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ON JOURDAIN'S PRINCIPLE

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Abstract—This paper reexamines the effect of the validity, or the lack thereof, of the commutation rule for kinematically possible, or admissible, and virtual variations, i.e. $d\delta(\cdots)$ vs $\delta d(\cdots)$, on the form of Jourdain's principle (JP), in general quasivariables. *It is shown that JP is independent of any such rule.* A similar independence holds in the derivation of the equations of motion from Lagrange's principle.

1. INTRODUCTION

In recent years the *differential* variational principle of Jourdain (JP), originally formulated by him in 1909 [1,2] as an “intermediate” principle between those of Lagrange and Gauss/Gibbs, and for long regarded by some as nothing more than an academic curiosity, has reemerged as a very versatile and powerful tool for the treatment of both finite and impulsive motion of constrained mechanical systems [3–8]. Unfortunately, and inexplicably, this principle seems to be very little known among English-speaking engineers; its brief statement in Kane and Levinson [9] does not correspond to the classical principle as encountered in the literature.

The formulation of JP assumes, explicitly or implicitly, the satisfaction of the *commutation* rule

$$d(\delta \mathbf{r}) - \delta(d\mathbf{r}) = \mathbf{0}, \quad (1a)$$

or

$$\frac{d}{dt}(\delta \mathbf{r}) = \delta\left(\frac{d\mathbf{r}}{dt}\right) \equiv \delta \mathbf{v}, \quad (1b)$$

where $\mathbf{r}$ = inertial *position* vector of a typical particle P , of the system under discussion S , $\mathbf{v}$ = inertial *velocity* of P , $d(\cdots)$ and $\delta(\cdots)$ = *kinematically possible* and *virtual* inertial differentials of $(\cdots)$. One solidly gets the impression that equations (1) are necessary not only for JP, but for the subsequent derivation of the correct equations of motion. This, however, is in sharp contrast with the corresponding *Lagrange's Principle* (LP). In the latter, as Hamel [10,11] *et al.*, have clearly demonstrated, the derivation of the equations of motion from LP and from its derivative *Central Equation* (CE—the “Zentralgleichung” of Heun and Hamel, which is a slight transformation of LP to *first-order* differential expressions), is *completely independent of any particular assumptions about*

$$d(\delta \mathbf{r}) - \delta(d\mathbf{r}) \quad \text{or} \quad (d\mathbf{r})' - \delta \mathbf{v}. \quad (2)$$

The following question then naturally arises: is JP also independent of commutation rules, like (1)? The purpose of this paper is to reexamine the theoretical foundations of JP and establish the necessity, or lack thereof, of the commutation rule (1). Section 2 summarizes the theory of LP and JP, while Section 3 investigates in detail the Jourdain form of the commutation rule. The results are summarized in the concluding Section 4.

2. THE PRINCIPLES OF LAGRANGE AND JOURDAIN

Consider a material system S subject to constraints that are bilateral, ideal, and *linear* in the (particle and/or system) velocities; these constraints may be genuinely *nonintegrable* (*nonholonomic*), or they may be *integrable* (*holonomic*) disguised in kinematical, or

differential, or velocity form. A typical particle P , of S , obeys Newton's "second law"

$$d\mathbf{m}\mathbf{a} = d\mathbf{F} + d\mathbf{R}, \quad (3)$$

where dm = mass of P , $\mathbf{a}$ = inertial acceleration of P , i.e. $d^2\mathbf{r}/dt^2$, $\mathbf{r}$ = inertial position vector and t = time; $d\mathbf{F}$ = total *impressed* force on P , and $d\mathbf{R}$ = total *constraint reaction* force on P . The derivation of *reactionless* equations of motion is based on *Lagrange's Principle (LP)*:

$$\delta' W^{(r)} \equiv \mathbf{S}(d\mathbf{R} \cdot \delta\mathbf{r}) = 0, \quad (4)$$

which, due to (3), takes the customary Lagrangean form:

$$\mathbf{S}[(d\mathbf{m}\mathbf{a} - d\mathbf{F}) \cdot \delta\mathbf{r}] = 0, \quad (5)$$

where $\delta\mathbf{r}$ represents the (inertial) *virtual* displacement of P (i.e. under $dt \rightarrow \delta t = 0$), and

$$\mathbf{S}(\cdot \cdot \cdot)$$

denotes *material* summation over all the elements (particles) of S (as in Stieltjes' integral). For details see Hamel [11, pp. 361–363].

Next taking the total differential of both sides of (5), with respect to time, as that equation holds for some time interval, and since

$$\mathbf{S}(\cdot \cdot \cdot) \quad \text{and} \quad \frac{d}{dt}(\cdot \cdot \cdot) \equiv (\cdot \cdot \cdot)' \quad \text{commute,}$$

we find:

$$\mathbf{S}\left[\frac{d}{dt}(d\mathbf{m}\mathbf{a} - d\mathbf{F}) \cdot \delta\mathbf{r}\right] + \mathbf{S}\left[(d\mathbf{m}\mathbf{a} - d\mathbf{F}) \cdot \frac{d}{dt}(\delta\mathbf{r})\right] = 0. \quad (6)$$

This leads to the following *differential* variational principle: the reactionless (or purely kinetic) equations of motion of S follow from the differential variational equation

$$\mathbf{S}[(d\mathbf{m}\mathbf{a} - d\mathbf{F}) \cdot (\delta\mathbf{r})'] = 0, \quad (7)$$

under the (fixed time *and* position) constraints

$$\delta t = 0 \quad (\text{as in LP}), \quad \text{and} \quad \delta\mathbf{r} = \mathbf{0}. \quad (8)$$

On the other hand, Jourdain's principle (JP), in its *original* enunciation by Jourdain [1], states that the (same) equations of motion of S follow from the variational principle

$$\left. \begin{aligned} \mathbf{S}[(d\mathbf{m}\mathbf{a} - d\mathbf{F}) \cdot \delta\mathbf{v}] &= 0, \\ \mathbf{v} \equiv d\mathbf{r}/dt &\equiv \dot{\mathbf{r}} \quad (= \text{inertial velocity of } P), \end{aligned} \right\} \quad (9)$$

under the constraints (8).

Clearly *if we assume* the commutation rule

$$\left. \begin{aligned} d(\delta\mathbf{r}) &= \delta(d\mathbf{r}), \\ \frac{d}{dt}(\delta\mathbf{r}) &= \delta\left(\frac{d\mathbf{r}}{dt}\right) \equiv \delta\mathbf{v}, \end{aligned} \right\} \quad (10)$$

then the principles (7) and (9) *coincide*, under the set of assumptions (8). The question is: can this equivalence happen under all circumstances, i.e. *without any special assumptions like* (10), that is, even if

$$\delta^*\mathbf{r} \equiv (\delta\mathbf{r})' - \delta(\dot{\mathbf{r}}) \neq \mathbf{0}.$$

The answer to this requires utilization of the *most general* expression for $\delta^*\mathbf{r}$. We now turn our attention to this problem.

3. THE TRANSITIVITY (OR COMMUTATIVITY) RELATIONS

In terms of the fundamental, generally nonholonomic, particle and system *basis* vectors $\{\mathbf{a}_i, i = m + 1, \dots, n; m = \text{number of velocity constraints}, n = \text{number of (initially independent) coordinates } q\text{—see e.g. [10–12]}\}$, the *particle* virtual displacement $\delta \mathbf{r}$ is expressed as the following linear combination:

$$\left. \begin{aligned} \delta \mathbf{r} &= \sum_{l=1}^n \mathbf{e}_l \delta q_l = \sum_i \mathbf{a}_i \delta \theta_i \quad \{i = m + 1, \dots, n; \mathbf{a}_i = \mathbf{a}_i(t, q)\}, \\ \text{where} \\ \delta \theta &= \{\delta \theta_1 = 0, \dots, \delta \theta_m = 0; \delta \theta_{m+1}, \dots, \delta \theta_n\} = \text{virtual independent system quasicoordinate variations,} \\ \mathbf{e}_l &\equiv \partial \mathbf{r} / \partial q_l \quad (l = 1, \dots, n), \quad \delta \theta_i = \sum_{l=1}^n a_{il} \delta q_l, \\ \omega_i &\equiv d\theta_i / dt = \text{independent system quasivelocities,} \\ \omega_i &= \sum_{l=1}^n a_{il} \dot{q}_l + a_i \quad (a_{il}, a_i: \text{known functions of } q, t). \end{aligned} \right\} \quad (11)$$

The first of (11) may also be viewed as the *definition* of the $\{\mathbf{a}_i\}$.

Since the $(n - m)$ vectors $\{\mathbf{a}_i\}$ are independent, the (second) Jourdain constraint (8) yields

$$\delta \mathbf{r} = \mathbf{0} \rightarrow \delta \theta_i = 0. \quad (12)$$

Differentiating the first of (11) in time, *while utilizing* (12), results in

$$\begin{aligned} (\delta \mathbf{r})' &= \sum_i (\dot{\mathbf{a}}_i \delta \theta_i + \mathbf{a}_i (\delta \theta_i)') \\ &= \sum_i \mathbf{a}_i (\delta \theta_i)' = \sum_i \left(\frac{\partial \mathbf{v}}{\partial \omega_i} \right) (\delta \theta_i)'. \end{aligned} \quad (13)$$

If we are to obtain something *nontrivial* from (6–8), then we should *not* assume that from (12) it follows that $(\delta \mathbf{r})' = \mathbf{0}$, and/or $(\delta \theta_i)' = 0$.

Let's now take for algebraic simplicity, but without loss of generality, the case of *scleronomic* systems, i.e. stationary constraints. Then a_i [last of (11)] $\rightarrow 0$, and

$$\mathbf{v} = \sum_l \mathbf{e}_l \dot{q}_l = \sum_i \mathbf{a}_i \omega_i. \quad (14)$$

Varying this à la Jourdain, *from now on to be denoted by* $\delta'(\dots)$, yields

$$\begin{aligned} \delta' \mathbf{v} &= \sum_i ((\delta' \mathbf{a}_i) \omega_i + \mathbf{a}_i (\delta' \omega_i)) = \sum_i \mathbf{a}_i (\delta' \omega_i) \\ &[\text{since for linear velocity constraints } \mathbf{a}_i = \mathbf{a}_i(t, q) \rightarrow \delta' \mathbf{a}_i = \mathbf{0}]. \end{aligned} \quad (15)$$

Therefore, from (13), (15), we obtain

$$(\delta \mathbf{r})' - \delta' \mathbf{v} = \sum_i \mathbf{a}_i \{(\delta \theta_i)' - \delta' \omega_i\}, \left[= \sum_l \mathbf{e}_l \{(\delta q_l)' - \delta \dot{q}_l\} \right],$$

or

$$(\delta \mathbf{r})' = \delta' \mathbf{v} + \sum_i \mathbf{a}_i \delta^* \theta_i, \quad (16)$$

$$\delta^* \theta_i \equiv \frac{d}{dt} (\delta \theta_i) - \delta' \omega_i \neq 0, \quad (17)$$

despite (12).

REMARK: Equation (16), or

$$\delta^* \mathbf{r} = \sum_i \mathbf{a}_i \delta^* \theta_i,$$

simply states that $\delta^* \mathbf{r} \equiv (\delta \mathbf{r})' - \delta'(\mathbf{r}) = (\delta \dot{\mathbf{r}})' - \delta' \dot{\mathbf{v}}$ lies in the *virtual plane* of that particle at the “point” (q, t) . Inserting now (16) into $d/dt(\text{LP}) \rightarrow$ equation (7), yields

$$\mathbf{S}[(d\mathbf{m}\mathbf{a} - d\mathbf{F}) \cdot \delta' \mathbf{v}] + \mathbf{S}\left[(d\mathbf{m}\mathbf{a} - d\mathbf{F}) \cdot \left(\sum_i \mathbf{a}_i (\delta^* \theta_i)\right)\right] = 0,$$

or

$$\mathbf{S}[(d\mathbf{m}\mathbf{a} - d\mathbf{F}) \cdot \delta' \mathbf{v}] + \sum_i \{\mathbf{S}[(d\mathbf{m}\mathbf{a} - d\mathbf{F}) \cdot \mathbf{a}_i]\} \delta^* \theta_i = 0, \quad (18)$$

from which—and this is the key step in the entire discussion—due to LP in quasivariabes (see e.g. [11,12]), i.e.

$$\mathbf{S}[(d\mathbf{m}\mathbf{a} - d\mathbf{F}) \cdot \mathbf{a}_i] = 0, \quad (19)$$

we obtain

$$\mathbf{S}[(d\mathbf{m}\mathbf{a} - d\mathbf{F}) \cdot \delta' \mathbf{v}] = 0, \quad (20)$$

under

$$\delta t = 0, \quad \text{and} \quad \delta \mathbf{r} = \mathbf{0},$$

but *not* $(\delta \mathbf{r})' = \mathbf{0}$; i.e. the original Jourdain formulation. In short, the derivation of JP from LP does *not* depend on the equality of $d(\delta \mathbf{r})$ with $\delta(d\mathbf{r})$; or, condition (10) is sufficient but not necessary for it.

Finally, substituting (15) into (20), and since the $(n - m)\delta' \omega_i$ are independent, reproduces the “raw” form of the fundamental kinetic (reactionless) equations (19):

$$\mathbf{S}(d\mathbf{m}\mathbf{a} \cdot \mathbf{a}_i) = \mathbf{S}(d\mathbf{F} \cdot \mathbf{a}_i). \quad (21)$$

Further transformation of (19,21) to *system* variables leads to such equations of motion as those by Appell, Maggi, and Boltzmann-Hamel (see e.g. [13]).

Remarks

(1) Equation (16) can also result from the general *transitivity* equations specialized for the Jourdain variation (8,12). The starting point is

$$(\delta \theta_k)' - \delta \omega_k = \sum_l a_{kl} \{(\delta q_l)' - \delta \dot{q}_l\} + \sum_{l,s} \gamma_{ls}^k \omega_s \delta \theta_l, \quad (22)$$

where the *three-index* (Hamel) symbols γ_{ls}^k are defined by (22); for notation and relevant theory see, e.g. [11, pp. 243, 475–477, equations (15a); 14].

Inverting (22) we obtain

$$(\delta q_k)' - \delta \dot{q}_k = \sum_l A_{kl} \{(\delta \theta_l)' - \delta \omega_l\} - \sum_{l,r,s} A_{kl} \{\gamma_{rs}^l \omega_s \delta \theta_r\}, \quad (23)$$

where $\{A_{kl}\}$ and $\{a_{kl}\}$ are inverse matrices.

Therefore, we have, successively,

$$\begin{aligned} (\delta \mathbf{r})' - \delta \dot{\mathbf{v}} &= \sum_k \mathbf{e}_k \{(\delta q_k)' - \delta \dot{q}_k\} \\ &= \sum_k \mathbf{e}_k \left\{ \sum_l A_{kl} [(\delta \theta_l)' - \delta \omega_l] - \sum_{l,r,s} A_{kl} \gamma_{rs}^l \omega_s \delta \theta_r \right\} \end{aligned} \quad (24)$$

$$\begin{aligned} &= \sum_{k,l} (\mathbf{e}_k A_{kl}) \delta^* \theta_l - \sum_{k,l,r,s} (\mathbf{e}_k A_{kl}) \gamma_{rs}^l \omega_s \delta \theta_r \\ &= \sum_l \mathbf{a}_l \delta^* \theta_l - \sum_{l,r,s} \mathbf{a}_l \gamma_{rs}^l \omega_s \delta \theta_r \quad \left(\text{since: } \mathbf{a}_l = \sum_k A_{kl} \mathbf{e}_k \right). \end{aligned} \quad (25)$$

But in a constrained system

$$\delta\theta_1, \dots, \delta\theta_m = 0, \quad (26)$$

while from the Jourdain requirement (8,12)

$$\delta\theta_{m+1}, \dots, \delta\theta_n = 0; \quad (27)$$

i.e. $\{\delta\theta_r; r = 1, \dots, n \text{ (}\& n + 1\text{)}\} = 0$, and (25) reduces here to

$$\delta^*\mathbf{r} = \sum_l \mathbf{a}_l \delta^*\theta_l \rightarrow \sum_{i=m+1}^n \mathbf{a}_i \delta^*\theta_i, \quad (28)$$

as before.

(2) If we had assumed that $\delta^*\mathbf{r} = \mathbf{0}$, i.e.

$$d(\delta\mathbf{r}) = \delta(d\mathbf{r}),$$

then (28) and (23), under the Jourdain constraints, would have led us to

$$\delta^*\theta_i = 0, \quad \text{or} \quad d(\delta\theta_i) = \delta(d\theta_i), \quad (29)$$

and

$$\delta^*q_k = \sum_i A_{ki} \delta^*\theta_i = 0,$$

or

$$d(\delta q_i) = \delta(dq_i),$$

and vice versa.

This is something that holds here *due to the Jourdain variation*.

In general, as (22,23) show, without the Jourdain constraints *either* $\delta^*q_i = 0$ or $\delta^*\theta_i = 0$ holds true; but not both.

(3) One can easily envision similar extensions of the above reasoning to the *higher-order* differential variational principles of *Gauss*, and *Mangeron-Deleanu*. For their basic theory see, e.g. [13, pp. 237–258].

4. CONCLUSIONS

We have demonstrated that Jourdain's Principle results naturally from the (total) time differentiation of Lagrange's Principle, and that like it, produces the correct equations of motion independently of any commutation assumptions regarding the $d(\dots)$ and $\delta(\dots)$ variations.

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