymbol{x}\mathrm{d}t &= 0,\\
-\iint_{\Omega_T} q \,\mathrm{div}\,\boldsymbol{u}_\varepsilon \,\mathrm{d}\boldsymbol{x}\mathrm{d}t &= 0,
\end{aligned} \tag{3.15}$$

for all $m^i \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))$, $\boldsymbol{v} \in \mathbf{L}^2(0,T;\mathbf{H}_0(\mathrm{div};\Omega))$, $\boldsymbol{z} \in \mathbf{L}^2(0,T;\mathbf{H}_0(\mathbf{curl};\Omega))$, $q \in \mathrm{L}^2(0,T;\mathrm{L}^2_0(\Omega))$.

4. Existence of weak solutions. We have shown in Section 3 that problem (2.7) admits a solution $(c_\varepsilon, s_\varepsilon, \boldsymbol{\omega}_\varepsilon, \boldsymbol{u}_\varepsilon, p_\varepsilon)$. The goal in this section is to send the regularisation parameter ε to zero in sequences of such solutions to obtain weak solutions of the original system (2.1)-(2.2). Observe that, for each fixed $\varepsilon > 0$, we have shown the existence of a solution $(c_\varepsilon, s_\varepsilon)$ to (2.7) such that

$$0 \le c_\varepsilon(\boldsymbol{x},t), s_\varepsilon(\boldsymbol{x},t) \le e^{\gamma t} k_m, \quad \gamma > 0, \quad \text{a.e. in } (\boldsymbol{x},t) \in \Omega_T. \tag{4.1}$$

Using the Brinkman equation in (2.7), it is easy to see that the estimates of (3.4) in Lemma 3.2 do not depend on ε, i.e., there exists $C_{12} > 0$ independent of ε such that

$$\|\boldsymbol{u}_\varepsilon\|_{\mathbf{H}(\mathrm{div};\Omega)} + \|\boldsymbol{\omega}_\varepsilon\|_{\mathbf{H}(\mathbf{curl};\Omega)} + \|p_\varepsilon\|_{0,\Omega} \le C_{12}\big(\|\boldsymbol{g}\|_{\infty,\Omega} + \|\boldsymbol{f}\|_{0,\Omega} + \|u_\partial\|_{-1/2,\partial\Omega} + \|\boldsymbol{\omega}_\partial\|_{-1/2,\partial\Omega}\big).$$

Taking $m^c = c_\varepsilon$ and $m^s = s_\varepsilon$ as test functions in (3.15) and working as in the proof of (ii) in Lemma 3.4, we readily obtain

$$\sup_{0\le t\le T}\int_\Omega |c_\varepsilon(\boldsymbol{x},t)|^2 \,\mathrm{d}\boldsymbol{x} + \sup_{0\le t\le T}\int_\Omega |s_\varepsilon(\boldsymbol{x},t)|^2 \,\mathrm{d}\boldsymbol{x} + \iint_{\Omega_T} |\nabla c_\varepsilon|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \iint_{\Omega_T} |\nabla s_\varepsilon|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \le C_{13},$$

for some constant $C_{13} > 0$ independent of ε. Repeating the steps of the proof of (iii) in Lemma 3.4, we derive the bound

$$\|\partial_t c_\varepsilon\|_{\mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega))')} + \|\partial_t s_\varepsilon\|_{\mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega))')} \le C_{14}, \tag{4.2}$$

for some constant $C_{14} > 0$. Then, combining (4.1)-(4.2) with standard compactness results (*cf.* [23]) we can extract subsequences, which we do not relabel, such that, as ε goes to 0,

$$\begin{cases}
(c_\varepsilon, s_\varepsilon) \to (c,s) \text{ weakly } -\star \text{ in } \mathrm{L}^\infty(\Omega_T, \mathbb{R}^2) \text{ and in } \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega,\mathbb{R}^2)) \text{ weakly},\\
(\boldsymbol{u}_\varepsilon, \boldsymbol{\omega}_\varepsilon, p_\varepsilon) \to (\boldsymbol{u}, \boldsymbol{\omega}, p) \text{ in } \mathbf{H}(\mathrm{div};\Omega) \times \mathbf{H}(\mathbf{curl};\Omega) \times \mathrm{L}^2_0(\Omega) \text{ weakly},\\
\partial_t c_\varepsilon \to \partial_t c \text{ weakly in } \mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega))'), \quad \text{and}\\
\partial_t s_\varepsilon \to \partial_t s \text{ weakly in } \mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega))').
\end{cases} \tag{4.3}$$

It is then evident that c_ε and s_ε are uniformly bounded in the set $\mathcal{S}$ defined in (3.14). Then, from the compact embedding $\mathcal{S} \hookrightarrow \mathrm{L}^2(\Omega_T)$ we deduce that there exist subsequences of c_ε and s_ε such that

$$c_\varepsilon \to c \text{ and } s_\varepsilon \to s \text{ strongly in } \mathrm{L}^2(\Omega_T) \text{ and a.e. in } \Omega_T. \tag{4.4}$$

This result, together with the weak-$\star$ convergences in $\mathrm{L}^\infty(\Omega_T)$ of c_ε and s_ε to c and s, respectively, yields

$$c_\varepsilon \to c \text{ and } s_\varepsilon \to s \text{ strongly in } \mathrm{L}^p(\Omega_T) \text{ for } 1 \le p < \infty, \tag{4.5}$$

which in turn implies that

$$G_{c,\varepsilon}(c_\varepsilon, s_\varepsilon) \to G_c(c,s), \qquad G_{s,\varepsilon}(c_\varepsilon, s_\varepsilon) \to G_s(c,s), \tag{4.6}$$

a.e. in Ω_T and strongly in $\mathrm{L}^p(\Omega_T)$ for $1 \le p < \infty$. As a consequence of (4.3), (4.4), (4.5) and (4.6), it is clear that as $\varepsilon \to 0$, we can identify the limit as the weak solution from Definition 2.1.

5. Uniqueness of weak solutions. The following result completes the well-posedness analysis for the reaction-diffusion-Brinkman system.

THEOREM 5.1. *Assume (2.3)-(2.5) hold, and let $(c_1, s_1, \boldsymbol{u}_1, \boldsymbol{\omega}_1, p_1)$ and $(c_2, s_2, \boldsymbol{u}_2, \boldsymbol{\omega}_2, p_2)$ be two weak solutions to (2.1)-(2.2), associated with the data $c_0 = c_{1,0}$, $s_0 = s_{1,0}$, $\boldsymbol{g} = \boldsymbol{g}_1$, $\boldsymbol{f} = \boldsymbol{f}_1$ and $c_0 = c_{2,0}$, $s_0 = s_{2,0}$, $\boldsymbol{g} = \boldsymbol{g}_2$, $\boldsymbol{f} = \boldsymbol{f}_2$, respectively. Then, for any $t \in [0,T]$ there exists $C > 0$ such that*

$$\begin{aligned}&\iint_{\Omega_t} \Big(|\mathcal{C}|^2 + |\mathcal{S}|^2 + |\mathcal{U}|^2 + |\mathcal{W}|^2 + |\mathcal{P}|^2 \Big) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma \\ &\quad \le C \left(\int_\Omega \Big(|c_{1,0}(\boldsymbol{x}) - c_{2,0}(\boldsymbol{x})|^2 + |s_{1,0}(\boldsymbol{x}) - s_{2,0}(\boldsymbol{x})|^2 \Big) \,\mathrm{d}\boldsymbol{x} + \iint_{\Omega_t} \Big(|\boldsymbol{g}_1 - \boldsymbol{g}_2|^2 + |\boldsymbol{f}_1 - \boldsymbol{f}_2|^2 \Big) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma \right),\end{aligned} \tag{5.1}$$

where $\mathcal{C} = c_1 - c_2$, $\mathcal{S} = s_1 - s_2$, $\mathcal{U} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, $\mathcal{W} = \boldsymbol{\omega}_1 - \boldsymbol{\omega}_2$, and $\mathcal{P} = p_1 - p_2$. In particular, there exists at most one weak solution to the reaction-diffusion-Brinkman system (2.1)-(2.2).

Proof. First we observe that the pair $(\mathcal{U}, \mathcal{W}, \mathcal{P})$ satisfies

$$\begin{aligned}&\iint_{\Omega_t} \mathbb{K}^{-1}\mathcal{U} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma + \sqrt{\mu} \iint_{\Omega_t} \boldsymbol{v} \cdot \mathbf{curl}\, \mathcal{W} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma - \iint_{\Omega_t} \mathcal{P} \,\mathrm{div}\, \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma \\ &\qquad = \iint_{\Omega_t} (s_1 \boldsymbol{g}_1 - s_2 \boldsymbol{g}_2) \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma + \iint_{\Omega_t} (\boldsymbol{f}_1 - \boldsymbol{f}_2) \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma\,, \\ &\sqrt{\mu} \iint_{\Omega_t} \mathcal{U} \cdot \mathbf{curl}\, \boldsymbol{z} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma - \iint_{\Omega_t} \mathcal{W} \cdot \boldsymbol{z} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma = 0, \\ &- \iint_{\Omega_t} q \,\mathrm{div}\, \mathcal{U} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma = 0,\end{aligned} \tag{5.2}$$

for $t \in (0,T)$ and for all $\boldsymbol{v} \in \mathbf{L}^2(0,T; \mathbf{H}_0(\mathrm{div};\Omega))$, $\boldsymbol{z} \in \mathbf{L}^2(0,T; \mathbf{H}_0(\mathbf{curl};\Omega))$, $q \in \mathrm{L}^2(0,T; \mathrm{L}^2_0(\Omega))$.

After substituting $\boldsymbol{v} = \mathcal{U}$, $z = \mathcal{W}$, $q = \mathcal{P}$ in (5.2) and adding the resulting equations, we can invoke the continuous dependence on the data (3.4) to establish the *a priori* bound

$$\begin{aligned}&\iint_{\Omega_t} |\mathcal{U}|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma + \iint_{\Omega_t} |\mathcal{W}|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma + \iint_{\Omega_t} |\mathcal{P}|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma \\ &\quad \le C_{14} \iint_{\Omega_t} |\mathcal{S}|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma + C_{15} \left(\iint_{\Omega_t} |\boldsymbol{g}_1 - \boldsymbol{g}_2|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma + \iint_{\Omega_t} |\boldsymbol{f}_1 - \boldsymbol{f}_2|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma \right),\end{aligned} \tag{5.3}$$

for some constant $C_{15} > 0$. Next, note that $(\mathcal{C}, \mathcal{S})$ satisfies

$$\begin{aligned}&- \int_0^t \langle \mathcal{C}, \partial_t m^c \rangle \,\mathrm{d}\sigma + \iint_{\Omega_t} (D_c(c_1)\nabla c_1 - D_c(c_2)\nabla c_2) \cdot \nabla m^c \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma - \iint_{\Omega_t} c_1 \mathcal{U} \cdot \nabla m^c \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma \\ &\qquad - \iint_{\Omega_t} \mathcal{C} \boldsymbol{u}_2 \cdot \nabla m^c \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma = \int_\Omega \mathcal{C}_0(\boldsymbol{x}) m^c(\boldsymbol{x},0) \,\mathrm{d}\boldsymbol{x} + \iint_{\Omega_t} (G_c(c_1,s_1) - G_c(c_2,s_2)) m^c \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma\,, \\ &- \int_0^t \langle \mathcal{S}, \partial_t m^s \rangle \,\mathrm{d}\sigma + \iint_{\Omega_t} (D_s(s_1)\nabla s_1 - D_s(s_2)\nabla s_2) \cdot \nabla m^s \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma - \iint_{\Omega_t} s_1 \mathcal{U} \cdot \nabla m^s \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma \\ &\qquad - \iint_{\Omega_t} \mathcal{S} \boldsymbol{u}_2 \cdot \nabla m^s \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma = \int_\Omega \mathcal{S}_0(\boldsymbol{x}) m^c(\boldsymbol{x},0) \,\mathrm{d}\boldsymbol{x} + \iint_{\Omega_t} (G_s(c_1,s_1) - G_s(c_2,s_2)) m^s \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma\,,\end{aligned} \tag{5.4}$$

for $t \in (0,T)$ and for all $m^i \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))$ and $\partial_t m^i \in \mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega))')$ with $m^c(\cdot,T)=0$, $i=c,s$. Now, for $t_0 \in (0,T)$ we take

$$m^i(\boldsymbol{x},t) = \begin{cases} \displaystyle\int_t^{t_0} (\mathcal{D}_i(i_1) - \mathcal{D}_i(i_2))\,\mathrm{d}\sigma & \text{for } 0 \le t < t_0, \\ 0 \text{ for } t_0 \le t \le T, \end{cases}$$

with $\mathcal{D}_i(i) = \displaystyle\int_0^i D_i(r)\,dr$, for $i=c,s$. Using these relations in (5.4), and recalling that the function $\mathcal{D}$ is strictly increasing, we get

$$-\int_0^T \langle \mathcal{C}, \partial_t m^c\rangle\,\mathrm{d}\sigma = \int_0^{t_0} \langle \mathcal{C}, (\mathcal{D}_c(c_1) - \mathcal{D}_c(c_2))\rangle\,\mathrm{d}\sigma \ge C_{\mathcal{D}} \iint_{\Omega_{t_0}} |\mathcal{C}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma\,, \tag{5.5}$$

$$\iint_{\Omega_t} (D_c(c_1)\nabla c_1 - D_c(c_2)\nabla c_2)\cdot\nabla m^c\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma = \frac{1}{2}\int_\Omega \left|\nabla \int_0^{t_0} (\mathcal{D}_c(c_1) - \mathcal{D}_c(c_2))\,\mathrm{d}\sigma\right|^2 \mathrm{d}\boldsymbol{x},$$

$$\int_\Omega \mathcal{C}_0(\boldsymbol{x}) m^c(\boldsymbol{x},0)\,\mathrm{d}\boldsymbol{x} = \int_t^{t_0}\int_\Omega \mathcal{C}_0(\boldsymbol{x})(\mathcal{D}_c(c_1) - \mathcal{D}_c(c_2))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma \le C_{16}\int_\Omega |\mathcal{C}_0(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} + \frac{C_{\mathcal{D}}}{4}\iint_{\Omega_{t_0}} |\mathcal{C}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma\,,$$

for some constants $C_{\mathcal{D}}, C_{16} > 0$. Moreover,

$$\iint_{\Omega_t} c_1 \mathcal{U}\cdot\nabla m^c\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma \le C_{17}\int_0^{t_0}\iint_{\Omega_t} |\mathcal{U}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma\,\mathrm{d}t + \frac{1}{8}\int_\Omega \left|\nabla\int_0^{t_0} (\mathcal{D}_c(c_1) - \mathcal{D}_c(c_2))\,\mathrm{d}\sigma\right|^2 \mathrm{d}\boldsymbol{x}, \tag{5.6}$$

$$\iint_{\Omega_t} \mathcal{C}\boldsymbol{u}_2\cdot\nabla m^c\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma \le C_{18}\int_0^{t_0}\iint_{\Omega_t} |\mathcal{C}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma\,\mathrm{d}t + \frac{1}{8}\int_\Omega \left|\nabla\int_0^{t_0} (\mathcal{D}_c(c_1) - \mathcal{D}_c(c_2))\,\mathrm{d}\sigma\right|^2 \mathrm{d}\boldsymbol{x}, \tag{5.7}$$

for some constants $C_{17}, C_{18} > 0$ depending on $\|c_1\|_{\mathrm{L}^\infty(\Omega_T)}$ and $\|\boldsymbol{u}_2\|_{\mathrm{L}^\infty(\Omega_T,\mathbb{R}^3)}$ (the latter being bounded thanks to classical maximal regularity results for Stokes equations *cf.* [25]). Next, we use the L^∞-bounds of (c_1,s_1) and (c_2,s_2), leading to

$$\begin{aligned}\iint_{\Omega_t} (G_s(c_1,s_1) - G_s(c_2,s_2)) m^s\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma \le C_{19}\int_0^{t_0}\Big(\iint_{\Omega_t} (|\mathcal{C}|^2 + |\mathcal{S}|^2)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma\Big)\,\mathrm{d}t \\ + \frac{C_{\mathcal{D}}}{4}\iint_{\Omega_{t_0}} |\mathcal{C}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma\,,\end{aligned} \tag{5.8}$$

for some constant $C_{19} > 0$. We proceed to collect the results (5.3) and (5.5)-(5.8), to infer that

$$\begin{aligned}\frac{C_{\mathcal{D}}}{2}\iint_{\Omega_{t_0}} |\mathcal{C}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma \le C_{16}\int_\Omega |\mathcal{C}_0(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} + C_{20}\Big(\iint_{\Omega_t} |\boldsymbol{g}_1 - \boldsymbol{g}_2|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma + \iint_{\Omega_t} |\boldsymbol{f}_1 - \boldsymbol{f}_2|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma\Big) \\ + C_{21}\int_0^{t_0}\Big(\iint_{\Omega_t} (|\mathcal{C}|^2 + |\mathcal{S}|^2)\,\mathrm{d}\boldsymbol{x}\mathrm{d}\sigma\Big)\,\mathrm{d}t,\end{aligned} \tag{5.9}$$

for some constants $C_{20}, C_{21} > 0$. Similarly we get for the equation of $\mathcal{S}$

$$\begin{aligned}\frac{C_{\mathcal{D}}}{2}\iint_{\Omega_{t_0}} |\mathcal{S}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma \le C_{22}\int_\Omega |\mathcal{S}_0(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} + C_{23}\Big(\iint_{\Omega_t} |\boldsymbol{g}_1 - \boldsymbol{g}_2|^2\,\mathrm{d}\boldsymbol{x}\mathrm{d}\sigma + \iint_{\Omega_t} |\boldsymbol{f}_1 - \boldsymbol{f}_2|^2\,\mathrm{d}\boldsymbol{x}\mathrm{d}\sigma\Big) \\ + C_{24}\int_0^{t_0}\Big(\iint_{\Omega_t} (|\mathcal{C}|^2 + |\mathcal{S}|^2)\,\mathrm{d}\boldsymbol{x}\mathrm{d}\sigma\Big)\,\mathrm{d}t,\end{aligned} \tag{5.10}$$

for some constants $C_{22}, C_{23}, C_{24} > 0$. Thus (5.9) and (5.10) imply

$$\begin{aligned}\iint_{\Omega_{t_0}} \Big(|\mathcal{C}|^2 + |\mathcal{S}|^2\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\sigma \le C_{25}\int_\Omega \Big(|\mathcal{C}_0(\boldsymbol{x})|^2 + |\mathcal{S}_0(\boldsymbol{x})|^2\Big)\,\mathrm{d}\boldsymbol{x} + C_{26}\int_0^{t_0}\Big(\iint_{\Omega_t} (|\mathcal{C}|^2 + |\mathcal{S}|^2)\,\mathrm{d}\boldsymbol{x}\mathrm{d}\sigma\Big)\,\mathrm{d}t \\ + C_{27}\Big(\iint_{\Omega_t} |\boldsymbol{g}_1 - \boldsymbol{g}_2|^2\,\mathrm{d}\boldsymbol{x}\mathrm{d}\sigma + \iint_{\Omega_t} |\boldsymbol{f}_1 - \boldsymbol{f}_2|^2\,\mathrm{d}\boldsymbol{x}\mathrm{d}\sigma\Big),\end{aligned} \tag{5.11}$$

for some constants $C_{25}, C_{26}, C_{27} > 0$. Then, an application of Gronwall's inequality to (5.11) combined with (5.3) proves (5.1). ∎

6. Numerical approximation. In this section, we construct a primal-mixed fully-discrete scheme for the approximation of the coupled system (2.1). The finite element spaces yielding unique solvability of the semi-discrete problem are then specified, and a proof of convergence to the unique weak solution (presented in Definition 2.1) is outlined.

6.1. The semi-discrete scheme. Let $\mathcal{T}_h$ be a regular family of triangulations of $\bar{\Omega}$ by tetrahedra K of maximum diameter h. Given an integer $k \geq 0$ and $S \subset \mathbb{R}^3$, by $\mathcal{P}_k(S)$ we denote the space of polynomial functions defined in S of total degree up to k, and define the following finite element subspaces

$$\begin{aligned}&\mathrm{V}_h = \{m_h \in \mathrm{H}^1(\Omega) : m_h|_K \in \mathcal{P}_1(K)\,\forall K \in \mathcal{T}_h\},\ \mathbf{H}_h = \{\boldsymbol{v}_h \in \mathbf{H}(\mathrm{div};\Omega) : \boldsymbol{v}_h|_K \in \mathbf{RT}_0(K)\,\forall K \in \mathcal{T}_h\},\\ &\mathbf{Z}_h = \{\boldsymbol{z}_h \in \mathbf{H}(\mathbf{curl},\Omega) : \boldsymbol{z}_h|_K \in \mathbf{ND}_1(K)\,\forall K \in \mathcal{T}_h\},\ \mathrm{Q}_h = \{q_h \in \mathrm{L}^2_0(\Omega) : q_h|_K \in \mathcal{P}_0(K)\,\forall K \in \mathcal{T}_h\},\end{aligned} \tag{6.1}$$

where for any $K \in \mathcal{T}_h(\Omega)$, the local lowest order Raviart-Thomas and edge space of Nédélec type, are defined as $\mathbf{RT}_0(K) = \mathcal{P}_0(K)^3 \oplus \mathcal{P}_0(K)\boldsymbol{x}$, and $\mathbf{ND}_1(K) = \mathcal{P}_0(K)^3 \oplus \mathcal{P}_0(K)^3 \times \boldsymbol{x}$, respectively.

Then, a Galerkin semi-discretisation associated to the formulation introduced in Definition 2.1 reads: For $t \in (0,T]$ find $(c_h(t), s_h(t), \boldsymbol{u}_h(t), \boldsymbol{\omega}_h(t), p_h(t)) \in \mathrm{V}_h \times \mathrm{V}_h \times \mathbf{H}_h \times \mathbf{Z}_h \times \mathrm{Q}_h$ such that

$$\begin{aligned}\int_\Omega \partial_t c_h(t) m_h^c \,\mathrm{d}\boldsymbol{x} + \int_\Omega (D_c(c_h(t))\nabla c_h(t) - c_h(t)\boldsymbol{u}_h)\cdot\nabla m_h^c \,\mathrm{d}\boldsymbol{x} &= \int_\Omega G_c(c_h(t), s_h(t)) m_h^c \,\mathrm{d}\boldsymbol{x},\\ \int_\Omega \partial_t s_h(t) m_h^s \,\mathrm{d}\boldsymbol{x} + \int_\Omega (D_s(s_h(t))\nabla s_h(t) - s_h(t)\boldsymbol{u}_h)\cdot\nabla m_h^s \,\mathrm{d}\boldsymbol{x} &= \int_\Omega G_s(c_h(t), s_h(t)) m_h^s \,\mathrm{d}\boldsymbol{x},\\ \int_\Omega \mathbb{K}^{-1}\boldsymbol{u}_h(t)\cdot\boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} + \sqrt{\mu}\int_\Omega \mathbf{curl}\,\boldsymbol{v}_h\cdot\boldsymbol{\omega}_h(t)\,\mathrm{d}\boldsymbol{x} - \int_\Omega p_h(t)\,\mathrm{div}\,\boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} &= \int_\Omega (s_h(t)\boldsymbol{g} + \boldsymbol{f})\cdot\boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x},\\ \sqrt{\mu}\int_\Omega \mathbf{curl}\,\boldsymbol{u}_h(t)\cdot\boldsymbol{z}_h \,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{\omega}_h(t)\cdot\boldsymbol{z}_h \,\mathrm{d}\boldsymbol{x} &= 0,\\ -\int_\Omega q_h \,\mathrm{div}\,\boldsymbol{u}_h(t)\,\mathrm{d}\boldsymbol{x} &= 0,\end{aligned} \tag{6.2}$$

for all $m_h^c, m_h^s \in \mathrm{V}_h$, $\boldsymbol{v}_h \in \mathbf{H}_h$, $\boldsymbol{z}_h \in \mathbf{Z}_h$, and $q_h \in \mathrm{Q}_h$.

6.2. Euler time discretisation. Let $c_h^0 = \mathbb{P}_{\mathrm{V}_h}(c_0)$, $s_h^0 = \mathbb{P}_{\mathrm{V}_h}(s_0)$ be appropriate projections of the initial data, and consider the following fully discrete method arising after backward Euler time discretisation using a fixed time step $\Delta t = T/N$: For $n \in \{1,\ldots,N\}$, find $(c_h^n, s_h^n, \boldsymbol{u}_h^n, \boldsymbol{\omega}_h^n, p_h^n) \in \mathrm{V}_h \times \mathrm{V}_h \times \mathbf{H}_h \times \mathbf{Z}_h \times \mathrm{Q}_h$ such that

$$\begin{aligned}\int_\Omega \frac{c_h^n - c_h^{n-1}}{\Delta t} m_h^c \,\mathrm{d}\boldsymbol{x} + \int_\Omega (D_c(c_h^n)\nabla c_h^n - c_h^n\boldsymbol{u}_h^n)\cdot\nabla m_h^c \,\mathrm{d}\boldsymbol{x} &= \int_\Omega G_c(c_h^{n-1}, s_h^{n-1}) m_h^c \,\mathrm{d}\boldsymbol{x},\\ \int_\Omega \frac{s_h^n - s_h^{n-1}}{\Delta t} m_h^c \,\mathrm{d}\boldsymbol{x} + \int_\Omega (D_s(s_h^n)\nabla s_h^n - s_h^n\boldsymbol{u}_h^n)\cdot\nabla m_h^s \,\mathrm{d}\boldsymbol{x} &= \int_\Omega G_s(c_h^{n-1}, s_h^{n-1}) m_h^s \,\mathrm{d}\boldsymbol{x},\\ \int_\Omega \mathbb{K}^{-1}\boldsymbol{u}_h^n\cdot\boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} + \sqrt{\mu}\int_\Omega \boldsymbol{v}_h\cdot\mathbf{curl}\,\boldsymbol{\omega}_h^n \,\mathrm{d}\boldsymbol{x} - \int_\Omega p_h^n \,\mathrm{div}\,\boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} &= \int_\Omega (s_h^n\boldsymbol{g} + \boldsymbol{f})\cdot\boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x},\\ \sqrt{\mu}\int_\Omega \boldsymbol{u}_h^n\cdot\mathbf{curl}\,\boldsymbol{z}_h \,\mathrm{d}\boldsymbol{x}\mathrm{d}t - \int_\Omega \boldsymbol{\omega}_h^n\cdot\boldsymbol{z}_h \,\mathrm{d}\boldsymbol{x} &= 0,\\ -\int_\Omega q_h \,\mathrm{div}\,\boldsymbol{u}_h^n \,\mathrm{d}\boldsymbol{x} &= 0,\end{aligned} \tag{6.3}$$

for all $m_h^c, m_h^s \in \mathrm{V}_h$, $\boldsymbol{v}_h \in \mathbf{H}_h$, $\boldsymbol{z}_h \in \mathbf{Z}_h$, $q_h \in \mathrm{Q}_h$, where the forcing term in the momentum equation is discretised explicitly. We also stress that, owing to the choice (6.1), the discrete velocities $\boldsymbol{u}_h$ generated using either (6.2) or (6.3) are exactly divergence-free, that is, $\mathrm{div}\,\boldsymbol{u}_h(t) = 0$ and $\mathrm{div}\,\boldsymbol{u}_h^n = 0$ in Ω (*cf.* [3]).

6.3. Sketched convergence analysis. The convergence of solutions generated by (6.3) is next established following two main lemmas. The first one states the non-negativity and boundedness of the discrete concentrations (s_h, c_h).

LEMMA 6.1. *Let (s_h^n, c_h^n) be part of the solution of (6.3). Then there exists a constant $M > 0$ depending on $\|c_0\|_{\mathrm{L}^\infty(\Omega)}$ and $\|s_0\|_{\mathrm{L}^\infty(\Omega)}$ such that, for all $n = 1, \ldots, N$*

$$0 \le s_h^n, c_h^n \le M.$$

Proof. Let us show by induction in n that $c_h^n, s_h^n \ge 0$. The claim is true for $n = 0$. We assume that $c_h^k, s_h^k \ge 0$ for $k = 1, \ldots, n-1$. Testing the first equation of (6.3) by $-c_h^{n-} = \dfrac{c_h^n - |c_h^n|}{2}$, we readily get

$$-\int_\Omega \frac{c_h^n - c_h^{n-1}}{\Delta t} c_h^{n-} \, \mathrm{d}\boldsymbol{x} + D^{\min} \int_\Omega \left|\nabla c_h^{n-}\right|^2 \mathrm{d}\boldsymbol{x} - \int_\Omega c_h^{n-} \boldsymbol{u}_h^n \cdot \nabla c_h^n \, \mathrm{d}\boldsymbol{x} \le -\int_\Omega G_c(c_h^{n-1}, s_h^{n-1}) c_h^{n-} \, \mathrm{d}\boldsymbol{x}. \quad (6.4)$$

Now, we use that $\operatorname{div} \boldsymbol{u}_h^n = 0$ in Ω and relation (2.5) to deduce

$$-\int_\Omega c_h^{n-} \boldsymbol{u}_h^n \cdot \nabla c_h^n \, \mathrm{d}\boldsymbol{x} = \frac{1}{2} \int_\Omega \boldsymbol{u}_h^n \cdot \nabla (c_h^{n-})^2 \, \mathrm{d}\boldsymbol{x} = 0, \quad \text{and} \quad -\int_\Omega G_c(c_h^{n-1}, s_h^{n-1}) c_h^{n-} \, \mathrm{d}\boldsymbol{x} \le 0.$$

Thus, according to the positivity of the second term in left hand side of (6.4), we obtain

$$-\int_\Omega (c_h^n - c_h^{n-1}) c_h^{n-} \, \mathrm{d}\boldsymbol{x} \le 0. \quad (6.5)$$

Next we employ the identity $c_h^n = c_h^{n+} - c_h^{n-}$, the nonnegativity of c_h^{n-1}, and (6.5) to deduce that

$$\int_\Omega \left|c_h^{n-}\right|^2 \mathrm{d}\boldsymbol{x} = 0.$$

This implies the nonnegativity of c_h^n. Much in the same way, we get $s_h^{n-} = 0$ for $k = 1, \ldots, n$, concluding the proof of our first claim.

Now we prove (by induction) that there exists a constant $M > 0$ such that $s_h^n, c_h^n \le M$ for $n = 0, \ldots, N$. The statement clearly holds for $n = 0$. We then assume that $c_h^k, s_h^k \le M$ for $k = 1, \ldots, n-1$. The main idea consists in realising that (H_h^n) is a super-solution of the second equation for (c_h^n, s_h^n) in (6.3). Indeed, thanks to (2.5) we have

$$\begin{cases} H_h^0 = \max\{\|c_0\|_{\mathrm{L}^\infty(\Omega)}, \|c_0\|_{\mathrm{L}^\infty(\Omega)}\} \ge c_h^0, s_h^0, \\ \dfrac{H_h^n - H_h^{n-1}}{\Delta t} = \max\{C_s, C_c\}(1 + |c_h^{n-1}| + |s_h^{n-1}|) \ge \begin{cases} G_s(c_h^{n-1}, s_h^{n-1}), \\ G_c(c_h^{n-1}, s_h^{n-1}), \end{cases} \end{cases} \quad (6.6)$$

for all $n = 1, \ldots, N$. Therefore we can prove by induction that $c_k^k, s_h^k \le H_h^k$ for all $k = 1, \ldots, N$. This relation is readily verified for $k = 0$ (recall that $H_h^0 = \max\{\|c_0\|_{\mathrm{L}^\infty(\Omega)}, \|c_0\|_{\mathrm{L}^\infty(\Omega)}\}$). Assuming it holds true for $k = n-1$, and after subtracting the first equation in (6.3) from (6.6) for H_h^n, it follows that

$$\begin{aligned} &\int_\Omega \frac{c_h^n - H_h^n}{\Delta t} m_h^c \, \mathrm{d}\boldsymbol{x} + \int_\Omega (D_c(c_h^n) \nabla c_h^n - c_h^n \boldsymbol{u}_h^n) \cdot \nabla m_h^c \, \mathrm{d}\boldsymbol{x} = \\ &\int_\Omega \frac{c_h^{n-1} - H_h^{n-1}}{\Delta t} m_h^c \, \mathrm{d}\boldsymbol{x} + \int_\Omega \Big(G_c(c_h^{n-1}, s_h^{n-1}) - \max\{C_s, C_c\}(1 + |c_h^{n-1}| + |s_h^{n-1}|)\Big) m_h^c \, \mathrm{d}\boldsymbol{x}, \end{aligned} \quad (6.7)$$

for all $m_h^c \in V_h$. Observe that as a consequence of (2.5), the locally divergence free velocity condition, and the relation $c_h^{n-1} - H_h^{n-1} \le 0$, we obtain

$$\int_\Omega (D_c(c_h^n) \nabla c_h^n - c_h^n \boldsymbol{u}_h^n) \cdot \nabla (c_h^n - H_h^n)^+ \, \mathrm{d}\boldsymbol{x} \ge 0,$$

$$\int_\Omega \frac{c_h^{n-1} - H_h^{n-1}}{\Delta t} (c_h^n - H_h^n)^+ \,\mathrm{d}\boldsymbol{x} \leq 0,$$

$$\int_\Omega \Big(G_c(c_h^{n-1}, s_h^{n-1}) - \max\{C_s, C_c\}(1 + |c_h^{n-1}| + |s_h^{n-1}|) \Big) (c_h^n - H_h^n)^+ \,\mathrm{d}\boldsymbol{x} \geq 0.$$

Combining these inequalities and substituting $m_h^c = (c_h^n - H_h^n)^+$ in (6.7), implies that $(c_h^n - H_h^n)^+ \leq 0$. Similarly, we get $s_h^k \leq H_h^k$ for all $k = 1, \ldots, N$. Using this and (6.6), we deduce

$$H_h^k \leq (1 + 2\Delta t \max\{C_s, C_c\}) H^{k-1} + \Delta t \max\{C_s, C_c\},$$

for $k = 1, \ldots, N$. This implies that

$$\sup_{N \in \mathbb{N}} \max_{1 \leq n \leq N} H_h^n \leq C(\max\{\|c_0\|_{\mathrm{L}^\infty(\Omega)}, \|s_0\|_{\mathrm{L}^\infty(\Omega)}\}, T) < +\infty,$$

which concludes the proof. □

The next step consists in deriving stability estimates. Their proof follows closely those presented in Sections 3 and 4. Namely, we employ the same kind of test functions in the same combination of equations; whereas the chain rule for time derivatives is now replaced by the convexity inequality

$$a^n(a^n - a^{n-1}) \geq \frac{|a^n|^2}{2} - \frac{|a^{n-1}|^2}{2},$$

to treat the finite difference discretisation of the time derivatives. Proceeding in this way, we can obtain estimates (uniform in both h and Δt) for the discrete Brinkman solutions

$$\|\boldsymbol{u}_h\|_{\mathbf{L}^2(0,T;\mathbf{H}(\mathrm{div};\Omega))} + \|\boldsymbol{\omega}_h\|_{\mathbf{L}^2(0,T;\mathbf{H}(\mathbf{curl};\Omega))} + \|p_h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2_0(\Omega))} \leq C, \tag{6.8}$$

and for the advection-reaction-diffusion system

$$\|c_h\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))} + \|s_h\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))} + \|\nabla c_h\|_{\mathrm{L}^2(\Omega_T)} + \|\nabla s_h\|_{\mathrm{L}^2(\Omega_T)} \leq C, \tag{6.9}$$

for some constant $C > 0$.

The next goal is to establish the relative compactness in $\mathrm{L}^2(\Omega_T)$ of the sequences (c_h, s_h), which is achieved by constructing space and time translates and using the *a priori* estimates given above.

Lemma 6.2. *There exists a constant $C > 0$ depending on Ω, T, c_0 and s_0 such that*

$$\iint_{\Omega_{\boldsymbol{r}} \times (0,T)} \left[|c_h(\boldsymbol{x} + \boldsymbol{r}, t) - c_h(\boldsymbol{x}, t)|^2 + |s_h(\boldsymbol{x} + \boldsymbol{r}, t) - s_h(\boldsymbol{x}, t)|^2 \right] \mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq \mathcal{C}\,|\boldsymbol{r}|^2, \tag{6.10}$$

$$\iint_{\Omega \times (0,T-\tau)} \left[|c_h(\boldsymbol{x}, t+\tau) - c_h(\boldsymbol{x}, t)|^2 + |s_h(\boldsymbol{x}, t+\tau) - s_h(\boldsymbol{x}, t)|^2 \right] \mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq C(\tau + \Delta t). \tag{6.11}$$

for all $\boldsymbol{r} \in \mathbb{R}^3$ and for all $\tau \in (0, T)$, where $\Omega_{\boldsymbol{r}} = \{\boldsymbol{x} \in \Omega,\ [\boldsymbol{x}, \boldsymbol{x} + \boldsymbol{r}] \subset \Omega\}$.

Proof. Let us introduce the space translates

$$(J^{\boldsymbol{r}} c_h)(\boldsymbol{x}, \cdot) = c_h(\boldsymbol{x} + \boldsymbol{r}, \cdot) - c_h(\boldsymbol{x}, \cdot), \quad \text{and} \quad (J^{\boldsymbol{r}} s_h)(\boldsymbol{x}, \cdot) = s_h(\boldsymbol{x} + \boldsymbol{r}, \cdot) - s_h(\boldsymbol{x}, \cdot).$$

From the $\mathrm{L}^2(0, T; \mathrm{H}^1(\Omega))-$estimate for c_h and s_h, the bound

$$\int_0^T \int_{\Omega_{\boldsymbol{r}}} |J^{\boldsymbol{r}} c_h|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^T \int_{\Omega_{\boldsymbol{r}}} |J^{\boldsymbol{r}} s_h|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq \mathcal{C} |\boldsymbol{r}|^2, \tag{6.12}$$

easily follows. It is then clear that the right-hand side in (6.12) vanishes as $|\boldsymbol{r}| \to 0$, uniformly in h, which yields (6.10).

Next we introduce the time translates

$$(T^h c_h)(\cdot, t) := c_h(\cdot, t + \tau) - c_h(\cdot, t) \text{ and } (T^h s_h)(\cdot, t) := s_h(\cdot, t + \tau) - s_h(\cdot, t),$$

and notice that for all $t \in [0, T-\tau]$, these functions assume values in V_h (*cf.* (6.1)). Therefore they can be used as test functions in the fully-discrete scheme (6.3). Moreover, the previously proved uniform bounds for (c_h, s_h) and $(\nabla c_h, \nabla s_h)$ in $\mathrm{L}^2(\Omega_T, \mathbb{R}^2)$ imply analogous bounds for the translates $T^\tau c_h, T^\tau s_h$ and $\nabla T^\tau c_h, \nabla T^\tau s_h$ in $\mathrm{L}^2((0, T-\tau) \times \Omega)$.

Let us now define $(\bar{c}_h, \bar{s}_h)$ as the piecewise affine in t function in $W^{1,\infty}([0;T]; V_h)$ interpolating the states $(c_h^n, s_h^n)_{n=0,\ldots,N} \subset V_h$ at the points $(n\Delta t)_{n=0,\ldots,N}$ (recall the interpolation scheme $\overline{n}_h = n_h^{n-1} + \dfrac{t - t_{n-1}}{\Delta t}(n_h^n - n_h^{n-1})$ for $n = c, s$). Then we have

$$\begin{aligned} \int_\Omega \partial_t \bar{c}_h \, m_h^c \, \mathrm{d}\boldsymbol{x} + \int_\Omega (D_c(c_h)\nabla c_h - c_h \boldsymbol{u}_h) \cdot \nabla m_h^c \, \mathrm{d}\boldsymbol{x} &= \int_\Omega G_c(c_h, s_h) m_h^c \, \mathrm{d}\boldsymbol{x}, \\ \int_\Omega \partial_t \bar{s}_h \, m_h^s \, \mathrm{d}\boldsymbol{x} + \int_\Omega (D_s(s_h)\nabla s_h - s_h \boldsymbol{u}_h) \cdot \nabla m_h^s \, \mathrm{d}\boldsymbol{x} &= \int_\Omega G_s(c_h, s_h) m_h^s \, \mathrm{d}\boldsymbol{x}, \end{aligned} \tag{6.13}$$

for all $m_h^c, m_h^s \in \mathrm{V}_h$. We integrate (6.13) with respect to time $\sigma \in [t, t+\tau]$ (with $0 < \tau < T$) and consider $m_h^c = T^\tau c_h$ and $m_h^s = T^\tau s_h$ as test functions in the resulting equations. Therefore

$$\begin{aligned} &\int_0^{T-\tau} \int_\Omega \left|(T^h \bar{c}_h)(\boldsymbol{x}, t)\right|^2 \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \int_0^{T-\tau} \int_\Omega \left|(T^h \bar{s}_h)(\boldsymbol{x}, t)\right|^2 \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\ &= \int_0^{T-\tau} \int_\Omega \Big(\int_t^{t+\tau} \partial_\sigma \bar{c}_h(\boldsymbol{x}, \sigma) \, \mathrm{d}\sigma \Big) (T^h c_h)(\boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \int_0^{T-\tau} \int_\Omega \Big(\int_t^{t+\tau} \partial_\sigma \bar{s}_h(\boldsymbol{x}, \sigma) \, \mathrm{d}\sigma \Big) (T^h s_h)(\boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\ &= - \int_0^{T-\tau} \int_\Omega \int_t^{t+\tau} (D_c(c_h)\nabla c_h(\boldsymbol{x}, \sigma) - c_h(\boldsymbol{x}, t))\boldsymbol{u}_h(\boldsymbol{x})) \cdot \nabla (T^h c_h)(\boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\sigma \ \mathrm{d}t \\ &\qquad - \int_0^{T-\tau} \int_\Omega \int_t^{t+\tau} (D_s(s_h)\nabla s_h(\boldsymbol{x}, \sigma) - s_h(\boldsymbol{x}, t))\boldsymbol{u}_h(\boldsymbol{x})) \cdot \nabla (T^h s_h)(\boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\sigma \ \mathrm{d}t \\ &\qquad + \int_0^{T-\tau} \int_\Omega \int_t^{t+\tau} G_c(c_h(\boldsymbol{x}, t), s_h(\boldsymbol{x}, t))(T^h c_h)(\boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\sigma \ \mathrm{d}t \\ &\qquad + \int_0^{T-\tau} \int_\Omega \int_t^{t+\tau} G_s(c_h(\boldsymbol{x}, t), s_h(\boldsymbol{x}, t))(T^h s_h)(\boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\sigma \ \mathrm{d}t \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Now, we examine these integrals separately. For I_1 we have

$$|I_1| \le C \left[\int_0^{T-\tau} \int_\Omega \Big(\int_t^{t+\tau} |\nabla c_h(\boldsymbol{x}, \sigma)|^2 \, \mathrm{d}\sigma \Big)^2 \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \right]^{\frac{1}{2}} \times \left[\int_0^{T-\tau} \int_\Omega \left|\nabla (T^h c_h)(\boldsymbol{x}, t)\right|^2 \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \right]^{\frac{1}{2}} \le C\,\tau,$$

and similarly $|I_2| \le C\,\tau$, for some constant $C > 0$. Herein we have used the Fubini theorem (recall that $\int_t^{t+\tau} \, \mathrm{d}\sigma = \tau = \int_{\sigma-\tau}^{\sigma} dt$), the divergence-free condition for the discrete velocity, the Hölder inequality and the L^2−bounds for (c_h, s_h), $(\nabla c_h, \nabla s_h)$ and $(\nabla T^h c_h, \nabla T^h s_h)$. Keeping in mind the growth assumptions on G_c, G_s, we can apply the Hölder inequality (with $p = 2$, $p' = 2$) to deduce that $|I_3| + |I_4| \le C\,\tau$, for some constant $C > 0$. Collecting these inequalities we readily get

$$\int_0^{T-\tau} \int_\Omega \Big(|\bar{c}_h(\cdot, t+\tau) - \bar{c}_h(\cdot, t)|^2 + |\bar{s}_h(\cdot, t+\tau) - \bar{s}_h(\cdot, t)|^2 \Big) \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \le \mathcal{C}\,\tau.$$

Furthermore, the definition of $(\bar{c}_h, \bar{s}_h)$ together with (6.9) eventually implies that

$$\begin{aligned} \|\bar{c}_h - c_h\|^2_{\mathrm{L}^2(\Omega_T)} &\le \sum_{n=1}^N \Delta t \|c_h^n - c_h^{n-1}\|^2_{\mathrm{L}^2(\Omega)} \le \mathcal{C}(\Delta t) \ \to 0 \ \text{ as } \Delta t \to 0, \\ \|\bar{s}_h - s_h\|^2_{\mathrm{L}^2(\Omega_T)} &\le \sum_{n=1}^N \Delta t \|s_h^n - s_h^{n-1}\|^2_{\mathrm{L}^2(\Omega)} \le \mathcal{C}(\Delta t) \ \to 0 \ \text{ as } \Delta t \to 0, \end{aligned}$$

which establishes (6.11), therefore finishing the proof. □

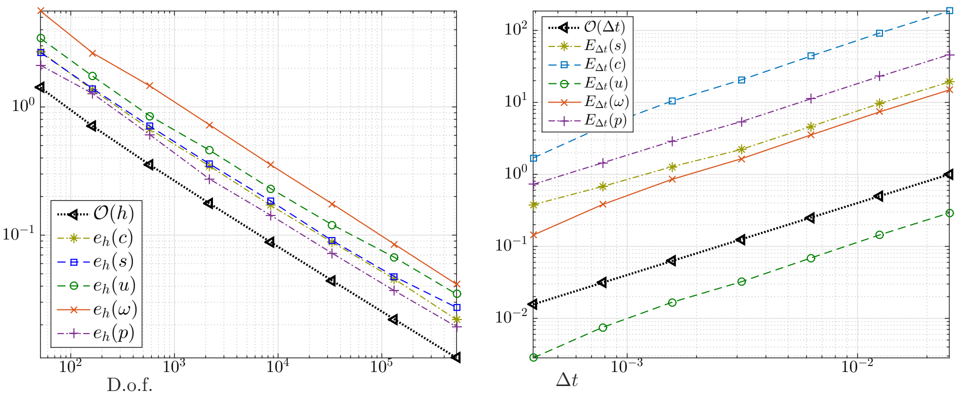

FIG. 7.1. *Example 1. Convergence tests for the spatial (left) and temporal (right) discretisation via mixed* $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{RT}_0 \times \mathbb{P}_1 \times \mathbb{P}_0$ *finite elements and backward Euler time stepping applied to* (2.1).

Note that as a consequence of (6.8)-(6.9), Lemma 6.2, and Kolmogorov's compactness criterion (*cf.* [6, Theorem IV.25]), we can assert that there exists a subsequence of $(c_h, s_h, \boldsymbol{u}_h, \boldsymbol{\omega}_h, p_h)$, not relabeled, such that, as $h \to 0$,

$$(c_h, s_h) \to (c, s) \text{ weakly } -\star \text{ in } \mathrm{L}^\infty(\Omega_T, \mathbb{R}^2) \text{ and in } \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega,\mathbb{R}^2)) \text{ weakly},$$
$$(\boldsymbol{u}_h, \boldsymbol{\omega}_h, p_h) \to (\boldsymbol{u}, \boldsymbol{\omega}, p) \text{ in } \mathbf{L}^2(0,T;\mathbf{H}(\mathrm{div};\Omega)) \times \mathbf{L}^2(0,T;\mathbf{H}(\mathbf{curl};\Omega)) \times \mathrm{L}^2(0,T;\mathrm{L}^2_0(\Omega)) \text{ weakly},$$
$$c_h \to c \text{ and } s_h \to s \text{ strongly in } \mathrm{L}^p(\Omega_T) \text{ for } 1 \le p < \infty.$$

These convergence properties allow us to identify the limit $(c, s, \boldsymbol{u}, \boldsymbol{\omega}, p)$ as the weak solution of (2.1), and the convergence result is summarised as follows.

THEOREM 6.3. *Assume that conditions* (2.3), (2.4) *and* (2.5) *hold. If* $c_0, s_0 \in \mathrm{L}^\infty(\Omega)$ *with* $c_0 \ge 0$ *and* $s_0 \ge 0$ *a.e. in* Ω, *then the finite element solution* $(c_h^n, s_h^n, \boldsymbol{u}_h^n, \boldsymbol{\omega}_h^n, p_h^n)$, *generated by* (6.3), *converges along a subsequence to* $(c, s, \boldsymbol{u}, \boldsymbol{\omega}, p)$ *as* $h, \Delta t \to 0$, *where* $(c, s, \boldsymbol{u}, \boldsymbol{\omega}, p)$ *is a unique weak solution of* (2.1)-(2.2) *in the sense of Definition 2.1.*

We observe that the error generated by the fully discrete scheme (6.3) has two components: one due to the spatial discretisation and depending on the mesh size h, and the error due to the time discretisation depending on the timestep Δt. Given the approximation properties of the employed finite element spaces and the time stepping method (see e.g. [10]), we can expect the following convergence rates for the proposed method

$$\|(c(\cdot,t_n), s(\cdot,t_n), \boldsymbol{u}(\cdot,t_n), \boldsymbol{\omega}(\cdot,t_n), p(\cdot,t_n)) - (c_h^n, s_h^n, \boldsymbol{u}_h^n, \boldsymbol{\omega}_h^n, p_h^n)\| \le C_1 h + C_2 \Delta t, \tag{6.14}$$

with $t_n = n\Delta t$, for $n = 1, \ldots, N$ and $C_1, C_2 > 0$ are constants independent of h and Δt. Here $(c_h^n, s_h^n, \boldsymbol{u}_h^n, \boldsymbol{\omega}_h^n, p_h^n)$ denotes the sequence generated by (6.3) for all $n = 1, \ldots, N$. A rigorous derivation of this space-time error estimate is part of ongoing developments.

7. Numerical tests.

We finally present a set of examples to illustrate the properties of the model and of the proposed finite element method. The coupling between Brinkman and reaction-diffusion equations will be implemented either via: a) a fully monolithic solution, b) an iterative Picard method splitting linear Brinkman and fully explicit reaction-diffusion equations, and c) an iterative Picard method with an embedded Newton algorithm for the linearisation of the reaction-diffusion system. Efficient coupling strategies and thorough comparisons will be reported elsewhere (see [17]).

Example 1: Spatio-temporal accuracy. Let us consider the spatio-temporal domain $\Omega_T = (0,1)^2 \times [0, 0.1]$ and define the following exact solutions to (2.1) in the case of constant diffusivities D_c, D_s:

$$\begin{aligned}
&c(\boldsymbol{x},t) = \exp(-4D_c\pi^2 t)[\cos(2\pi x) + \cos(2\pi y)], \;\; s(\boldsymbol{x},t) = \exp(-4D_s\pi^2 t)[\cos(2\pi x) + \cos(2\pi y)],\\
&\boldsymbol{u}(\boldsymbol{x},t) = 3\sin(t)x^2(x-1)^2y^2(y-1)^2\big(x(x-1)(2y-1), -(2x-1)y(y-1)\big)^T, \quad p(\boldsymbol{x},t) = x^4 - y^4,\\
&\boldsymbol{\omega}(\boldsymbol{x},t) = -6\sin(t)\sqrt{\mu}[x(5x^3 - 10x^2 + 6x - 1)y^3(y-1)^3 + y(5y^3 - 10y^2 + 6y - 1)x^3(x-1)^3].
\end{aligned} \tag{7.1}$$

The model parameters are set as $\mu = 1$, $D_c = 0.01$, $D_s = 0.5$, $\boldsymbol{g} = (0, -1)^T$, and $\mathbb{K} = \mathbf{I}$; the reaction kinetics are specified by

$$G_j(c, s) = 6\pi \sin(t) \exp(-4D_j\pi^2 t)x^2(x-1)^2y^2(y-1)^2 \\ \times [(2x-1)y(y-1)\sin(2\pi y) - x(x-1)(2y-1)\sin(2\pi x)], \quad j = c, s,$$

whereas $\boldsymbol{f}$ is computed using (7.1) and the momentum equation in (2.1). Note that these solutions satisfy the mass conservation equation, while boundary and initial conditions (2.2) are imposed according to (7.1). Also, a non-homogeneous source term is incorporated on the right hand side of the reaction-diffusion equations and constructed using the manufactured solutions.

Errors associated to splitting algorithms are avoided by invoking the exact full Jacobian and performing Newton iterations until convergence at each time step. The mesh convergence is investigated by fixing Δt =1.6e-4 and computing errors between the exact solutions from (7.1) and approximations obtained with our mixed-primal method on a sequence of successively refined structured triangulations of Ω, at the final adimensional time T =5e-3. That is, for each field η, we compute $e_h(\eta) := \|\eta^N - \eta_h^N\|$, where $\eta^N = \eta(\cdot, T)$, and $\|\cdot\|$ is either the H^1, $\mathrm{H}(\mathrm{div})$, or L^2-norm. On the other hand, the error history associated to the time discretisation is studied by fixing a small meshsize h = 2.43e-5 and computing accumulative errors up to $T = 0.1$, and decreasing Δt. That is, we measure errors in the $\ell^\infty(0, t; \cdot)-$norm: $E_{\Delta t}(\eta) := \sum_{n=0}^N \|\eta^n - \eta_h^n\|_{(\cdot)}$. Figure 7.1 depicts a convergence analysis for the propose method with respect to meshsize and timestep. All plots indicate an overall first order convergence in both space and time, as expected from the involved approximations yielding (6.14).

Example 2: Bacterial bioconvection. Next we focus on the interaction between bacteria (with concentration c) and oxygen (described by s) in a small chamber, as studied in e.g. [16]. As it stands, this model does not fall exactly in the framework analysed in Section 3, since a cross-diffusion term for the bacteria is utilised $\mathrm{div}(D_c\nabla c) + \mathrm{div}(D_c'\nabla s)$, with $D_c'(c, s) = cr(s)$, $r(s) = -\frac{\alpha}{2}(1 + \frac{s-s^*}{\sqrt{(s-s^*)^2+\varepsilon^2}})$, where s^* is an oxygen threshold, below which the chemotactic convection is turned off. We consider a fully explicit approximation of the cross diffusion, the reaction part for the oxygen conservation equation is given by $G_s(c, s) = \beta cr(s)$, and the remaining model functions and adimensional parameters are set as follows: $\mathbb{K} = K_0 + K_0\eta(x)$ (with $K_0 = 7700$ and η a uniformly distributed random field), $g = (0, -\gamma)^T$, $s^* = 0.3$, $D_c = 0.01$, $D_s = 0.25$, $\varepsilon = h$. The computational domain is a disk of radius $\frac{1}{2}$ centred at $(\frac{1}{2}, \frac{1}{2})$, discretised into an unstructured mesh of 13972 points and 27942 triangular elements, and a timestep of Δt =1e-3 is employed. For this example we solve via fixed point iterations the coupling between the Brinkman problem and the set of reaction-diffusion equations. The system is initially at rest (zero velocity, vorticity, and pressure), having a concentration of bacteria near the top of the disk $c_0 = 1 - (1 + \exp(-50\sqrt{(x-0.5)^2 + (y-0.9)^2}))^{-1}$, and an homogeneous oxygen content $s_0 = 1$. In Figure 7.2 we show three snapshots (at advanced time) of the obtained numerical solutions on different regimes characterised by α, β, γ. In all cases we observe the species heading to the bottom of the disk, generating vortical flow around the zones of high concentrations of oxygen and bacteria. The plots indicate that transitional flow occurs (mainly due to the triggering of unstable modes), which can be captured in a robust manner by the numerical method, even in the most unstable regime.

Example 3: FitzHugh-Nagumo dynamics in a 3D porous cavity. This test emphasises the effects of high contrast permeabilities on the propagation of travelling waves dictated by the well-known FitzHugh-Nagumo reactions $G_c(c, s) = k(s + c(c-a)(c-1))$, and $G_s(c, s) = d_1 c - s$. In this prototype model of excitable systems, c and s represent a membrane voltage, and a recovery variable. We consider a polygonal domain $\Omega = (0, 1.8) \times (0, 1) \times (0, 0.6)$ filled with 60 large particles (of radii 1.5e-3 and randomly distributed on Ω) with a permeability 100 times higher than in the rest of the porous matrix.

The forcing term is specified by $\boldsymbol{g} = d_2(1, 1, 1)^T$ and $\boldsymbol{f} = \mathbf{0}$. Moreover, the conductivity of the medium is assumed anisotropic and affected by the heterogeneity of the permeability field $(D_c)_{1,1} = 40\mathbb{K}$, $(D_c)_{2,2} = 5\mathbb{K}$, D_s =1e-3, the fluid viscosity is $\mu = 0.01$, and the remaining parameters are chosen as $a = 0.25$, $d_1 = 0.16875$, $d_2 = 250\sqrt{3}$, $k = -100$. The domain is discretised into an unstructured mesh of 126034 tetrahedral elements sharing 21952 nodes, and we set a timestep of Δt =1e-3. All sides of the box are provided with zero-fluxes for voltage and recovery fields, zero tangential vorticity, and slip velocities, and as initial condition we excite the bottom left part of the domain. This time we use the method of characteristics to treat the convective terms, a Newton algorithm is used to linearise the reaction-

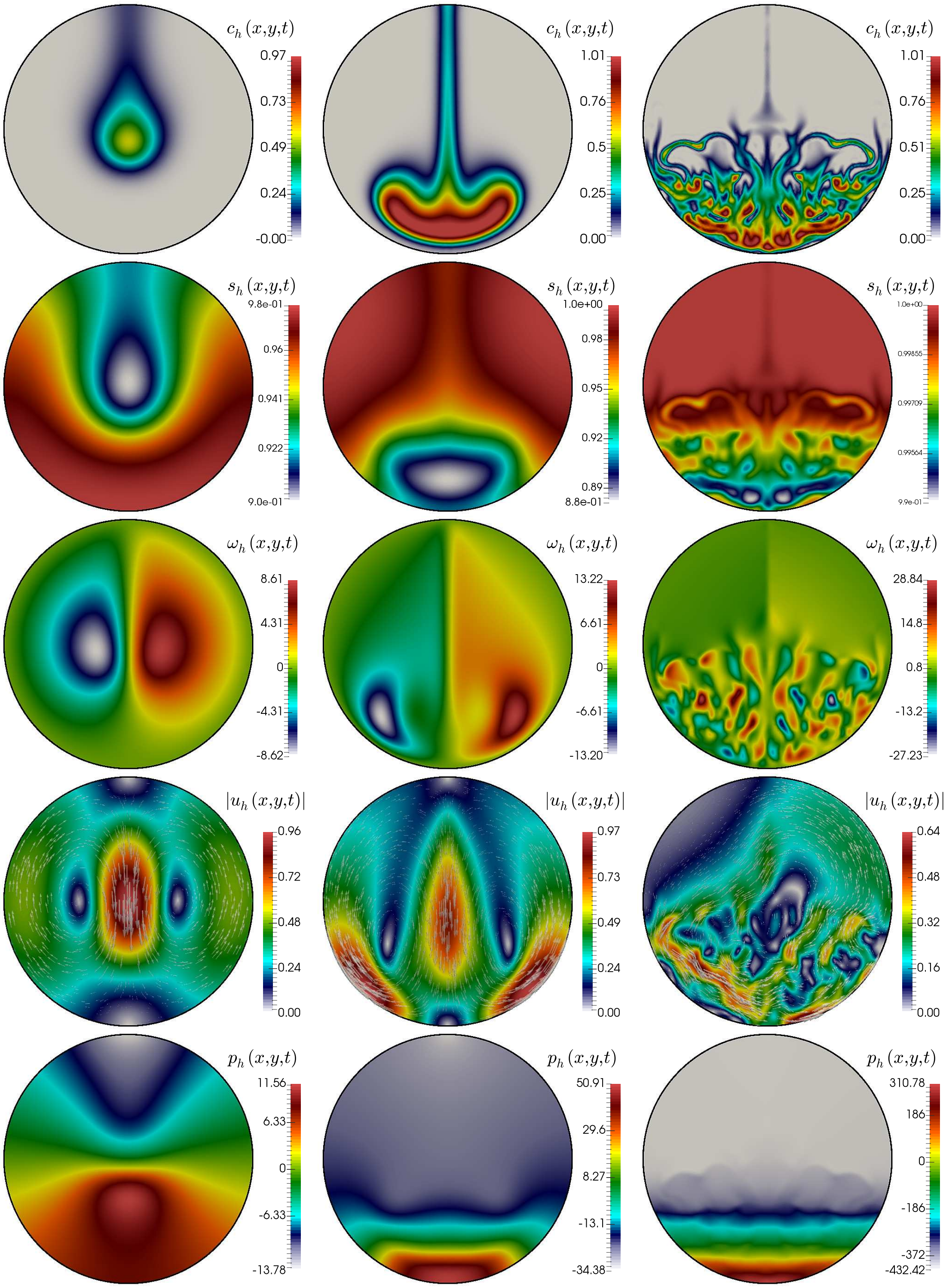

FIG. 7.2. *Example 2: snapshots at $t = 0.5$ of the bioconvection dynamics for three different regimes characterised by $\alpha = \beta = 0.1, \gamma = 41.8$ (left), $\alpha = 0.25, \beta = 2.5, \gamma = 418$ (centre), and $\alpha = \beta = 5, \gamma = 4180$ (right). Computed solutions from top to bottom: bacteria concentration, amount of oxygen, vorticity, velocity, and pressure.*

diffusion equations, and outer Picard iterations are applied to couple that system together with the set of Brinkman equations. The obtained numerical solutions are depicted in Figure 7.3. A propagating front for the potential moves towards the positive $x-$axis, followed by the slower recovery variable front. As a consequence of the heterogeneity of conductivities and permeabilities, preferential velocity patterns

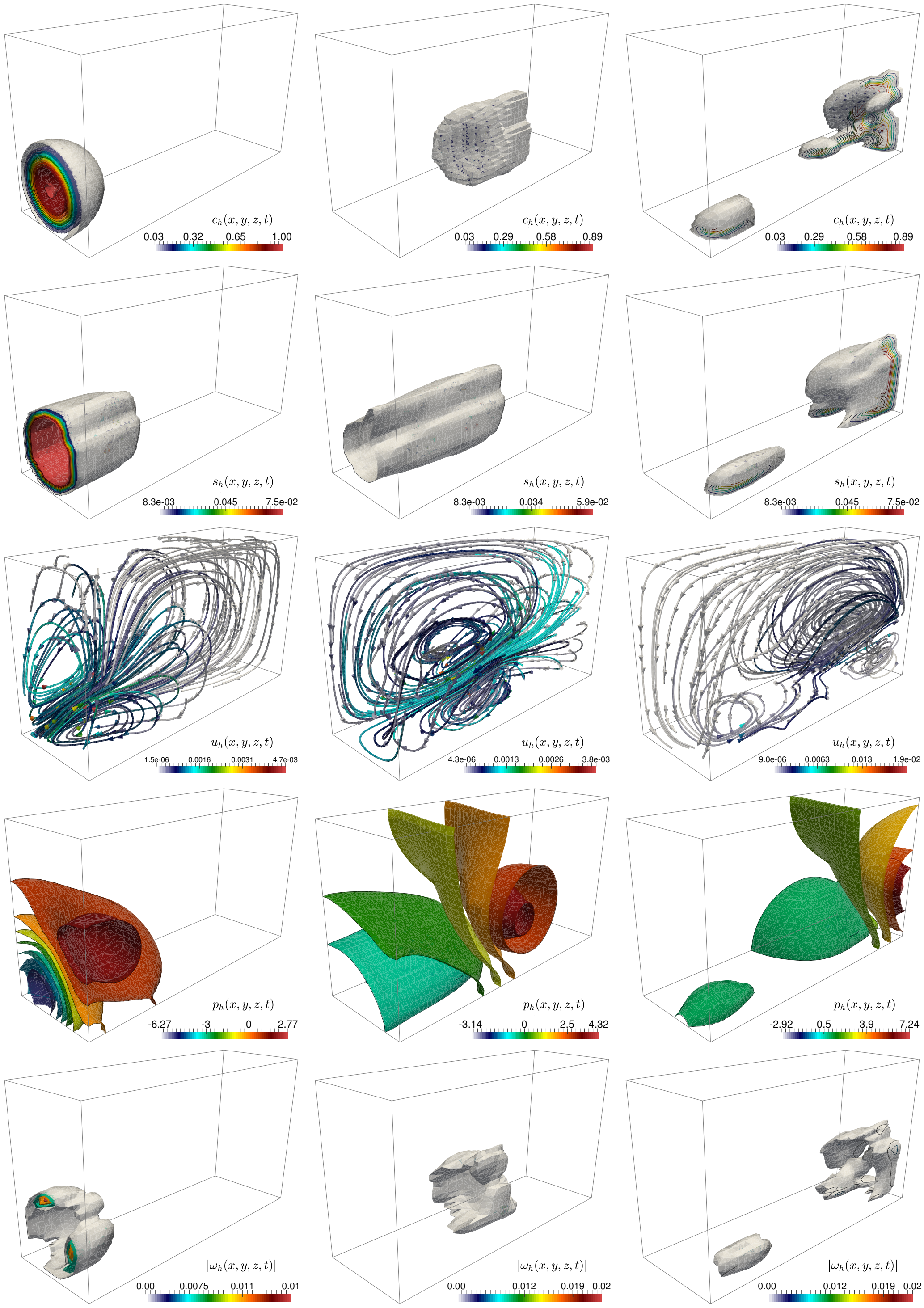

FIG. 7.3. *Example 3: Computed solutions (voltage, recovery variable, vorticity, velocity, and pressure) for the FitzHugh-Nagumo dynamics on a porous mixture at early (left), mid (centre), and advanced (right) times.*

start to form. We also notice that vorticity (in this case, we show only its magnitude) clearly marks the regions of contact between high gradients of potential and recovery field.

Example 4: Intracellular calcium-induced calcium release. In closing this section we present a simulation of the interaction between two species c, s (the concentrations of cytosolic and sarcoplasmic calcium, respectively) inside a cardiac cell. This phenomenon has been studied in terms of the reacting

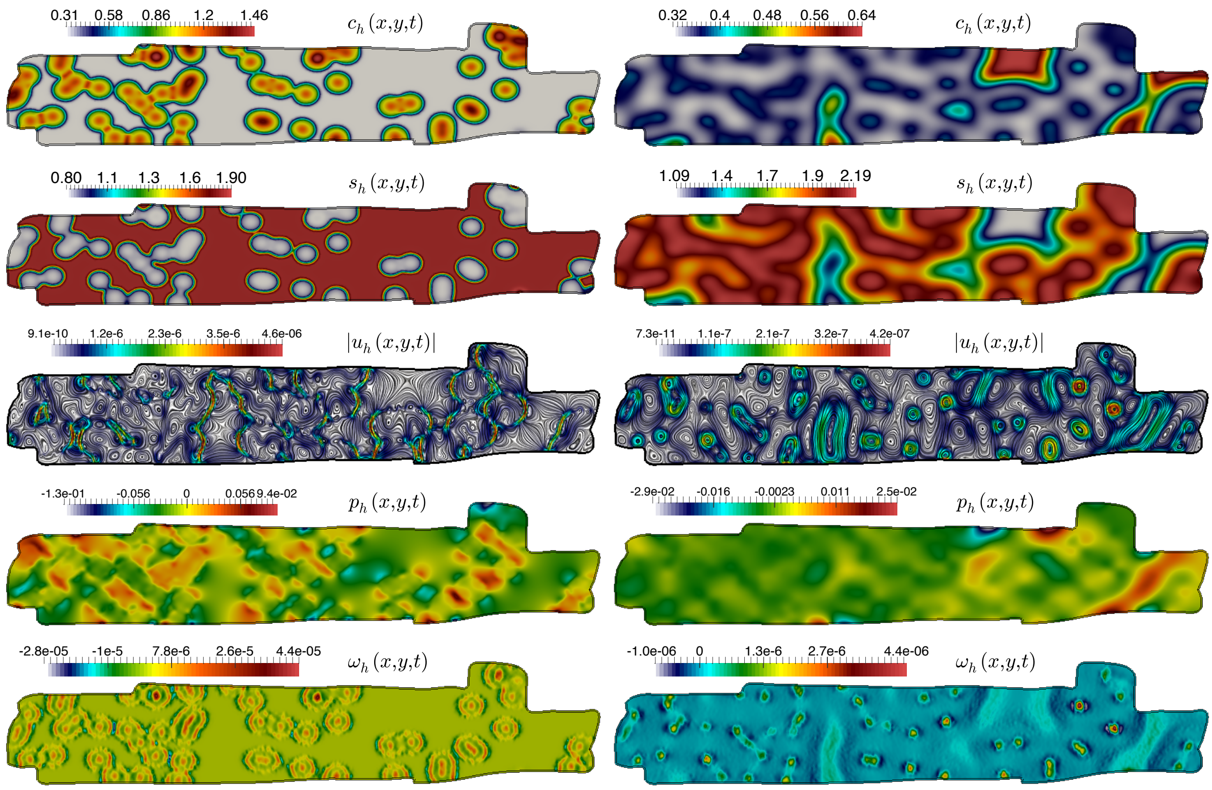

FIG. 7.4. *Example 4: Computed solutions (cytosolic calcium, sarcoplasmic calcium, vorticity, velocity, and pressure) for the intracellular calcium dynamics at early (left) and advanced (right) times.*

species alone (*cf.* [14, 18]), and also including the active cell contraction while solving for the underlying finite elasticity equations [20, 26]. In contrast, here we assume that the species interact with an interstitial fluid occupying the sarcoplasmic reticulum, which is in turn pictured as a porous medium with a non-homogeneous permeability distribution. A 2D geometry reconstructed from confocal images is used and a triangular mesh of 46610 elements and 23741 vertices is generated. The time advancing algorithm uses a fixed time step of $\Delta t = 0.01$. The process consists in opening 80 channels of cytosolic calcium located randomly within the myocyte, and observing how these propagate thorough plasma into the whole cell. We also consider that the permeability is higher in the vicinity of these channels. The minimal reaction-diffusion model proposed in [14] involves the following specialisation for constant and species-dependent coefficients:

$$D_c = (1 + \frac{1}{4}\eta(\boldsymbol{x}))\begin{pmatrix} 0.6 & 0 \\ 0 & 0.3 \end{pmatrix},\ D_s = 0,\ G_c(c,s) = \nu_1 - \nu_2\frac{c^2}{k_2 + c^2} + \nu_3\frac{c^4 s^2}{(k_3 + s^2)(k_4 + c^4)} - \nu_4 c,$$
$$G_s(c,s) = \nu_2\frac{c^2}{k_2 + c^2} - \nu_3\frac{c^4 s^2}{(k_3 + s^2)(k_4 + c^4)} - \nu_5 s,$$

where η is a step function assuming the value 1 on disks centred at uniformly distributed random locations corresponding to the channels, and zero elsewhere. The influence of calcium patterns into the flow behaviour of the plasma is encoded in the forcing term for the momentum equation $\boldsymbol{f} = \gamma_0|s|(\boldsymbol{f}_0 \otimes \boldsymbol{f}_0)\nabla s$, which might be regarded as the flow-counterpart to the active force proposed in [19]. Here $\boldsymbol{f}_0 = (1,0)^T$ is a given preferential direction of plasma displacement, and the remaining parameters are $\gamma_0 = -0.12$, $\nu_1 = 1.58$, $\nu_2 = 16$, $\nu_3 = 91$, $\nu_4 = 2$, $\nu_5 = 0.2$, $k_2 = 1$, $k_3 = 4$, $k_4 = 0.75$, $\mathbb{K} = 1\text{e-}4(1+10\eta(\boldsymbol{x}))$. Snapshots of the numerical solutions at an early and advanced time steps are collected in Figure 7.4. Starting from the open channels, the cytosolic calcium starts to propagate towards other channels, also following a higher diffusion in the preferential direction $\boldsymbol{f}_0$ (aligned with the $x-$axis).

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