



Fractional spaces and conservation laws

Pierre Castelli, Pierre-Emmanuel Jabin, Stéphane Junca

► **To cite this version:**

Pierre Castelli, Pierre-Emmanuel Jabin, Stéphane Junca. Fractional spaces and conservation laws. XVI International Conference on Hyperbolic Problems: Theory, Numerics, Applications: Hyp2016, Klingenberg C.; Westdickenberg M., Aug 2016, Aachen, Germany. pp.285-293, 10.1007/978-3-319-91545-6_23. hal-01407099

HAL Id: hal-01407099

<https://hal.archives-ouvertes.fr/hal-01407099>

Submitted on 1 Dec 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Fractional spaces and conservation laws

Pierre CASTELLI, Pierre-Emmanuel JABIN, Stéphane JUNCA

Abstract In 1994, Lions, Perthame and Tadmor conjectured the maximal smoothing effect for multidimensional scalar conservation laws in Sobolev spaces. For strictly smooth convex flux and the one-dimensional case we detail the proof of this conjecture in the framework of Sobolev fractional spaces $W^{s,1}$, and in fractional BV spaces: BV^s . The BV^s smoothing effect is more precise and optimal. It implies the optimal Sobolev smoothing effect in $W^{s,1}$ and also in $W^{s,p}$ with the optimal $p = 1/s$. Moreover, the proof expounded does not use the Lax-Oleinik formula but a generalized one-sided Oleinik condition.

1 Introduction

In this short note we first prove a remark on the optimal smoothing effect in Sobolev spaces given by Lions, Perthame and Tadmor [LPT] for strictly convex flux with a power-law type of degeneracy. Then we improve this result in fractional BV spaces: BV^s . The smoothing effect in BV^s with the optimal s yields to the optimal Sobolev regularity in $W^{s,p}$ with the same exponent s and the optimal p . This BV^s regularity also gives traces property of the entropy solution ([BM, DOW, DR, P1, P2]). We do not use the Lax-Oleinik formula ([E]) as suggested in [LPT] and used in [BGJ, CJ3] but the one-sided Lipschitz condition on the velocity of the entropy solution (2) below. In all the paper, the following one-dimensional scalar conservation law are considered:

P. Castelli
Lycée Audibert, 63 bd Wilson, 06600, Antibes France, e-mail: Pierre.Castelli@ac-nice.fr

P-E Jabin
CSCAMM and Dept. of Mathematics, University of Maryland, College Park, MD 20742, USA.
e-mail: pjabin@umd.edu

S. Junca
Université Côte d'Azur, Inria, CNRS, LJAD, Nice, France, e-mail: Stephane.JUNCA@unice.fr

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(f(u)) = 0 & \text{on } \mathbb{R} \times]0, +\infty[\\ u(x, 0) = u_0(x) & \text{on } \mathbb{R} \end{cases}, \quad (1)$$

where the flux f is strictly convex and $f \in C^1$, that is to say that the velocity $a(u) = f'(u)$ is strictly increasing. Lax ([La]) and Oleinik ([O]) obtained the optimal smoothing effect in BV for an uniformly convex flux: $\inf f'' > 0$. The BV regularity is known to be lost if the second derivative of the flux vanishes at only one point ([Ch1]). Nevertheless we can obtain BV^s regularity with $0 < s < 1$. BV^s spaces are essentially fractional BV spaces with similar properties as for example $BV = BV^1$. First applications of such fractional BV spaces for a 2×2 system are already obtained in [BGJP].

We do not speak about SBV regularity ([AGV2, AD, B]) since this related to an uniform convex flux and the BV smoothing effect. On the other hand there are many other studies of the regularizing effect ([AGV1, BM, Cc, Ja]) which can be interpreted in the framework of fractional BV spaces but are beyond the scope of this short note.

The paper is organized as follow. In Section 2 a one-sided Lipschitz condition is recalled and linked with the degeneracy of the flux. Then, the smoothing effect is stated in $W^{s,p}$ and BV^s . Section 3 gives a very simple proof of the smoothing effect in BV^s and some consequences as the optimal $W^{s,p}$ regularity. We also present in the final Section 4 an alternative proof of the optimal Sobolev regularity in $W^{s,1}$.

2 One-sided condition and degeneracy of the flux

The proof uses the following generalized one-sided Oleinik inequality due to Dafermos ([D]) (see also [Ch2]) under convex condition on the flux and for almost all $x < y$:

$$a(u(y,t)) - a(u(x,t)) \leq \frac{y-x}{t} \quad (2)$$

There is a geometric interpretation of Inequality (2). If there is no shock issued from $(x,0)$ and $(y,0)$ in the (x,t) plane then u is constant on the two characteristics issued from $(x,0)$ and $(y,0)$ and the constant difference of the velocities is less than the constant velocity needed to meet the characteristics at time t . Inequality (2) is usually interpreted as an entropy condition: the jump of velocity can only be negative when $y \rightarrow x$. A complete argument can be given with the generalized characteristics of Dafermos ([D]).

Since any strictly C^1 convex flux can be approached in C^1 by a sequence of C^2 uniformly convex flux, inequality (2) is still valid for entropy solutions almost everywhere. If the flux is uniformly convex then $m|u-v| \leq |a(u)-a(v)|$ for a positive constant m and Inequality (2) yields to the one-sided Oleinik condition:

$u(y,t) - u(x,t) \leq \frac{y-x}{mt}$ which can be rewritten with $(v)_+ = \max(v, 0)$ for almost all $(x, y) \in \mathbb{R}^2$ and $t > 0$:

$$(u(y,t) - u(x,t))_+ \leq \frac{(y-x)_+}{mt}$$

For a non uniformly convex flux, the previous one-sided Lipschitz condition becomes a one-sided Hölder condition: Inequality (5) at the end of this section. Before, a precise definition of a nonlinear flux is needed.

A power-law type of nonlinear degeneracy is considered as in [BGJ]. The following condition is enough for a strictly convex flux with some regularity and power-law flux for instances.

Definition 1. Let $f \in C^1(K, \mathbb{R})$, where K is a closed interval of \mathbb{R} . We say that the degeneracy of f on K is at least $q > 0$ if the continuous derivative $a(u) = f'(u)$ satisfies :

$$\inf_{(u,v) \in (K \times K) \setminus \mathcal{D}_K} \frac{|a(u) - a(v)|}{|u - v|^q} > 0, \quad (3)$$

where $\mathcal{D}_K = \{(u, v) \in (K \times K) \mid u \neq v\}$. The lowest real number q , if there exists, is called the degeneracy of f on K and denoted p .

Inequality 3 is equivalent to:

$$\exists m > 0, \forall (u, v) \in K^2, |a(u) - a(v)| \geq m |u - v|^q. \quad (4)$$

In particular, (4) implies that $a : K \rightarrow \mathbb{R}$ is strictly monotonic, since a is injective and continuous.

Example 1. If $f(u) = |u|^{1+p}$, $p \geq 1$, then p is the degeneracy of f on any interval which contains 0.

Remark 1. 1) Suppose that for all $u, v \in K$, $|a(u) - a(v)| \geq m |u - v|^q$ and that a is differentiable at u_0 . Then $q \geq 1$, since $\infty > \left| \frac{a(u) - a(u_0)}{u - u_0} \right| \geq m |u - u_0|^{q-1}$.

2) If f is smooth on K , then f has a degeneracy p which is a positive integer. ([BGJ])

Replacing $f(u)$ by $-f(-u)$ if necessary, we will assume subsequently that $a(u) = f'(u)$ is strictly increasing, so f is strictly convex.

Now, one-sided condition (2) on the velocity is interpreted as the following one-sided Hölder condition for almost all x, y :

$$(u(y,t) - u(x,t))_+ \leq K \frac{[(y-x)_+]^s}{t^s}, \quad (5)$$

where $s = \frac{1}{p}$, $K = m^{-s}$, $(v)_+ = \max(v, 0)$ since $m(u-v)^p = m((u-v)_+)^p \leq a(u) - a(v)$ for $u \geq v$. Again, the exponent s naturally appears to estimate the positive variation of the entropy solution u .

Now, we can state the smoothing effect for entropy solutions.

Theorem 1. *Let u_0 belongs to $L^\infty(\mathbb{R})$ function, K be the convex hull of $u_0(\mathbb{R})$, f a C^1 strictly convex flux with degeneracy p on K . Then the associated entropy solution of the conservation law (1) have got the following regularity in space for all positive time t and for all $\varepsilon > 0$: $u(\cdot, t) \in W_{loc}^{s-\varepsilon, 1/s}(\mathbb{R}) \cap BV_{loc}^s(\mathbb{R})$.*

The regularity in $W_{loc}^{s-\varepsilon, 1}(\mathbb{R})$ was conjectured in [LPT] and proved in [Ja]. The regularity in BV^s was first proved in [BGJ]. The regularity in $W_{loc}^{s-\varepsilon, 1/s}$ is the consequence of the BV_{loc}^s regularity. The optimality in Sobolev spaces can be found in [DW] and in the fractional BV spaces in [CJ1].

The main originality of this note is hence to give new and simpler proofs for this smoothing effect. Our proofs are based on the key BV estimate of the velocity $a(u)$ for positive time given by (2). This is obviously a nonlinear regularity estimate on the entropy solution u which can easily be translated into more traditional regularity estimates.

The spaces BV^s are in particular well adapted for this, leading to a very simple proof in the next section. From this BV^s regularity it is straightforward to deduce fractional Sobolev regularity as well. But of course one can also prove directly the Sobolev regularity; we give an example of such a proof in the last Section.

3 BV^s smoothing effect

We can define BV_+^s as $W_+^{s,1}$ and try to adapt the previous proof from Section 4.3. Unfortunately, the equality: $L^\infty \cap BV_+^s = BV^s$ for $0 < s < 1$ is an open problem. It is only known for $s = 1$ and, fortunately, it is enough to get the optimal BV^s regularity.

3.1 BV^s spaces

We recall briefly the definition and the main properties of fractional BV spaces.

Definition 2. Let I be an non-empty interval of \mathbb{R} and let $\mathcal{S}(I)$ be the set of subdivisions of I : $\mathcal{S}(I) = \{(x_0, x_1, \dots, x_n), n \geq 1, x_i \in I, x_0 < x_1 < \dots < x_n\}$.

For $0 < s \leq 1$ set

$$TV^s u[I] = \sup_{\mathcal{S}(I)} \sum_{i=1}^n |u(x_i) - u(x_{i-1})|^{\frac{1}{s}},$$

then $BV^s(I) := \{u, TV^s u[I] < \infty\}$.

For $s = 1$ we recover the space BV of functions of bounded variation. Functions in BV^s have always left and right traces, like in BV ([MO]). The exponent s is related to the fractional Sobolev derivative: $BV^s \subset W^{s-\varepsilon, 1/s}$ for all $\varepsilon > 0$ ([BGJ]). The space BV^s is also called the space of functions of bounded p -variation, with $p = \frac{1}{s}$.

3.2 The short proof of the BV^s smoothing

Set $TV_+u := \sup_{n \in \mathbb{N}^*} \sum_{i=1}^n (u(x_i) - u(x_{i-1}))_+$ and $BV_+ := \{u, TV_+u < \infty\}$
 $x_0 < x_1 < \dots < x_n$

then $a(u(\cdot, t)) \in BV_+$ from inequality (2). According to the Maximum principle $a(u(x, t)) \in L^\infty$ as the initial data u_0 . Since $BV_+ \cap L^\infty = BV$, then $a(u(\cdot, t)) \in BV$. Moreover, the velocity $a(\cdot)$ has at most a power law degeneracy:

$$|a(u) - a(v)| \geq m|u - v|^p,$$

thus $u \in BV^s$, where $s = \frac{1}{p}$.

This proof is very short and shortens the proof given in [BGJ]. Moreover, it gives more information about the singularity of u . For instance if the flux is a convex power law $f(u) = |u|^{1+p}$ and the convex hull K of $u_0(\mathbb{R})$ does not contain the singular point 0 then the entropy solution associated to the initial data u_0 belongs to BV . That means that the BV^s regularity is due to bigger oscillations around the state $u = 0$. Notice that since there is only a finite number of oscillations with any given positive strength thus the oscillations near state $u = 0$ has to be smaller and smaller and with infinitely many oscillations as optimal examples given in [CJ1, DW].

4 Optimal smoothing effect in Sobolev spaces $W^{s,1}$

The best smoothing effect in Sobolev spaces $W^{s,1}$ was suggested in [LPT] with the Lax-Oleinik formula, bounded in [DW] and proved in [Ja] with a kinetic formulation and a BV assumption on the velocity. In this short note an another proof is proposed.

We recall a classical result for $W_+^{1,1}$ and BV_+ . Then a similar result in $W_+^{s,1}$ is proved and used to get the maximal smoothing effect for conservation laws.

4.1 Usual results in $W^{1,1}$

$W_+^{1,1}(\mathbb{R})$ is the set of functions u such that the semi-norm $|u|_1^+ = \int_{\mathbb{R}} (\partial_x u)_+ dx$ is finite, where $(v)_+ = \max(v, 0)$.

Lemma 1. $W_+^{1,1}(\mathbb{R}) \cap L^1(\mathbb{R}) \subset BV(\mathbb{R})$.

Notice that $W_+^{1,1}(\mathbb{R}) \cap L^1(\mathbb{R})$ is bigger than $W^{1,1}(\mathbb{R})$: for instance consider the BV function $u(x) = x \chi_{[0,1]}(x)$ where χ_I is the indicator function of the set I .

The lemma follows from the equality $\int_{\mathbb{R}} |\partial_x u| dx = 2|u|_1^+$, which is valid for any smooth compactly supported function, since $|x| = 2x_+ - x$ and then $\int_{\mathbb{R}} |\partial_x u| dx = 2 \int_{\mathbb{R}} (\partial_x u)_+ dx - \int_{\mathbb{R}} \partial_x u dx = 2 \int_{\mathbb{R}} (\partial_x u)_+ dx$.

The space BV is better fitted through the control of the positive variation since

$$BV_+(\mathbb{R}) \cap L^\infty(\mathbb{R}) = BV(\mathbb{R})$$

where $BV_+(\mathbb{R})$ is the space of function u such that

$$TV_+u = \sup_{n \in \mathbb{N}, x_0 < x_1 < \dots < x_n} \sum_{i=1}^n (u(x_i) - u(x_{i-1}))_+ < \infty.$$

4.2 The $W_+^{s,1}(\mathbb{R})$ case

The usual Sobolev semi-norm in $W^{s,1}(\mathbb{R})$ for $s \in]0, 1[$ is :

$$|u|_s = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x) - u(y)|}{|x - y|^{1+s}} dx dy = \int \int \frac{|u(x+h) - u(x)|}{|h|^{1+s}} dx dh.$$

The Sobolev semi-norm in $W_+^{s,1}$ for $s \in]0, 1[$ is :

$$|u|_s^+ = \int_{\mathbb{R}} \int_{h>0} \frac{[u(x+h) - u(x)]_+}{|h|^{1+s}} dx dh.$$

The definition of the set $W_+^{s,1}(\mathbb{R})$ is:

Definition 3. $u \in W_+^{s,1}(\mathbb{R})$ if $\iint_{x>y} \frac{[u(x) - u(y)]_+}{|x - y|^{1+s}} dx dy < +\infty$.

Proposition 1. For all $0 < s < 1$, $L^1(\mathbb{R}) \cap W_+^{s,1}(\mathbb{R}) = W^{s,1}(\mathbb{R})$.

For $s < 1$ the estimates for the "one-sided" semi-norm is almost enough to stay in the associated Sobolev space. This is not true for $s = 1$ (Lemma 1). Proposition 1 is also valid locally: $L_{loc}^1(\mathbb{R}) \cap W_{+,loc}^{s,1}(\mathbb{R}) = W_{loc}^{s,1}(\mathbb{R})$.

Proof. The bound of the Sobolev semi-norm $2I = \iint \frac{|u(x) - u(y)|}{|x - y|^{1+s}} dx dy < +\infty$ is obtained thanks to the integral $I = \iint_{x>y} \frac{|u(x) - u(y)|}{|x - y|^{1+s}} dx dy$, which is a priori singular. Consider for $\varepsilon > 0$ the well-defined integrals :

$$\begin{aligned} I_\varepsilon^+ &= \iint_{x>y} \frac{[u(x) - u(y)]_+}{\varepsilon + |x - y|^{1+s}} dx dy \\ I_\varepsilon^- &= \iint_{x>y} \frac{[u(x) - u(y)]_-}{\varepsilon + |x - y|^{1+s}} dx dy \end{aligned} ,$$

$$I_\varepsilon = \iint_{x>y} \frac{|u(x) - u(y)|}{\varepsilon + |x - y|^{1+s}} dx dy = I_\varepsilon^+ + I_\varepsilon^-$$

$$J_\varepsilon = \iint_{x>y} \frac{u(x) - u(y)}{\varepsilon + |x - y|^{1+s}} dx dy = I_\varepsilon^+ - I_\varepsilon^-.$$

Since

$$\begin{aligned} J_\varepsilon &= \int u(x) \int_{x>y} \frac{1}{\varepsilon + |x - y|^{1+s}} dy dx - \int u(y) \int_{x>y} \frac{1}{\varepsilon + |x - y|^{1+s}} dx dy \\ &= \int u(x) \int_{x>y} \frac{1}{\varepsilon + |x - y|^{1+s}} dy dx - \int u(x) \int_{x<y} \frac{1}{\varepsilon + |x - y|^{1+s}} dy dx, \\ &= \int u(x) \int_{h>0} \frac{1}{\varepsilon + |h|^{1+s}} dh dx - \int u(x) \int_{h<0} \frac{1}{\varepsilon + |h|^{1+s}} dh dx \\ &= 0 \end{aligned}$$

it follows that $I_\varepsilon^+ = I_\varepsilon^-$ and then $I_\varepsilon = 2I_\varepsilon^+ \leq 2I_0^+ < +\infty$. The monotone convergence theorem yields $\lim_{\varepsilon \rightarrow 0} I_\varepsilon = I = 2I_0^+$, so that $u \in W^{s,1}(\mathbb{R})$.

4.3 Optimal smoothing effect in fractional Sobolev spaces

Thanks to Proposition (1), it suffices to get a $W_+^{\sigma,1}$ estimate to have the $W^{\sigma,1}$ regularity $\forall \sigma < s$. The information is propagated with a finite speed for conservation laws so we work only locally with local Sobolev semi-norm as in [Ju].

The well known embedding from Holder spaces to Sobolev spaces: $W_{loc}^{s,\infty} \subset W_{loc}^{\sigma,1}, \forall \sigma < s$, is simply extended to $W_{+,loc}^{s,\infty}$:

Lemma 2. $W_{+,loc}^{s,\infty} \subset W_{+,loc}^{\sigma,1}, \forall \sigma < s$,

Proof. Assume that $(u(x+h) - u(x))_+ \leq Ch^s$ for $h > 0$.

Let h_0 and A be some positive constants to estimate the $W_{+,loc}^{\sigma,1}$ semi-norm on the interval $]-A, A+h_0[$:

$$|u|_{\sigma,loc}^+ = \int_{|x|<A} \int_{h_0>h>0} \frac{[u(x+h) - u(x)]_+}{h^{1+\sigma}} dx dh \leq \int_{|x|<A} \int_{h_0>h>0} Ch^{s-\sigma-1} dx dh$$

$$|u|_{\sigma,loc}^+ \leq 2AC \int_{h_0>h>0} h^{s-\sigma-1} dh$$

which is finite if and only if $\sigma < s$.

Finally, the one-sided Holder condition for an entropy solution is already given by Inequality (5) and the Sobolev regularity follows since an entropy solution is bounded in L^∞ as its initial data and $W_{+,loc}^{\sigma,1} \cap L^\infty \subset W_{loc}^{\sigma,1}$.

Acknowledgements We thanks the support of the project SlowDyn (team leader Bruno Lombard): an interdisciplinary CNRS project based on the LMA (CNRS, UPR 7051, Marseille). P-E

Jabin acknowledges the support of NSF Grants DMS 1312142 and 1614537 and by NSF Grant RNMS (Ki-Net) 1107444.

References

- [AGV1] Adimurthi, S. S. Ghoshal, G. D. Veerappa Gowda. *Structure of entropy solutions to scalar conservation laws with strictly convex flux*. J. Hyperbolic Differ. Equ. 9 (2012), no. 4, 571611.
- [AGV2] Adimurthi, S. S. Ghoshal, G. D. Veerappa Gowda. *Finer regularity of an entropy solution for 1-d scalar conservation laws with non uniform convex flux*. Rend. Semin. Mat. Univ. Padova 132 (2014), 124.
- [AD] L. Ambrosio, C. De Lellis. *A note on admissible solutions of 1D scalar conservation laws and 2D Hamilton-Jacobi equations*. J. Hyperbolic Differ. Equ. 1 (2004), no. 4, 813826.
- [B] S. Bianchini. *SBV regularity for scalar conservation laws*. Hyperbolic problemstheory, numerics and applications. Volume 1, 311, Ser. Contemp. Appl. Math. CAM, 17, World Sci. Publishing, Singapore, 2012.
- [BM] S. Bianchini, E. Marconi. *On the structure of L^∞ -entropy solutions to scalar conservation laws in one-space dimension*. Preprint SISSA 43/2016/MATE.
- [BGJ] C. Bourdarias, M. Gisclon, S. Junca. *Fractional BV spaces and applications to scalar conservation laws*. J. Hyperbolic Differential Equations, 11, 4, (2014), 655-677.
- [BGJP] C. Bourdarias, M. Gisclon, S. Junca, Y.-J Peng. *Eulerian and Lagrangian formulations in BV^s for gas-solid chromatography*. Commun. Math. Sci. 14 (2016), no. 6, 1665–1685.
- [CJ1] P. Castelli, S. Junca. *Oscillating waves and the maximal smoothing effect for one-dimensional nonlinear conservation laws*. AIMS Ser. on Appl. Math. 8, (2014), 709-716.
- [CJ2] P. Castelli, S. Junca. *On the maximal smoothing effect for multidimensional scalar conservation laws*. Preprint: hal-01290871, (2016), 14 pp.
- [CJ3] P. Castelli, S. Junca. *Smoothing effect in $BV - \Phi$ for scalar conservation laws*. Preprint: hal-01133725, version 2, (2016), 30 pp.
- [Ch1] K. S. Cheng. *The space BV is not enough for hyperbolic conservation laws*. J. Math. Anal. App., 91, 2, (1983), 559-561.
- [Ch2] K. S. Cheng. *A regularity theorem for a nonconvex scalar conservation law*. J. Differential Equations 61, no. 1, (1986), 79–127.
- [Cc] C. Cheverry. *Regularizing effects for multidimensional scalar conservation laws*. Ann. Inst. H. Poincaré Anal. Non Linéaire 17 (2000), no. 4, 413472.
- [D] C. Dafermos. *Regularity and large time behavior of solutions of a conservation law without convexity*. Proc. Royal Soc. Edinburgh 99 A, (1985), 201-239.
- [DOW] C. De Lellis, F. Otto, M. Westdickenberg. *Structure of entropy solutions for multidimensional scalar conservation laws*. Arch. Ration. Mech. Anal. 170 (2003), no. 2, 137-184.
- [DR] C. De Lellis, T. Riviere. *The rectifiability of entropy measures in one space dimension*. J. Math. Pures Appl. (9) 82 (2003), no. 10, 1343-1367.
- [DW] C. De Lellis, M. Westdickenberg. *On the optimality of velocity averaging lemmas*. Ann. Inst. H. Poincaré, Anal. Nonlinéaire 20 (2003), no. 6, 1075-1085.
- [E] Laurence C. Evans. *Partial differential equations*. American Mathematical Society, (1998).
- [Ja] P.-E. Jabin. *Some regularizing methods for transport equations and the regularity of solutions to scalar conservation laws*. [2008-2009], Exp. No. XVI, Smin. Equ. Driv. Partielles, Ecole Polytech., Palaiseau, (2010).
- [Ju] S. Junca. *High frequency waves and the maximal smoothing effect for nonlinear scalar conservation laws*. SIAM Math. Anal. 46 (2014), no. 3, 2160-2184.
- [La] P. D. Lax. *Hyperbolic systems of conservation laws. II*. Comm. Pure Appl. Math. (1957), 10, 537-566.
- [LPT] P.-L. Lions, B. Perthame, E. Tadmor. *A kinetic formulation of multidimensional scalar conservation laws and related equations*. J. Amer. Math. Soc. 7, (1994), 169-192.

- [O] O. Oleinik. *Discontinuous solutions of nonlinear differential equations*. Usp. Mat. Nauk., 12, (1957), 3-73. English transl. in Am. Math. Soc. Transl. Ser. 2 (1963), no. 26, 95-172.
- [P1] E.Yu. Panov. *Existence of strong traces for generalized solutions of multidimensional scalar conservation laws*. J. Hyperbolic Differ. Equ. 2 (2005), no. 4, 885908.
- [P2] E.Yu. Panov. *Existence of strong traces for quasi-solutions of multidimensional conservation laws*. J. Hyperbolic Differ. Equ. 4 (2007), no. 4, 729770.
- [MO] J. Musielak, W. Orlicz. *On generalized variations*. I. Studia Math. 18 (1959), 11-41.